Non-perturbative Overlaps in JT gravity

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based on
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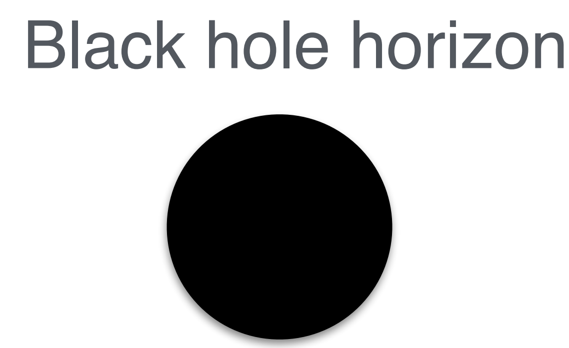
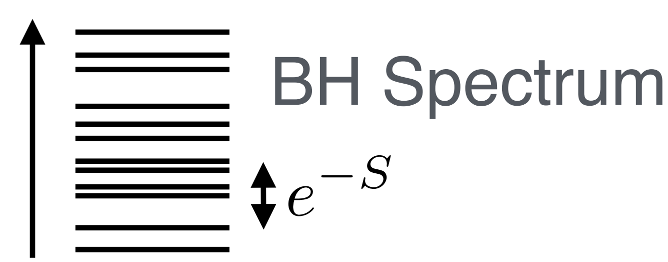
Finite Black Hole Entropy

Bekenstein-Hawking Entropy: [Bekenstein (1973)][Hawking (1975)]
BH has finite entropy given by

$$S_{\mathrm{BH}} = \frac{\mathrm{Area}[\mathrm{Horizon}]}{4G_N}$$



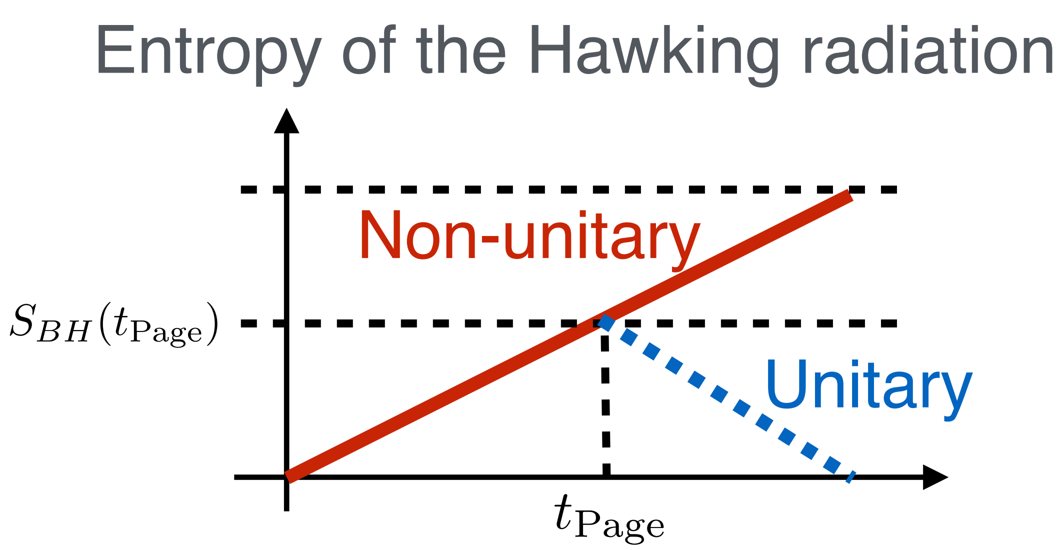
- This implies that the number of BH microstates is finite, and therefore the spectrum is discrete



- Thus QFT on curved background including gravitons, which usually have an infinite local degrees of freedom, must be modified

Tension

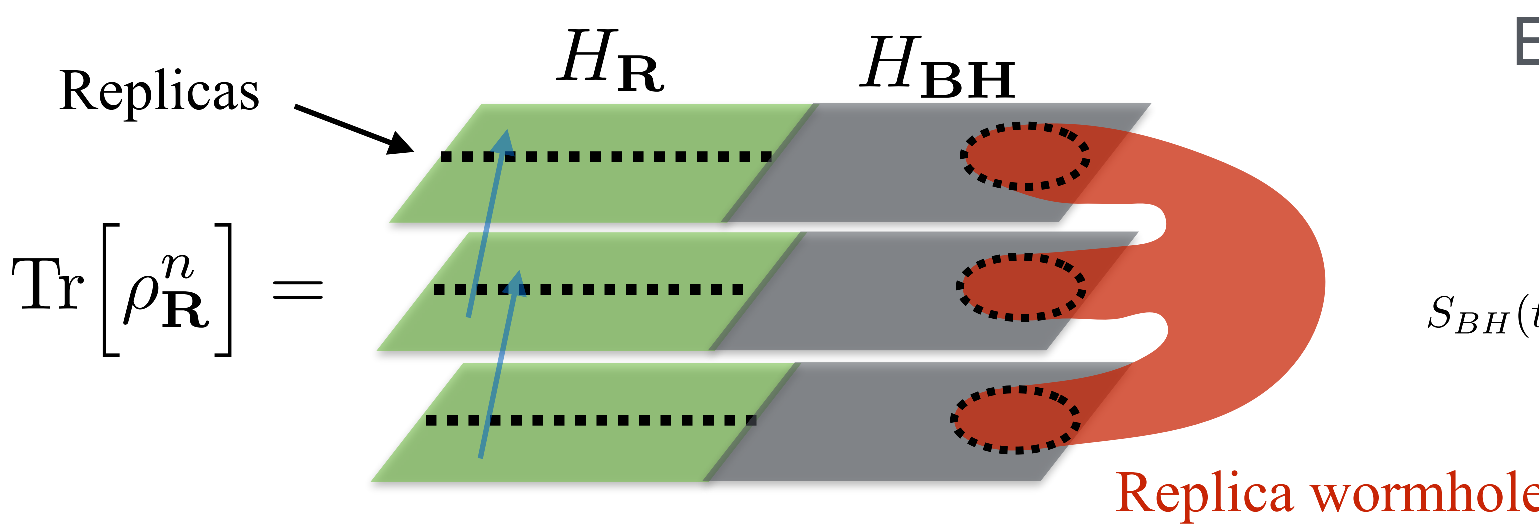
Black hole information paradox: [Hawking (1974)]
Hawking radiation entangled with the BH, appears to have much larger entropy than allowed; more than BH entropy

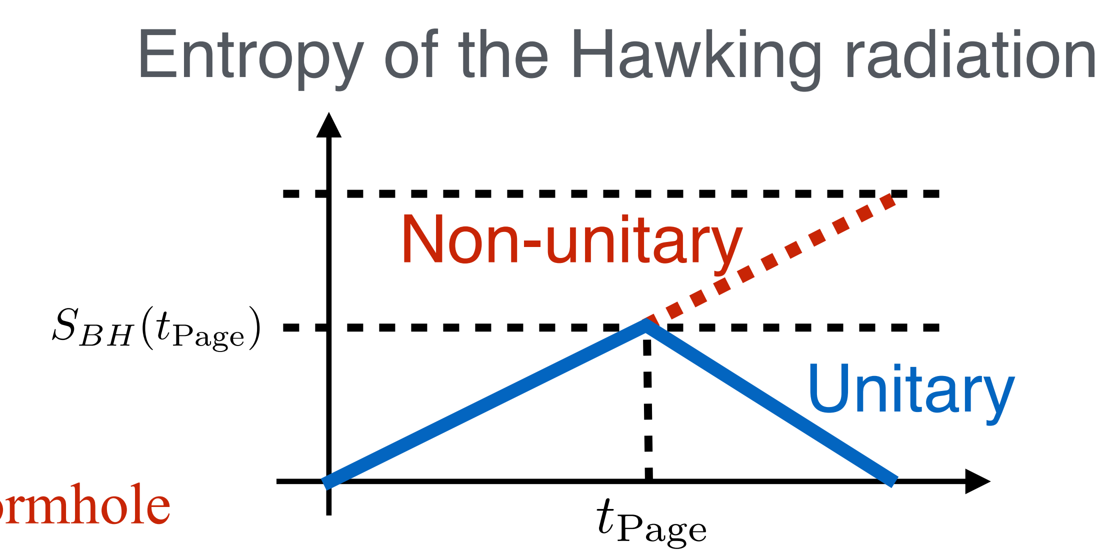


Fixing the gravitational path-integral: Euclidean Wormholes

Discreteness from Euclidean Wormholes:

Adding Euclidean wormholes to the gravitational path-integral was found to give unitary Page curve, resolving the paradox for the Hawking radiation entropy





[Penington (2019)][Almheiri, Engelhardt, Markolf, Maxfield (2019)][Almheiri, Mahajan, Maldacena, Zhao (2019)]
[Penignton, Shenker, Stanford, Yang(2019)][Almheiri, Hartman, Maldacena, Shaghoulian, Tajdini (2019)]

Why adding Euclidean wormholes works?

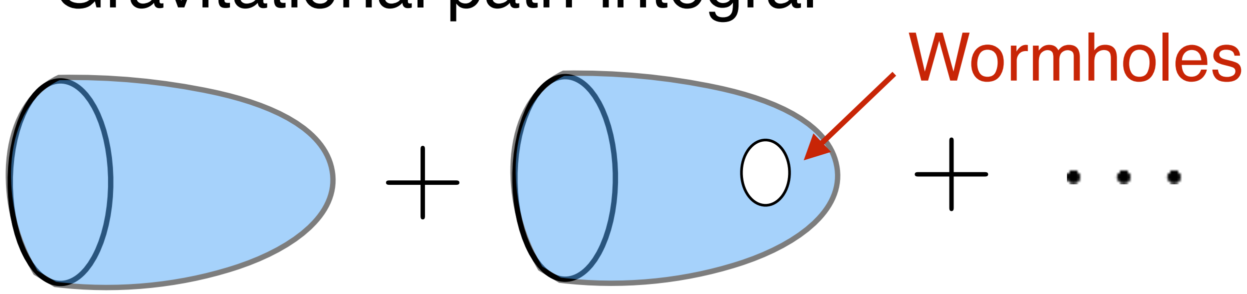
[Saad, Shenker, Stanford (2018)][Saad, Shenker, Stanford (2019)]
[Bousso, Tomasevic (2019)][Bousso, Wildenhain(2020)]

Ensemble Averaging:
In AdS/CFT, the dual of the gravitational path-integral with Euclidean wormholes is an ensemble of microscopic theories, each of which corresponds to microscopic gravity theory

