Black Hole Multi-Entropy Curves

[arXiv:2412.07549]

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Page curve of outgoing Hawking radiation

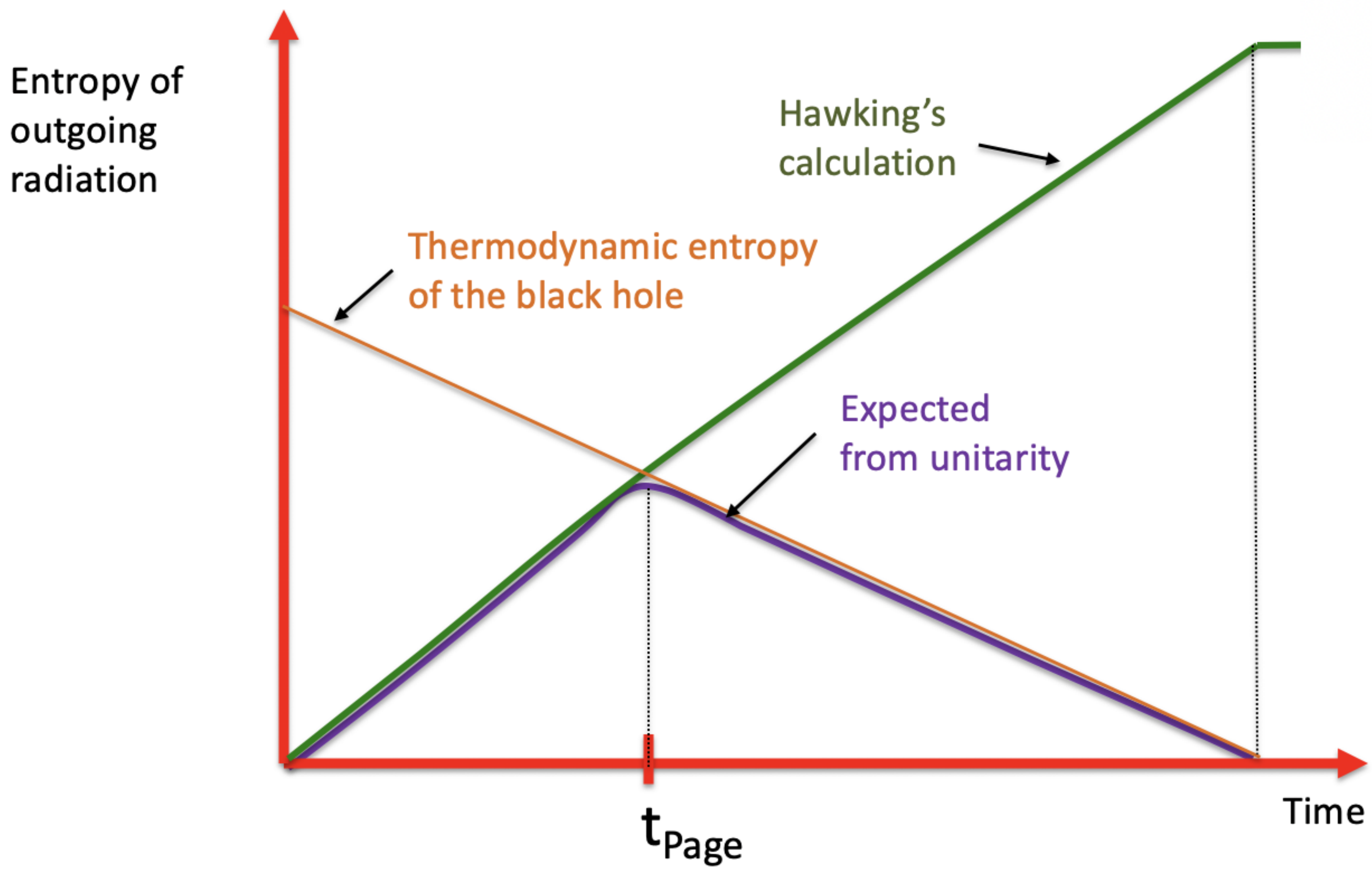


Figure from
[A. Almheiri,
T. Hartman,
J. Maldacena,
E. Shaghoulian,
A. Tajdini, 2020]

Hawking's calculation indicates that entropy of Hawking radiation increases until BH evaporates.

But, from unitarity, entropy of Hawking radiation is expected to decreases from Page time (Page curve).

