Recent development in black holes and quantum gravity
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Resurgence and $T\bar{T}$-deformed Torus Partition function

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Introduction and Motivation

- $T\bar{T}$ deformation: "integrable" deformations of 2d QFTs （Smirnov, Zamolodchikov, 17）

$$\partial_\lambda S = \int d^2x\, Det\, T^\lambda_{ij}(x)$$

- RG irrelevant: drastic modification of UV properties
- Spectral flow: $\partial_\lambda E_n(\lambda) = E_n(\lambda)\partial_R E(\lambda) + \frac{1}{R} P_n(R)^2$
- Exact solutions in CFTs:

$$E_n(\lambda) = \frac{2R}{\lambda}\left(\sqrt{1 + \frac{\lambda E_n}{R} + \frac{\lambda^2 P_n^2}{4R^2}} - 1\right)$$

- Suggest UV finiteness: (i) $\lambda > 0$ Hagedorn growth; (ii) $\lambda < 0$ UV instabilities (truncation?)

Introduction and Motivation

- $T\bar{T}$ deformation in holographic CFTs:

  (i) $\lambda < 0$: finite cut-off surface (H.Verlinde et al, 18)

  (ii) $\lambda > 0$: glue-on surface? (Wei Song et al, 24)

- As a double-trace deformation: mixed boundary condition (Guica and Monten, 21)
- UV finiteness revealed by UV-sensitive observables, e.g. correlation functions, entanglement entropy, etc. (Donnelly et al, 18; Chen et al, 18, …)
- Conformal perturbation theory: $S(\lambda) = S_{CFT} + \lambda \int d^2x\, T(x)\, \bar{T}(x) + \cdots$

Introduction and Motivation

- UV finiteness not accessible to perturbative analysis
- Toy model example: resolving "UV" divergence at $x = 0$

$$\frac{1}{x} \to \frac{1}{x+\epsilon} = \frac{1}{x} - \frac{\epsilon}{x^2} + \frac{\epsilon^2}{x^3} + \cdots$$

- Re-summation required to see the resolution
- Interesting phenomenon encoded in the non-perturbative aspects.
- This is difficult for correlation functions, entanglement entropies, etc.
- Thermodynamics: $T\bar{T}$-deformed partition function (torus with $\tau = \tau_1 + i\tau_2$)

$$Z(\tau, \bar{\tau}, \lambda) = \sum_n e^{-\tau_2 \, E_n(\lambda) - i \, \tau_1 P_n}$$

Introduction and Motivation

- Perturbative expansion: $Z(\tau,\bar{\tau},\lambda) = Z_{CFT}(\tau,\bar{\tau}) + \lambda\, Z_1(\tau,\bar{\tau}) + \lambda^2\, Z_2(\tau,\bar{\tau}) + \cdots$
- Can we access non-perturbative effects from perturbative expansion?
- 0^{th} order question: is there any non-perturbative effects in $Z(\tau,\bar{\tau},\lambda)$?
- Equivalent question: is the perturbation series convergent?
  - (i) YES: then NO;
  - (ii) NO: then YES…
- Non-convergent series are called asymptotic series, whose meaningful re-summation involves non-perturbative corrections. Examples: instanton corrections in QFTs, etc

Introduction and Motivation

- Goal of our work:
  ① Check convergence of the perturbation series $Z(\tau,\bar{\tau},\lambda) = Z_{CFT}(\tau,\bar{\tau}) + \lambda\, Z_1(\tau,\bar{\tau}) + \lambda^2\, Z_2(\tau,\bar{\tau}) + \cdots$
  ② If it is an asymptotic series, find the properties of the non-perturbative effects
  ③ Explain the origin of the non-perturbative effects, like instantons in Yang-Mill theory.
  ④ Explore physical implications of the non-perturbative effects
- To be explicit, we will work with $T\bar{T}$-deformed free boson/free fermion.
- To simplify analysis, we study $Z(\tau,\bar{\tau},\lambda)$ with $\tau = i\tau_2$

Outline

- Resurgence: a quick review
- Series expansion of $Z(\tau,\bar{\tau},\lambda)$: recursive method
- Saddle-point analysis
- Stokes phenomenon
- Discussions

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Resurgence: a quick review

- Perturbation series $Z(\lambda) =? \sum_{n\geq 0} Z_n \lambda^n$: convergent/asymptotic
- Convergent series: finite radius of convergence r

$$Z(\lambda) = \text{perturbation series for } |\lambda| < r$$

- Asymptotic series: zero radius of convergence

① Exact result $Z(\lambda) \neq$ perturbation series for all λ

② Optimal approximation truncated at a particular finite order

③ In terms of coefficients: $\lim_{n\to\infty} \frac{Z_n}{Z_{n+1}} = r \to 0$, e.g. $Z_n \sim n!$

④ Exact result $Z(\lambda)$ v.s. perturbation series: non-perturbative corrections

Resurgence: a quick review

- Usually: $Z(\lambda)$ defined by (path-)integral, e.g. QFTs

$$Z(\lambda) = \int D\phi \; e^{-\lambda^{-1} I(\phi)}$$

- Series in small λ: semi-classical approximation

  ① Dominated by saddle-point configurations ϕ^* satisfying $\delta I(\phi^*) = 0$

  ② Perturbative vacuum: $I(\phi^*) = 0$

  ③ Gaussian integral about ϕ^*: $\int D\delta\phi \; e^{-\frac{1}{2\lambda}\delta\phi\, I''(\phi^*)\delta\phi + \cdots} = Z_0 + Z_1\lambda + \ \ldots$

- In general: there could be multiple saddles $\{\phi_i^*\}$
- Gaussian integral about each saddle ϕ_i^*:

$$e^{-\frac{I(\phi_i^*)}{\lambda}} \int D\delta\phi \;\; e^{-\frac{1}{2\lambda}\delta\phi\, I''(\phi_i^*)\delta\phi + \cdots} = e^{-\frac{I(\phi_i^*)}{\lambda}} (Z_0^i + Z_1^i \lambda + \cdots)$$

Resurgence: a quick review

- Summing over all saddles, we expect:

  ① Full result $Z(\lambda) = \sum_n Z_n \lambda^n + c^i \sum_i e^{-\frac{I(\phi^*)}{\lambda}} (\sum_n Z_n^i \lambda^n)$ ← "trans-series".