Jackiew-Teitelboim gravity: 1+1 dilation gravity on AdS

Gravitational path-integral

Matrix integral



$$+ \cdots = \int dH\, e^{-\mathrm{tr}[V(H)]}$$

Sum over all possible smooth geometries

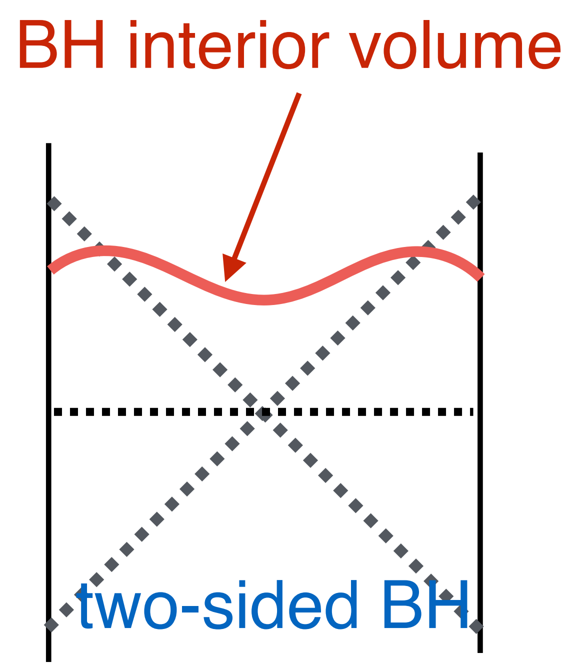
H: microscopic Hamiltonian at boundary

- After ensemble averaging, the spectrum becomes smooth

Two Tensions

Continuous spectrum in perturbative gravity:
When we canonically quantize gravity, variables like induced metric are continuous and unbounded, appears to conflict with finite BH entropy (in microcanonical ensemble)

- This is particularly relevant in the two-sided BH and its interior length, since the interior length can be arbitrarily long

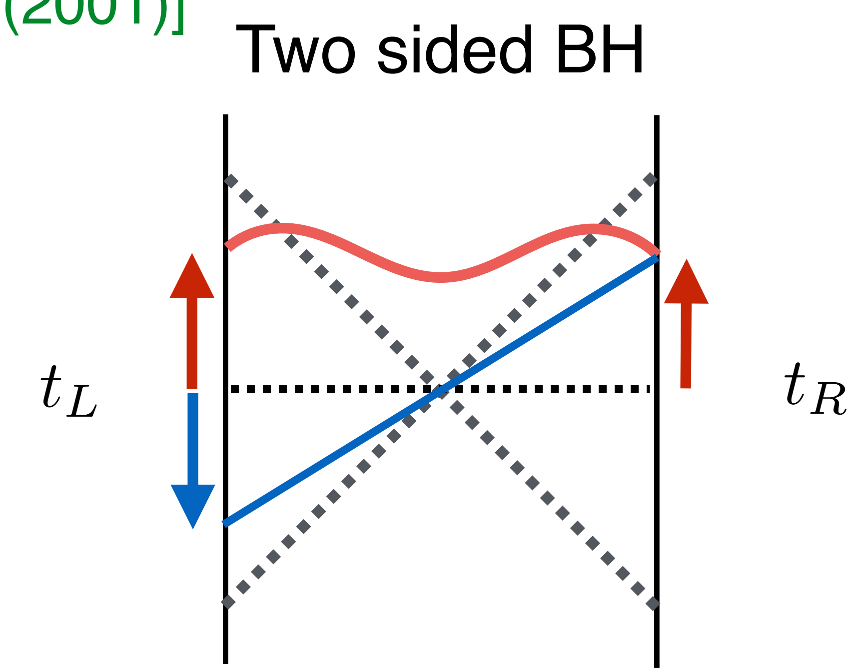


Two-sided black hole

Dual to Thermofield double state: [Maldacena (2001)]

$$|\mathrm{TFD}\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_i e^{-\beta E_i/2}|E_i^L\rangle|E_i^R\rangle$$

- Two sides are connected via smooth horizon and non-traversable wormhole

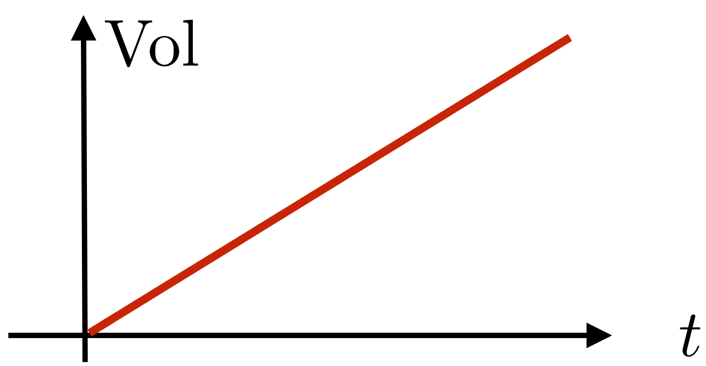


Non-trivial time evolution:
Invariant under $H^L - H^R$
Non-trivial under $H = H^L + H^R$

$$|\mathrm{TFD}(t)\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_i e^{-(\beta+4it)E_i/2}|E_i^L\rangle|E_i^R\rangle$$

Classically, it has linearly growing interior volume

[Susskind (2013)][Susskind, Stanford (2014)]



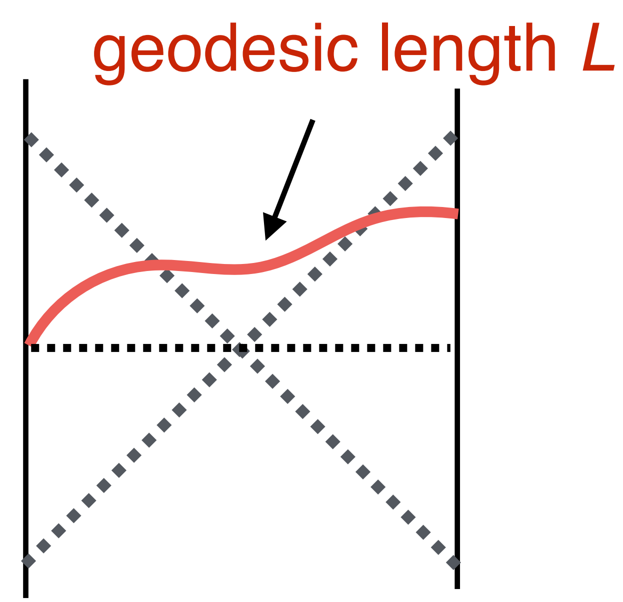
Length state without wormholes

In JT gravity, we know the quantum state corresponding to fixed geodesic length state

- Hamiltonian

$$H = \frac{P^2}{2} + 2e^{-L}$$

$P = \dot{L}$ is the conjugate momentum



- Overlap between the energy and the length eigenstate

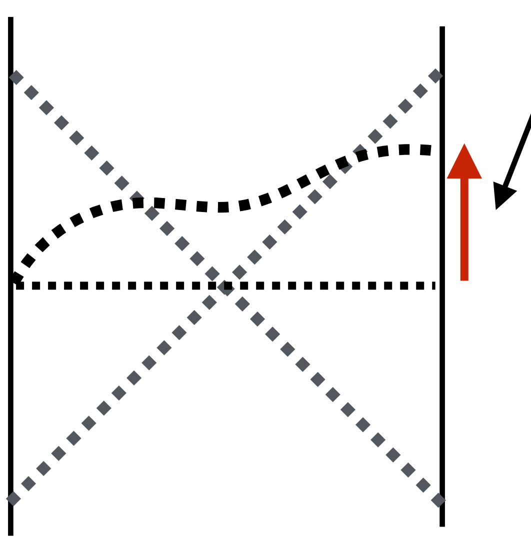
$$\langle l|E\rangle = e^{-S_0/2} 2^{3/2} K_{i2\sqrt{E}}(2e^{-l/2})$$

Here *E* denotes twice the single sided energy

Fixed timeshift state without wormholes

In JT gravity, we know the quantum state corresponding to fixed timeshift state

- Hamiltonian is the canonical momentum of δ



- Overlap between the energy and the timeshift eigenstate

$$\langle E|\delta\rangle = \frac{e^{-i\delta E}}{\sqrt{e^{S_0} D_{\text{Disk}}(E)}}$$

Goal of this talk ①

Non-perturbative length/timeshift states:
We consider non-perturbative realization of these states

Non-perturbative overlaps:
Overlaps with TFD states gives a “probability” for length and timeshift

geodesic length *L* $$P(l,t) = |\langle \mathrm{TFD}(t)|l\rangle|^2$$

timeshift *δ* $$P(\delta,t) = |\langle \mathrm{TFD}(t)|\delta\rangle|^2$$

Generating function:
We also compute the generating functions

geodesic length *L* $$\langle e^{-\alpha l}\rangle_t = \int dl P(l,t) e^{-\alpha l}$$

