# Quantum error correction and black hole interior in a gravitating bath system

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Recent Developments in
Black Holes and Quantum Gravity

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Based on joint work (to appear) with Tomonori Ugajin (Rikkyo university)

# Problem we address in this talk

- From the island formula, we believe that, after the Page time, information inside the horizon is complicatedly encoded into the Hawking radiation.

- We can not easily change the interior information from operations on the Hawking radiation.

→Information inside the horizon is protected against some operations (errors) on the Hawking radiation.


But, to what extent is the interior information protected against errors in the Hawking radiation?

Also, if we consider gravitational backreaction from the error, what happens?

# Outline of this talk

1. Introduction
    - Brief review on black hole, Hawking radiation, island formula, etc.
    - Brief review on QEC
    - QEC in an evaporating black hole with a non-gravitating bath
    - PSSY model
2. QEC in an evaporating black hole with a gravitating bath (Without backreaction)
3. QEC in an evaporating black hole with a gravitating bath (With backreaction)
4. Summary and future works

# 1. Introduction

# Entropy of Hawking radiation and Island

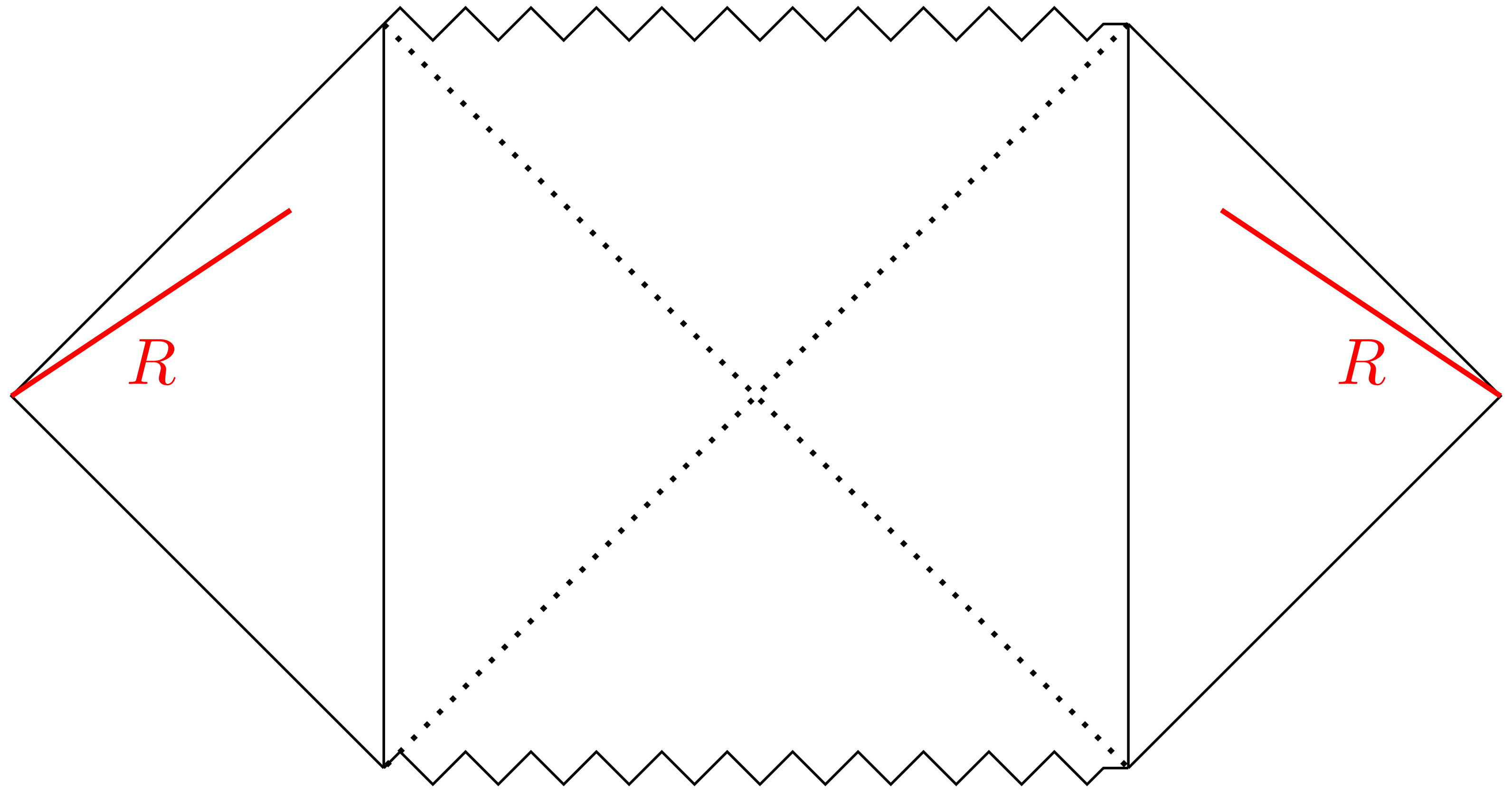
- In quantum gravity with unitarity, the entanglement entropy of Hawking radiation is expected to obey the Page curve.

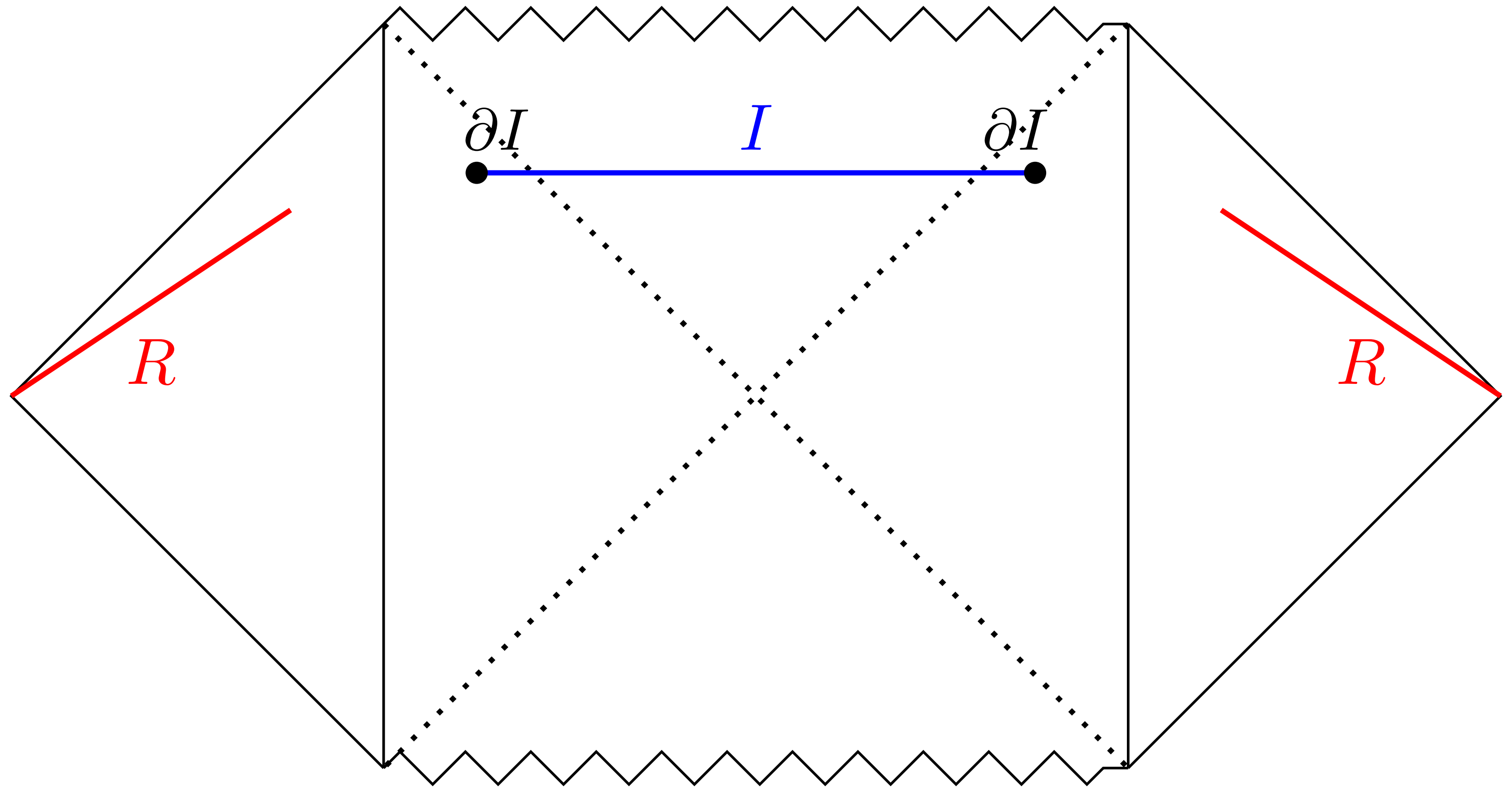
- The island formula [Penington '19, Almheiri-Engelhardt-Marolf-Maxfield '19, ⋯] compute a *correct* entropy of the Hawking radiation obeying the Page curve;

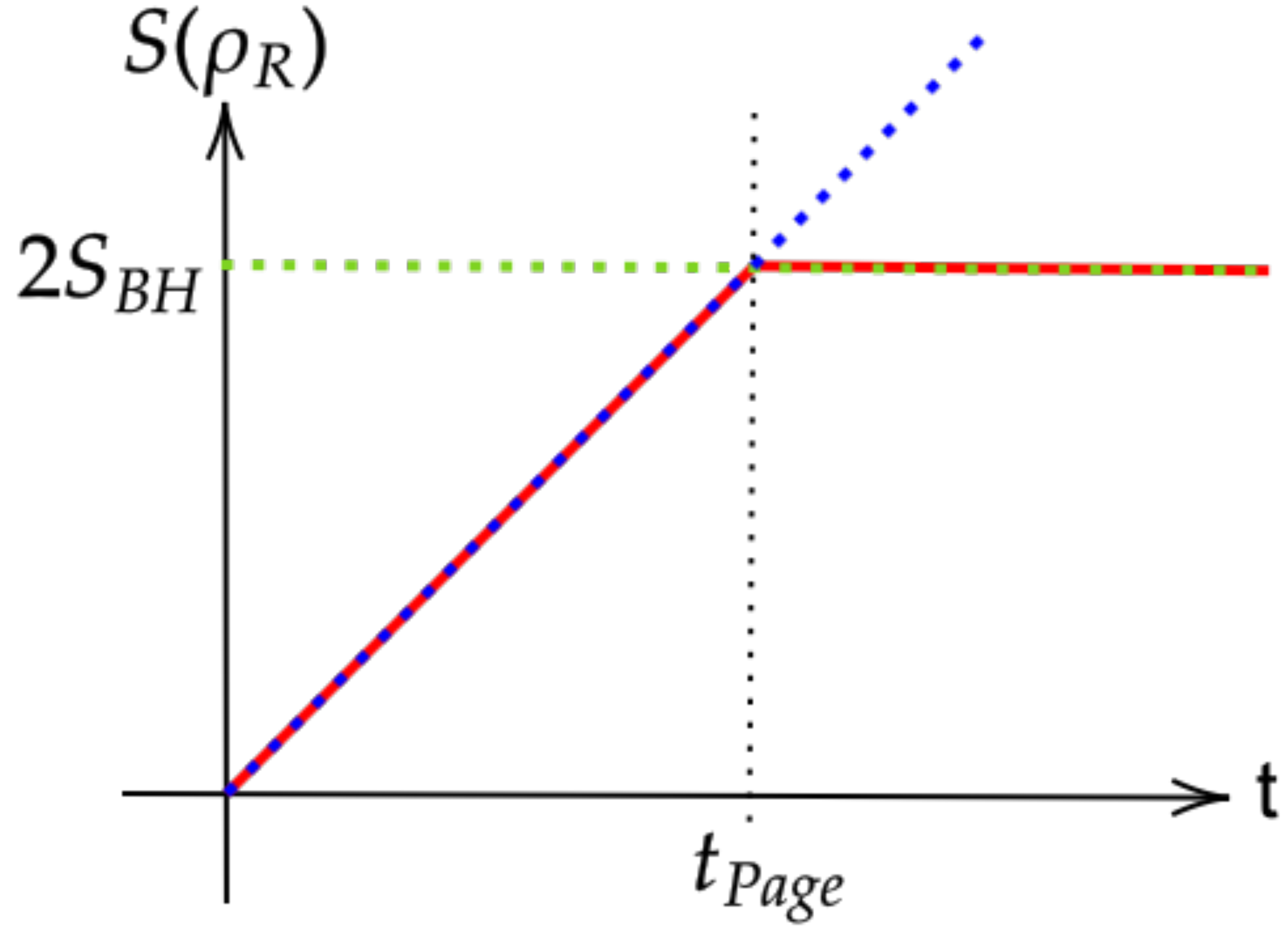
$$S_{\text{Island}}(R) = \min_I \left\{ \text{ext}_I \left[ \frac{\text{Area}[\partial I]}{4G_N} + S_{\text{QFT}}[R \cup I] \right] \right\} \qquad (I: \text{ island})$$

- Before the Page time, the black hole interior ⊂ EW[Black hole] → Linear growth

- After the Page time, the black hole interior ⊂ EW[HR] →Bekenstein-Hawking entropy