[D. N. Page, "Average entropy of a subsystem", 1993]
[D. N. Page, "Information in black hole radiation", 1993]

Our interest

Multi-partite generalization of Page curve

- Entanglement between 1 BH subsystem and $\mathrm{q}-1$ Hawking radiation subsystems
- Many multi-partite measures have been proposed such as entanglement negativity and reflected entropy.
- We are interested in a generalization that reduces to entanglement entropy if $\mathrm{q}=2$.

Our result

[N. Iizuka, S. Lin, MN, 2024]

- We study a special new measure called multi-entropy.

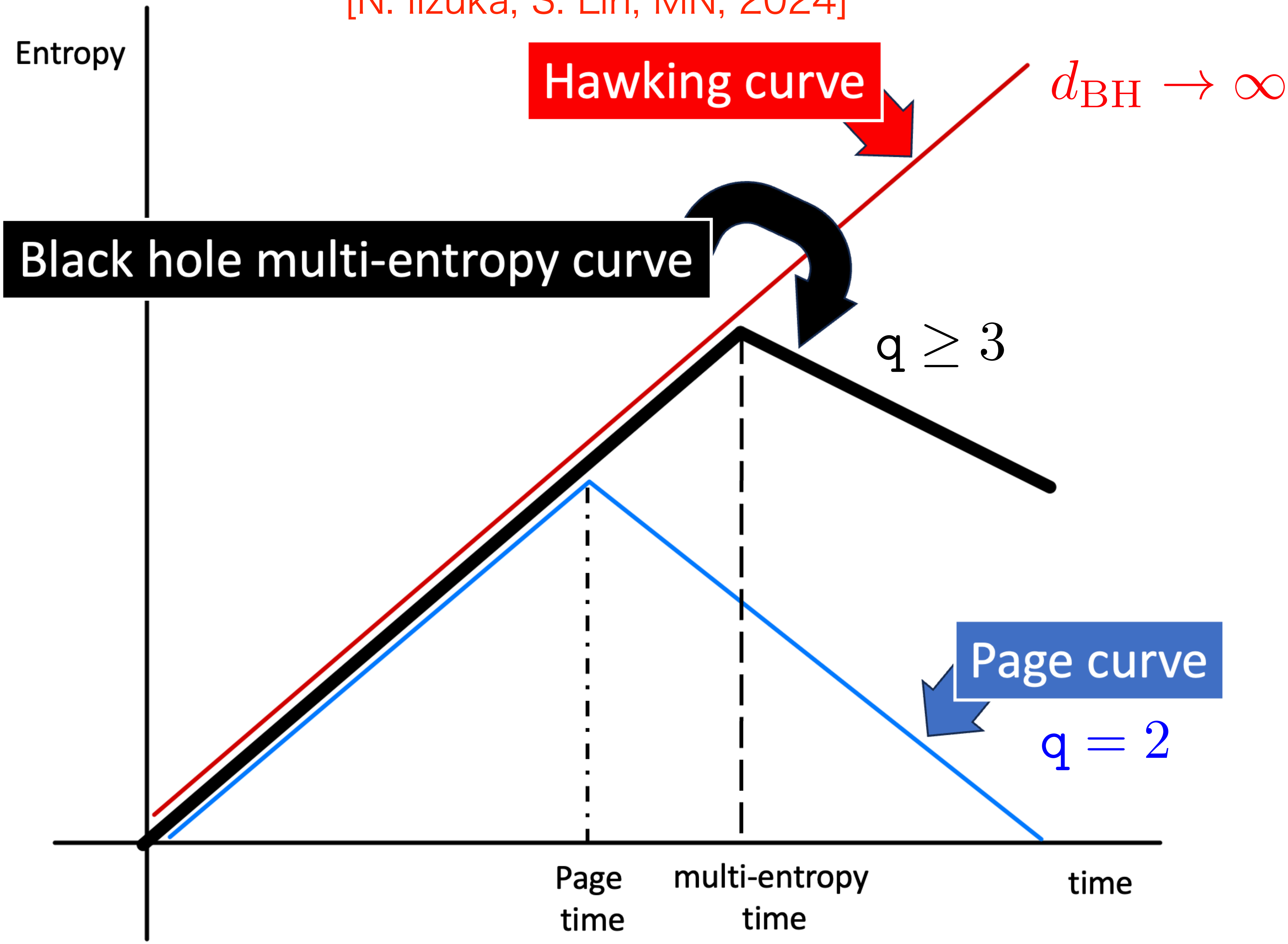
  [A. Gadde, V. Krishna, T. Sharma, 2022]

- We define a black hole multi-entropy curve, which describes how multi-entropy changes during BH evaporation.