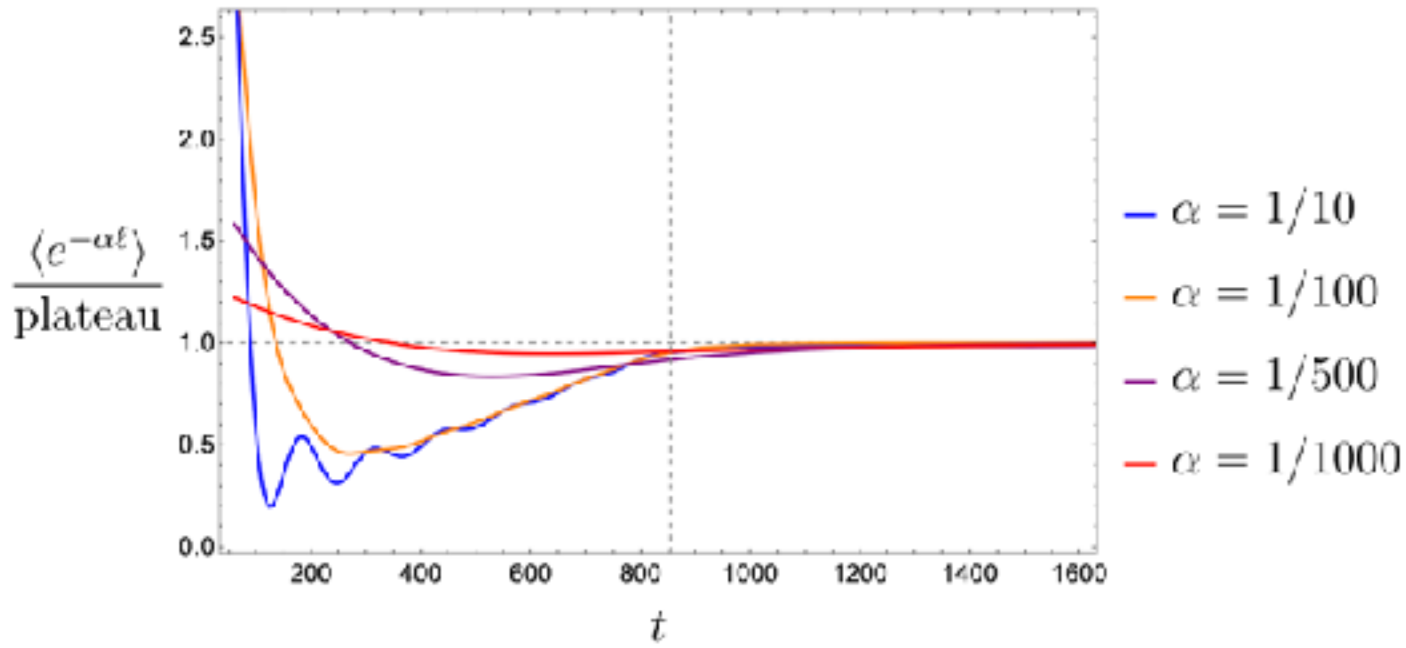
timeshift *δ* $$\langle e^{-\alpha\delta}\rangle_{+,t} = \int_0^\infty d\delta P(\delta,t) e^{-\alpha\delta} \qquad \langle e^{\alpha\delta}\rangle_{-,t} = \int_{-\infty}^0 d\delta P(\delta,t) e^{\alpha\delta}$$

Goal of this talk ②

Generating function as probe for chaotic spectrum:
These probes are similar to spectral form factor; they exhibit dip-ramp-plateau structure at the Heisenberg time

Heisenberg time $T_H := 2\pi\rho(E)$

$$\langle e^{-\alpha l}\rangle_t = \int dl P(l,t) e^{-\alpha l}$$



- In terms of spectrum,

$$\langle e^{-\alpha l}\rangle_t \sim \sum_{E_1, E_2} \frac{\alpha \cos\left((E_1 - E_2)t\right)}{(E_1 - E_2)^2 + 2(E_1 + E_2)\alpha^2}$$

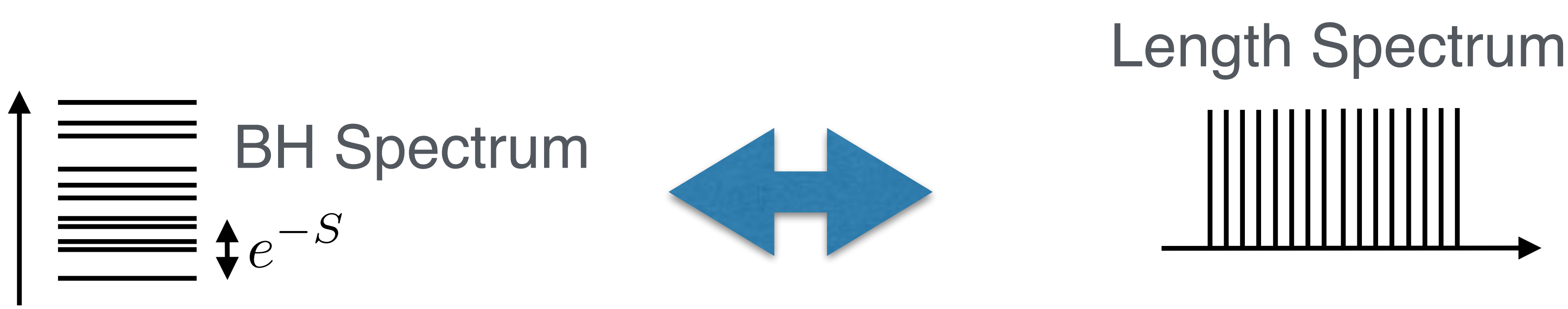
which can be used to probe “interior length” for any system

The aim of this talk ③

Non-orthogonality:
The distribution P(l,t) does not give a probability distribution because length states are not orthogonal

- We perform Gram-Schmidt orthogonalization to construct an orthonormal basis. From this basis we construct non-perturbative length operator

However, once we include the Euclidean wormholes, we will find discrete spectrum, consistent with the finiteness of the BH entropy!

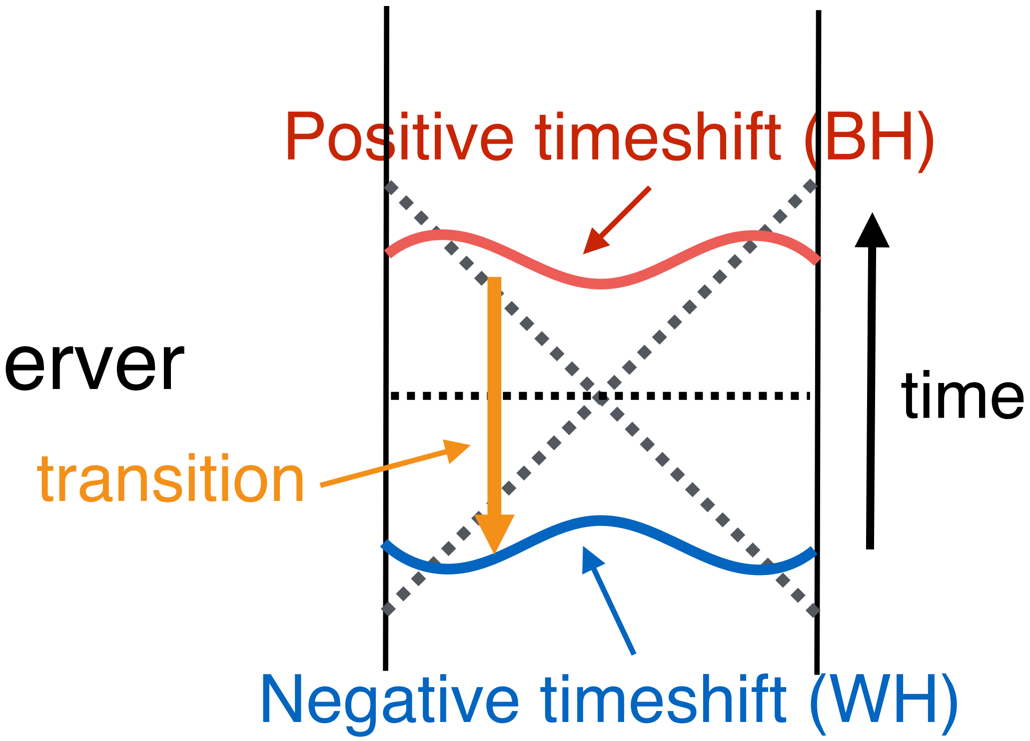


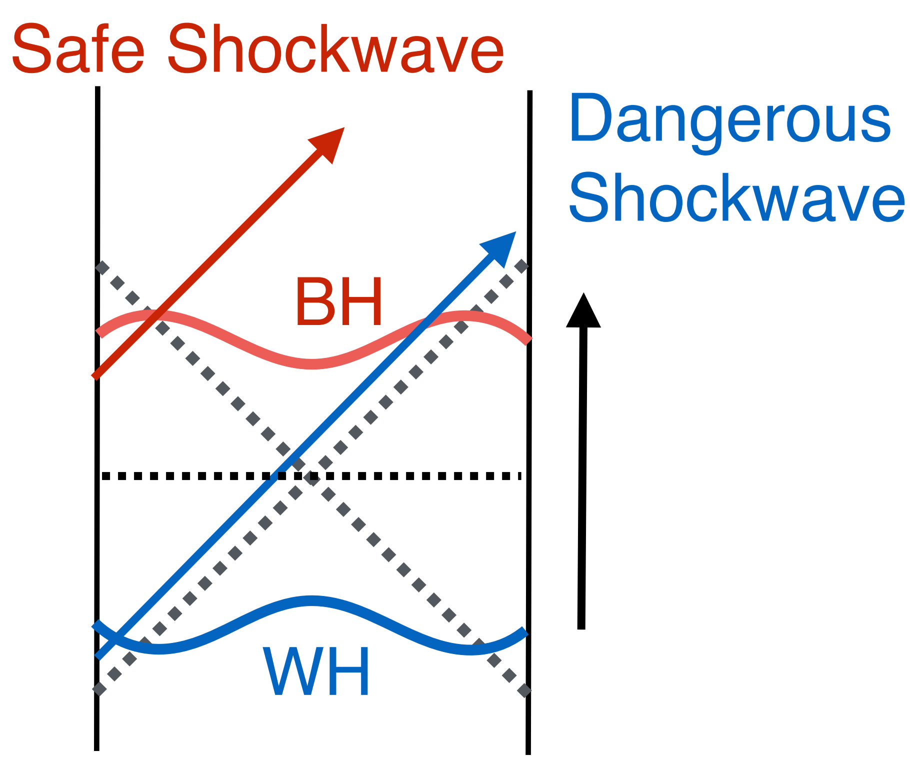
Applications to Bulk Physics?