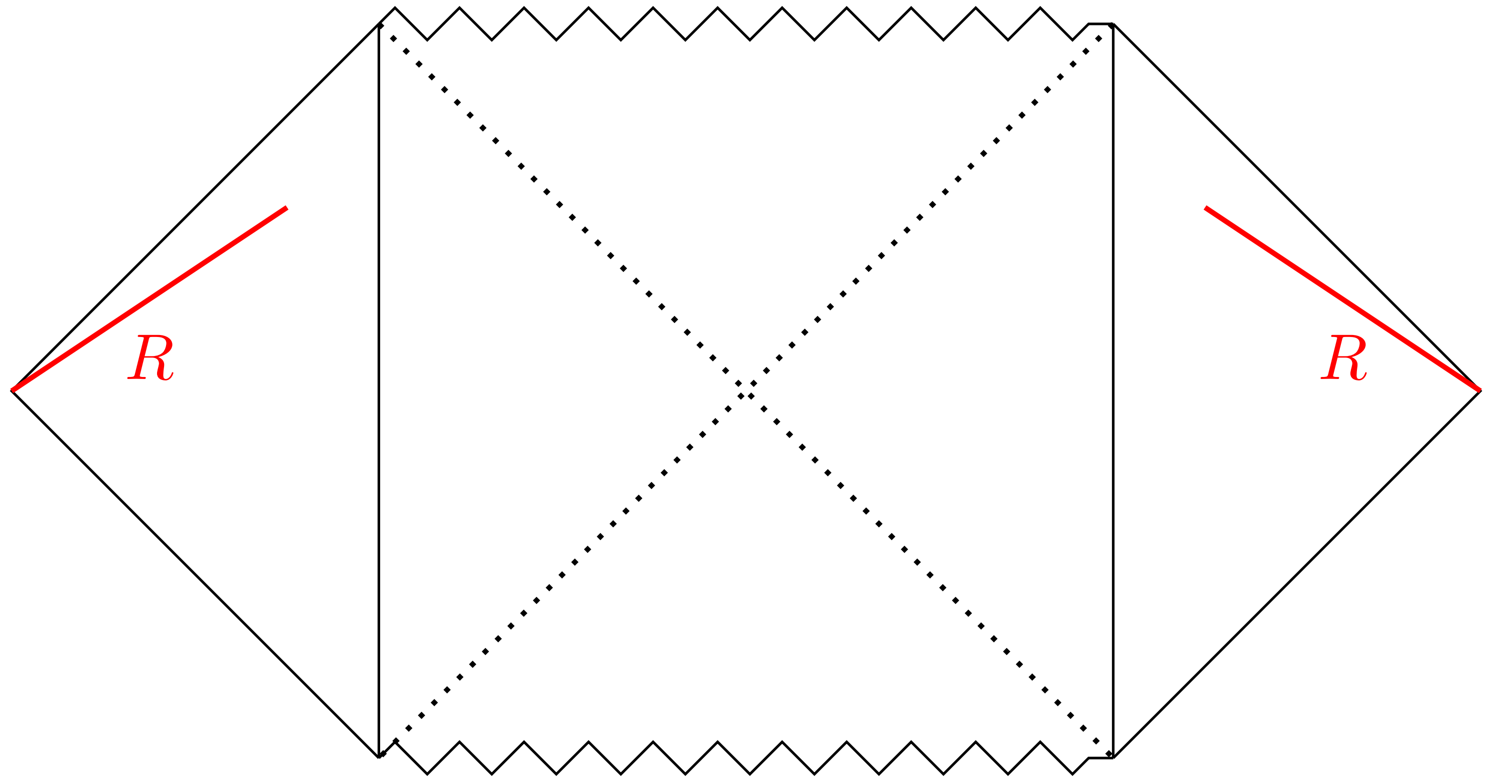



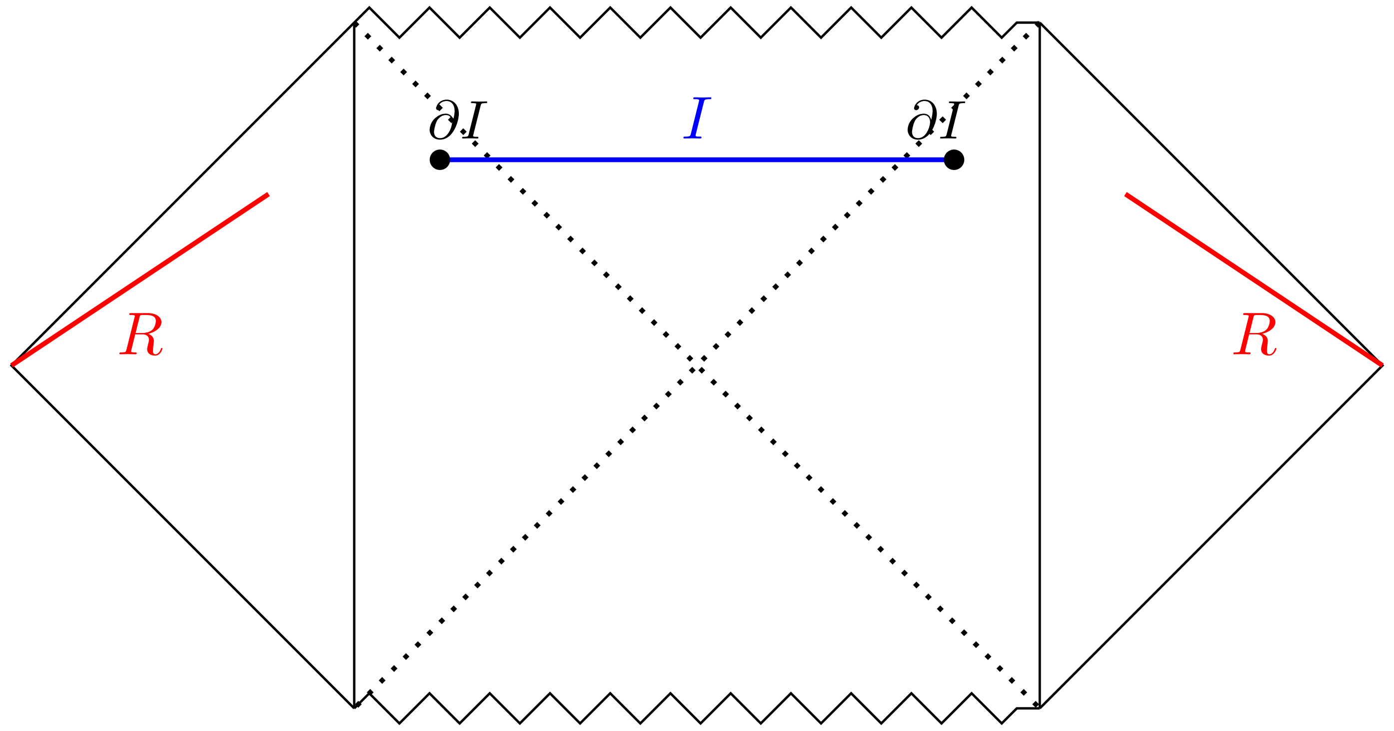
# Entanglement wedge reconstruction

- The entanglement wedge reconstruction [Dong-Harlow-Wall '16, $\cdots$] states that information on an entanglement wedge of some boundary region $A$, EW[$A$], can be reconstructed from the boundary region $A$.

- This implies that, after the Page time, information on the black hole interior, $I$, can be reconstructed from the Hawking radiation $R \rightarrow$ the interior information is encoded into the Hawking radiation.

(Before the Page time, the information is encoded into the complement region $BH = \overline{R}$)



# Robust encoding property

- The encoding of the black hole interior information has the quantum error correction (QEC) property like the usual AdS/CFT case.

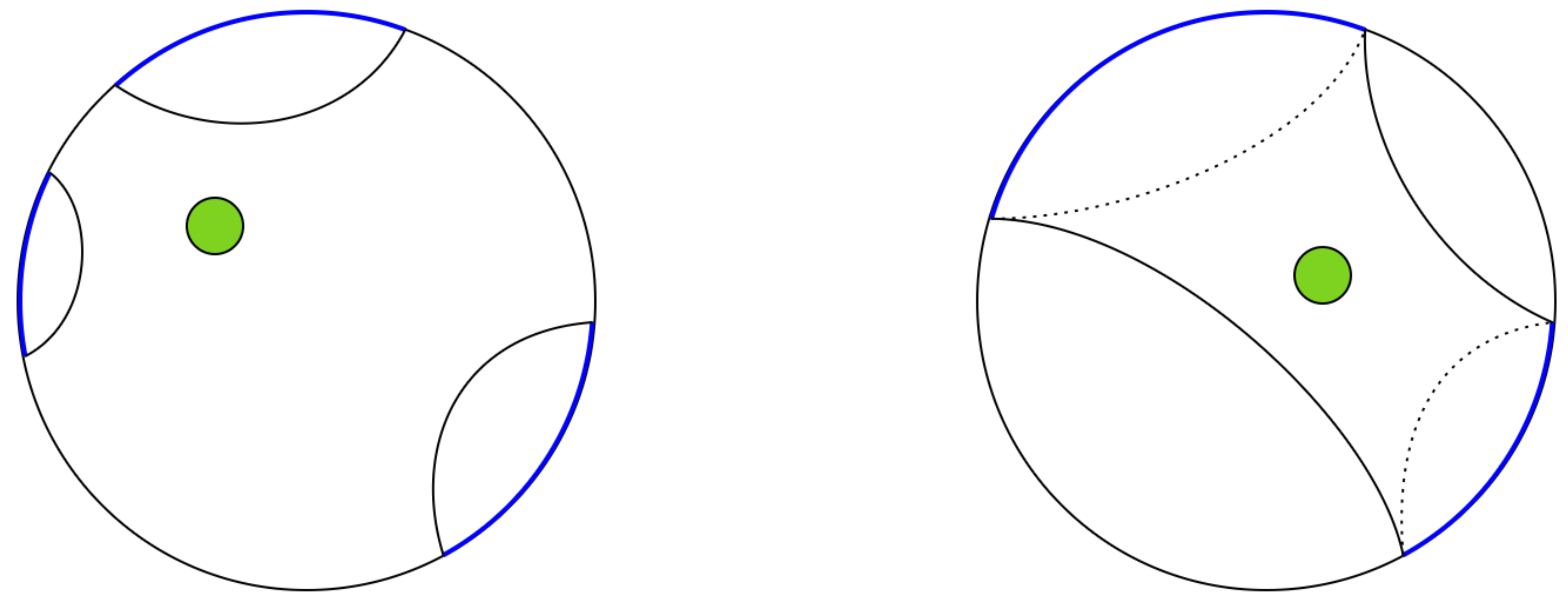
- QEC in AdS/CFT case [Almheiri-Dong-Harlow ’14, ⋯]

  - AdS deep interior regions are robust against erasure (errors) of small boundary sub-regions.

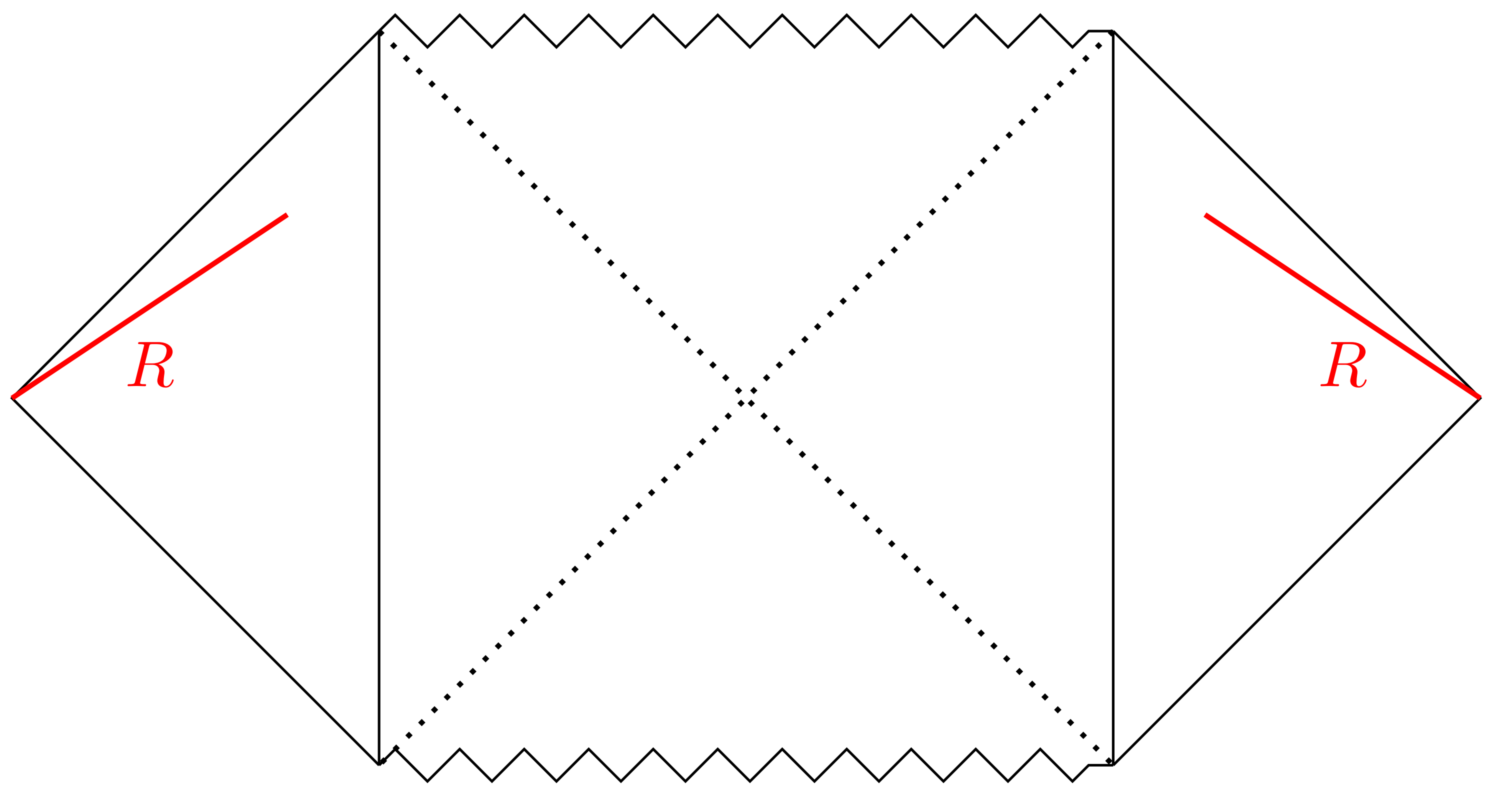
    = Information on the deep interior regions are protected against such erasures.

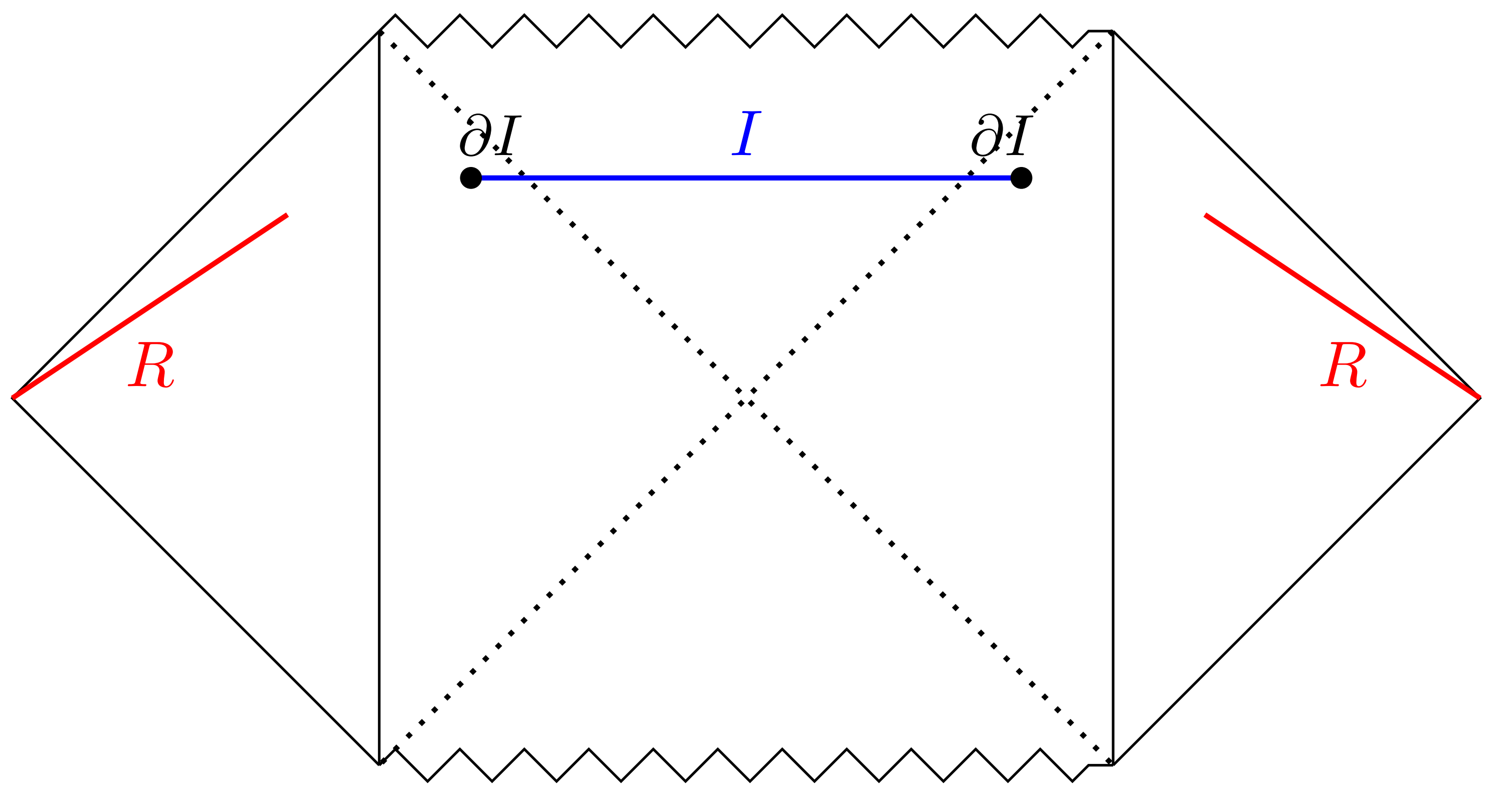
  - Relatedly, such AdS deep interior regions are robust agains local operations associated with such small boundary subregions (~”simple operations”). To affect them, we need to consider “complex operations”.

    = Information on the deep interior regions are protected against simple errors (operations).



- Similar to the AdS/CFT case, after the Page time, the interior of the black hole is complicatedly encoded into the Hawking radiation. As a result, it is not easy to access the interior information from the Hawking radiation, implying that it is robust against an error acting on the Hawking radiation.




- But, how complicated is it? How robust the encoding is?