- For $\mathtt{q}$-partite black hole multi-entropy curves, we compute Rényi multi-entropy of a single random tensor with $\mathtt{q}$ indices.

Typical behavior of a black hole multi-entropy curve

[N. Iizuka, S. Lin, MN, 2024]



Outline

1. Review of Rényi multi-entropy $S_n^{(\mathtt{q})}$

2. Review of a single random tensor

3. Black hole Rényi multi-entropy curve

4. Early time and late time behaviors

1. Review of Rényi multi-entropy $S_n^{(\mathtt{q})}$

[A. Gadde, V. Krishna, T. Sharma, 2022]

- Rényi multi-entropy $S_n^{(\mathtt{q})}$ is a natural generalization of Rényi entanglement entropy $S_n^{(2)}$ for $\mathtt{q}$ -partite systems.

- $S_n^{(\mathtt{q})}$ is defined by contracting indices of density matrices in different ways for each subsystem.

- $S_n^{(\mathtt{q})}$ cannot be determined by eigenvalues of reduced density matrix.

Rényi entanglement entropy $S_n^{(2)}$

Pure state $|\psi\rangle$ on bi-partite system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

Reduced density matrix $\rho := \mathrm{Tr}_{\mathcal{H}_2} |\psi\rangle\langle\psi|$

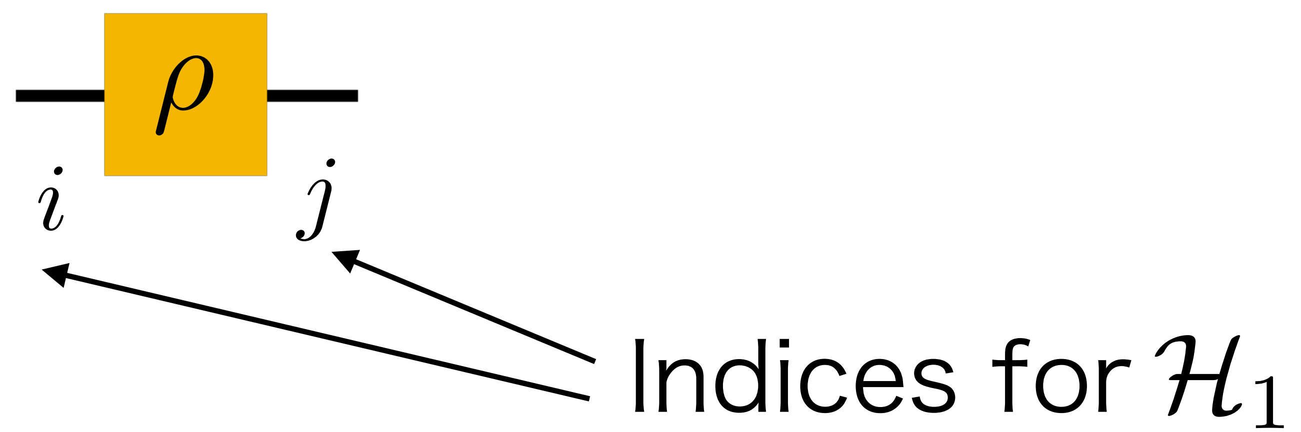
Replica partition function $Z_n^{(2)} := \mathrm{Tr}_{\mathcal{H}_1} \rho^n$

Rényi entanglement entropy $S_n^{(2)} := \dfrac{1}{1-n} \log \dfrac{Z_n^{(2)}}{\left(Z_1^{(2)}\right)^n}$