Transition from BH to white hole(WH) [Stanford, Yang (2022)]
We can investigate the transition probability from BH into WH

- BH can get *younger*, by emitting baby universes

- Further investigation may uncover observer experience in the interior, like *firewall*





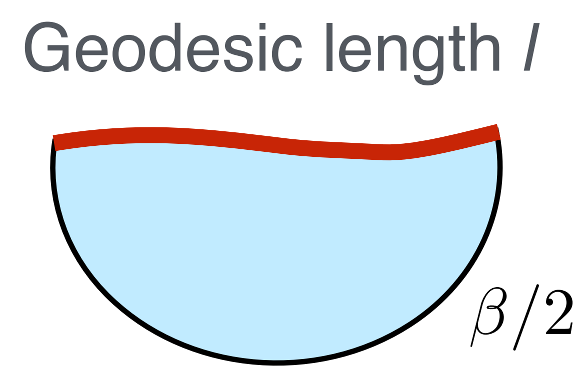
Construction of Non-perturbative length & timeshift States

Bulk quantum state from Hartle-Hawking prescription

Generalized Hartle-Hawking prescription:
The bulk wavefunction is given by the sum over all possible geometries

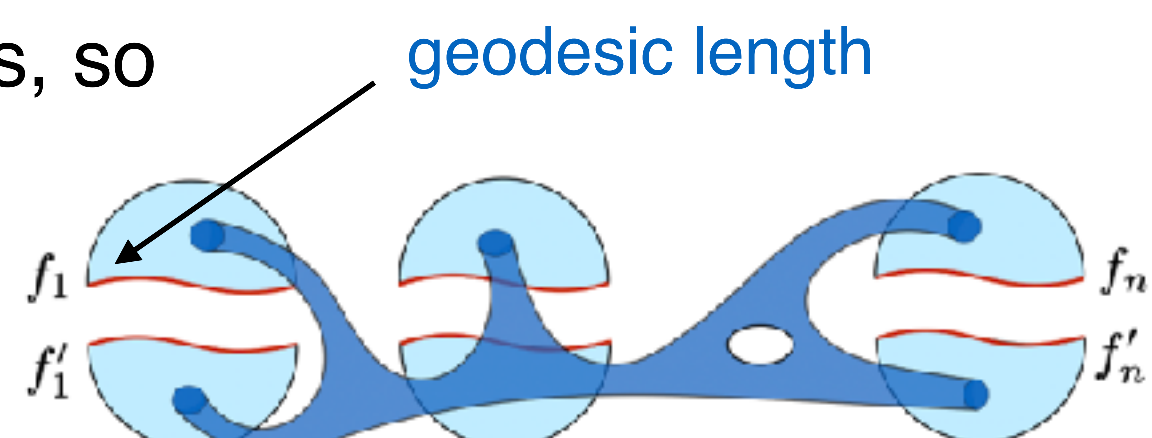
- We seek for non-perturbative quantum state $|l\rangle$ which satisfies

$$\mathbb{E}\Big[\langle l|\mathrm{TFD}(\beta/2)\rangle\Big] =$$



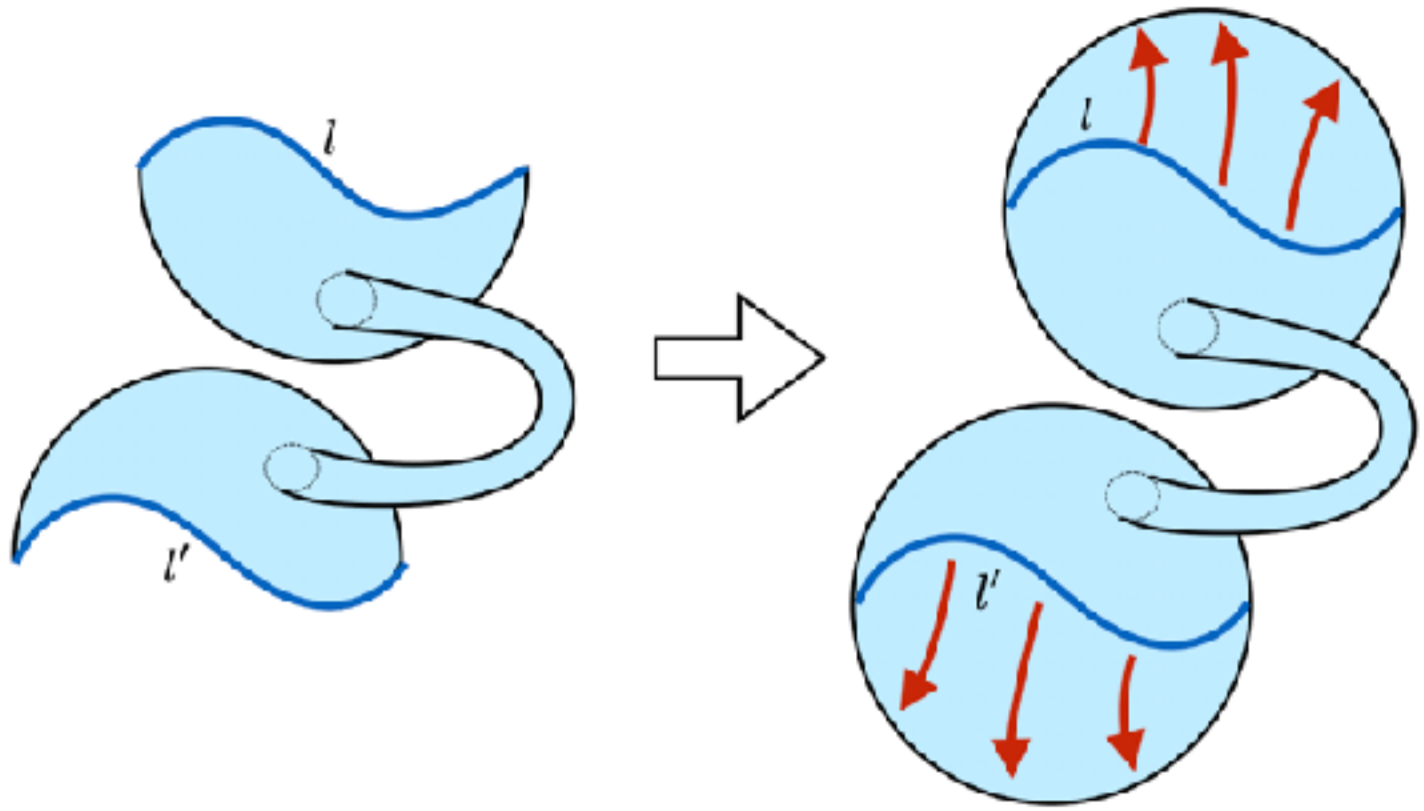
with all possible wormhole corrections, so

$$\mathbb{E}\Big[\langle l_1|\mathrm{TFD}(\beta_1/2)\rangle\langle\mathrm{TFD}(\beta_2/2)|l_2\rangle\cdots\Big] = \sum_{\mathrm{Wormholes}}$$



Simplification in JT

Direct connection to the multi-boundary partition function:
Since there is an unique geodesic homologous to the boundary, the relation to the multi-boundary partition function is simple



- This relation allows us to write the wavefunction using disk wavefunction

Geodesic length over-complete basis

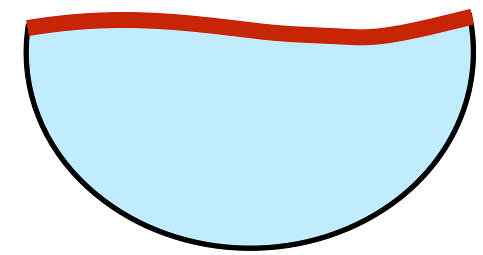
Fixed geodesic length state is given in JT by [Iliesiu et.al. (2024)]

$$|l\rangle = \sum_{|E_i - E_0| < \Delta E'/2} \psi_{\text{Disk}}(E_i, l)|E_i\rangle$$

Microcanonical

JT gravity wave function with
fixed length geodesic and energy

Microscopic
energy eigenstate
(Two-sided energy)



- The wavefunction is

$$\langle l|E\rangle = e^{-S_0/2} 2^{3/2} K_{i2\sqrt{E}}(2e^{-l/2})$$

- We take the microcanonical ensemble and the Hilbert space dimension is finite

$$N' = e^{S_0} D(E_0) \Delta E'$$

Timeshift

Timeshift $\hat{\delta}$:
Average of time at the left and right boundaries, conjugate to the Hamiltonian

- Positive timeshift state corresponds to BH (expanding interior), while negative timeshift state corresponds to WH (contracting interior)

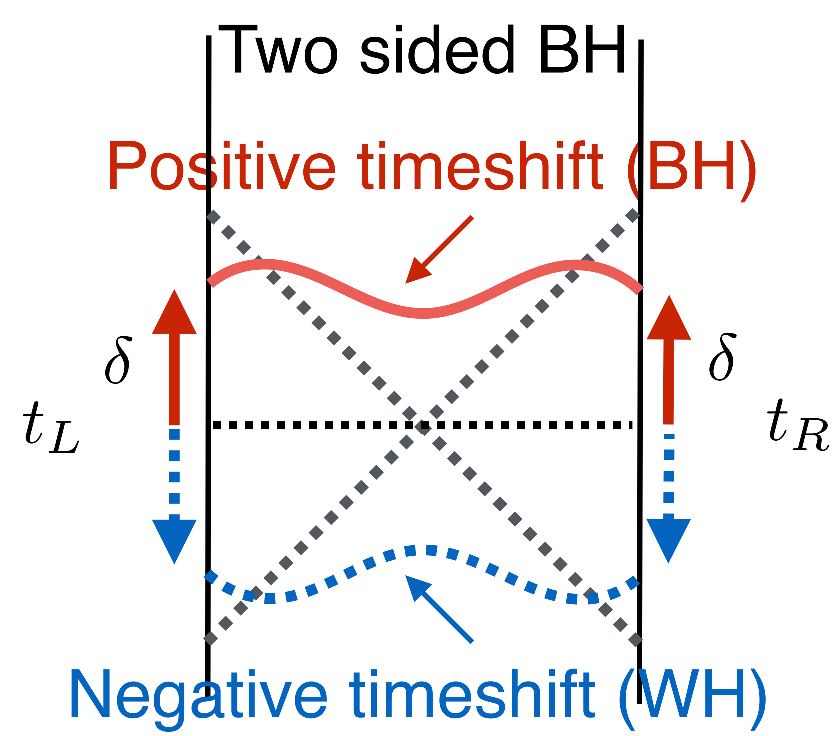
$$|\delta\rangle = \sum_{|E_i - E_0| < \Delta E'/2} \psi_{\text{Disk},\delta}(E_i, \delta)|E_i\rangle$$

Microcanonical

Microscopic energy eigenstate

Wavefunction

$$\psi_{\text{Disk},\delta}(E, \delta) = \frac{e^{-i\delta E}}{\sqrt{e^{S_0} D_{\text{Disk}}(E)}}$$



Non-perturbative Overlaps and Probe of Chaos

Length and Timeshift distribution

Overlaps:
If length/timeshift states are orthogonal, the following overlaps define probability distribution for length/timeshift

geodesic length *L*

$$P(l,t) = |\langle \mathrm{TFD}(t)|l\rangle|^2 = \frac{e^{2S_0}}{Z}\int dE_1 \int dE_2\ e^{-i(E_1-E_2)t}\psi_{E_1}(\ell)\psi_{E_2}(\ell)\langle D(E_1)D(E_2)\rangle$$

timeshift *δ*

$$P(\delta,t) = |\langle \mathrm{TFD}(t)|\delta\rangle|^2 = \frac{e^{2S_0}}{Z^2}\int dE_1 \int dE_2\ e^{-i(E_1-E_2)(t-\delta)}\langle D(E_1)D(E_2)\rangle$$