→Focus on questions about these quantum error correction properties

# Quantum error correction

- To explain QEC more systematically, we need some preparations.
  - We focus on a CPTP error (quantum channel) $\mathscr{E}$: map from density matrix to density matrix having the properties;

    1. Completely-positive (CP);
       $\mathscr{D}(H) \ni \rho > 0 \rightarrow \mathscr{E}[\rho] > 0, \mathscr{E}[\rho] \in \mathscr{D}(H)$

    2. Trace-preserving (TP); $\mathrm{tr}[\rho] = 1 \rightarrow \mathrm{tr}[\mathscr{E}[\rho]] = 1$

- A CPTP error has the Kraus representation; $\mathscr{E}[\rho] = \sum_m E_m \rho E_m^\dagger$

  $E_m$ : Kraus (error) operator obeying the condition $\sum_m E_m^\dagger E_m = I$

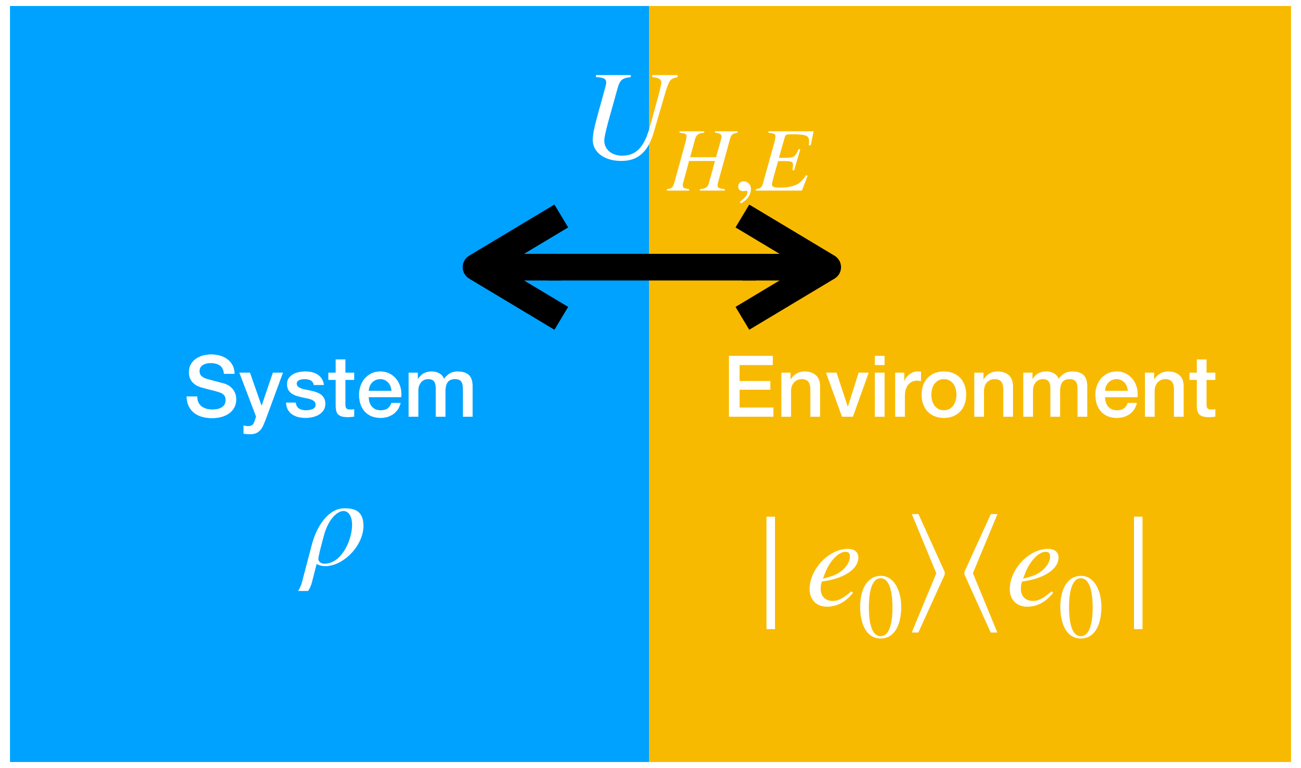
- A CPTP error has the Stinespring dilation; $\exists \, |e_0\rangle_E$ : initial state of a environment system, $\exists \, U_{H,E}$ : unitary acting on the original system $H$ and the environment system $E$ such that $\mathscr{E}[\rho] = \mathrm{tr}_E \left[ U_{H,E} \left( \rho \otimes |e_0\rangle_E \langle e_0| \right) U_{H,E}^\dagger \right]$.

- By taking a orthonormal basis of the environment system, $\{|e_m\rangle_E\}_{m=1}^{d_E}$, the two representations are related by

$$\mathscr{E}[\rho] = \mathrm{tr}_E \left[ U_{H,E} \left( \rho \otimes |e_0\rangle_E \langle e_0| \right) U_{H,E}^\dagger \right] = \sum_{m=1}^{d_E} {}_E\langle e_m| \, U_{H,E} \left( \rho \otimes |e_0\rangle_E \langle e_0| \right) U_{H,E}^\dagger \, |e_m\rangle_E = \sum_{m=1}^{d_E} E_m \rho E_m^\dagger$$

- Generally, a error $\mathscr{E} : \mathscr{D}(H) \to \mathscr{D}(H)$ changes input states $\rho$ to $\mathscr{E}[\rho]$, the input information might be lost in the view point of the system $H$.

$\to$Input information on $H$ might flow to the environment system $E$, and some environment information (noise) might flow to the original system $H$.




In this case, without accessing the environment system $E$, the original input information can not be recovered.

- However, there is a possibility that

  there exists some subspace called code subspace $H_{code} \subset H$ and a recovery map $\mathscr{R}$ such that for $\sigma_{code} \in \mathscr{D}(H_{code})$ ,

$$\mathscr{R}\left[\mathscr{E}[\sigma_{code}]\right] = \sigma_{code}$$

→The error $\mathscr{E}$ is called correctable or recoverable.

One can use the Petz map [Petz' 88] as a recovery map.

- If we embed (or put) important information onto the code subspace, we can protect the information against the error $\mathscr{E}$.

  →Basic idea of quantum error correction.

Question: When does the error become correctable? Or, what error is correctable?

→ The correctability is characterized by an error correcting condition.

There are many equivalent conditions, e.g.,

- Decoupling condition [Schumacher-Nielsen ’96, ⋯] ← Focus this
- Knill-Laflamme condition [Knill-Laflamme ‘97]
- Sufficiency condition [Petz ’88],
- ⋯ etc.

- To this end, we consider a physical state $|\Psi_i\rangle_{phs} \in H$ which we embed the code information $|i\rangle_{code}$ into by an embedding map $V$, $|\Psi_i\rangle_{phs} = V|i\rangle_{code}$, and its entangled state with a reference system $Ref$;

$$\frac{1}{\sqrt{d_{code}}}\sum_{i=1}^{d_{code}} |i\rangle_{Ref} \otimes |\Psi_i\rangle_{phs} \, .$$

- Then, the error acts on this state;

$$|\Psi'\rangle = \frac{1}{\sqrt{d_{code}}}\sum_{i=1}^{d_{code}} |i\rangle_{Ref} \otimes \sum_{m=1}^{d_E} U_{H,E}\left(|\Psi_i\rangle_{phs} \otimes |e_0\rangle_E\right) = \frac{1}{\sqrt{d_{code}}}\sum_{i=1}^{d_{code}}\sum_{m=1}^{d_E} |i\rangle_{Ref} \otimes E_m|\Psi_i\rangle_{phs} \otimes |e_m\rangle_E.$$

- The decoupling condition is given by $\rho_{Ref,E} \overset{?}{=} \rho_{Ref} \otimes \rho_E$ , $\rho_\alpha = \mathrm{tr}_{\bar{\alpha}}\left[|\Psi'\rangle\langle\Psi'|\right]$

→No correlation between $Ref$ and $E$

If the (not so small) correlation exists and we can not access to the environment $E$, then some fraction of the reference information (~code information) is sent to the environment $E$, meaning the lost of the information.

- The condition $\rho_{Ref,E} \stackrel{?}{=} \rho_{Ref} \otimes \rho_E$ can be measured by evaluating the mutual information, $I(Ref;E) = S(\rho_{Red}) + S(\rho_E) - S(\rho_{Ref,E})$.

  If $I(Ref;E) \neq 0$, the error is not correctable (for the code subspace).

  If $I(Ref;E) = 0$, the error is correctable.

- Apply this treatment to an evaporating BH setup.

# Setup of QEC for an evaporating BH

- Consider the identification
  - Interior and Exterior semi-classical excitations $|i, i'\rangle_{in,ex} \quad (i = 1,2,\cdots, d_{in}, i' = 1,2,\cdots, d_{ex})$ → Code information
  - Reference system for the Interior semi-classical excitations → $Ref(in)$
  - Reference system for the Exterior semi-classical excitations → $Ref(ex)$
  - Entangled state between black hole and Hawking radiation with semi-classical excitations in the state $|i, i'\rangle_{in,ex}$

    → Physical state encoding the code information $|\Psi_{i,i'}\rangle = V|i, i'\rangle_{in,ex}$
- An error acts on the Hawking radiation. → Environment $E$ interacts with the Hawking radiation.
- Under the setup, we are interested in whether the black hole interior is protected or not. Then, we need to consider the decoupling condition between $Ref(in)$ and $Ref(ex) \cup E$ ;
  - if $I(Ref(in) : Ref(ex) \cup E) \neq 0$, the error is not correctable,
  - If $I(Ref(in) : Ref(ex) \cup E) = 0$, then correctable.

**→Investigate this decoupling condition in the PSSY model!**

# PSSY (or West-coast) model

[Penington-Shenker-Stanford-Yang ’19]

- The model consists of the two-dimensional Jackiw-Teitelboim (JT) gravity and end-of-the-world (EoW) branes with tension $\mu$,

$$I_{PSSY} = -S_0\chi - \frac{1}{4\pi}\left[\frac{1}{2}\int_{\mathscr{M}} \phi(R+2) + \int_{\partial\mathscr{M}} \sqrt{h}\phi K\right] + \mu\int_{\text{brane}} ds,$$

$S_0$ : Extremal entropy of the black hole, $\chi$ : Euler character of $\mathscr{M}$,

$K$ : Extrinsic curvature of $\partial\mathscr{M}$, $h$: boundary induced metric on $\partial\mathscr{M}$

with the boundary conditions

$$ds^2\Big|_{\partial\mathscr{M}} = \frac{du^2}{\epsilon^2}, \quad \phi\Big|_{\partial\mathscr{M}} = \frac{\phi_r}{\epsilon},\ \partial_n\phi\,|_{\text{brane}} = \mu, \quad K|_{\text{brane}} = 0$$

- The EOW brane is located deep inside the black hole and has $k$-internal states labeled by $\alpha = 1, \cdots, k$.
  $\rightarrow$ Mimic states of the interior partner of the (early) Hawking radiation.

- Semi-classical excitations propagate on the black hole spacetime $A$ with the EOW brane.

  $\rightarrow$ This introduces states $|\psi^{\alpha}_{i,i'}\rangle_A$

  with $\overline{\langle\psi^{\alpha}_{i,i'}|\psi^{\beta}_{j,j'}\rangle}_A = \delta_{\alpha,\beta}\,\delta_{ij}\,\delta_{i',j'}\,e^{S_0}Z_1$ under the gravitational path integral

- We entangle the black hole $A$ with an bath system $B$, which is not gravitating and stores the Hawking radiation;

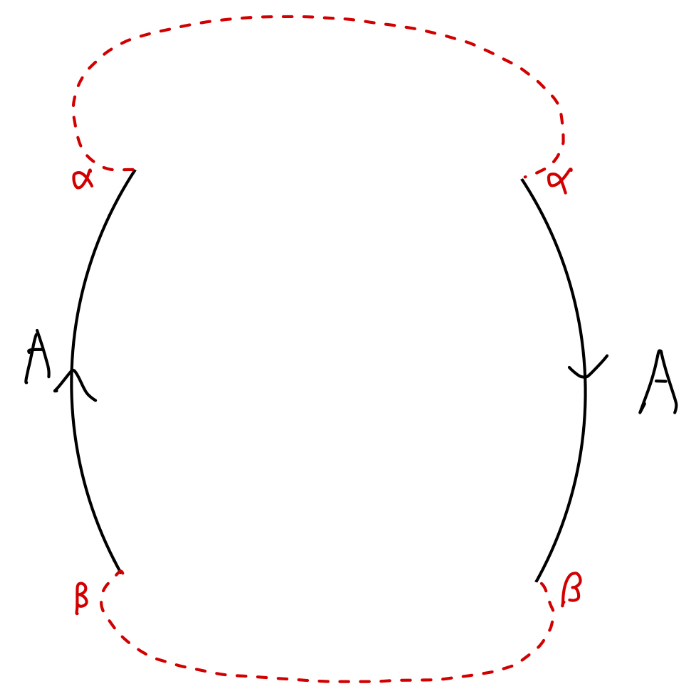
$$|\Psi_{i,i'}\rangle \propto \sum_{\alpha=1}^{k} |\psi^{\alpha}_{i,i'}\rangle_A \otimes |\alpha\rangle_B$$

$A \leftrightarrow$ Black hole $\qquad B \leftrightarrow$ Bath

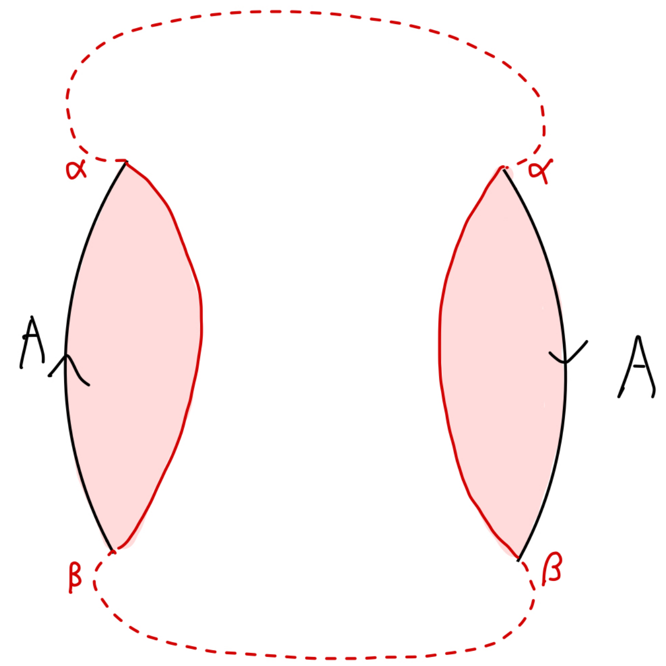
- At first, for simplicity, we basically focus on topological contributions from the topological term in the PSSY action, and do not focus on the dynamical contribution of the PSSY model, i.e., $Z_n$ $(n = 1,2,\cdots)$.

- We can compute the entropy of the Hawking radiation. For simplicity, we focus on the Renyi-two case;

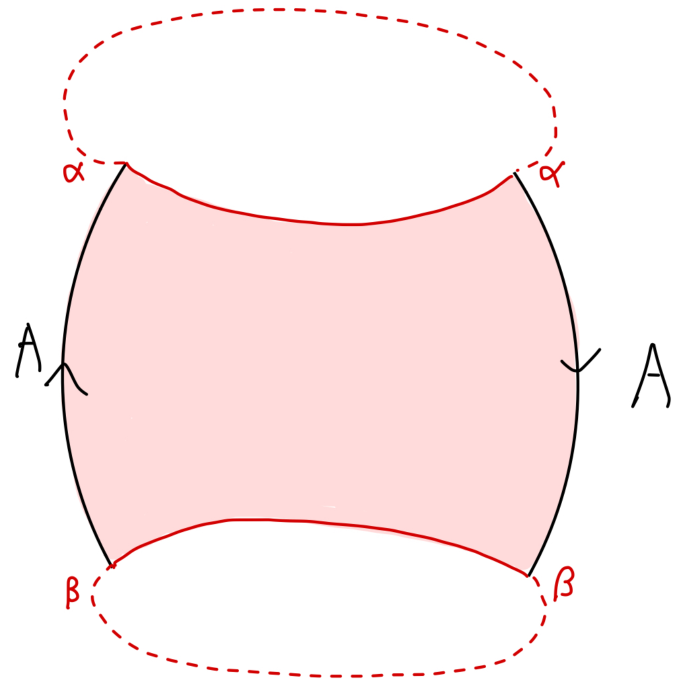
$$S^{(2)}(\rho_B) = -\log\left(\mathrm{tr}\left[\rho_B^2\right]\right) \text{ with } \mathrm{tr}\rho_R^2 = \frac{1}{(k\,e^{S_0})^2}\sum_{\alpha,\beta=1}^{k}\langle\psi^\alpha|\psi^\beta\rangle_A\cdot\langle\psi^\beta|\psi^\alpha\rangle_A.$$




- To evaluate this quantity, we need to consider the gravitational path integral of the quantity, resulting in two possible dominant contributions: disconnected saddle and replica wormhole saddle;




$\to ke^{2S_0}$ ,




$\to k^2 e^{S_0}$

- From the diagram, we can read off the factors, $\overline{\mathrm{tr}\rho_B^2} \approx \frac{1}{k} + \frac{1}{e^{S_0}} = \frac{1}{k} + \frac{1}{d_{BH}}$, where $d_{BH} \approx e^{S_0}$.