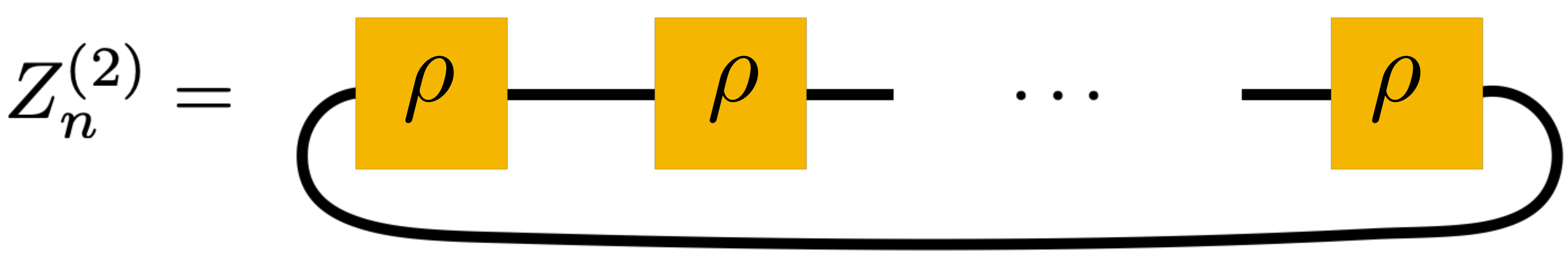
Entanglement entropy $S^{(2)} := \lim_{n\to 1} S_n^{(2)}$

Graphical representation of $Z_n^{(2)}$

Graphical representation of $\rho := \mathrm{Tr}_{\mathcal{H}_2} |\psi\rangle\langle\psi|$



Graphical representation of $Z_n^{(2)} := \mathrm{Tr}_{\mathcal{H}_1} \rho^n$

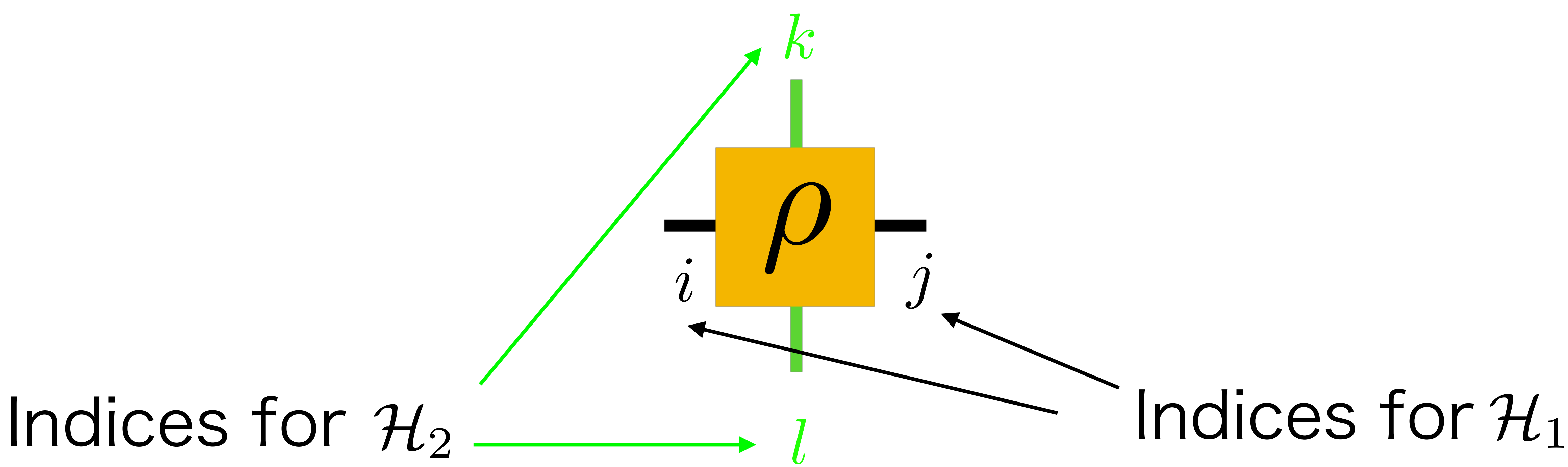


This figure represents the contraction pattern of ρ for Rényi entanglement entropy.