  ② The coefficients c^i could be zero: saddle-point ϕ_i^* does not contribute

  ③ Depends its relation to the integration contour

  ④ If $c^i \neq 0$ for any $I(\phi_i^*) \neq 0$: non-perturbative effects → $Z(\lambda) \neq$ perturbation series

- Asymptotic series ↔ non-perturbative corrections ↔ additional saddles
- *Resurgence: non-perturbative corrections leave their "footprints" in the perturbation series!*
- (very) roughly speaking: perturbation series at large orders "probe" far away in configuration space to "know about" other saddles

Resurgence: a quick review

- Stokes' phenomenon: non-perturbative discontinuity in complex coupling plane

① $Z(\lambda) = \int_\Gamma D\phi \; e^{-I(\phi)/\lambda} = \sum_m c_m \int_{\Gamma_m} e^{-I(\phi)/\lambda} = \sum_m c_m \; e^{-\frac{I(\phi_m)}{\lambda}} (Z_0^m + Z_1^m \lambda + \cdots)$

② Steepest descent contour Γ_m through saddle point ϕ_m: Lefschetz thimbles

③ Integral coefficients c_m determined by topological decomposition: $\Gamma = \cup_m \; c_m \, \Gamma_m$

④ Fixing Γ (physical contour), varying $\lambda = |\lambda| e^{i\theta}$, Γ_m also deform accordingly

⑤ ***<u>Stokes' phenomena</u>*** **:** Critical values θ^* across which $c_m \to c_m \pm 1$

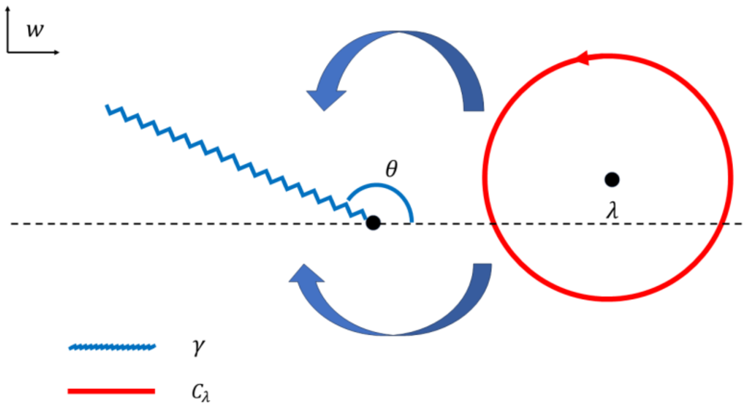
⑥ Stokes' ray $\gamma = \{\mathrm{Arg}(\lambda) = \theta^*\}$: branch-cut of $Z(\lambda)$ with $Disc_\gamma \, Z(\lambda) \sim e^{-\frac{I(\phi_m)}{\lambda}} = Z_{np}(\lambda)$

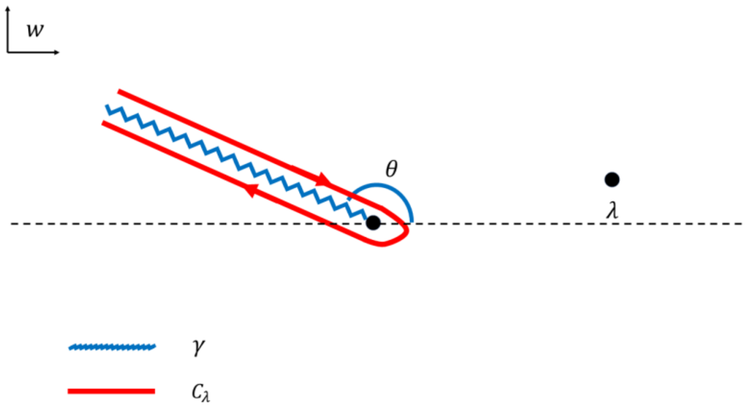
⑦ Criterion for Stokes' ray: $\mathbf{Im}\left(\frac{I(\phi_m)}{\lambda}\right) = \mathbf{Im}\left(\frac{I(\phi_0)}{\lambda}\right)$; $\mathbf{Re}\left(\frac{I(\phi_m)}{\lambda}\right) > \mathbf{Re}\left(\frac{I(\phi_0)}{\lambda}\right)$

Resurgence: a quick review

- Stokes' phenomenon: relating $Z_{np}(\lambda)$ to perturbation series $\sum_n Z_n \lambda^n$

- Dispersion relation: $Z(\lambda) = \oint_{C_\lambda} \frac{Z(w)dw}{w-\lambda} = \int_\gamma \frac{Disc_\gamma Z(w)dw}{w-\lambda} = \int \frac{Z_{np}(w)dw}{w-\lambda} = \int_0^\infty \frac{e^{-\frac{I(\phi_m)}{w}}(Z_0^m + Z_1^m\lambda + \cdots)}{w-\lambda}$





Resurgence: a quick review

- Stokes' phenomenon: relating $Z_{np}(\lambda)$ to perturbation series $\sum_n Z_n \lambda^n$
- Dispersion relation: $Z(\lambda) = \oint_{C_\lambda} \frac{Z(w)dw}{w-\lambda} = \int_\gamma \frac{Disc_\gamma Z(w)dw}{w-\lambda} = \int \frac{Z_{np}(w)dw}{w-\lambda} = \int_0^\infty \frac{e^{-\frac{I(\phi_m)}{w}}(Z_0^m + Z_1^m w + \cdots)}{w-\lambda}$
- Expanding both sides in small λ: $\sum_n Z_n \lambda^n = \sum_n \lambda^n \int_0^\infty Z_{np}(w)\, w^{1-n}\, dw$
- Identifying coefficients: $Z_n = \int_0^\infty Z_{np}(w)\, w^{1-n}\, dw,\ Z_{np}(w) \sim e^{-I(\phi_m)/w}$
- At large order n: saddle-point approximation → $Z_n \sim \frac{n!}{A^n}(1+\cdots),\ A = I(\phi_m)$
- Non-perturbative action can be extracted from $\lim_{n\to\infty} s_n = \frac{Z_{n+1}}{n\, Z_n} = A$

Resurgence: a quick review

- Resurgence analysis can progress in both ways:
    - Knowledge of saddle-points → (prediction for) large order perturbation series
    - Knowledge of large order perturbation series → (clues for) saddle-point origins of non-perturbative effects
- Our goal: non-perturbative aspects of $T\bar{T}$-deformed partition function
- Strategy: compute $Z(\tau, \bar{\tau}, \lambda)$ to large order in λ; using resurgence analysis to help identifying non-perturbative effects