- This results in the Renyi-two entropy, $\overline{S^{(2)}(\rho_B)} \approx \begin{cases} \log k & \log k \ll \log d_{BH} \\ \log d_{BH} & \log d_{BH} \ll \log k \end{cases}$, consistent with the Page curve.

# QEC in the PSSY model

[Balasubramanian-Kar-Li-Parikkar '22]

In the PSSY model, we can investigate the QEC properties for an CPTP error with Kraus representation $\{K_m\}$ acting on the Hawking radiation by considering the state,

$$|\Psi'\rangle \propto \sum_{i=1}^{d_{in}} \sum_{i'=1}^{d_{ex}} \sum_{\alpha=1}^{k} \sum_{m=1}^{d_E} |i\rangle_{ref(in)} \otimes \left|i'\right\rangle_{ref(ex)} \otimes \left|\psi_{i,i'}^{\alpha}\right\rangle_A \otimes \left(K_m|\alpha\rangle_B\right) \otimes \left|e_m\right\rangle_E$$

We assume $d_{in}, d_{ex} \ll k, d_{BH}$.

The correctability of the black hole interior can be characterized by the mutual information $I(ref(in) : ref(ex) \cup E)$.

For simplicity, let us focus on the Renyi-two case, given by

$$I^{(2)}(ref(in), ref(ex) \cup E) = S^{(2)}(\rho'_{ref(in)}) + S^{(2)}(\rho'_{ref(ex),E}) - S^{(2)}(\rho'_{ref(in),\, ref(ex),E}).$$

- Focus on the second term $S^{(2)}(\rho'_{ref(ex),E})$ and evaluate it.

$$S^{(2)}(\rho'_{ref(ex),E}) = -\log \mathrm{tr}\left[(\rho'_{ref(ex),E})^2\right]$$

$$\mathrm{tr}\left(\rho'_{ref(ex),E}\right)^2 = \frac{1}{(d_{in}d_{ex}kd_{BH})^2}\sum_{i=1}^{d_{in}}\sum_{i'=1}^{d_{ex}}\sum_{\alpha,\beta=1}^{k}\sum_{m=1}^{d_E}\left\langle \psi^{\beta_1}_{i_1,i'_1} \mid \psi^{\alpha_1}_{i_1,i'_2}\right\rangle_A \left\langle \psi^{\beta_2}_{i_2,i'_2} \mid \psi^{\alpha_2}_{i_2,i'_1}\right\rangle_A$$
$$\times \left\langle \alpha_1 \right| K^{\dagger}_{m_2}K_{m_1}\left|\beta_1\right\rangle_B \left\langle \alpha_2 \right| K^{\dagger}_{m_1}K_{m_2}\left|\beta_2\right\rangle_B$$

Gravitational path integral of this quantity gives

$$\overline{\mathrm{tr}\left(\rho'_{ref(ex),E}\right)^2} = \frac{1}{d_{ex}}\mathrm{tr}\left[(\tau_E)^2\right] + \frac{1}{d_{in}d_{ex}}\cdot\frac{1}{d_{BH}}\cdot\mathrm{tr}\left[(\tau_{\mathrm{Bath}})^2\right]$$

coming from Hawking saddle

coming from Replica wormhole saddle

where $\tau_E = \sum_{m,n=1}^{d_E}\frac{\mathrm{tr}_R\left\{K_mK_n^{\dagger}\right\}}{k}\left|e_m\right\rangle_E\left\langle e_n\right|$, $\tau_{\mathrm{Bath}} = \sum_{m=1}^{d_E}K_m\left(\frac{I_R}{k}\right)K_m^{\dagger}$, and $\log d_{BH} \approx S_0$

Similarly, for the other two terms,

$$\overline{\mathrm{tr}\left(\rho'_{ref(in)}\right)^2} = \frac{1}{d_{in}} + \frac{1}{kd_{BH}} \approx \frac{1}{d_{in}},$$

$$\overline{\mathrm{tr}\left(\rho'_{ref(in),ref(ex),E}\right)^2} = \frac{1}{d_{in}d_{ex}}\,\mathrm{tr}\left[\left(\tau_E\right)^2\right] + \frac{1}{d_{ex}}\cdot\frac{1}{d_{BH}}\cdot\mathrm{tr}\left[\left(\tau_{\mathrm{Bath}}\right)^2\right],$$

Then, the Renyi-two mutual information is given by

$$\overline{I^{(2)}(ref(in) : ref(ex) \cup E)}$$
$$= \begin{cases} 0 & \text{for } -\log d_{BH} + \log d_{in} \ll S^{(2)}\left(\tau_R\right) - S^{(2)}\left(\tau_E\right) \\ \left(-\log d_{BH} + \log d_{in}\right) - \left(S^{(2)}\left(\tau_R\right) - S^{(2)}\left(\tau_E\right)\right) & \text{for } -\log d_{BH} - \log d_{in} \ll S^{(2)}\left(\tau_R\right) - S^{(2)}\left(\tau_E\right) \ll -\log d_{BH} + \log d_{in} \\ 2\log d_{in} & \text{for } S^{(2)}\left(\tau_R\right) - S^{(2)}\left(\tau_E\right) \ll -\log d_{BH} - \log d_{in} \end{cases}$$

$$\tau_E = \sum_{m,n=1}^{d_E} \frac{\mathrm{tr}_R\left\{K_m K_n^\dagger\right\}}{k}\left|e_m\right\rangle_E\left\langle e_n\right|,\ \tau_R = \sum_{m=1}^{d_E} K_m\left(\frac{I_R}{k}\right)K_m^\dagger.$$

Similarly, for the other two terms,

$$\overline{\mathrm{tr}\left(\rho'_{ref(in)}\right)^2} = \frac{1}{d_{in}} + \frac{1}{kd_{BH}} \approx \frac{1}{d_{in}},$$

$$\overline{\mathrm{tr}\left(\rho'_{ref(in),ref(ex),E}\right)^2} = \frac{1}{d_{in}d_{ex}}\,\mathrm{tr}\left[\left(\tau_E\right)^2\right] + \frac{1}{d_{ex}}\cdot\frac{1}{d_{BH}}\cdot\mathrm{tr}\left[\left(\tau_{\mathrm{Bath}}\right)^2\right].$$

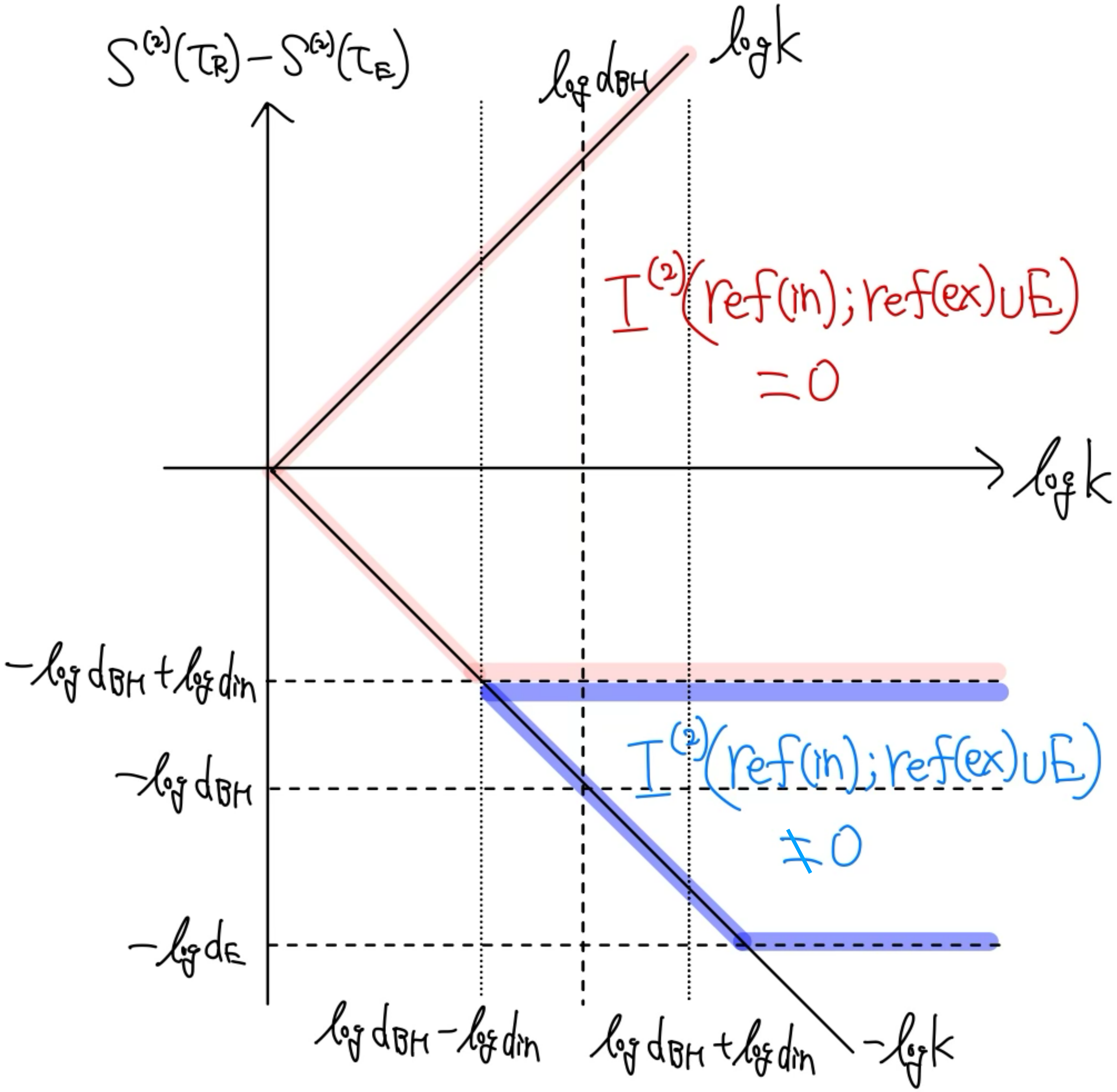
Then, the Renyi-two mutual information is given by

(Renyi-two) Coherent information

$$\overline{I^{(2)}(ref(in):ref(ex)\cup E)}$$
$$= \begin{cases} 0 & \text{for } -\log d_{BH} + \log d_{in} \ll S^{(2)}\left(\tau_R\right) - S^{(2)}\left(\tau_E\right) \\ \left(-\log d_{BH} + \log d_{in}\right) - \left(S^{(2)}\left(\tau_R\right) - S^{(2)}\left(\tau_E\right)\right) & \text{for } -\log d_{BH} - \log d_{in} \ll S^{(2)}\left(\tau_R\right) - S^{(2)}\left(\tau_E\right) \ll -\log d_{BH} + \log d_{in} \\ 2\log d_{in} & \text{for } S^{(2)}\left(\tau_R\right) - S^{(2)}\left(\tau_E\right) \ll -\log d_{BH} - \log d_{in} \end{cases}$$

$$\tau_E = \sum_{m,n=1}^{d_E} \frac{\mathrm{tr}_R\left\{K_m K_n^\dagger\right\}}{k} \left|e_m\right\rangle_E \left\langle e_n\right|,\ \tau_R = \sum_{m=1}^{d_E} K_m\left(\frac{I_R}{k}\right)K_m^\dagger.$$

The schematic phase diagram of the mutual information on the $\log k$ -$\left(S^{(2)}\left(\tau_R\right) - S^{(2)}\left(\tau_E\right)\right)$ plane




The lower bound on $\left(S^{(2)}\left(\tau_R\right) - S^{(2)}\left(\tau_E\right)\right)$ comes from the weak subadditivity;

$$S^{(n)}(\rho_{AB}) \leq S^{(0)}(\rho_A) + S^{(n)}(\rho_B) \qquad \text{for } n \in (0,1) \cup (1,\infty)$$

In this case, it means

$$-\max\{\log k, \log d_E\} \leq S^{(2)}\left(\tau_R\right) - S^{(2)}\left(\tau_E\right).$$

# 2. QEC in an evaporating black hole with a gravitating bath (Without backreaction)

# Outline of our Research

- We focus on the gravitating bath setup (doubled PSSY model) that the bath system $B$ is also gravitating and includes a black hole [Anderson-Parrikar-Soni '21].

- We assume that the black hole $A$ has semi-classical excitations, but the gravitating bath $B$ does not.

- In this system, an CPTP error with Kraus representation $\{E_m\}$ acts on the gravitating bath $B$.

- First, we IGNORE gravitational backreactions from the error. Later, we include it.

- In this setup, we study the QEC properties by evaluating the (Renyi-two) mutual information.

# Doubled PSSY model

[Anderson-Parrikar-Soni ‘21]

- In the doubled PSSY model, we consider the state

$$|\Psi_{i,i'}\rangle \propto \sum_{\alpha=1}^{k} |\psi^{\alpha}_{i,i'}\rangle^{*}_{A} \otimes |\psi^{\alpha}\rangle_{B},$$

where $A \leftrightarrow$ Black hole with semi-classical excitations, $B \leftrightarrow$ Gravitating bath.

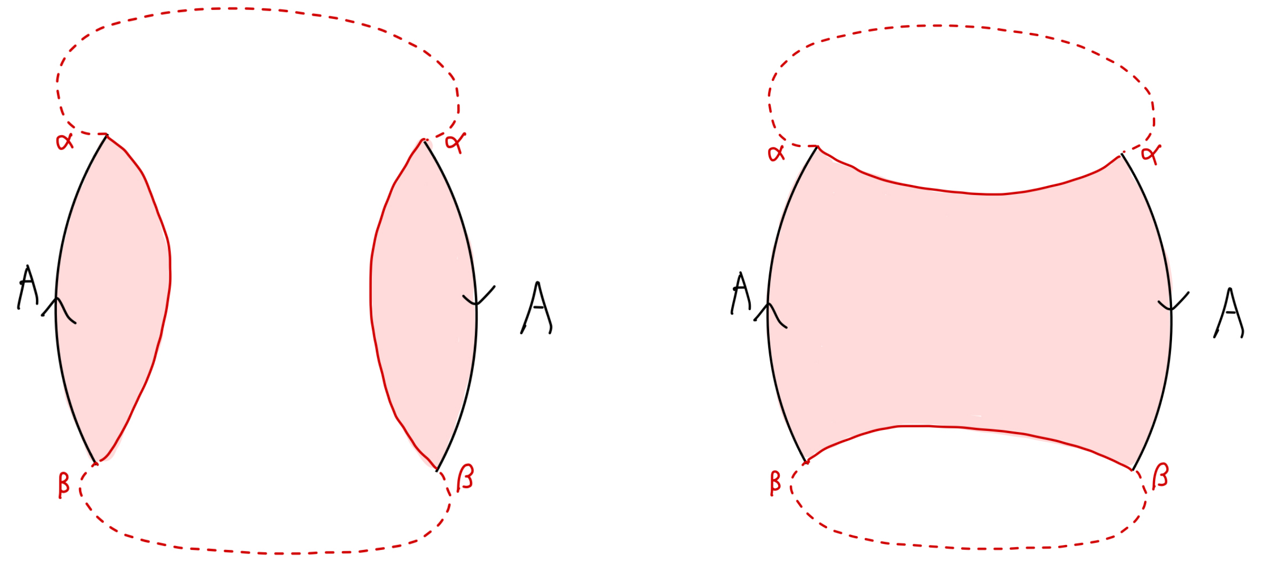
- For simplicity, we assume that the black holes on the two systems $A, B$ have the same black hole entropy. Also, we again assume $d_{in}, d_{ex} \ll k, d_{BH}$.

- The introduction of the gravitating bath changes the gravitational path integrals.