We compute these overlaps by using sine-kernel for density of state two point function, corresponding to all genus contributions

$$\langle D(E_i)D(E_j)\rangle \approx D_{\mathrm{Disk}}(E_i)D_{\mathrm{Disk}}(E_j) + e^{-S_0}\delta(E_i-E_j)D_{\mathrm{Disk}}(E_i) - e^{-2S_0}\frac{\sin^2(\pi e^{S_0}D_{\mathrm{Disk}}(E_i)(E_i-E_j))}{\pi^2(E_i-E_j)^2}$$

Length distribution

Length:
At large *l* and sufficiently large *ΔE*, we can approximate

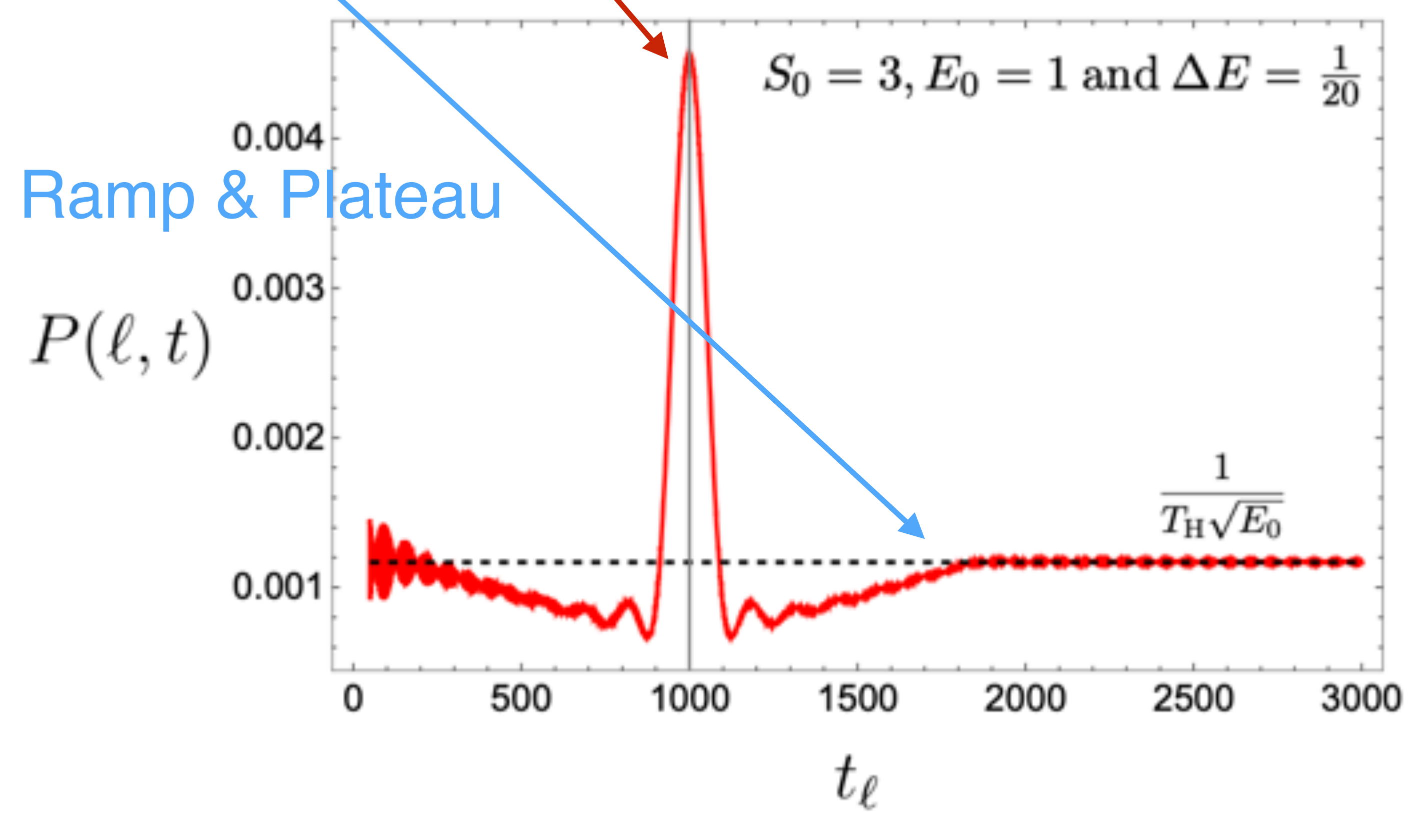
$$\frac{\langle \mathrm{TFD}(t)|l\rangle\langle l|\mathrm{TFD}(t)\rangle}{\langle l|l\rangle} \to \frac{2}{\Delta E \Delta E'}\left[\frac{\sin^2\left((t_l - t)\frac{\Delta E}{2}\right)}{\left(t_l - t\right)^2} + \frac{\sin^2\left((t_l + t)\frac{\Delta E}{2}\right)}{\left(t_l + t\right)^2}\right]$$

$$+\frac{\pi}{T_H^2 \Delta E'}\left[\mathrm{Min}\left[T_H,\ |t_l - t|\right] + \mathrm{Min}\left[T_H,\ |t_l + t|\right]\right]$$

Classical peak

Classical relation between time and the length :

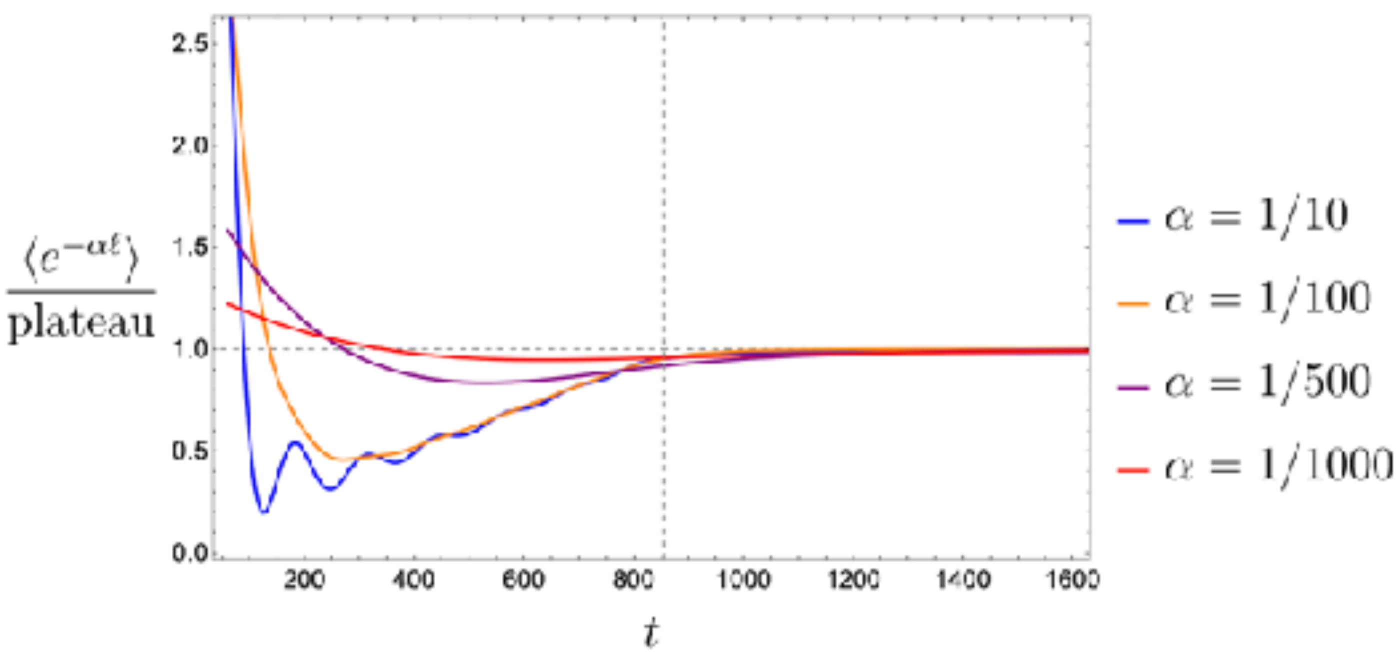
$$t_l := \frac{l + \log(4E_0)}{2\sqrt{E_0}}$$

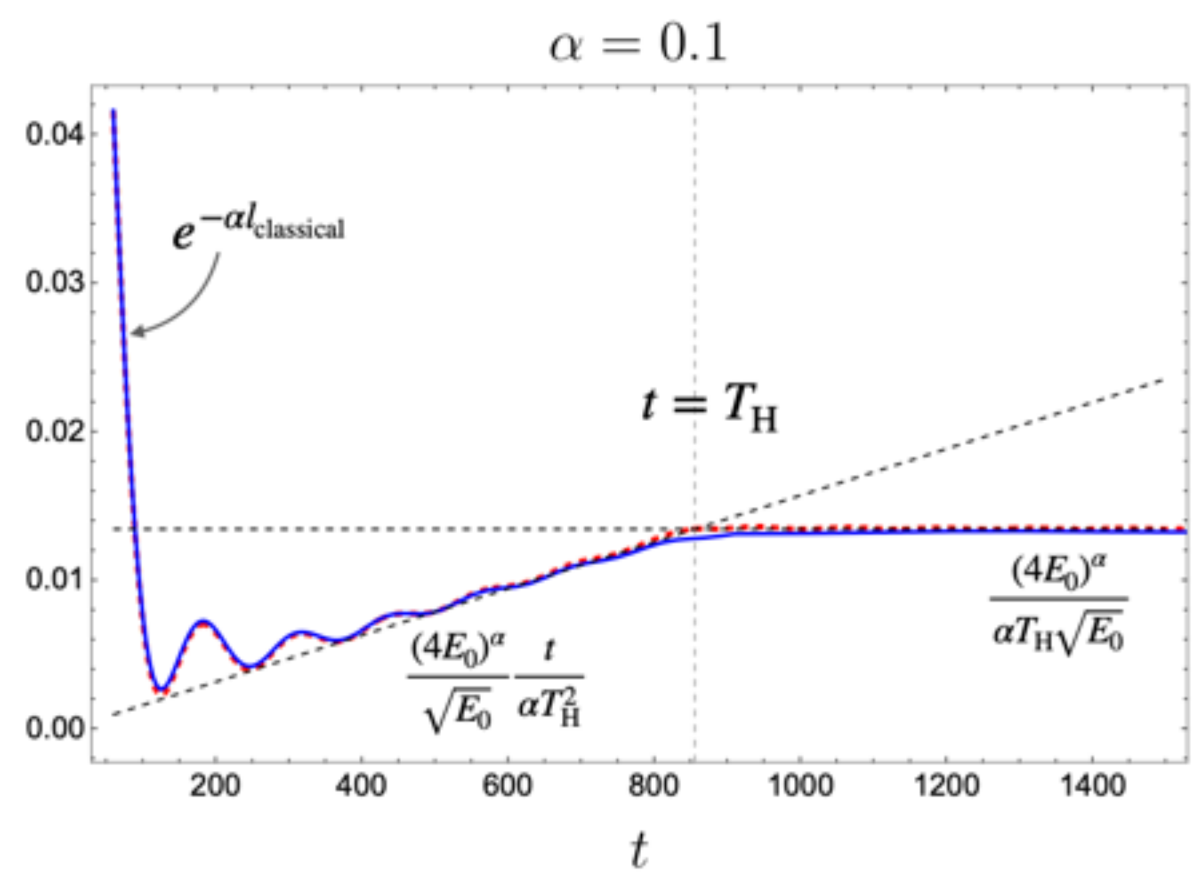


Length Generating Function

Dip-ramp-plateau behavior:
Generating function exhibits dip-ramp-plateau behavior, reaching plateau at the Heisenberg time

$$\langle e^{-\alpha l}\rangle_t = \int dl P(l,t) e^{-\alpha l}$$





- As we take smaller α, it diverges and the ramp disappears, i.e. early exponential decay is followed immediately by the plateau