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Series expansion of $\boldsymbol{Z(\tau, \bar{\tau}, \lambda)}$: recursive method

- Hard way: conformal perturbation theory

Perturbative expansion for the action: $\partial_\lambda S^\lambda = \int d^2x \, Det \, T^\lambda_{ij}(x) \rightarrow \; S^\lambda = S_{CFT} + \lambda \int d^2x \, Det \, T_{ij}(x) + \cdots$

Perturbative expansion for partition function: $Z(\tau, \bar{\tau}, \lambda) = \int D\phi \, e^{-\int_T d^2x \; S^\lambda(\phi)}$

- Short-cut: deformed partition function from deformed spectrum

$$Z(\tau, \bar{\tau}, \lambda) = \sum_n e^{-\tau_2 \, E_n(\lambda) + i \, \tau_1 P_n}, \qquad E_n(\lambda) = \frac{R}{2\lambda}\left(\sqrt{1 + \frac{4\lambda E_n}{R} + \frac{4\lambda^2 P_n^2}{R^2}} - 1\right)$$

- Spectral flow equation → Partition function satisfies flow equation (Cardy 18; Datta and Jiang 18)

$$\partial_\lambda \, Z(\tau, \bar{\tau}, \lambda) = \left[\tau_2 \, \partial_\tau \partial_{\bar{\tau}} + \frac{1}{2}\left(\partial_{\tau_2} - \frac{1}{\tau_2}\right) \lambda \partial \lambda\right] Z(\tau, \bar{\tau}, \lambda)$$

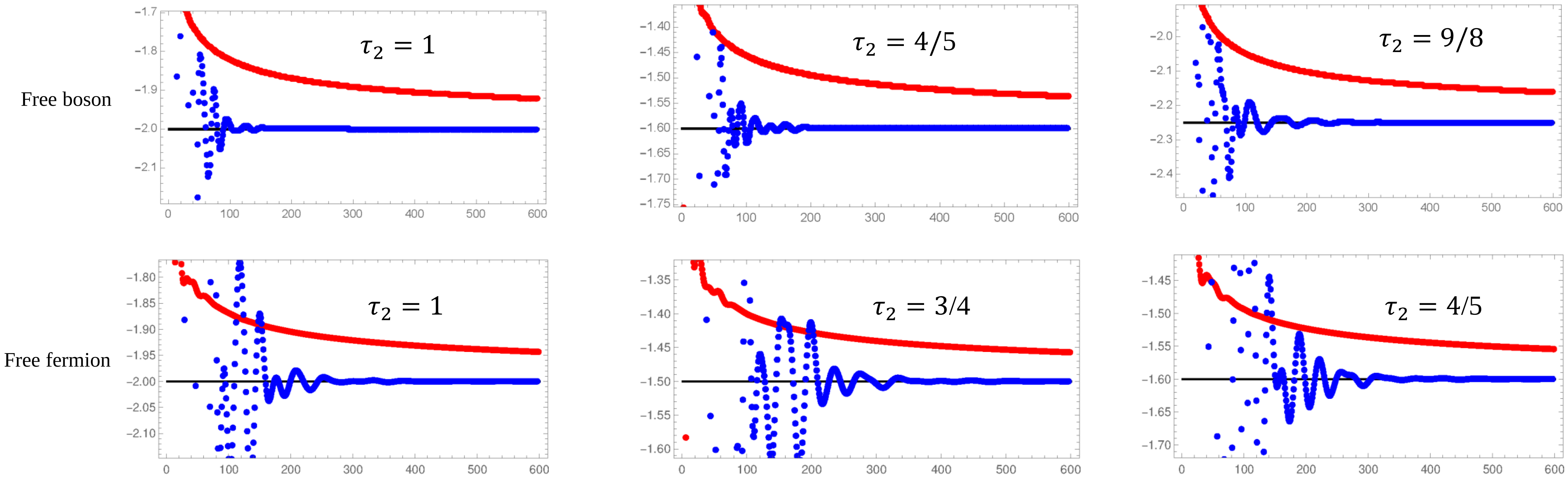
Series expansion of $\boldsymbol{Z(\tau,\bar{\tau},\lambda)}$: recursive method

- Modular "covariance": $Z(\tau,\bar{\tau},\lambda)=Z\left(\frac{a\tau+b}{c\tau+d},\frac{a\bar{\tau}+b}{c\bar{\tau}+d},\frac{\lambda}{|c\tau+d|^2}\right)$ (Aharony et al 19)
- Recursive relation for perturbation coefficients: $Z(\tau,\bar{\tau},\lambda)=\sum_n Z_n(\tau,\bar{\tau})\lambda^n$
  - $Z_{k+1}(\tau,\bar{\tau})=\frac{\tau_2}{k+1}\left(D_\tau^{(k)}D_{\bar{\tau}}^{(k)}-\frac{k(k+1)}{4\tau_2^2}\right)Z_k(\tau,\bar{\tau}),\quad Z_0(\tau,\bar{\tau})=Z_{CFT}(\tau,\bar{\tau})$
  - Ramanujan-Serre (RS) derivatives: $D_\tau^{(k)}=\partial_\tau-\frac{ik}{2\tau_2}, D_{\bar{\tau}}^{(k)}=\partial_{\bar{\tau}}+\frac{ik}{2\tau_2}$
- Efficient algorithm to generate the coefficients **(credits to collaborators: J.G and Y.Jiang)**
  - $Z_k(\tau,\bar{\tau})$ are elements of a differential ring closed under RS derivatives
  - Free boson: generated by $\{\eta^{-1}(\tau),\tilde{E}_2(\tau,\bar{\tau}),E_4(\tau),E_6(\tau)\}+c.c$; $\tilde{E}_2(\tau,\bar{\tau})=E_2(\tau)-\frac{3}{\pi\tau_2}$
  - Free fermion: generated by $\{\eta^{-1}(\tau),\ \theta_2(\tau),\Theta_{34}(\tau),\ \tilde{E}_2(\tau,\bar{\tau}),E_4(\tau),E_6(\tau)\}+c.c$
  - Many clever tricks to streamline the computation!