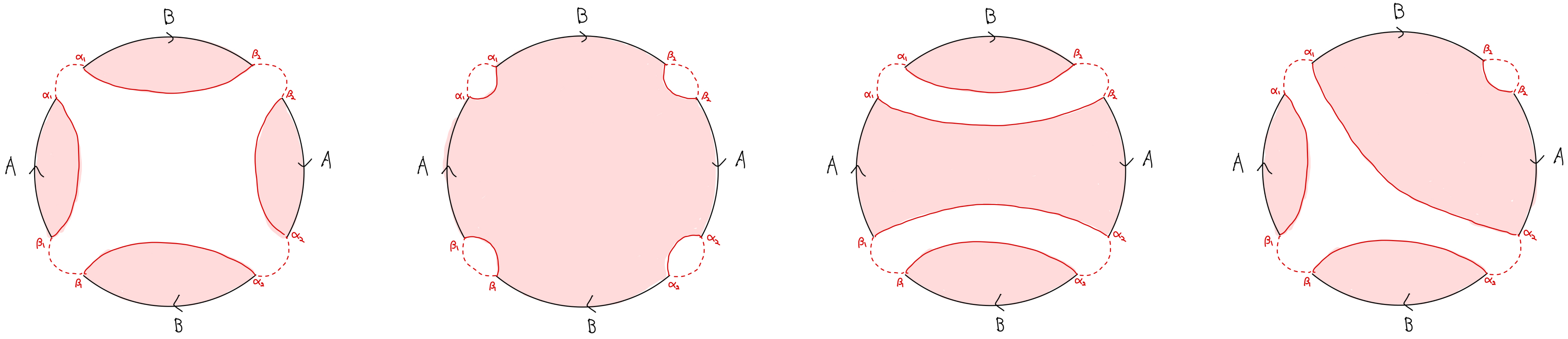
Ex. In evaluating the gravitational path integral of the Renyi-two entropy of the Hawking radiation, we can see the difference between them.

Ex. Renyi-two entropies of the Hawking radiation for the gravitating and non-gravitating cases. (Ignoring the semi-classical excitation indices)

- $\mathrm{tr}\left[\rho_B^2\right] \propto \sum_{\alpha,\beta=1}^{k} \langle \psi^{\alpha} | \psi^{\beta} \rangle_A \cdot \langle \psi^{\beta} | \psi^{\alpha} \rangle_A$ (non-gravitating)




- $\mathrm{tr}\left[\rho_B^2\right] \propto \sum_{\alpha_1,\alpha_2,\beta_1,\beta_2=1}^{k} \langle \psi^{\alpha_1} | \psi^{\beta_1} \rangle_A \cdot \langle \psi^{\beta_1} | \psi^{\alpha_2} \rangle_B \cdot \langle \psi^{\alpha_2} | \psi^{\beta_2} \rangle_A \cdot \langle \psi^{\beta_2} | \psi^{\alpha_1} \rangle_B$ (gravitating)




$, \cdots$

- The Renyi-two entropies of the Hawking radiation for the gravitating case is the same as that for the non-gravitating case at the leading order; $\overline{S^{(2)}(\rho_B)} \approx \begin{cases} \log k & \log k \ll \log d_{BH} \\ \log d_{BH} & \log d_{BH} \ll \log k \end{cases}.$

# QEC in the doubled PSSY model

- In the doubled PSSY model, we investigate the QEC properties by considering the decoupling condition.
- To this end, we consider the state

$$|\Psi'\rangle \propto \sum_{i=1}^{d_{in}} \sum_{i'=1}^{d_{ex}} \sum_{\alpha=1}^{k} \sum_{m=1}^{d_E} |i\rangle_{ref(in)} \otimes \left|i'\right\rangle_{ref(ex)} \otimes \left|\psi^{\alpha}_{i,i'}\right\rangle^{*}_{A} \otimes \left(E_m|\psi^{\alpha}\rangle_B\right) \otimes \left|e_m\right\rangle_E$$

, and evaluate the Renyi-two mutual information

$$I^{(2)}(Ref(in); Ref(ex) \cup E)$$

to see whether the black hole interior is protected or not.

# Renyi-two entropies

- In evaluating the Renyi-two mutual information, we need to consider the gravitational path integral of the following purities;

$$\mathrm{tr}\left[\left(\rho_{ref(in),ref(ex),E}\right)^2\right] \propto \sum_{\boldsymbol{i}=1}^{d_{in}} \sum_{\boldsymbol{i}'=1}^{d_{ex}} \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}=1}^{k} \sum_{\boldsymbol{m}=1}^{d_E} \left\langle \psi^{\beta_1}_{i_1,i'_1} \mid \psi^{\alpha_1}_{i_2,i'_2} \right\rangle_A \left\langle \psi^{\beta_2}_{i_2,i'_2} \mid \psi^{\alpha_2}_{i_1,i'_1} \right\rangle_A \left\langle \psi^{\alpha_1} \left| E^\dagger_{m_2} E_{m_1} \right| \psi^{\beta_1} \right\rangle_B \left\langle \psi^{\alpha_2} \left| E^\dagger_{m_1} E_{m_2} \right| \psi^{\beta_2} \right\rangle_B$$

$$\mathrm{tr}\left[\left(\rho_{ref(ex),E}\right)^2\right] \propto \sum_{\boldsymbol{i}=1}^{d_{in}} \sum_{\boldsymbol{i}'=1}^{d_{ex}} \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}=1}^{k} \sum_{\boldsymbol{m}=1}^{d_E} \left\langle \psi^{\beta_1}_{i_1,i'_1} \mid \psi^{\alpha_1}_{i_1,i'_2} \right\rangle_A \left\langle \psi^{\beta_2}_{i_2,i'_2} \mid \psi^{\alpha_2}_{i_2,i'_1} \right\rangle_A \left\langle \psi^{\alpha_1} \left| E^\dagger_{m_2} E_{m_1} \right| \psi^{\beta_1} \right\rangle_B \left\langle \psi^{\alpha_2} \left| E^\dagger_{m_1} E_{m_2} \right| \psi^{\beta_2} \right\rangle_B$$

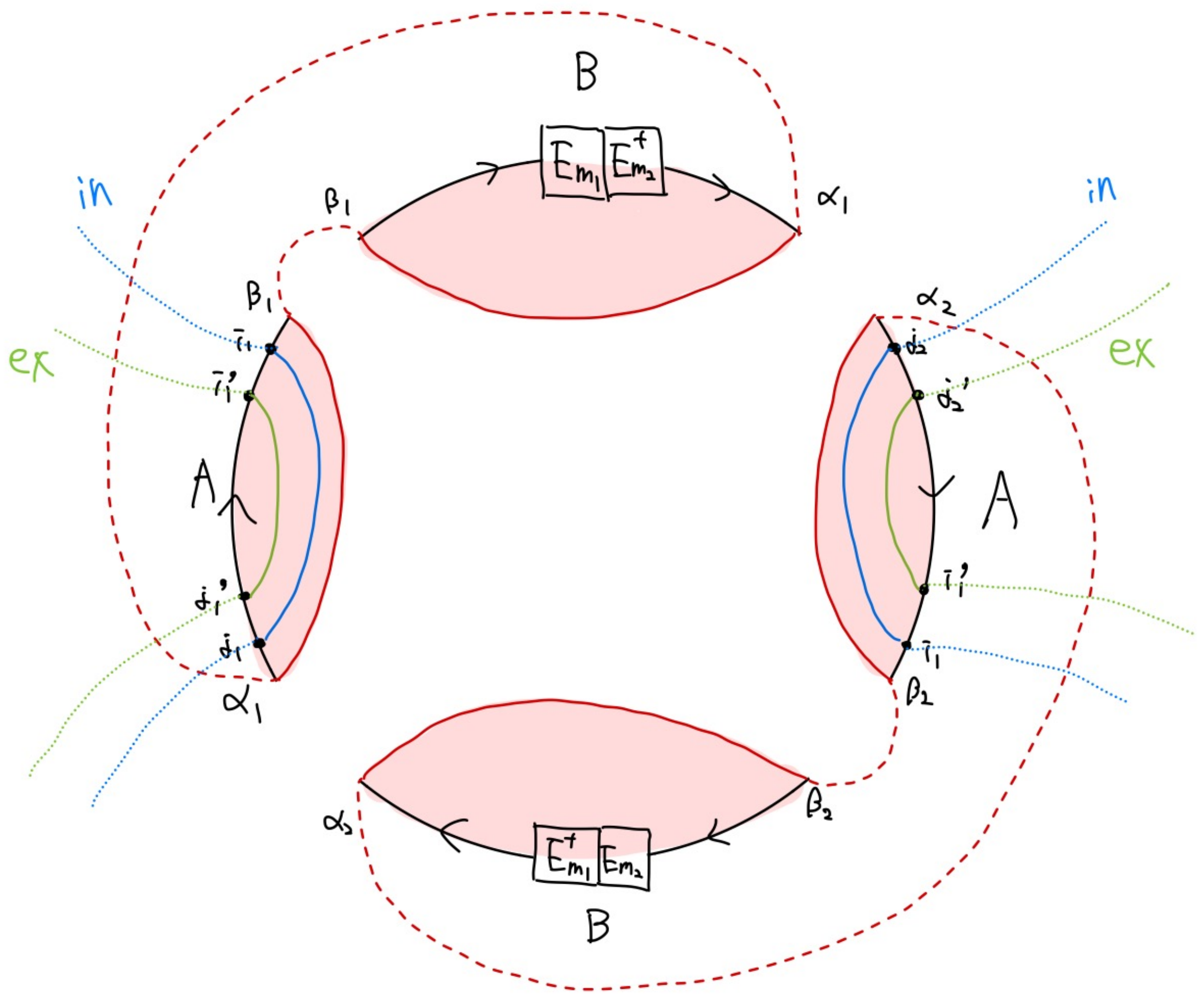
$$\mathrm{tr}\left[\left(\rho_{ref(in)}\right)^2\right] \propto \sum_{\boldsymbol{i}=1}^{d_{in}} \sum_{\boldsymbol{i}'=1}^{d_{ex}} \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}=1}^{k} \left\langle \psi^{\beta_1}_{i_1,i'_1} \mid \psi^{\alpha_1}_{i_2,i'_1} \right\rangle_A \left\langle \psi^{\beta_2}_{i_2,i'_2} \mid \psi^{\alpha_2}_{i_1,i'_2} \right\rangle_A \left\langle \psi^{\alpha_1} \mid \psi^{\beta_1} \right\rangle_B \left\langle \psi^{\alpha_2} \mid \psi^{\beta_2} \right\rangle_B$$

- Let us evaluate the gravitational path integral of

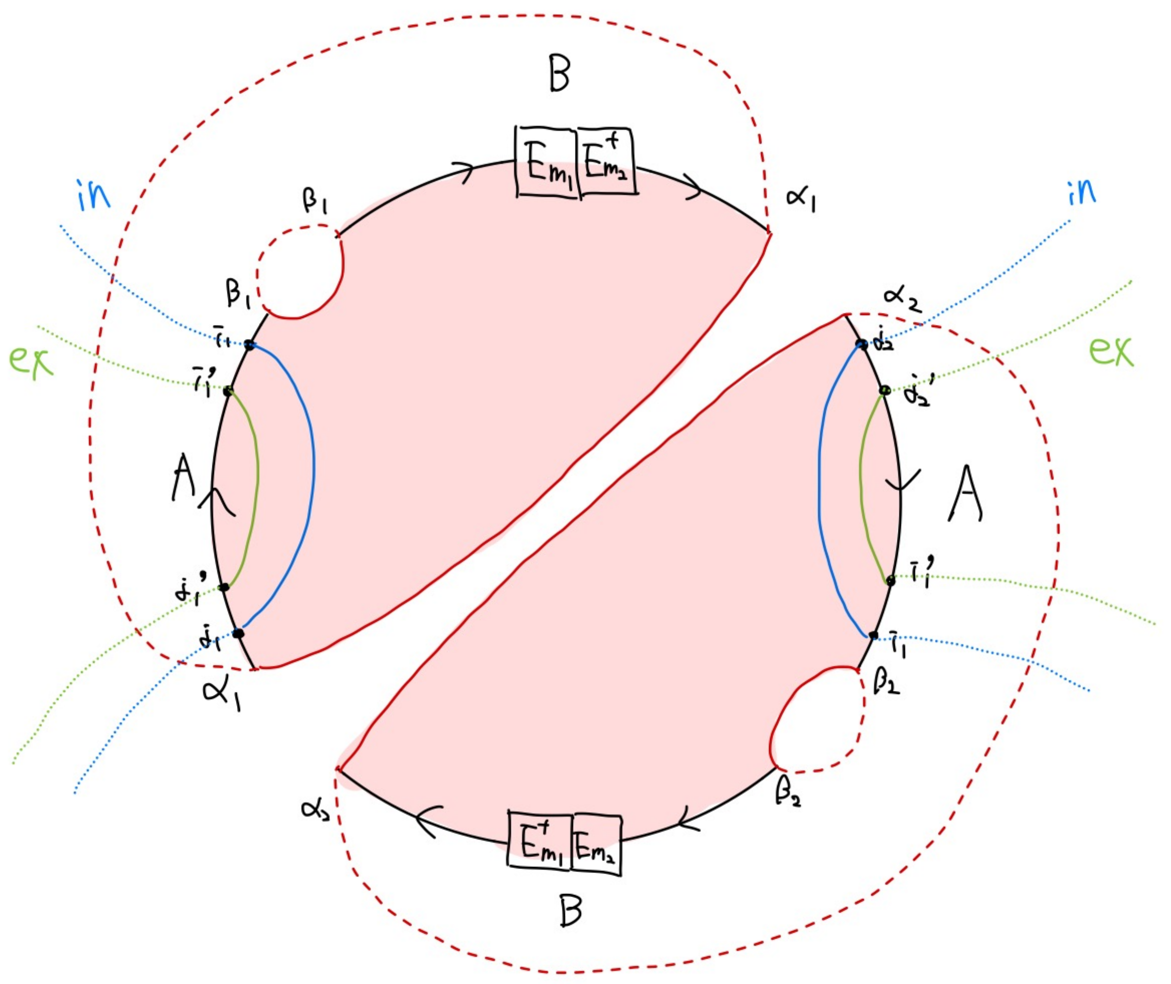
$$\mathrm{tr}\left[\left(\rho_{ref(in),ref(ex),E}\right)^2\right] \propto \sum_{\boldsymbol{i}=1}^{d_{in}} \sum_{\boldsymbol{i'}=1}^{d_{ex}} \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}=1}^{k} \sum_{\boldsymbol{m}=1}^{d_E} \left\langle \psi^{\beta_1}_{i_1,i'_1} \mid \psi^{\alpha_1}_{i_2,i'_2} \right\rangle_A \left\langle \psi^{\beta_2}_{i_2,i'_2} \mid \psi^{\alpha_2}_{i_1,i'_1} \right\rangle_A \left\langle \psi^{\alpha_1} \left| E^\dagger_{m_2} E_{m_1} \right| \psi^{\beta_1} \right\rangle_B \left\langle \psi^{\alpha_2} \left| E^\dagger_{m_1} E_{m_2} \right| \psi^{\beta_2} \right\rangle_B$$

- We can evaluate it by writing down possible 14 gravitational saddles (diagrams), and reading off their resulting factors. For simplicity, we focus on very early (pre-Page) times and very late (post-Page) times.