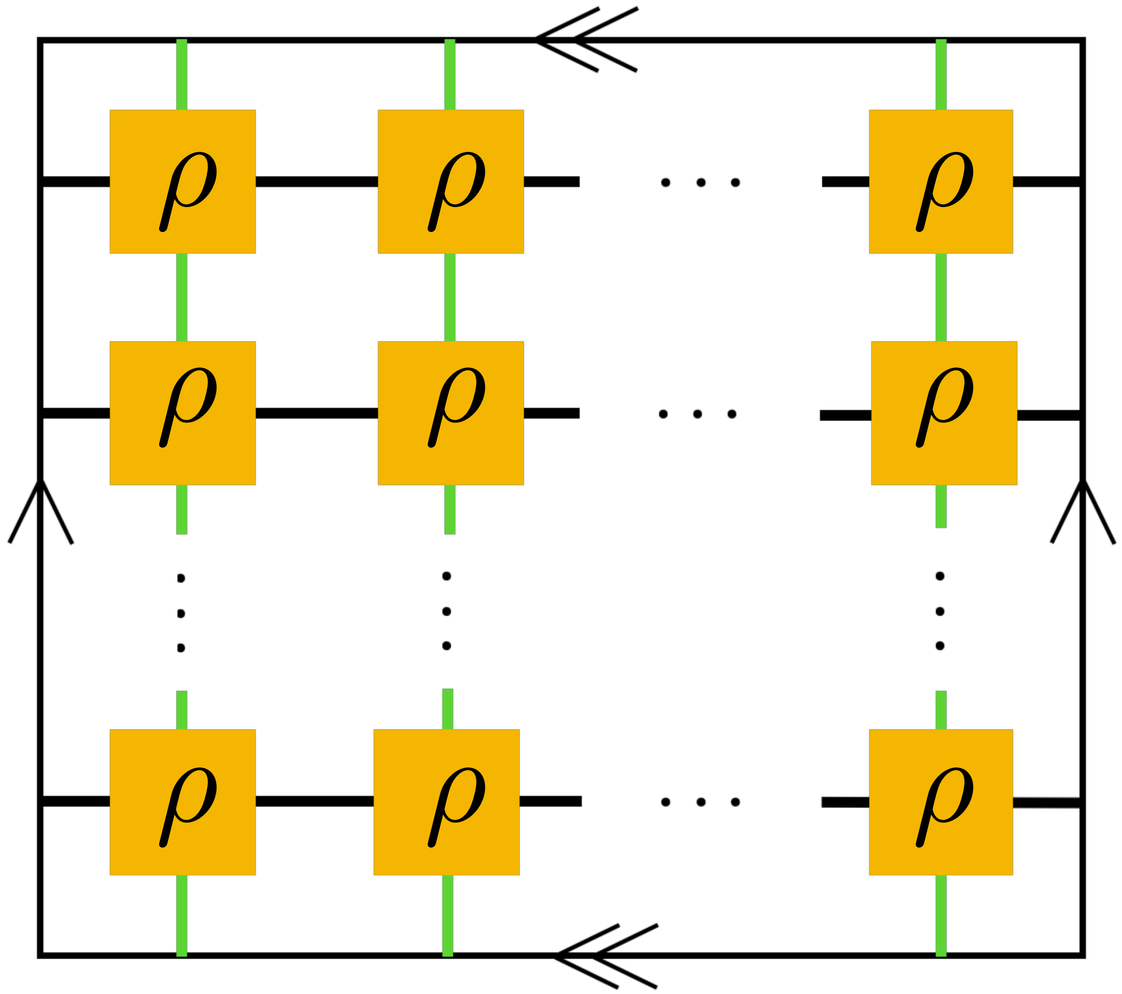
Graphical representation for $\mathtt{q} = 3$ tri-partite systems

Pure state $|\psi\rangle$ on tri-partite system $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$

Graphical representation of $\rho := \mathrm{Tr}_{\mathcal{H}_3} |\psi\rangle\langle\psi|$



Graphical representation of replica partition function $Z_n^{(3)}$

$Z_n^{(3)} =$ 

Number of ρ

$n^{\mathtt{q}-1} = n^2$

Contraction for $\mathcal{H}_1$ is horizontal.

Contraction for $\mathcal{H}_2$ is vertical.

We can generalize $Z_n^{(\mathtt{q})}$ for $\mathtt{q}$ -partite systems.

$(\mathtt{q}, n)$ Rényi multi-entropy $S_n^{(\mathtt{q})}$

$$S_n^{(\mathtt{q})} := \frac{1}{1-n} \log \frac{Z_n^{(\mathtt{q})}}{(Z_1^{(\mathtt{q})})^{n^{\mathtt{q}-1}}}$$

[A. Gadde, V. Krishna, T. Sharma, 2022]

- $S_n^{(2)}$ reduces to Rényi entanglement entropy if $\mathtt{q} = 2$.
- $S_n^{(\mathtt{q})}$ treats all subsystems symmetrically.
- Symmetric construction is crucial for monotonicity.
  [A. Gadde, S. Jain, V. Krishna, H. Kulkarni, T. Sharma, 2023]
  [A. Gadde, S. Jain, H. Kulkarni, 2024]
- The holographic dual for multi-entropy $S^{(\mathtt{q})} := \lim_{n \to 1} S_n^{(\mathtt{q})}$ has been proposed.