Series expansion of $Z(\tau, \bar{\tau}, \lambda)$: recursive method

- To be explicit, we focus on the free boson and free fermion CFTs
  - $Z_0^B(\tau, \bar{\tau}) = \frac{1}{\sqrt{\tau_2}\eta(\tau)\eta(\bar{\tau})}$, $Z_0^F(\tau, \bar{\tau}) = \sum_{i=2,3,4} \left|\frac{\theta_i(\tau)}{\eta(\tau)}\right|$
- Compute numerical series for fixed modular parameter: $\tau = -\bar{\tau} = i\tau_2$
- Extract the coefficient ratios at large orders: $s_n = \frac{Z_{n+1}}{n\, Z_n}$
- Using the available computing power, we generated up to $n = 600$ orders
- Large n expansion of s_n: Richardson transformation (faster convergence)
- So, let's show some plots!

Series expansion of $Z(\tau,\bar{\tau},\lambda)$: recursive method



- Numerical observation in both free boson/fermion: $\lim_{n\to\infty} s_n = -2\tau_2$

Series expansion of $Z(\tau, \bar{\tau}, \lambda)$: recursive method

- Recall the resurgence analysis:

$$\lim_{\lambda \to 0} Z_{np}(\lambda) \sim e^{-\frac{A}{\lambda}} \to \lim_{n \to \infty} Z_n \sim \frac{n!}{A^n} \to \lim_{n \to \infty} s_n \sim A$$

- Numerical observation suggests:

$$\lim_{\lambda \to 0} Z_{np}(\tau, \bar{\tau}, \lambda) \sim e^{2\tau_2/\lambda}$$

- Can we explain such a non-perturbative contribution?
  - Path-integral representation of $Z(\tau, \bar{\tau}, \lambda)$
  - Coupling constant is λ
  - Find saddle-point contribution that accounts for $Z_{np}(\tau, \bar{\tau}, \lambda)$

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Saddle-point analysis:

- Alternative formulation of $T\bar{T}$-deformed CFTs
  - CFT coupled to flat JT-gravity (coupling constant λ) (Dubovsky et al 18)
  - CFT as a non-critical string theory (string tension λ) (H.Verlinde et al 20; Kutasov et al 20)
  - Insight from both: "dynamical coordinates"
- Integral representation of torus partition function:
  - $Z(\tau,\bar{\tau},\lambda) = \frac{\tau_2}{\pi\lambda}\int_{H_+}\frac{d^2\zeta}{\zeta_2^2}\, e^{-\frac{|\zeta-\tau|^2}{\lambda\zeta_2}} Z_{CFT}(\zeta,\bar{\zeta})$
  - Dynamical modular parameter $(\zeta,\bar{\zeta})$
  - Integration contour: upper-half-plane $H_+ = \{\zeta_1 \in \mathbb{R},\ \zeta_2 \in \mathbb{R}^+\}$
  - Semi-classical limit $\lambda \to 0$: saddle-point approximation
  - λ only enters the $T\bar{T}$-deformation "kernel": $K_{T\bar{T}}(\tau,\bar{\tau},\zeta,\bar{\zeta},\lambda) = e^{-\frac{|\zeta-\tau|^2}{\lambda\zeta_2}}$

Saddle-point analysis:

- Saddle-point analysis is easy:
  - $\partial_{\zeta,\bar{\zeta}}\left(\frac{|\zeta-\tau|^2}{\zeta_2}\right) = 0 \;\rightarrow\; \zeta_1^* = \tau_1,\; \zeta_2^* = \pm\tau_2$
  - Perturbation series from expansion about the "physical" saddle point $\zeta_1^* = \tau_1,$ $\zeta_2^* = \tau_2$: $Z_{pert} = Z_{CFT}(1 + b_1\lambda + \cdots)$
  - Non-perturbative contribution from $\zeta_1^* = \tau_1,\; \zeta_2^* = -\tau_2$: $Z_{np} \sim e^{\frac{4\tau_2}{\lambda}}$
  - Inconsistent with the numerical observation: $Z_{np} \sim e^{\frac{2\tau_2}{\lambda}}$!
- There must be additional saddle-point contributions to $Z(\tau,\bar{\tau},\lambda)$
- Puzzle: how can they arise given the simple λ-dependence of the integrand?

Saddle-point analysis:

- Revisit saddle-point analysis:
  - $Z(\tau,\bar{\tau},\lambda) = \frac{\tau_2}{\pi\lambda}\int_{H_+}\frac{d^2\zeta}{\zeta_2^2}\, e^{-I(\lambda)},\ I(\lambda) = -\frac{|\zeta-\tau|^2}{\lambda\zeta_2} + F_{CFT}(\zeta,\bar{\zeta}),\ Z_{CFT} = e^{-F_{CFT}}$
  - $F_{CFT}(\zeta,\bar{\zeta})$ is sub-leading in small λ: saddle-points independent of F_{CFT}
  - A caveat: $F_{CFT}(\zeta,\bar{\zeta})$ could diverge at $\zeta^{sing},\bar{\zeta}^{sing}$
  - Near $\zeta^{sing},\bar{\zeta}^{sing}$, F_{CFT} could be "leading order" effectively
  - $\partial_{\zeta,\bar{\zeta}}\left(-\frac{|\zeta-\tau|^2}{\lambda\zeta_2} + F_{CFT}(\zeta,\bar{\zeta})\right)_{\zeta^*,\bar{\zeta}^*} = 0,\quad (\zeta^*,\bar{\zeta}^*) = (\zeta^{sing},\bar{\zeta}^{sing}) + O(\lambda^\alpha)$
- A subtle mechanism of saddle-point: "supported" by singularities of $F_{CFT}(\zeta,\bar{\zeta})$
- To identify them $\leftarrow$ analytic structure of $F_{CFT}(\zeta,\bar{\zeta})$

Saddle-point analysis:

- Universal divergences of F_{CFT}:
  - low temperatures: $\lim_{\zeta,\bar{\zeta}\to\infty} F_{CFT}(\zeta,\bar{\zeta}) \sim \frac{\pi c}{12}(\zeta+\bar{\zeta})$
  - high temperatures: $\lim_{\zeta,\bar{\zeta}\to 0} F_{CFT}(\zeta,\bar{\zeta}) \sim \frac{\pi c}{12}\left(\frac{1}{\zeta}+\frac{1}{\bar{\zeta}}\right)$
  - Related by modular (inversion) invariance
- For integrable/rational CFTs, chiral divergences from ζ and $\bar{\zeta}$ independently
  - Finite rep. under modular transformation: $\chi_\alpha(\zeta) = \sum_\beta S^{\alpha\beta}\chi_\beta(1/\zeta) \to \lim_{\zeta\to 0,\infty}\chi_\alpha(\zeta) \to \infty$
  - Finite sum over characters: $Z(\zeta,\bar{\zeta}) = \sum_{\alpha,\beta} n^{\alpha\beta}\chi_\alpha(\zeta)\bar{\chi}_\beta(\bar{\zeta})$
  - $Z(\zeta,\bar{\zeta})$ diverges for $\zeta \to 0,\infty$ and $\bar{\zeta} \to 0,\infty$ independently