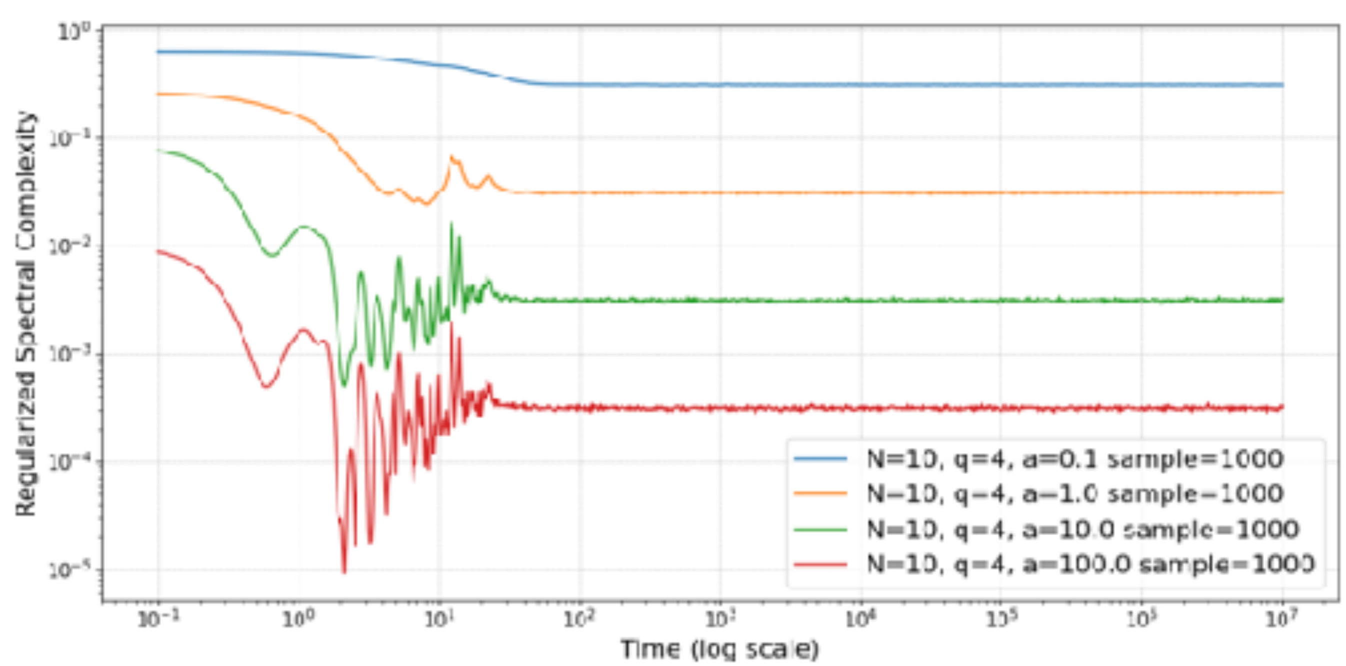
Probe for Chaotic Spectrum

Generating function as probe for chaotic spectrum:
The length generating function can be written as

$$\langle e^{-\alpha l} \rangle_t \sim \sum_{E_1, E_2} \frac{\alpha \cos\left((E_1 - E_2)t\right)}{(E_1 - E_2)^2 + 2(E_1 + E_2)\alpha^2}$$

We can apply this quantity in any system to probe "internal length"

- We studied SYK model and indeed find the dip-ramp-plateau behavior



Spectral Complexity and Length?

Limit:
Taking small α gives the spectral complexity [Gabor, Iliesiu, Mezei (2021)]

Spectral complexity — Divergence

$$\langle l \rangle_t \overset{?}{=} -\lim_{\alpha \to 0} \frac{d\langle e^{-\alpha l}\rangle_t}{d\alpha} \sim \boxed{\sum_{E_1, E_2} \frac{1 - \cos\left((E_1 - E_2)t\right)}{(E_1 - E_2)^2}} + \boxed{O\left(\frac{1}{\alpha^2 T_H}\right)}$$

However, there are several problems relating the length and the spectral complexity

- Length states are not orthogonal to each other
- Divergent in *α*
- Qualitatively different from finite α (only classical + plateau)

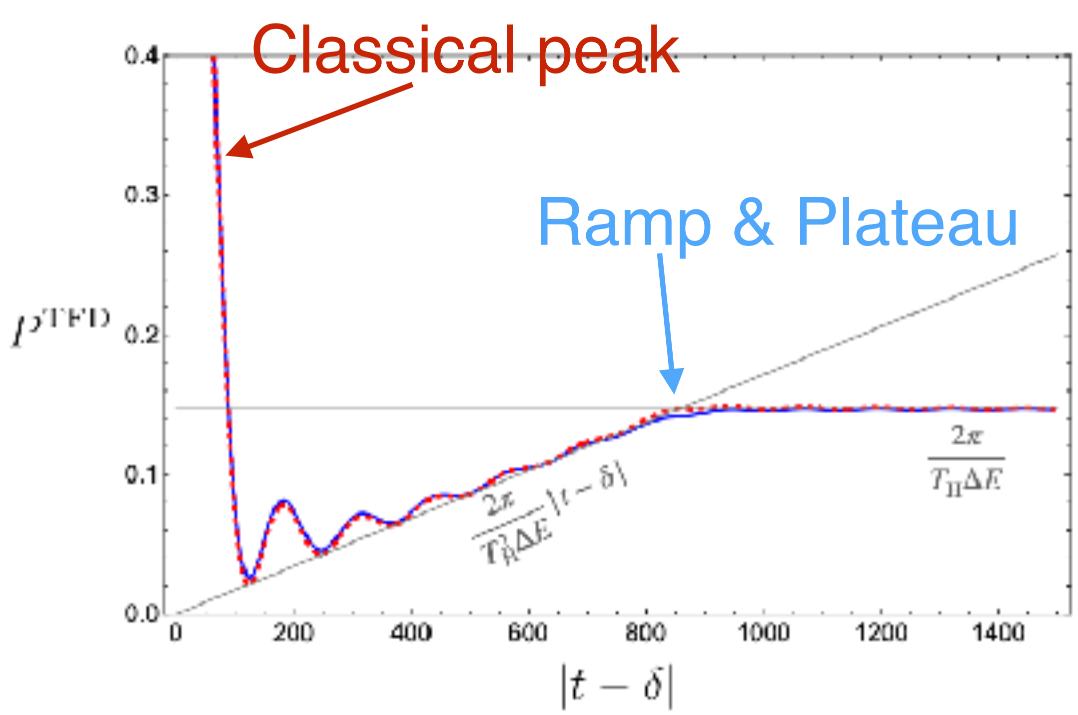
These suggest that the *interior length can be probed well only when α is sufficiently large*

Timeshift distribution

Timeshift:
It is directly related the spectral form factor

$$P^{\mathrm{TFD}}(\delta, t) := \langle \mathrm{TFD}(t)|\delta\rangle\langle\delta|\mathrm{TFD}(t)\rangle = \frac{|Z(0+i(t-\delta))|^2}{Z(0)^2} = \mathrm{SFF}\left(|t-\delta|\right)$$

- Because the timeshift state is also microcanonical TFD state



Timeshift Generating Function

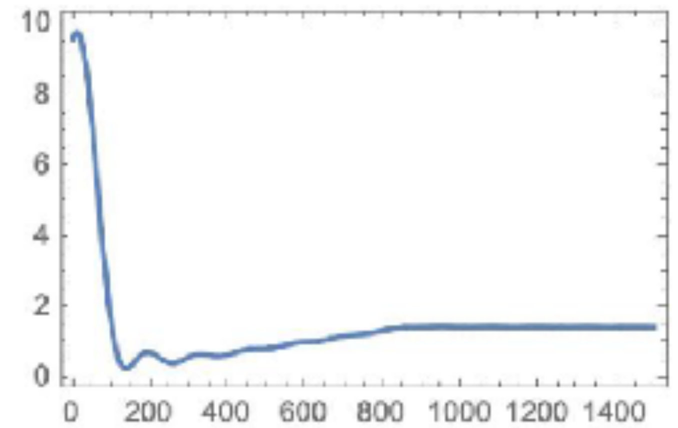
Positive and Negative part:
We divide into two parts;

timeshift δ

$$\langle e^{-\alpha\delta}\rangle_{+,t} = \int_0^{\infty} d\delta P(\delta,t)e^{-\alpha\delta} \qquad \langle e^{\alpha\delta}\rangle_{-,t} = \int_{-\infty}^{0} d\delta P(\delta,t)e^{\alpha\delta}$$

We again find the dip-ramp-plateau behavior, reaching plateau at the Heisenberg time

$\langle e^{-\alpha\delta}\rangle_{+,t}$



Spectral representation:
For any system

$$\langle e^{\mp\alpha\delta}\rangle_{\pm,t} := \sum_{E_i,E_j} e^{i(E_i-E_j)t}\frac{1}{\alpha \pm i(E_i-E_j)}$$

Timeshift?