$S_n^{(\mathrm{q})}$ contains broader information than bi-partite entanglement measures.

[A. Gadde, V. Krishna, T. Sharma, 2022]

$$|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}\Big(|001\rangle + |010\rangle + |100\rangle\Big) \qquad S_n^{(3)} = \log 9$$

$$|\mathrm{generalized\,GHZ}\rangle = \frac{1}{\sqrt{3}}\Big(|000\rangle + \sqrt{2}|111\rangle\Big) \qquad S_n^{(3)} = \log\frac{81}{17}$$

Their eigenvalues of reduced density matrices are the same, but $S_n^{(3)}$ are different.

$S_n^{(\mathrm{q})}$ cannot be determined by eigenvalues of reduced density matrix.

Recent developments on multi-entropy

Multi-entropy in random tensor networks and holographic states [G. Penington, M. Walter, F. Witteveen, 2022]

Multi-entropy in CFTs [A. Gadde, V. Krishna, T. Sharma, 2023] [J. Harper, T. Takayanagi, T. Tsuda, 2024]

Multi-entropy for topological ground states [B. Liu, J. Zhang, S. Ohyama, Y. Kusuki, S. Ryu, 2024]

Reflected multi-entropy for mixed states [M.-K. Yuan, M. Li, Y. Zhou, 2024]

Multi-invariants and bulk replica symmetry [A. Gadde, J. Harper, V. Krishna, 2024]

1. Review of Rényi multi-entropy $S_n^{(\mathtt{q})}$

[A. Gadde, V. Krishna, T. Sharma, 2022]

- Rényi multi-entropy $S_n^{(\mathtt{q})}$ is a natural generalization of Rényi entanglement entropy $S_n^{(2)}$ for $\mathtt{q}$ -partite systems.

- $S_n^{(\mathtt{q})}$ is defined by contracting indices of density matrices in different ways for each subsystem.

- $S_n^{(\mathtt{q})}$ cannot be determined by eigenvalues of reduced density matrix.

2. Review of single random tensor

[P. Hayden, S. Nezami, X.-L. Qi, N. Thomas, M. Walter, Z. Yang, 2016]
[C. Akers, T. Faulkner, S. Lin, P. Rath, 2021]

- Single random tensor is a good model to study BH evaporation as in Page's work.

- Random average can be represented by Wick contraction due to Gaussian property.

- By counting the number of loops in graphical representation, we can compute $S_n^{(\mathrm{q})}$.

Single random tensor in bi-partite systems

Two subsystems $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ $\dim \mathcal{H}_A = d_A, \dim \mathcal{H}_B = d_B$

Single random tensor state

$$|\psi\rangle_{AB} = \sum_{i_A=1}^{d_A} \sum_{i_B=1}^{d_B} c_{i_A,j_B} |i_A\rangle \otimes |j_B\rangle$$

c_{i_A,j_B} : complex random variable with Haar measure $d\mu_\psi$

Gaussian property

$$\int d\mu_\psi = 1 \quad \int d\mu_\psi c_{i_A,j_B} = 0 \quad \int d\mu_\psi c_{i_A,j_B} c^*_{i'_A,j'_B} = \frac{\delta_{i_A,i'_A} \delta_{j_B,j'_B}}{d_A d_B}$$

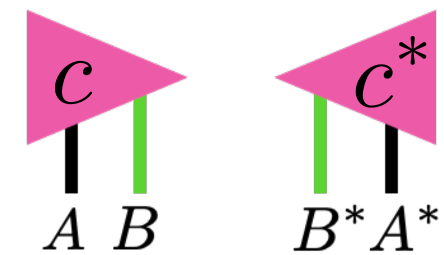
Page computed entanglement entropy of $|\psi\rangle_{AB}$
and use it to discuss entropy of Hawking radiation.