Saddle-point analysis:

- Toy model approximation: $F_{CFT}(\zeta,\bar{\zeta}) \approx F_{TM}(\zeta,\bar{\zeta}) = \frac{\pi c}{12}\left(\zeta + \frac{1}{\zeta}\right) + \frac{\pi \bar{c}}{12}\left(\bar{\zeta} + \frac{1}{\bar{\zeta}}\right)$
  - captures the leading order divergences near $\zeta \to (0,\infty)$ and $\bar{\zeta} \to (0,\infty)$ independently
  - sufficient for saddle-point analysis
- Additional saddle-point solutions:
  - Solving $\partial_{\zeta,\bar{\zeta}}\left(-\frac{|\zeta-\tau|^2}{\lambda\zeta_2} + F_{TM}(\zeta,\bar{\zeta})\right) = 0 \to \ \zeta_1^* = s''\sqrt{\frac{6\tau_2^2+2\pi c s\tau_2\lambda}{2\pi c\lambda}}\ ,\zeta_2^* = i\tau_2 s'\sqrt{\frac{3}{\pi c\lambda}},\ \ s,s',s'' = \pm 1$
  - For small $\lambda \to 0$: $\zeta^* \sim \pm\sqrt{\frac{\pi c\lambda}{12}} \to 0,\ \ \bar{\zeta}^* \sim \pm\sqrt{\frac{12}{\pi c\,\lambda}}\tau_2 \to \infty$, self-consistent with approximation!
  - Analytically continued away from the Euclidean section $\bar{\zeta} = \zeta^*$; in the “Regge” regime
  - Universal for integrable CFTs

Saddle-point analysis:

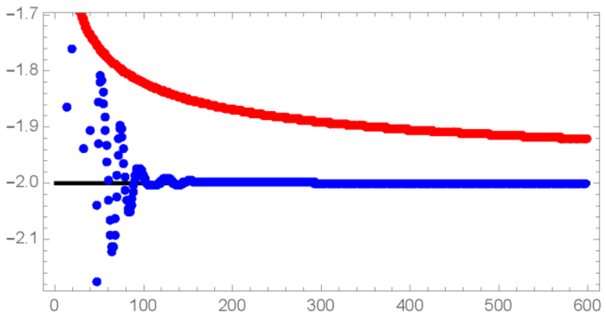
- Plugging them into the toy model effective action:
  - Non-perturbative contribution: $Z_{np}(\tau,\bar{\tau},\lambda) \sim \lambda^{\frac{1}{2}(\alpha+1)} e^{\frac{2\tau_2}{\lambda}+i\sqrt{\frac{4\pi c}{3\lambda}}(\pm\tau_2\pm 1)+O(\sqrt{\lambda})}$
  - Extracting the leading order action: $A=-2\tau_2$, consistent with numerical observation!
  - For free scalar: $\alpha=1, c=\frac{1}{2}$; for free fermion $\alpha=0, c=\frac{1}{4}$
  - Non-perturbative sector: expansion in $\sqrt{\lambda}$
- Checking higher order corrections:
  - General formula: $Z_{np} \sim \lambda^{-\nu} e^{-\frac{A}{\lambda}-\frac{B}{\sqrt{\lambda}}+O(\sqrt{\lambda})} \to s_n \sim A-\frac{B}{2}\sqrt{\frac{A}{n}}+\frac{B^2-8A\nu}{8n}+O(n^{-3/2})$
  - Richardson transformation on s_n for each order in large n
  - Let's look at more plots!

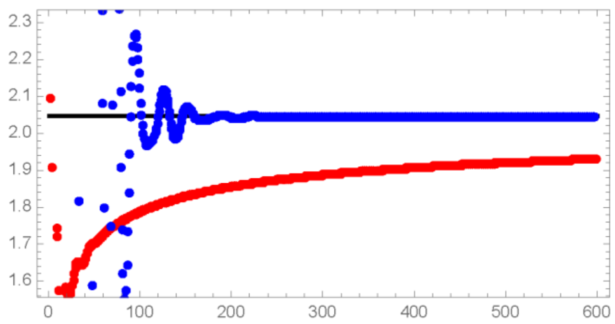
Saddle-point analysis:

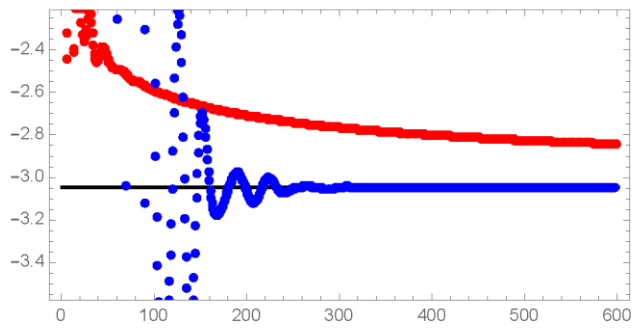
$O(1)$ $O(n^{-1/2})$ $O(n^{-1})$

Free boson

$\tau_2 = 1$

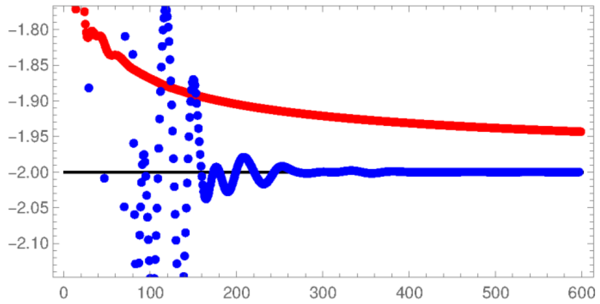


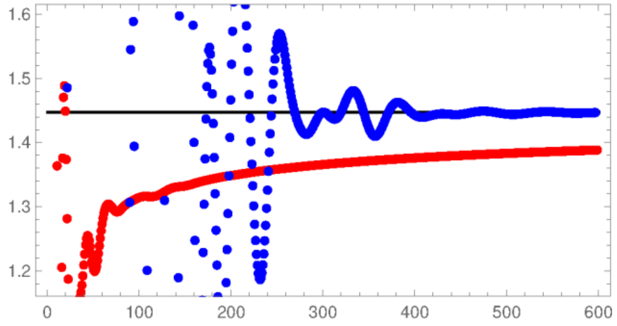


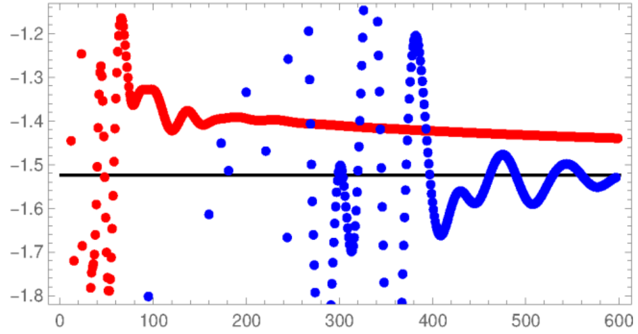


Free fermion

$\tau_2 = 1$





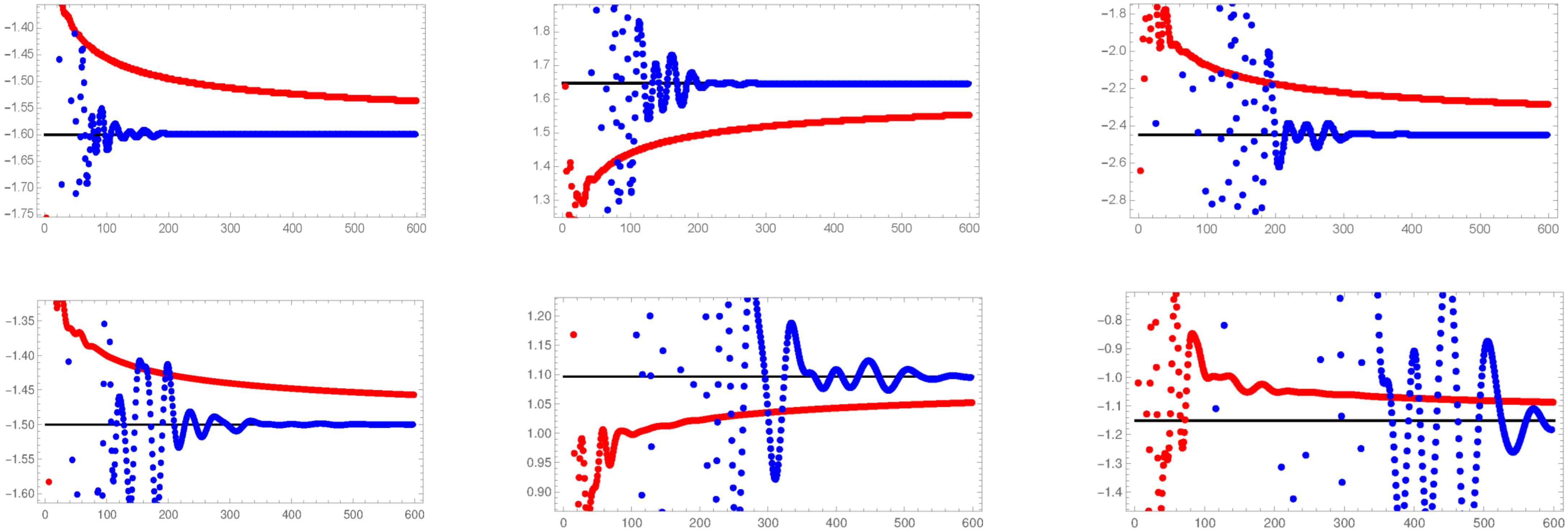


Saddle-point analysis:

$O(1)$ $O(n^{-1/2})$ $O(n^{-1})$

Free boson $\tau_2 = 4/5$

Free fermion $\tau_2 = 3/4$



Saddle-point analysis:

- Good matches up to order n^{-1} in s_n, strong evidence we are on the right track
- Higher orders do not match -- toy model result no longer accurate from $O(\sqrt{\lambda})$
- Free fermions converge more slowly
  - One possible explanation: define $\delta Z = Z_{CFT} - Z_{TM}$
  - Near saddle-points (Regge regime): $\delta Z \sim q$ for boson; $\delta Z \sim q^{1/4}$ for fermion, $q \sim e^{-\frac{K}{\sqrt{\lambda}}}$
- Exact CFT v.s. toy model: additional decaying oscillatory features
  - Full modular invariance of Z_{CFT} v.s. only inversion invariance of Z_{TM}
  - Modular images of the “Regge” singularities → additional non-perturbative corrections
  - Infinitely many!
  - Can check: they give decaying and oscillatory contributions to s_n

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Stokes phenomenon

- Resurgence analysis helps identifying (potential) non-perturbative corrections
- They may be absent, i.e. $\Gamma = \cup_m \; c_m \, \Gamma_m, \; c_m = 0, \; \Gamma = H_+$
- Stokes phenomenon: $c_m \to c_m \pm 1$ as coupling $\lambda = |\lambda| e^{i\theta}$ rotates in complex plane
- Stokes ray: critical θ_m s.t. $\mathbf{Im}\left(\frac{I(\phi_m)}{\lambda}\right) = \mathbf{Im}\left(\frac{I(\phi_0)}{\lambda}\right)$; $\mathbf{Re}\left(\frac{I(\phi_m)}{\lambda}\right) > \mathbf{Re}\left(\frac{I(\phi_0)}{\lambda}\right)$
- For $T\bar{T}$-deformed CFTs, very distinct physical properties for:

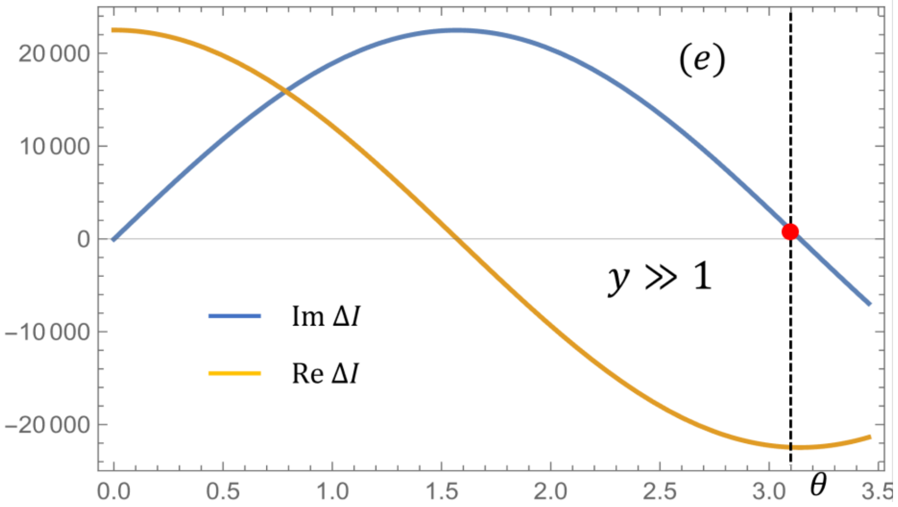
  (i) $\lambda > 0$ (Hagedorn density of states, stringy); (ii) $\lambda < 0$ (complex spectrum or UV cut-off)
- From $\lambda > 0$ to $\lambda < 0$: Stokes phenomenon?
- We will analyze using Z_{TM}:
  - $Z(\tau, \bar{\tau}, \lambda) = \frac{\tau_2}{\pi\lambda} \int_{H_+} \frac{d^2\zeta}{\zeta_2^2} \; e^{-\frac{|\zeta-\tau|^2}{\lambda\zeta_2}} Z_{TM}(\zeta, \bar{\zeta}) = \frac{\tau_2}{\pi\lambda} \int_{H_+} \frac{d^2\zeta}{\zeta_2^2} \; e^{-\frac{|\zeta-\tau|^2}{\lambda\zeta_2} + \frac{\pi c}{12}\left(\zeta + \frac{1}{\zeta} + \bar{\zeta} + \frac{1}{\bar{\zeta}}\right)}$

Stokes phenomenon

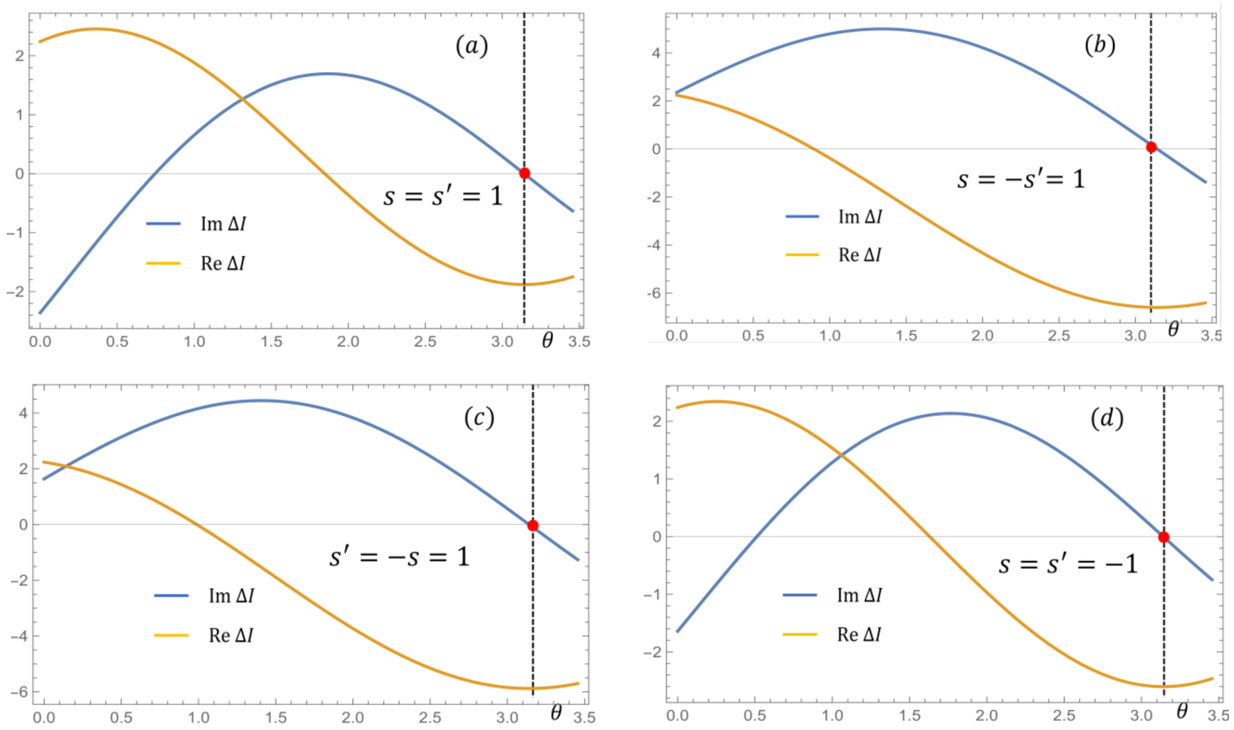
- For $\lambda > 0$ $(\theta = 0)$ and $\tau_2 > \tau_2^c = \sqrt{\frac{\pi c \lambda}{3}}$, can check that $\Gamma = \Gamma_{phys}$
  - Γ_{phys} : Lefschetz thimble through the physical saddle $\zeta_1^* = 0, \zeta_2^* = \tau_2$
  - The additional saddles do not contribute: $Z\,(\tau_2, \lambda > 0) \sim Z_{pert}(\tau_2, \lambda)$
  - $Z_{pert}(\tau_2, \lambda) = e^{\frac{2\tau_2}{\lambda} - \frac{2}{3\lambda}\sqrt{(3\tau_2^2 - \pi c \lambda)(3 - \pi c \lambda)}} \sim Z_{CFT}(\tau_2)(1 + O(\lambda))$
  - Singularity in τ_2 at τ_2^c : Hagedorn type?
- Rotate $\lambda = |\lambda| e^{i\theta},\ \theta \in [0, \pi]$ in the complex coupling plane:
  - Identify Stokes ray associated with the additional saddles
  - Compute $\Delta Re\ I(\theta) = Re\ I(\zeta^*, \bar{\zeta}^*) - Re\ I(\zeta_{phys}, \bar{\zeta}_{phys}); \Delta Im\ I(\theta) = Im\ I(\zeta^*, \bar{\zeta}^*) - Im\ I(\zeta_{phys}, \bar{\zeta}_{phys})$
  - Look for solutions with $\Delta\, Im\ I(\theta) = 0,\ \Delta\, Re\ I(\theta) < 0$