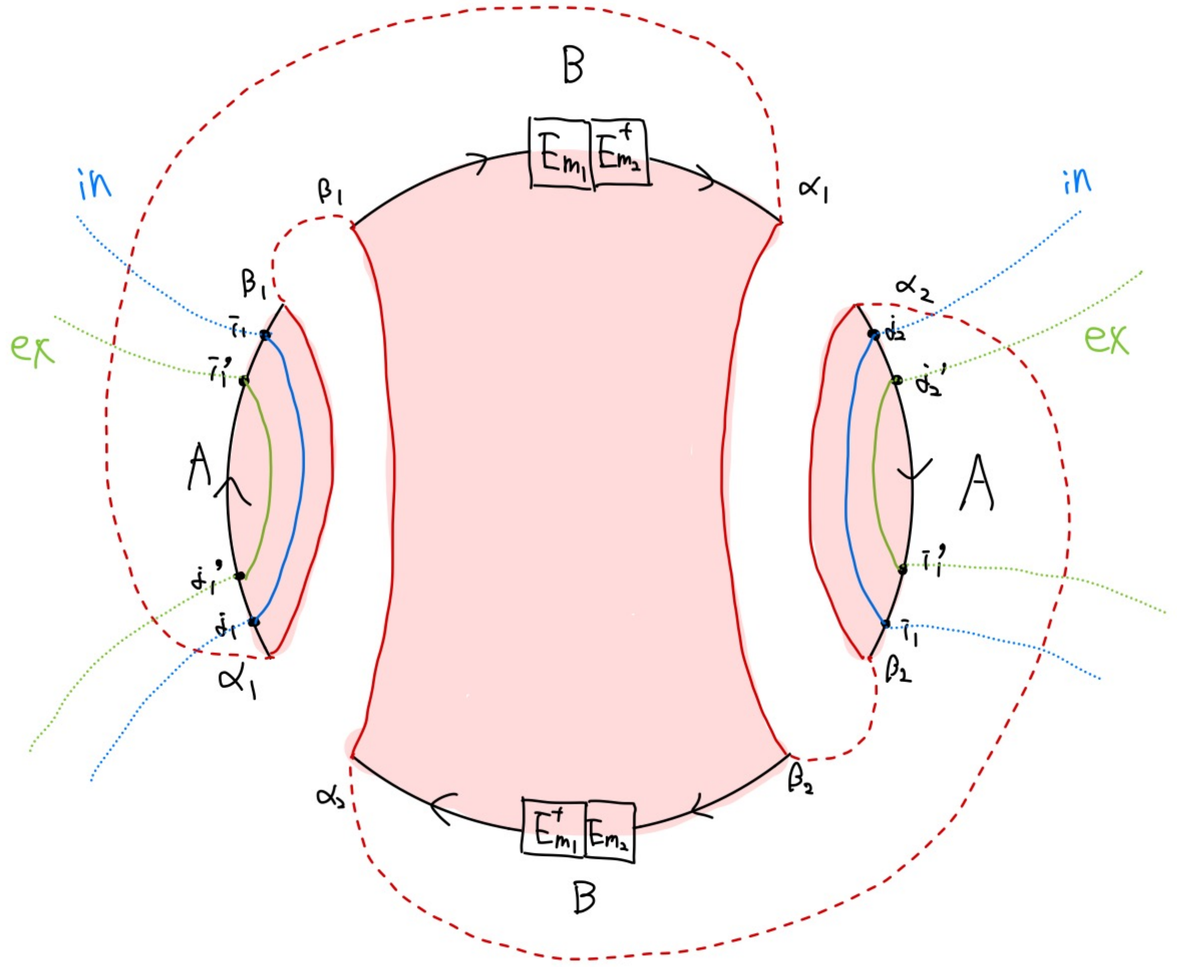
- In these time regimes, there are four dominant saddles;



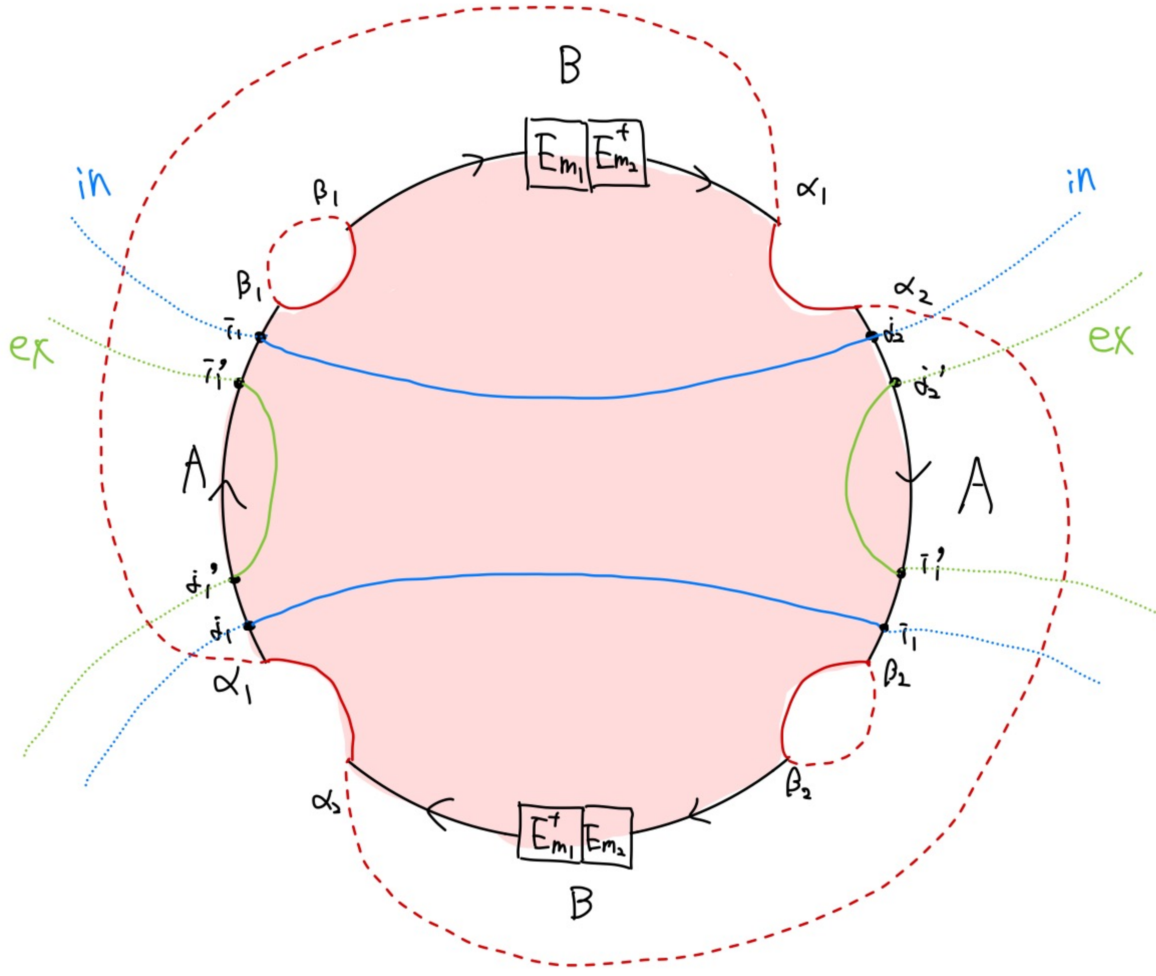
Hawking saddle



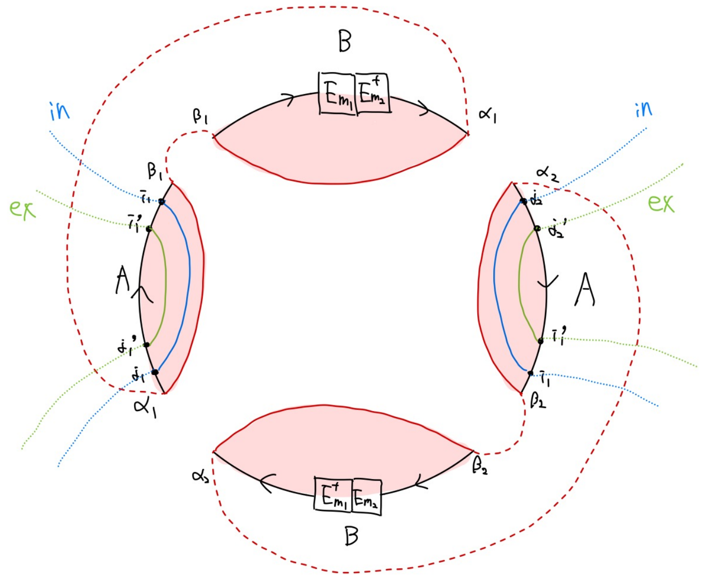
Two (A,B)-wormhole saddle

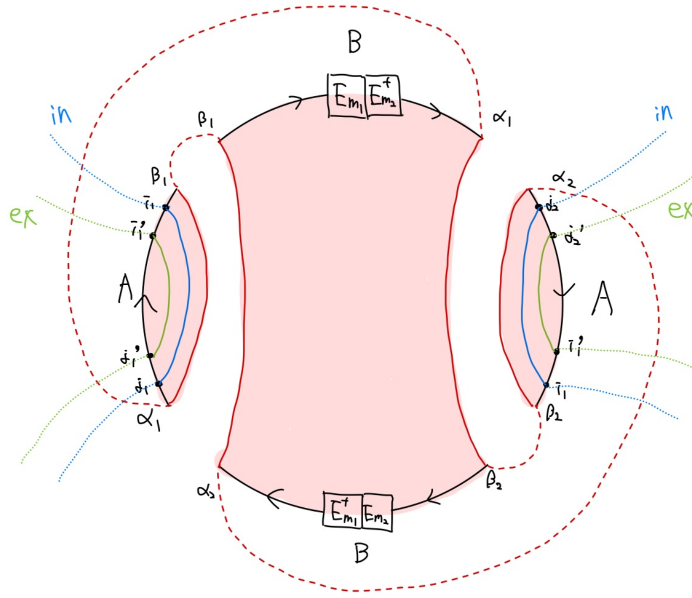


(B,B)-replica wormhole saddle



Fully connected saddle





- The four saddles leads to the following Renyi-two entropy;

For early times $k \ll d_{BH}$,

Hawking saddle

$$\overline{S^{(2)}\left(\rho_{ref(in),ref(ex),E}\right)} \approx \begin{cases} \log d_{in} + \log d_{ex} + S^{(2)}\left(\sigma_E\right) & \text{for } -\log k \ll S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \\ \log d_{in} + \log d_{ex} + \log k + S^{(2)}\left(\sigma_B\right) & \text{for } S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \ll -\log k \end{cases}$$

(B,B)-replica wormhole saddle

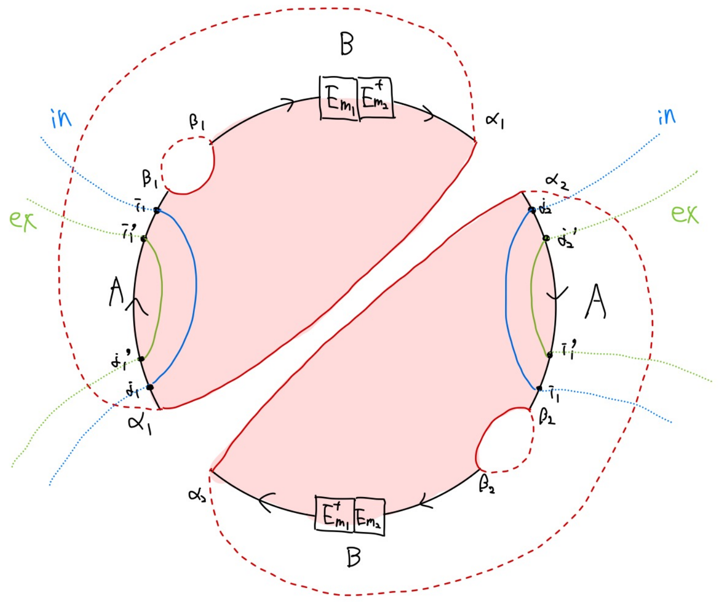
For late times $d_{BH} \ll k$,

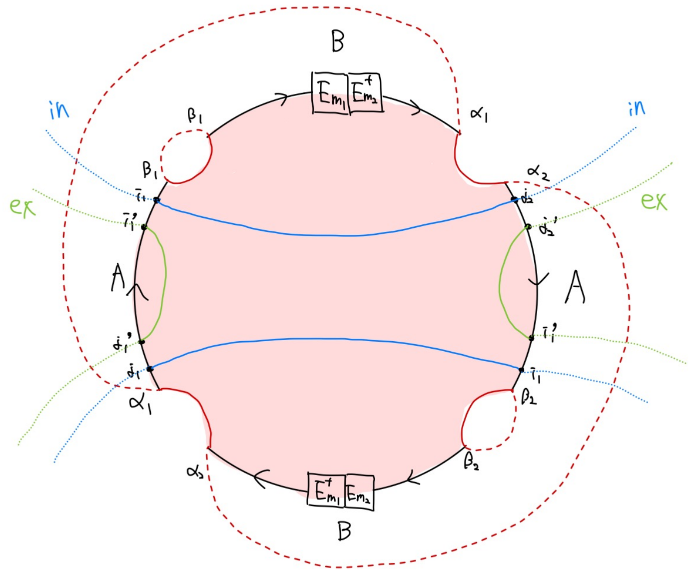
Two (A,B)-wormhole saddle

$$\overline{S^{(2)}\left(\rho_{ref(in),ref(ex),E}\right)} \approx \begin{cases} \log d_{in} + \log d_{ex} + S^{(2)}\left(\sigma_E\right) & \text{for } -\log k + \log d_{in} \ll S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \\ \log d_{ex} + \log k + S^{(2)}\left(\sigma_B\right) & \text{for } S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \ll -\log k + \log d_{in} \end{cases}$$

Fully connected saddle

where $\sigma_E = \sum_{m,n=1}^{d_E} \frac{\mathrm{tr}_{BH}\left\{E_m E_n^\dagger\right\}}{d_{BH}} \left|e_m\right\rangle_E \left\langle e_n\right|$, $\sigma_B = \sum_{m=1}^{d_E} E_m \left(\frac{I_{BH}}{d_{BH}}\right) E_m^\dagger$.





- First, regarding to the early times $k \ll d_{BH}$,

Hawking saddle

$$\overline{S^{(2)}\left(\rho_{ref(in),ref(ex),E}\right)} \approx \begin{cases} \log d_{in} + \log d_{ex} + S^{(2)}\left(\sigma_E\right) & \text{for } -\log k \ll S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \\ \log d_{in} + \log d_{ex} + \log k + S^{(2)}\left(\sigma_B\right) & \text{for } S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \ll -\log k \end{cases}$$

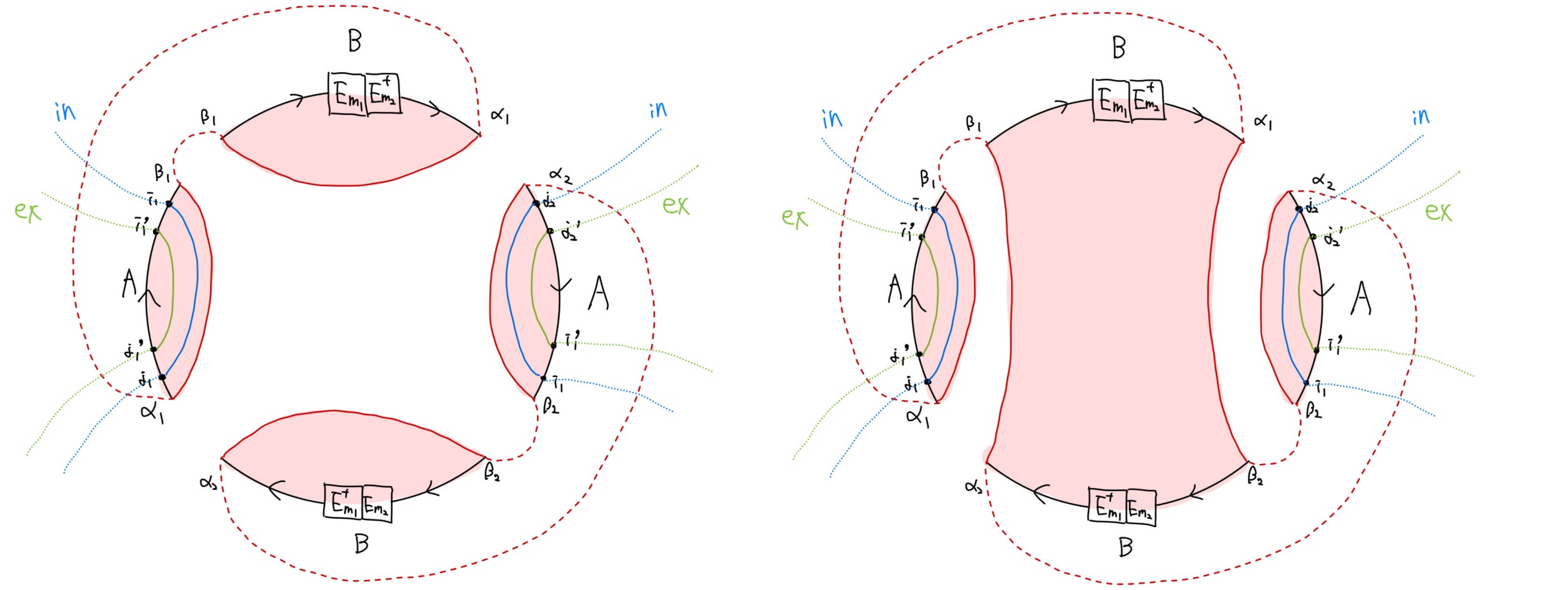
(B,B)-replica wormhole saddle

- The difference between the two cases is the entropy of the environment system;

for the first case, $S^{(2)}\left(\sigma_E\right) \ll \log k + S^{(2)}\left(\sigma_B\right)$,

for the second case, $S^{(2)}\left(\sigma_E\right) \gg \log k + S^{(2)}\left(\sigma_B\right)$.

Hawking saddle




(B,B)-replica wormhole saddle

- The large entropy of the environment system results in the replica wormhole connecting the two bath systems $B$.