Pathological Limit:
If we consider the natural definition

$$\langle \delta \rangle := -\lim_{\alpha \to +0} \frac{d}{d\alpha} \left(\langle e^{-\alpha\delta} \rangle_{+,t} + \langle e^{\alpha\delta} \rangle_{-,t} \right)$$

we arrive at

$$\langle \delta \rangle = 0$$

This is not reproducing the classical early time behavior $\langle \delta \rangle \sim t$

- Again this suggests the above limit does not lead to faithful bulk description, and highly dependent on regularization scheme
- Only when α is sufficiently large, we can use it to probe the bulk

Non-perturbative Length and Timeshift Operators

Non-orthogonality

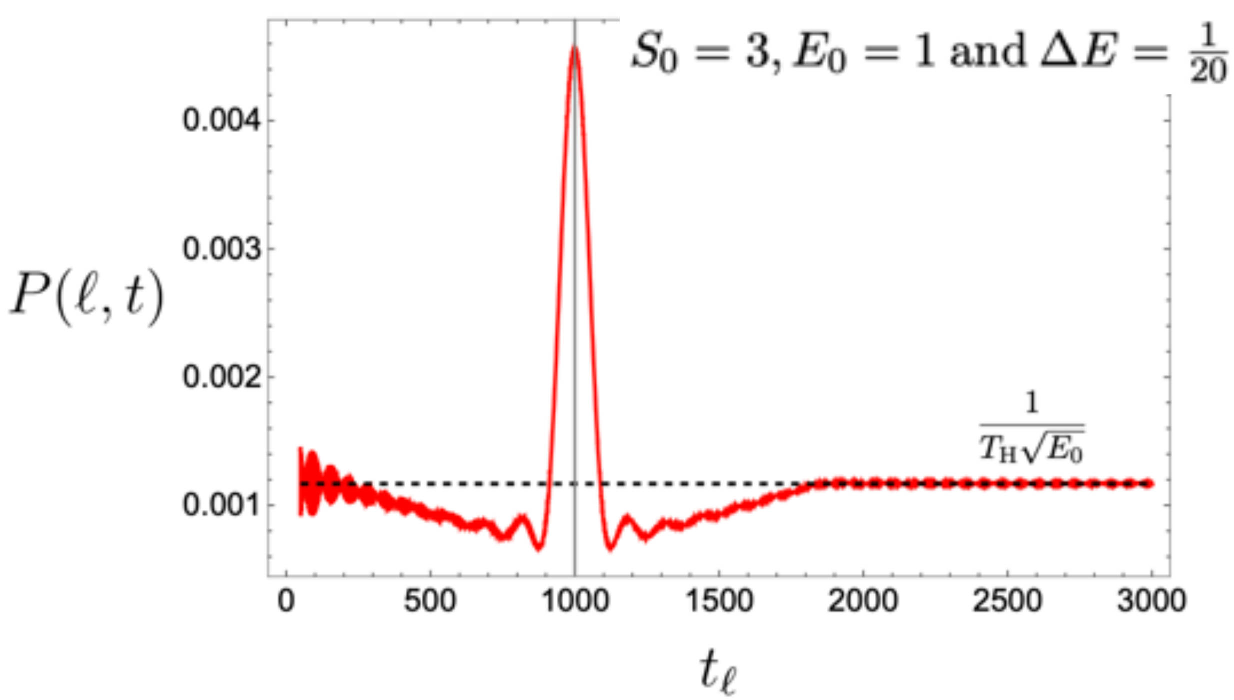
Non-zero Overlap for all states:
The overlap becomes constant for large $|t - t_l|$ or $|t - \delta|$
This implies that

- Length states cannot be eigenstates of an Hermitian operator. For example no Hermitian operator like

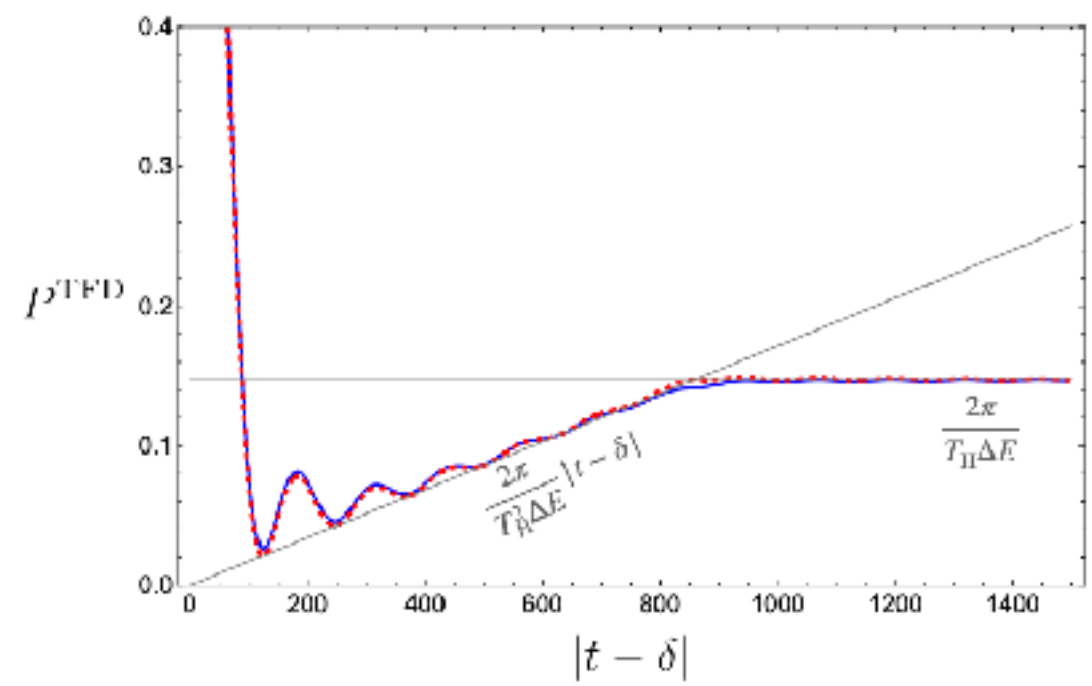
$$\hat{l}|l\rangle = l|l\rangle$$

- Thus P(l,t) and P(δ,t) are not probability distribution

geodesic length *L*



timeshift *δ*



Non-orthogonality makes naive resolution of identity ill-defined

[MM] [MM, Ruan, Shibuya, Yano (To appear)]

The length states give the resolution of identity on the disk

$$\int_{-\infty}^{\infty} dl \ |l\rangle\langle l| = e^{S_0} \int_{E_0-\Delta E/2}^{E_0+\Delta E/2} dE \ D_{\rm Disk}(E)|E\rangle\langle E| = \mathbb{I} \qquad \text{Disk}$$

However, with Euclidean wormholes

$$\int_{-\infty}^{\infty} dl \ |l\rangle\langle l| \neq \mathbb{I} \qquad \text{Wormholes}$$

The left hand side is actually divergent for TFD state

$$\langle {\rm TFD}| \Big[\int_{-\infty}^{\infty} dl |l\rangle\langle l| \Big] |{\rm TFD}\rangle = \infty$$

Construction of baby-universe corrected length state

We have non-orthogonal length state

$$|l_1\rangle,\ |l_2\rangle,\ |l_3\rangle, \cdots \ (l_1 < l_2 < l_3 < \cdots)$$

We identify corrected length states by removing shorter length states by Gram-Schmidt procedure

$$|l_1\rangle^{NP} = |l_1\rangle,\ |l_2\rangle^{NP} = \frac{|l_2\rangle - |l_1\rangle\langle l_1|l_2\rangle}{\sqrt{1-|\langle l_1|l_2\rangle|^2}},\ \cdots$$

Continuing this process many times, we will reach

$$|l_{N+1}\rangle = \sum_{i=1}^{N} c_i |l_i\rangle^{NP} = \sum_{i=1}^{N} d_i |l_i\rangle$$

in terms of shorter wormhole states

for dimension N of the microcanonical window

Orthogonalization via replica trick

This Gram-Schmidt procedure seems hard to perform, but we can do so by considering this manifestly positive operator

$$\hat{S}[m] := \sum_{i=1}^{m} |l_i\rangle\langle l_i|$$

whose nonzero positive eigenstates are spanned by

$$|l_1\rangle^{NP},\ |l_2\rangle^{NP}, \cdots, |l_m\rangle^{NP}\ (l_1 < l_2 < \cdots < l_m)$$

Thus the projector onto this subspace can be obtained via

$$\hat{P}[m] = \lim_{n \to +0} \hat{S}[m]^n$$

Then we obtain

$$\hat{P}[m] - \hat{P}[m-1] = |l_m\rangle^{NP}\langle l_m|^{NP}$$

Non-perturbative length operator

The corrected length operator is now given by

$$\hat{L}^{NP} = \sum_{i=1}^{N} l_i |l_i\rangle^{NP} \langle l_i|^{NP}$$

- The spectrum is unchanged, except it terminates at i=N

Length probability distribution is conveniently written as (assuming continuity)