[D. N. Page, “Average entropy of a subsystem”, 1993]
[D. N. Page, “Information in black hole radiation”, 1993]

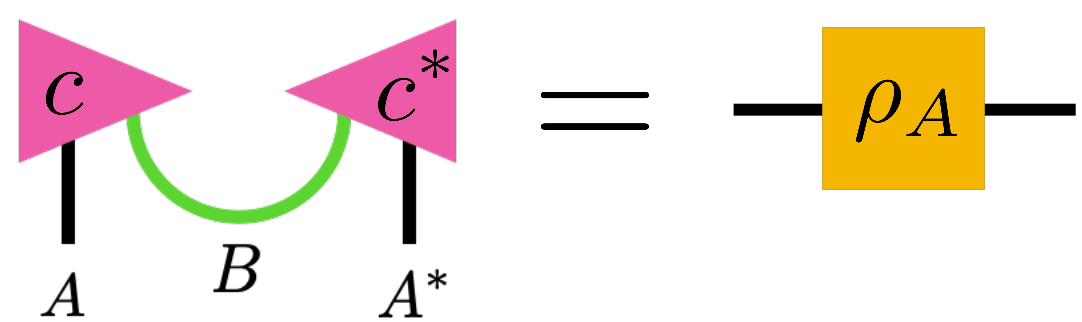
Graphical representation of

$$|\psi\rangle_{AB} = \sum_{i_A=1}^{d_A} \sum_{i_B=1}^{d_B} c_{i_A,j_B} |i_A\rangle \otimes |j_B\rangle$$

$|\psi\rangle_{AB} =$ 

$|\psi\rangle_{AB} \langle\psi|_{A^*B^*} =$ 

Graphical representation of reduced density matrix

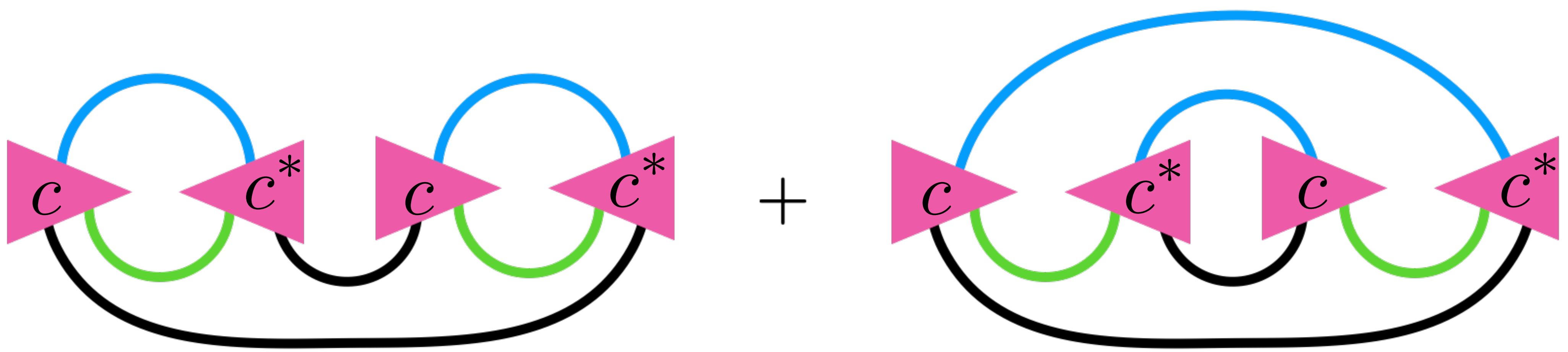
$\rho_A = \mathrm{Tr}_B(|\psi\rangle_{AB} \langle\psi|_{A^*B^*}) =$ 

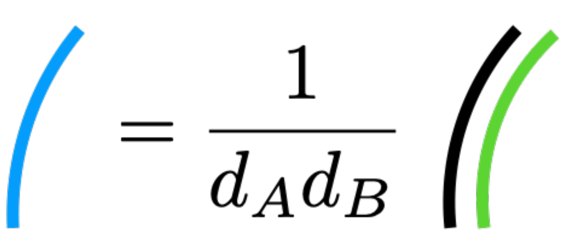
Graphical representation of random average

c_{i_A,j_B} has Gaussian property under the random average.

This random average causes Wick contraction of c_{i_A,j_B}.