Stokes phenomenon

$\tau_2 \gg \tau_2^c$

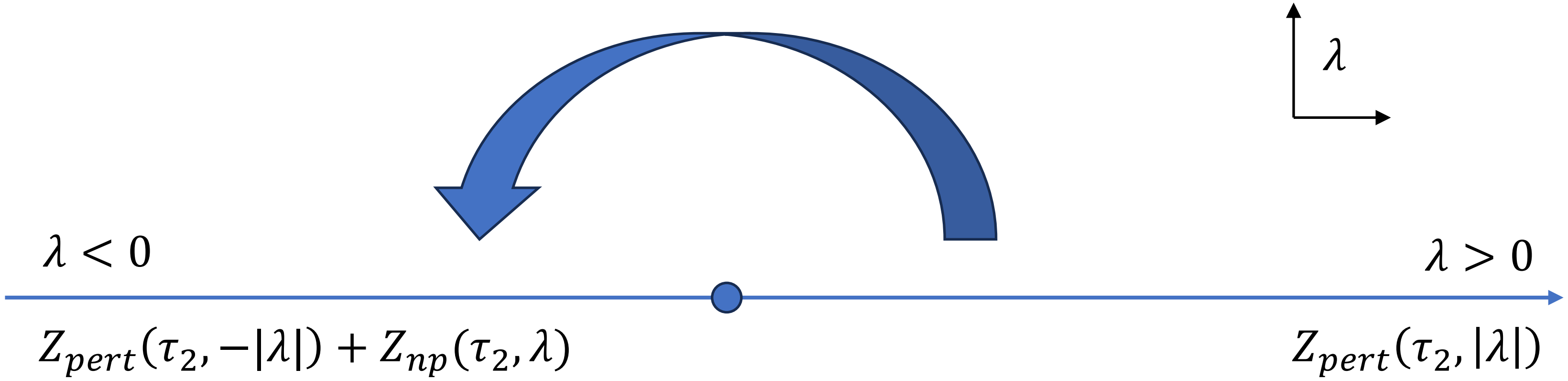


$\tau_2 \sim \tau_2^c$



Stokes phenomenon

- For any $\tau_2 > \tau_2^c$, Stokes phenomenon at $\theta = \pi$ ($\lambda < 0$) for all additional saddles
  - Receives non-perturbative corrections at $\lambda < 0$
  - $Z(\tau_2, \lambda < 0) = Z_{pert}(\tau_2, -|\lambda|) + Z_{n.p}(\tau_2, \lambda) \sim e^{-\frac{2\tau_2}{|\lambda|}+\frac{2}{3|\lambda|}\sqrt{(3\tau_2^2+\pi c|\lambda|)(3+\pi c|\lambda|)}} + e^{-\frac{2\tau_2}{|\lambda|}+\sqrt{\frac{4\pi c}{3|\lambda|}}+\cdots}$
  - Partition function non-singular and well-defined for all $\tau_2 > 0$



Stokes phenomenon

- Deformed density of states: $Z(\tau_2, \lambda < 0) = \int dE\, g(E,\lambda) e^{-\tau_2 E} = Z_{pert} + Z_{n.p}$
- Z_{pert} alone: complex density/spectrum at UV
- Consistent with $\lambda \to -|\lambda|$ of flow equation solution: $E_n(\lambda < 0) = \frac{2}{|\lambda|}\left(1 - \sqrt{1 - |\lambda| E_n + \frac{\lambda^2 P_n^2}{4}}\right)$
- $Z_{n.p}$ alone implies a UV cut-off Λ: $\int^{\Lambda} dE\, e^{-\tau_2 E}\, g(E) \sim e^{-\frac{2\tau_2}{|\lambda|} + \sqrt{\frac{4\pi c}{3|\lambda|}}} \to\ \Lambda = \frac{2}{|\lambda|},\ \ln g(E) = \sqrt{\frac{2\pi c E}{3}}$
- Tow folklores of $\lambda < 0$ theories: Z_{pert} v.s. $Z_{n.p}$

Stokes phenomenon

- However, this can't be the whole story
- Z_{pert} v.s $Z_{n.p}$: mutually incompatible picture
- $Z(\tau_2, \lambda < 0)$ for $\tau_2 < \tau_2^c$: order of (i) $\lambda \to -\lambda$; (ii) $\tau_2 \to \tau_2^{c^-}$
- Possibly requires resolving the Hagedorn transition at $\lambda > 0$, "winding mode condensation"
- Current integral rep of $Z(\tau_2, \lambda)$ derived assuming unit winding, need to consider all winding sectors? Future work…
- Exact CFT v.s. toy model: modular symmetry → infinitely many other non-perturbative corrections

Outline

- Resurgence: a quick review
- Series expansion of $Z(\tau, \bar{\tau}, \lambda)$: recursive method
- Saddle-point analysis
- Stokes phenomenon
- Discussions

Summary

- We studied large order expansion $Z_{pert}(\tau,\bar{\tau},\lambda) = \sum_n Z_n(\tau,\bar{\tau})\lambda^n$
- Recursive relation: $Z_{k+1}(\tau,\bar{\tau}) = \frac{\tau_2}{k+1}\left(D_\tau^{(k)} D_{\bar{\tau}}^{(k)} - \frac{k(k+1)}{4\tau_2^2}\right) Z_k(\tau,\bar{\tau})$
- Efficient implementation in terms of differential ring generated by modular forms
- Resurgence analysis of $s_n = \frac{Z_{n+1}}{n\, Z_n}$ at large orders reveals $Z_{n.p} \sim e^{\frac{2\tau_2}{\lambda}}$
- Using integral rep over dynamic modular parameter, novel saddle-point origin of Z_{np}:

  supported by "Regge" singularity of (integrable) $Z_{CFT}(\zeta \to 0, \bar{\zeta} \to \infty)$
- Stokes phenomenon: $Z(\lambda > 0) = Z_{pert} \to Z(\lambda < 0) = Z_{pert} + \mathrm{Z}_{n.p}$
- Implication for spectral property at $\lambda < 0$: Z_{pert} (complex spectrum) v.s. $Z_{n.p}$ (UV cut-off)

Outlook

- Other non-perturbative corrections in actual CFT results
- Hagedorn transition at $\lambda > 0$: summing over all winding sectors?
- Fully understand properties of $\lambda < 0$ theories
- Non-perturbative effects in other deformed observables, e.g: correlation functions; entanglement entropy; defect properties; etc.
- Deformed integrable CFTs → deformed chaotic CFTs → deformed holographic CFTs

Thank you!