- Next, regarding to the late times $d_{BH} \ll k$,

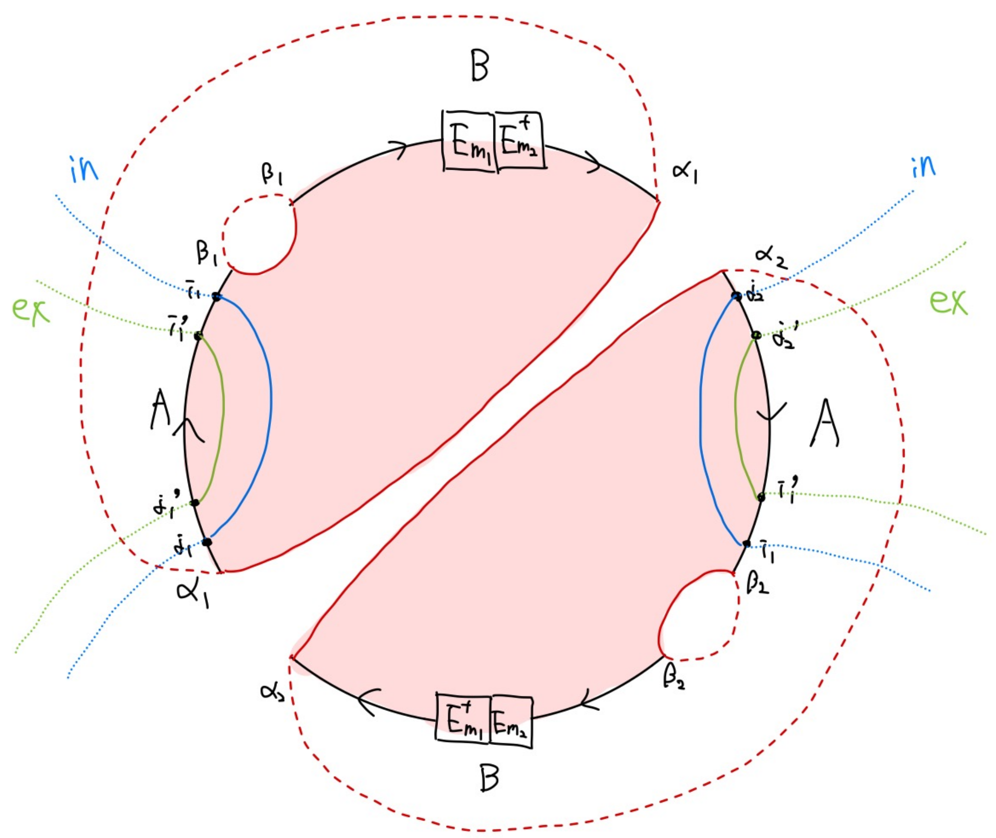
$$\overline{S^{(2)}\left(\rho_{ref(in),ref(ex),E}\right)} \approx \begin{cases} \log d_{in} + \log d_{ex} + S^{(2)}\left(\sigma_E\right) \text{ for } -\log k + \log d_{in} \ll S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \\ \log d_{ex} + \log k + S^{(2)}\left(\sigma_B\right) \text{ for } S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \ll -\log k + \log d_{in} \end{cases}$$

Two (A,B)-wormhole saddle

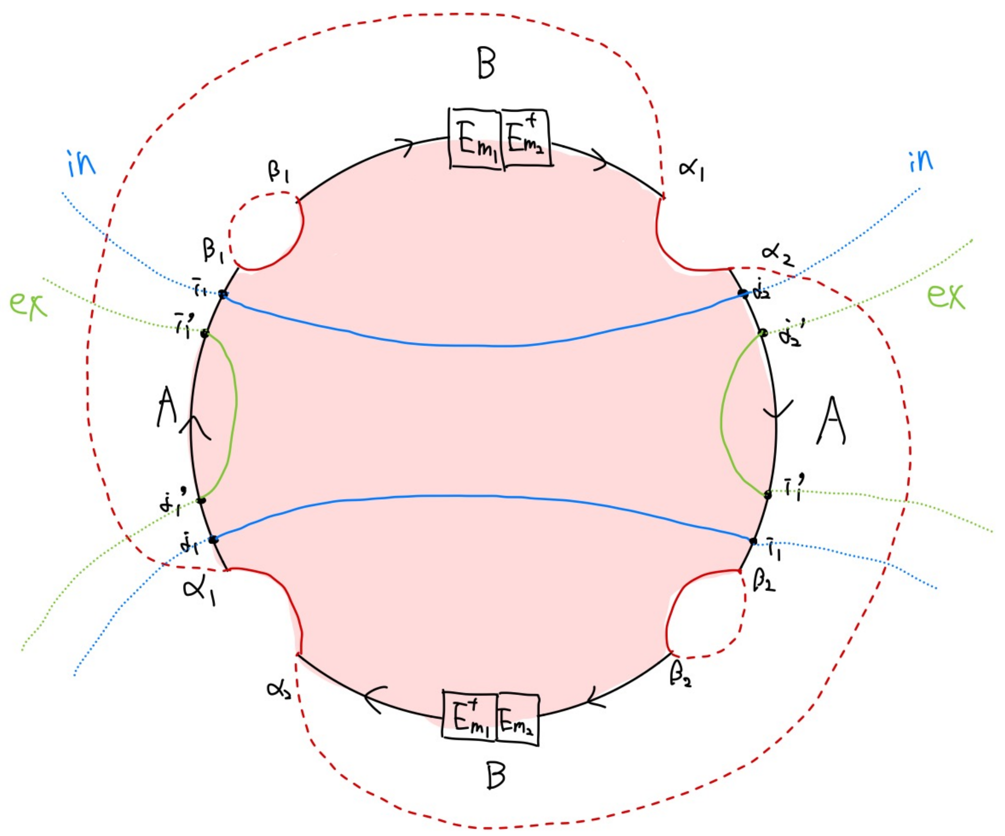
Fully connected saddle

- Since we consider the late times $d_{BH} \ll k$ implying the wormhole connecting the black hole $A$ and the bath system $B$.

- In addition to the wormhole, the large entropy of the environment system results in the replica wormhole connecting the two bath systems $B$, resulting in the fully connected wormhole.




Two (A,B)-wormhole saddle




Fully connected saddle

- Similarly, we can evaluate the quantities, $\mathrm{tr}\left[\left(\rho_{ref(ex),E}\right)^2\right]$ and $\mathrm{tr}\left[\left(\rho_{ref(in)}\right)^2\right]$.

  - For early times $k \ll d_{BH}$,

$$\overline{S^{(2)}\left(\rho_{ref(ex),E}\right)} \approx \begin{cases} \log d_{ex} + S^{(2)}\left(\sigma_E\right) & \text{for } -\log k \ll S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \\ \log d_{ex} + \log k + S^{(2)}\left(\sigma_B\right) & \text{for } S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \ll -\log k \end{cases}$$

Hawking saddle

(B,B)-replica wormhole saddle

$$\overline{S^{(2)}\left(\rho_{ref(in)}\right)} \approx \log d_{in}$$

Hawking saddle

  - For late times $d_{BH} \ll k$,

$$\overline{S^{(2)}\left(\rho_{ref(ex),E}\right)} \approx \begin{cases} \log d_{ex} + S^{(2)}\left(\sigma_E\right) & \text{for } -\log k - \log d_{in} \ll S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \\ \log d_{in} + \log d_{ex} + \log k + S^{(2)}\left(\sigma_B\right) & \text{for } S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \ll -\log k - \log d_{in} \end{cases}$$

Two (A,B)-wormhole saddle

Fully connected saddle

$$\overline{S^{(2)}\left(\rho_{ref(in)}\right)} \approx \log d_{in}$$

Two (A,B)-wormhole saddle

# Renyi-two mutual information

- Combining the Renyi-two entropies, we get the Renyi-two mutual information;

for early times $k \ll d_{BH}$,

$$\overline{I^{(2)}(ref(in); ref(ex) \cup E)} \approx 0 \quad \text{for } -\log k \ll S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right),$$

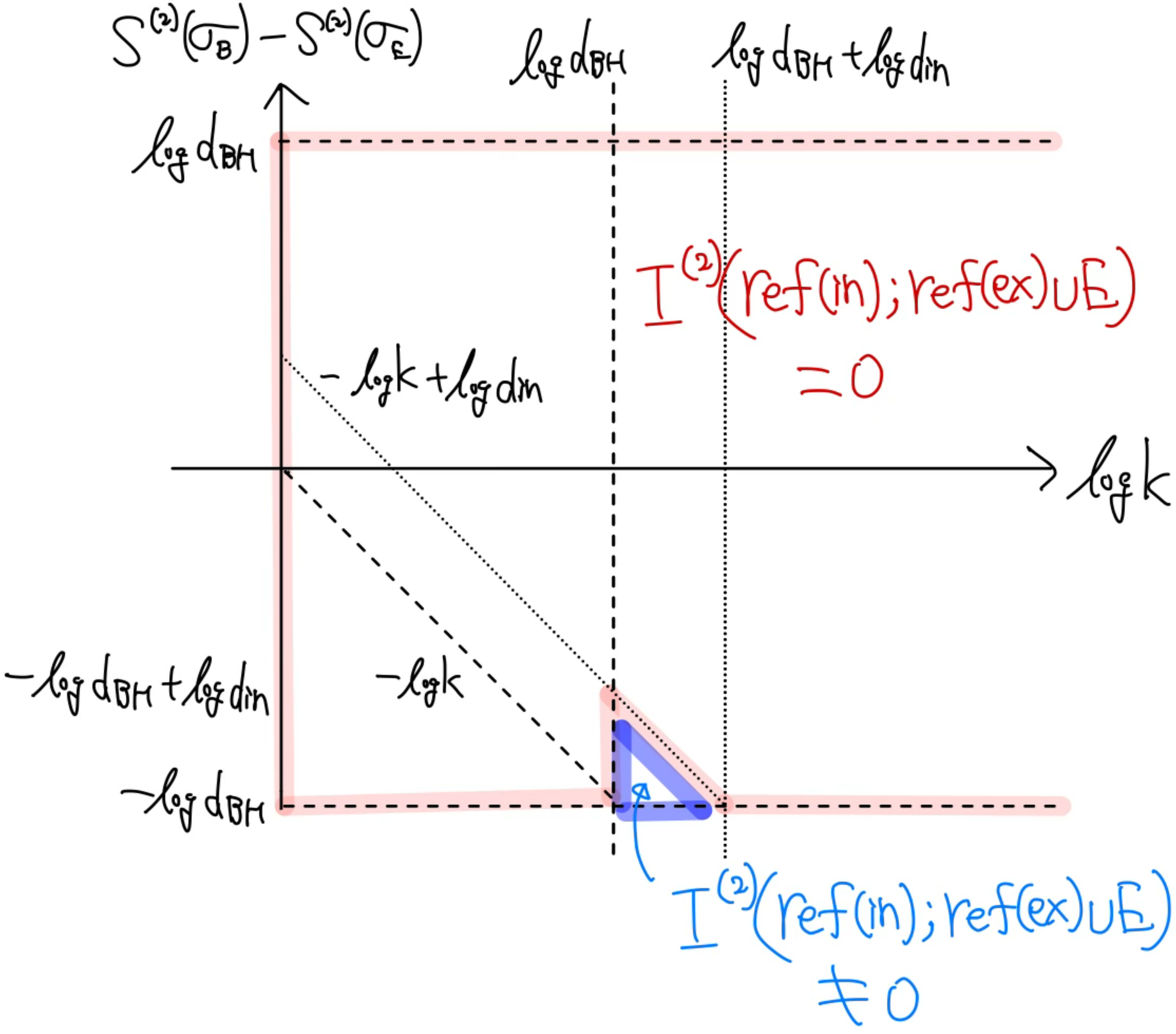
$$\text{and for } S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \ll -\log k,$$

for late times $d_{BH} \ll k$,

$$\overline{I^{(2)}(ref(in); ref(ex) \cup E)}$$

$$\approx \begin{cases} 0 \quad \text{for } \max\left\{-\log k + \log d_{\text{in,}} -\log d_{BH}\right\} \ll S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \\ \left(-\log k + \log d_{in}\right) - \left(S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right)\right) \\ \quad \text{for } d_{BH} \ll k \text{ with } -\log d_{BH} \leq S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \ll \max\left\{-\log k + \log d_{\text{in}}, -\log d_{BH}\right\} \end{cases}$$

The schematic phase diagram of the mutual information on the $\log k$ - $\left(S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right)\right)$ plane




- The lower and upper bound on $\left(S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right)\right)$ again comes from the weak subadditivity;

$$-\max\{\log d_{BH}, \log d_E\} \leq S^{(2)}(\sigma_B) - S^{(2)}(\sigma_E) \leq \log d_{BH}$$

- Due to the existence of the black hole in the bath, the effect of the error can not have a large effect.

- However, the parameter region, where the Renyi-two mutual information does not vanish, still exits. The parameter region is smaller than that for the non-gravitating case.

# 3. QEC in an evaporating black hole with a gravitating bath (With backreaction)

# Including gravitational back reaction from the error

- Next, we include a gravitational backreaction from the error onto the bath system, and check whether the resulting Renyi-two mutual information changes or not.

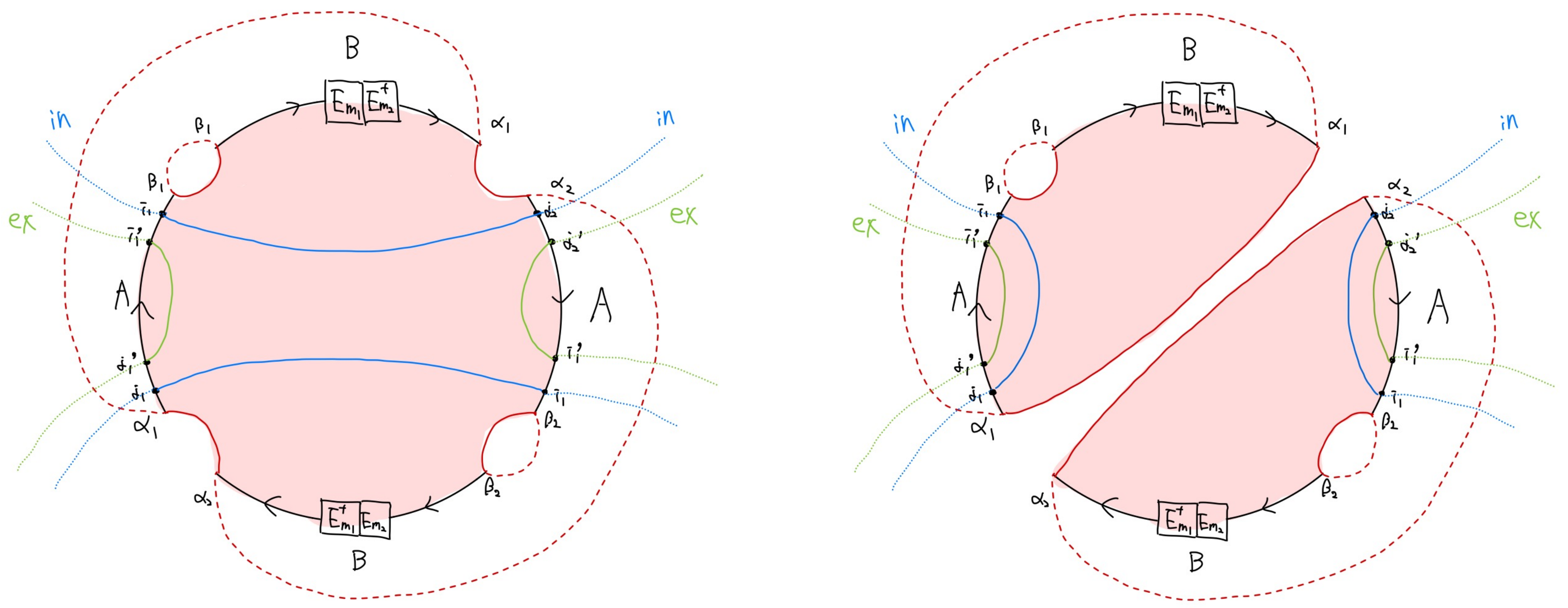
- There are infinitely many implementation of backreactions from the error onto the system. Focus on one of the implementations;

    - We regard the Kraus operator for the error as a local scaling operator with scaling dimension $\Delta$, causing the gravitational backreaction.