$$D(l) = \ \mathrm{Tr}\Big[\frac{d}{dl}\hat{P}[l]\Big]$$

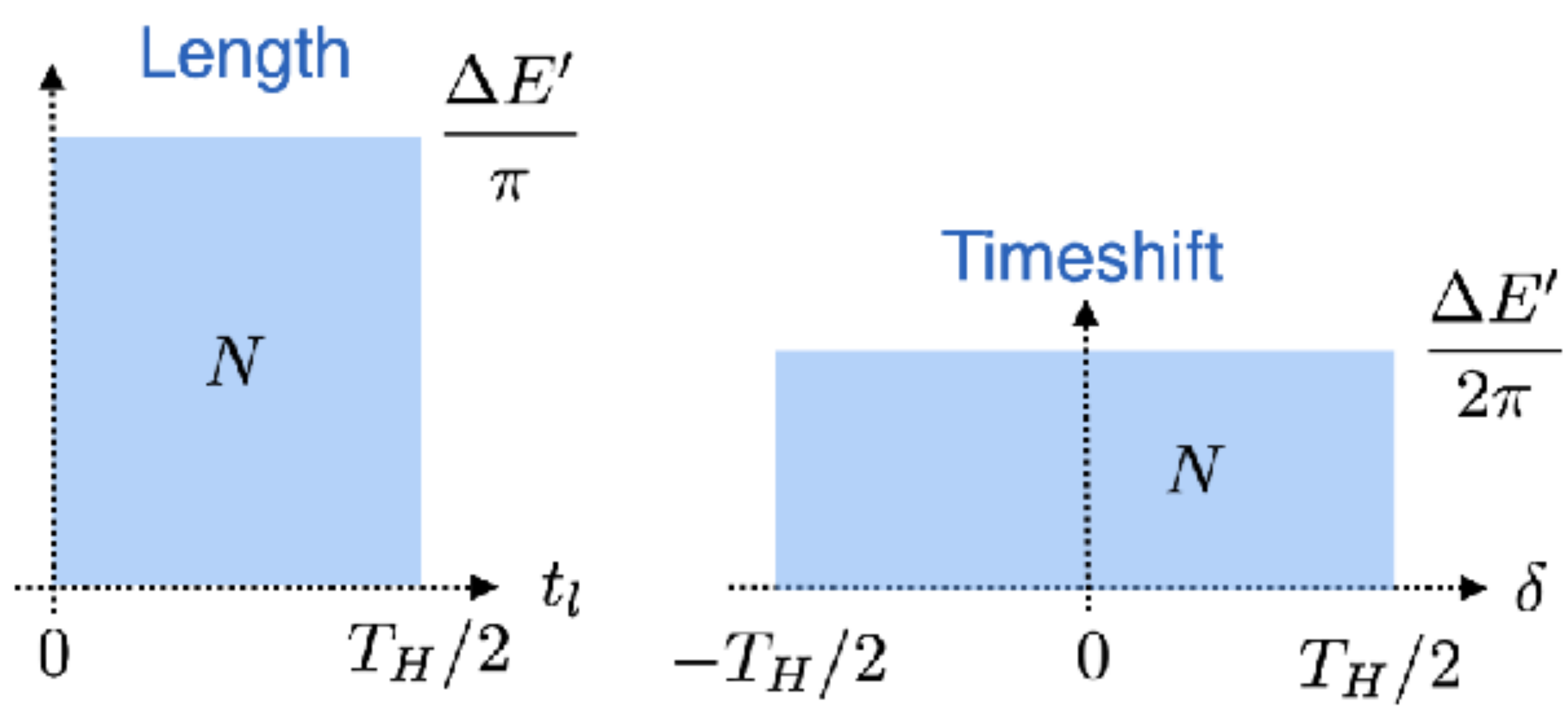
Length probability distribution $P(l)$ on TFD state is conveniently written as

$$P[l] = \langle \mathrm{TFD}(t)|\frac{d}{dl}\hat{P}[l]|\mathrm{TFD}(t)\rangle$$

Length/timeshift Spectrum and Probability

We consider perturbative expansion in terms of t_l and δ up to second order. The results are already highly non-trivial

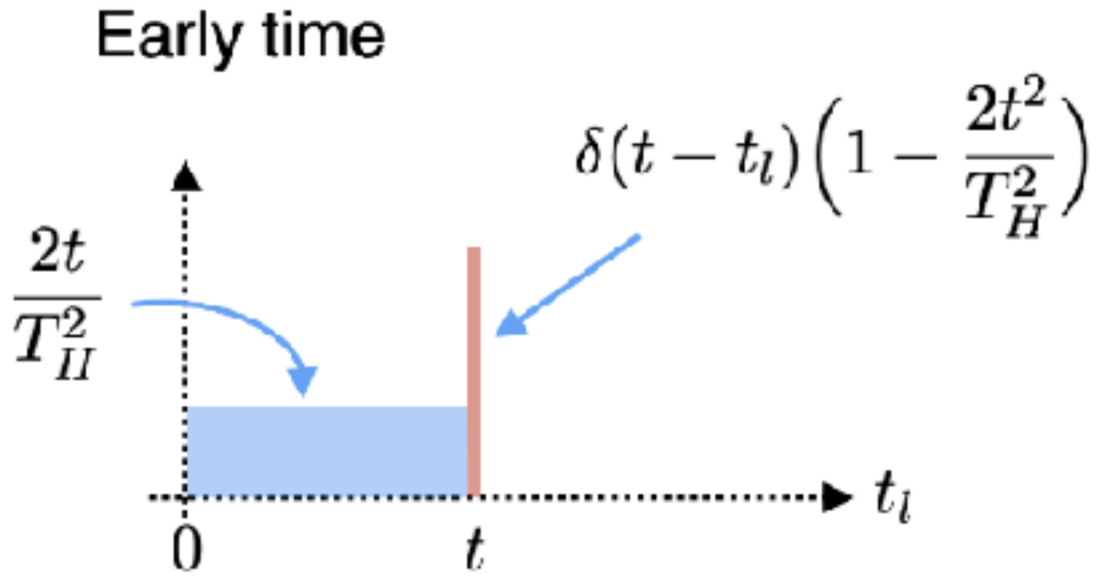
Spectrum:
Density of states turns out to be uniform but terminates at Heisenberg time (in unit of t_l and δ)



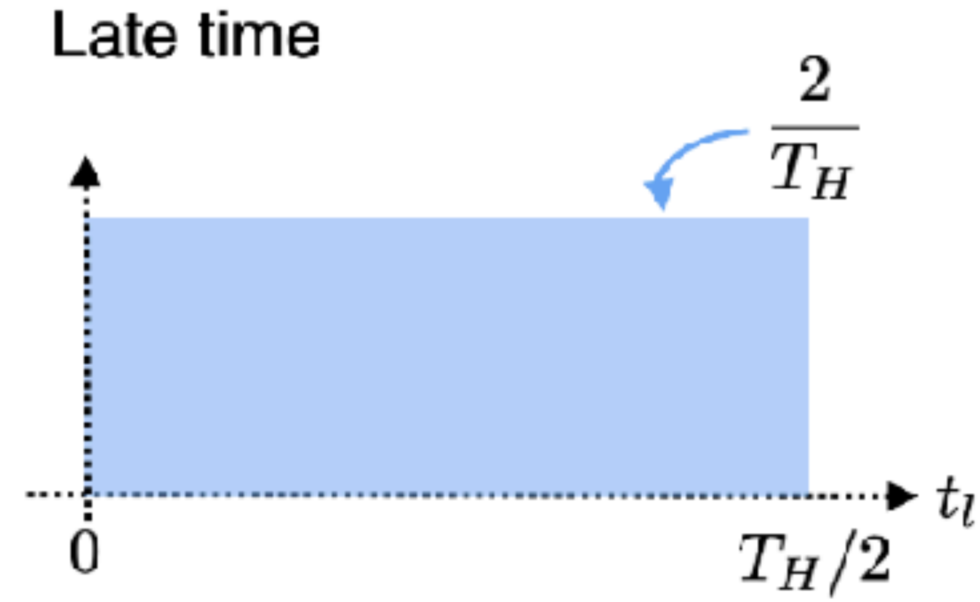
Length Probability

Probability distribution:
Early time: Classical peak + constant probability to have shorter interior length. Classical linear growth, and small variance
Late time: Uniform probability, no peak. Saturation but large variance

$t < T_H/2$



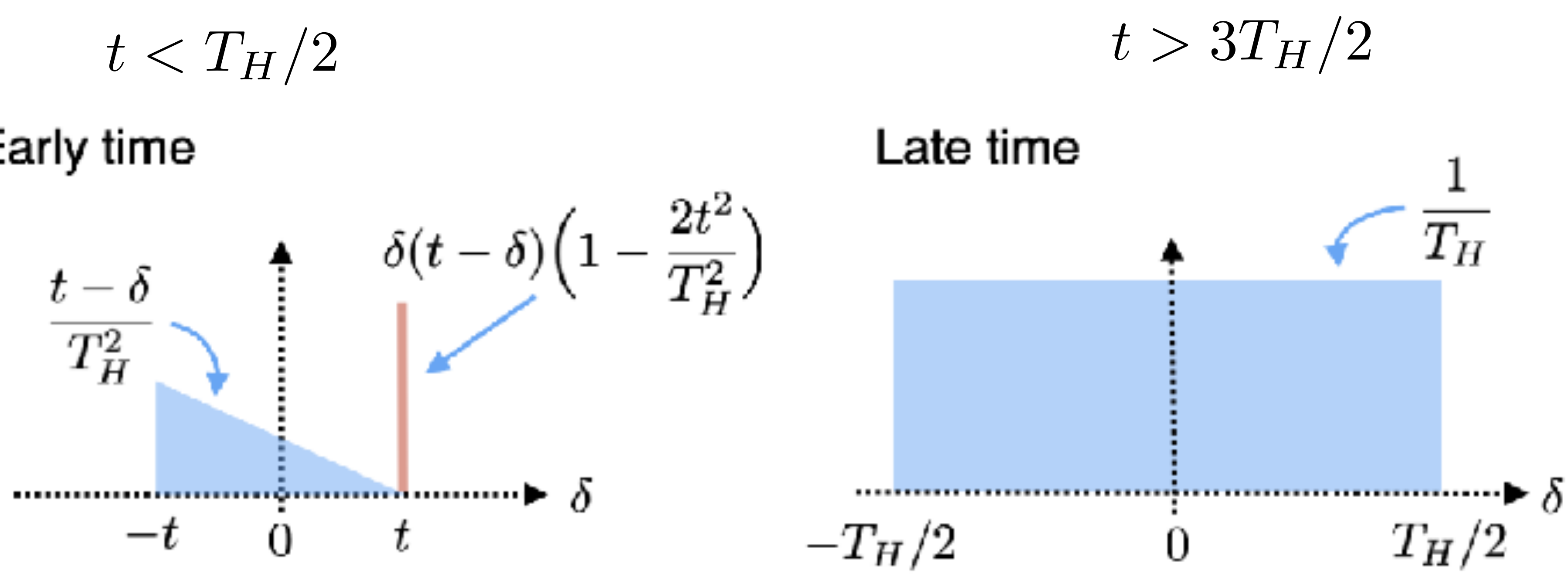
$t > 3T_H/2$



$$\mathbb{E}[t_l]^{(0+1)} = t - \frac{t^3}{T_H^2}, \ \sqrt{\mathbb{E}\left[(t_l - \mathbb{E}[t_l])^2\right]^{(0+1)}} = \frac{t^2}{T_H}\sqrt{\frac{2}{3} - \frac{t^2}{T_H^2}}$$

Timeshift Probability

Probability distribution:
Early time: Classical peak + constant probability to have smaller timeshift absolute value. Classical linear growth, and small variance
Late time: Uniform probability, no peak. Saturation but large variance



- In particular, it is equally possible to have BH and WH at late time (Susskind's grey hole)

$$P^{\mathrm{BH}}(t) = P^{\mathrm{WH}}(t) = \frac{1}{2}$$

Summary and Future Directions

[Summary]

- We constructed non-perturbative length and timeshift states and operators

- Proposed new quantities probing "interior length" in arbitrary systems

[Future directions]

- Full order calculation

- Non-perturbative Hilbert space in de Sitter space

- DSSYK model [Okuyama, M.M, Mori, *work in progress*]

- Relation to Krylov state complexity etc