    - The scaling dimension $\Delta$ depends on the details of the error. (Ex. $\Delta$ can be a function of $d_E$. )

- By this implementation, we can treat the backreaction as if it comes from a brane ending at the local scaling operators with tension $\Delta$ and the brane action [Goel-Lam-Turiaci-Vrlinde '19, ….]

$$I_{\text{Bulk Massive}} = \Delta \int_{\text{Bulk Massive}} ds$$



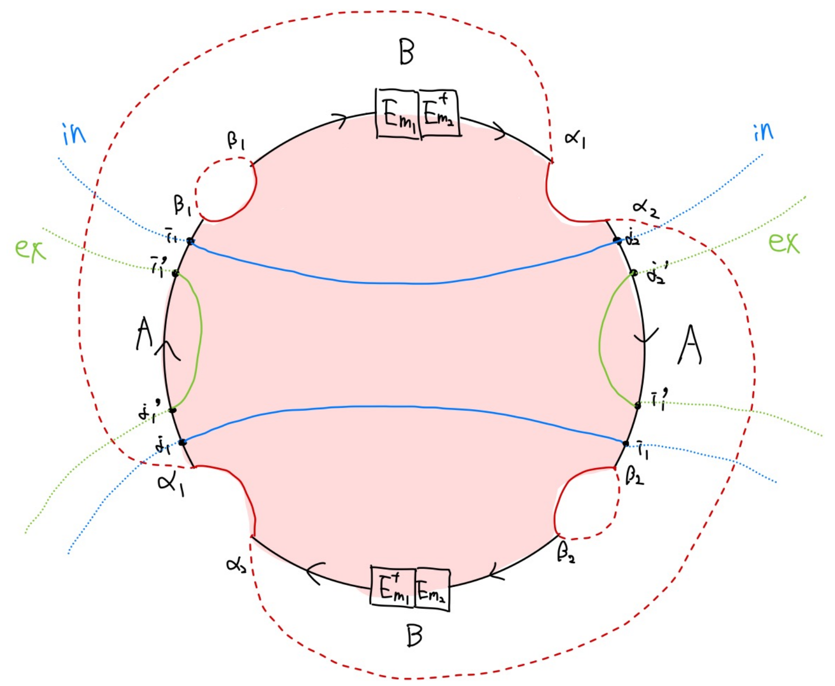
# Gravitational saddle with the local scaling operators

- We need to re-evaluate the gravitational path integral, but with inserting the local scaling operators.

- The saddles, which we need to focus, are those after the Page time since we are interested in the change of the mutual information.

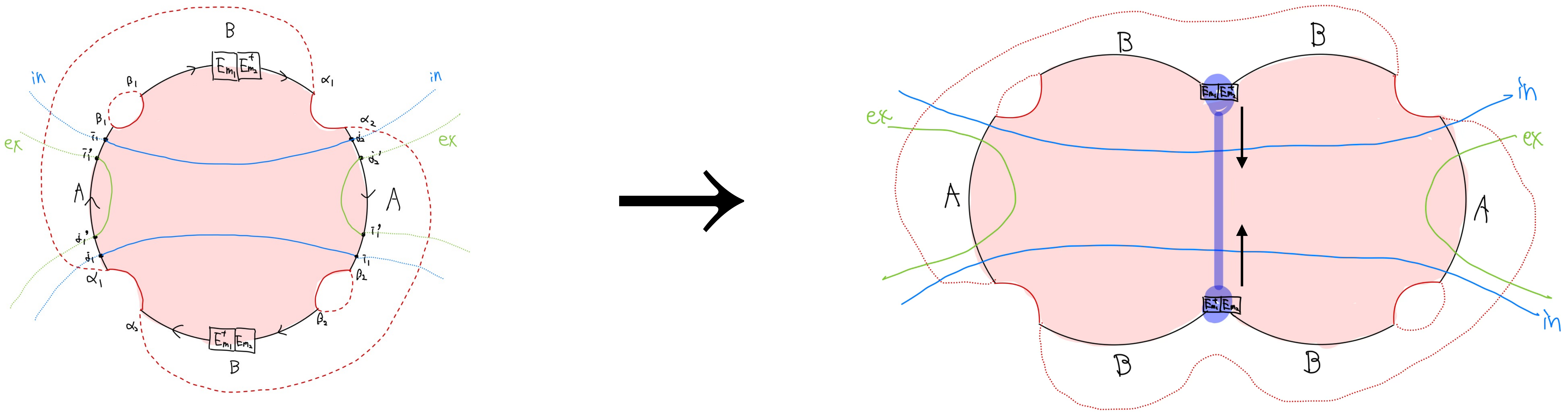
- Saddles connecting the boundaries of the universe B are affected by the backreactions

→Focus on the fully connected saddle with the backreactions, and check whether it can dominate over the other saddles.



# Cusp from the brane

- Due to the tension of the brane from the error, the geometry is deformed, and it starts having the cusp




→Briefly evaluate the on-shell gravitational action for the backreacted geometry

The boundary term of the (doubled) PSSY model capture the cusp contribution (amounting to the Hayward term)

$$-\frac{1}{4\pi}\int_{\partial\mathcal{M}}\sqrt{h}\phi K = -\frac{1}{4\pi}\int_{\text{usual AdS bdy.}}\sqrt{h}\phi K - \frac{1}{4\pi}\int_{\text{cusp}}\sqrt{h}\phi K \approx -\frac{1}{4\pi}\int_{\text{usual AdS bdy.}}\sqrt{h}\phi K + 2\Delta$$

from the cusp

- This results in a change of the Renyi-two entropy

$$\overline{S^{(2)}\left(\rho'_{ref(in),ref(ex),E}\right)}\Bigg|_{\text{Fully connected saddle}} \approx \log d_{ex} + \log k + S^{(2)}\left(\sigma_B\right) + 2\Delta \quad \text{for } d_{BH} \ll k$$

- Need to compare it with other saddles

$$\overline{S^{(2)}\left(\rho'_{ref(in),ref(ex),E}\right)}\Bigg|_{\text{After the Page time } d_{BH}\ll k}$$

$$= \min\left\{\overline{S^{(2)}\left(\rho'_{ref(in),ref(ex),E}\right)}\Bigg|_{\text{Two (A,B)}-\text{wormhole saddle}}, \overline{S^{(2)}\left(\rho'_{ref(in),ref(ex),E}\right)}\Bigg|_{\text{Fully Connected saddle}}\right\}$$

$$\approx \min\left\{\log d_{in} + \log d_{ex} + S^{(2)}\left(\sigma_E\right), \log d_{ex} + \log k + S^{(2)}\left(\sigma_B\right) + 2\Delta\right\}$$

From the cusp

- Depending on the scaling dimension $\Delta$, there are two possibilities.

# Small scaling dimension $2\Delta < \log d_{in}$

- After the Page time, the dominant saddle are the almost same as the non-backreacting one.

$$\overline{S^{(2)}\left(\rho'_{ref(in),ref(ex),E}\right)}$$

$$\approx \begin{cases} \log d_{in} + \log d_{ex} + S^{(2)}\left(\sigma_E\right) \\ \quad \text{for max } \left\{-\log k + \log d_{in} - 2\Delta, -\log d_{BH}\right\} \ll S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right), \\ \log d_{ex} + \log k + S^{(2)}\left(\sigma_B\right) + 2\Delta \\ \quad \text{for } -\log d_{BH} \leq S^{(2)}\left(\sigma_B\right) - S^{(2)}\left(\sigma_E\right) \ll \max\left\{-\log k + \log d_{in} - 2\Delta, -\log d_{BH}\right\}. \end{cases}$$

$$\rightarrow \quad \overline{I^{(2)}(ref(in)\,;\,ref(ex) \cup E)}$$

$$\approx \begin{cases} 0 & \text{for } \max\{-\log k + \log d_{in} - \Delta, -\log d_{BH}\} \ll S^{(2)}(\sigma_{\mathrm{B}}) - S^{(2)}(\sigma_E) \\ (-\log k + \log d_{in} - 2\Delta) - \left(S^{(2)}(\sigma_{\mathrm{B}}) - S^{(2)}(\sigma_E)\right) \\ & \text{for } d_{BH} \ll k \text{ with } -\log d_{BH} \leq S^{(2)}(\sigma_{\mathrm{B}}) - S^{(2)}(\sigma_E) \\ & \qquad \ll \max\{-\log k + \log d_{in} - 2\Delta, -\log d_{BH}\} \end{cases}$$

The Renyi-two mutual information still have a non-vanishing parameter region!

# Large scaling dimension $2\Delta > \log d_{in}$

- After the Page time, the fully connected saddle can not appear as a dominant saddle, and the two-(A,B) wormhole saddle dominates.

- Thus, after the Page time, the Renyi-two entropy is given by

$$\overline{S^{(2)}\left(\rho'_{ref(in),\, ref(ex),\, E}\right)} \approx \log d_{in} + \log d_{ex} + S^{(2)}(\sigma_E)$$

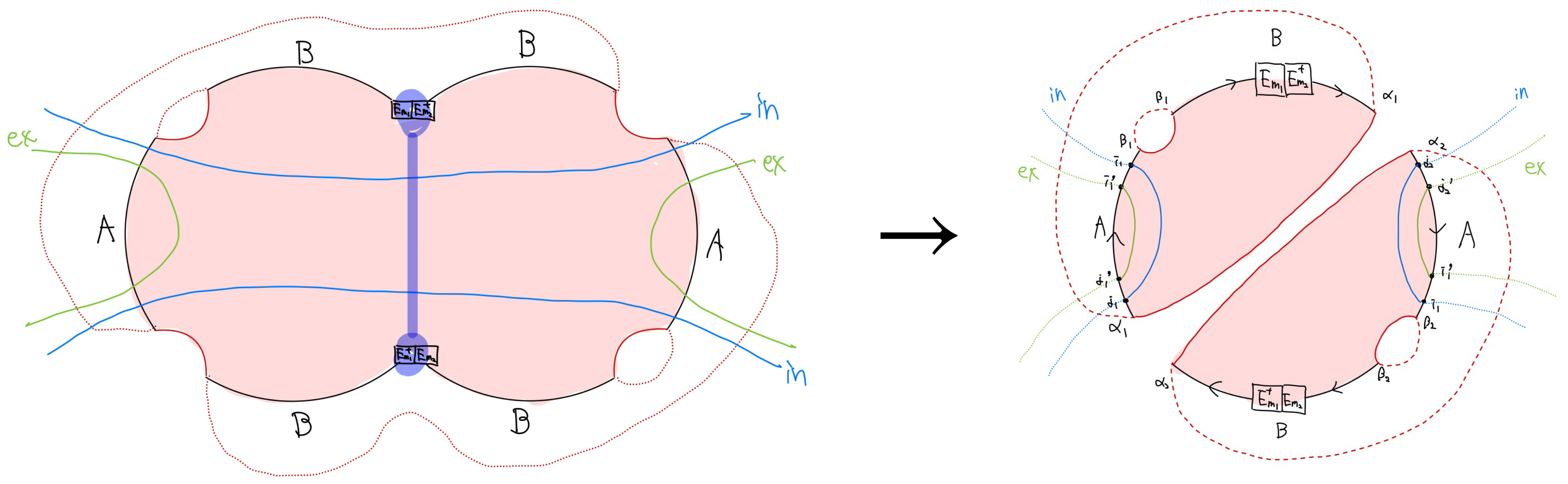
- Therefore, for this case, the mutual information vanishes.

$$\overline{I^{(2)}(ref(in); ref(ex) \cup E)} \approx 0$$

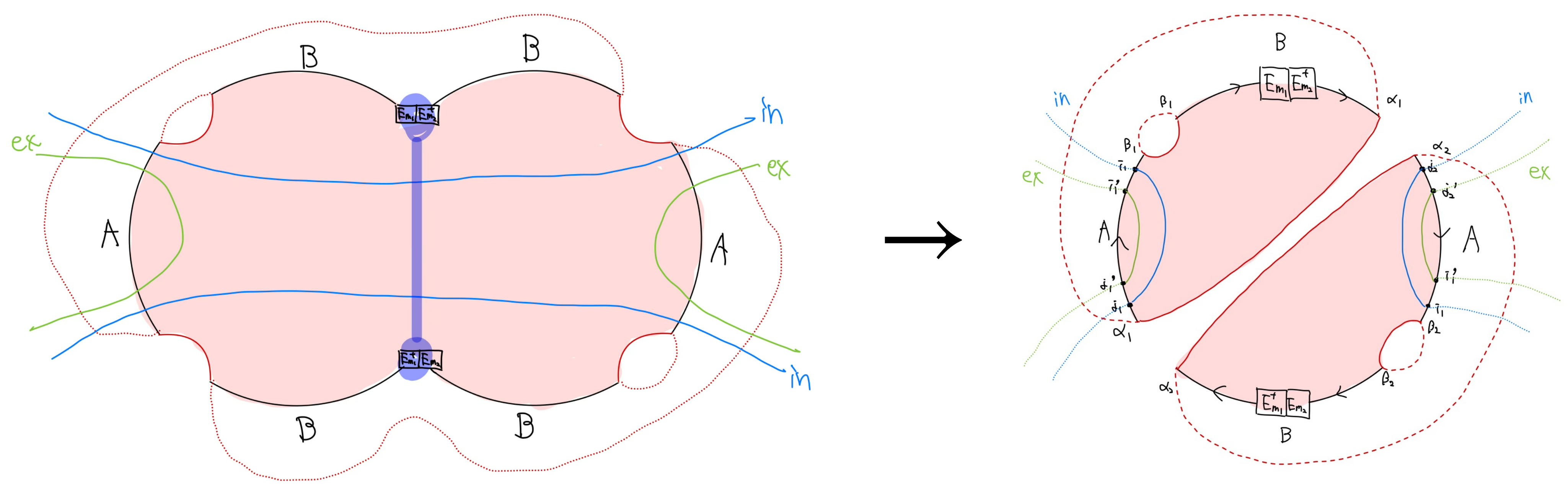
# Interpretation

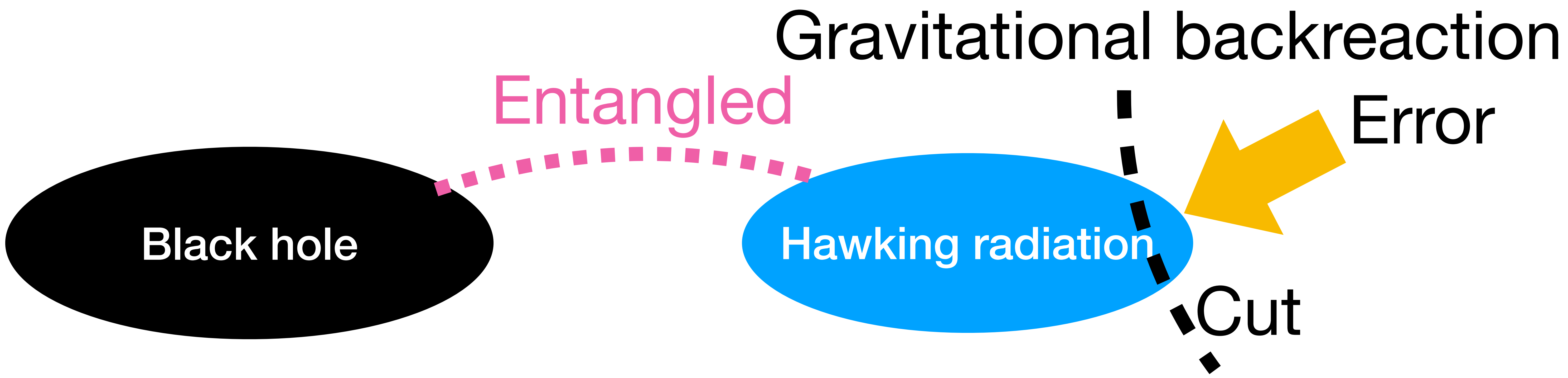
- The gravitational backreaction destroys the entanglement between the environment system and the other systems. It is similar to the Firewall [AMPS ‘12].

- Geometrically, this is represented by the dominance change of the saddles due to the gravitation backreaction from the error.



- In some sense, the wormhole connecting the boundaries of the universe B are cut by the gravitational backreaction.






- Due to this gravitational backreaction, the black hole interior is protected against the error with relatively small scaling dimension $2\Delta > \log d_{in}$ !

# Summary and future works

- If we do not consider gravitational backreaction from an error acting on the gravitating bath, there are similarities between the gravitating and non-gravitating bath cases.

- The gravitational backreaction from error is important, and due to the backreaction, the black hole interior can be protected against an error acting on gravitating bath.

Future directions

- Other error model

- Recovery map for the gravitating bath case (Petz map, Petz-lite, etc. )

- Mutual information by summing over all possible planar contributions

- QEC properties for the two identical black hole case, where both of them have semi-classical excitations.

- …

Thank you for your attention!!