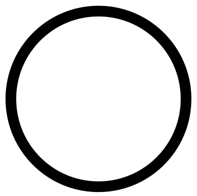
$$\int d\mu_\psi c_{i_A,j_B} c^*_{i'_A,j'_B} = \frac{\delta_{i_A,i'_A}\delta_{j_B,j'_B}}{d_A d_B}$$

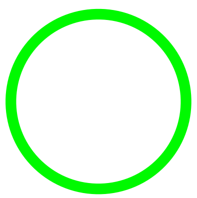
$\overline{\mathrm{Tr}\,\rho_A^2} =$ 



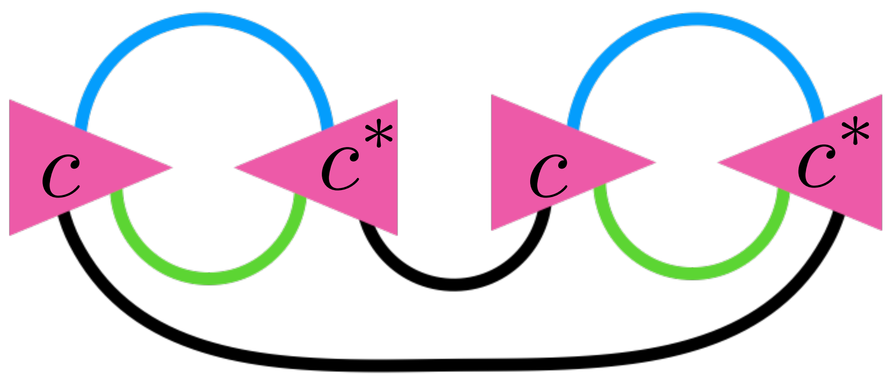
Normalization for $\overline{\mathrm{Tr}\rho_A} = 1$

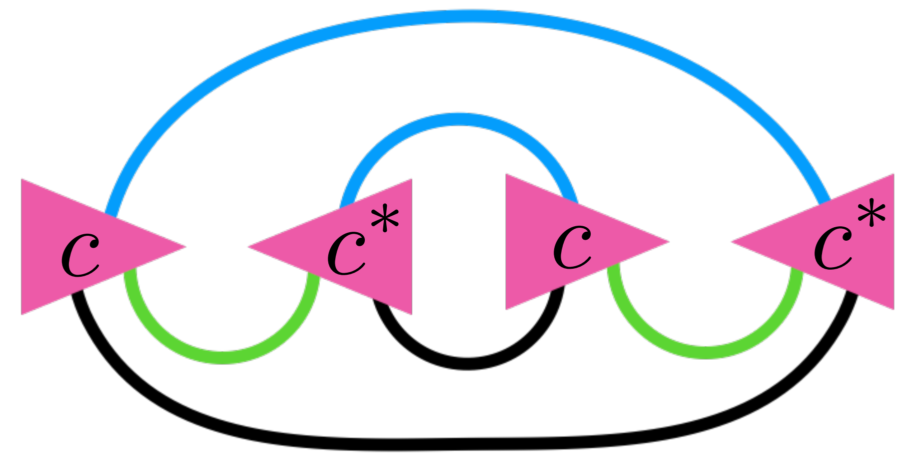
Counting the number of loops, we can compute $\overline{\mathrm{Tr}\rho_A^2}$.

 $= \mathrm{Tr}_{\mathcal{H}_A} 1 = d_A$

 $= \mathrm{Tr}_{\mathcal{H}_B} 1 = d_B$

 $= \frac{1}{d_A d_B}$ 

$\overline{\mathrm{Tr}\,\rho_A^2} =$ 

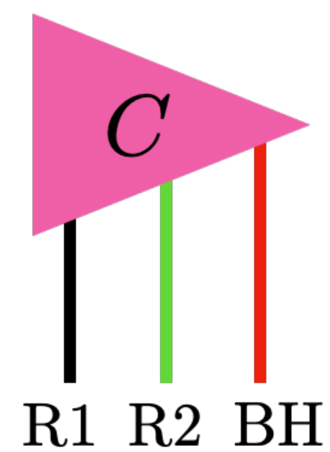
$+$ 

$$= \frac{d_A d_B^2}{d_A^2 d_B^2} + \frac{d_A^2 d_B}{d_A^2 d_B^2} = \frac{1}{d_A} + \frac{1}{d_B}$$

Single random tensor and evaporating black hole

We interpret single random tensor with $\mathtt{q}$ indices as 1 BH and $\mathtt{q}-1$ Hawking radiation.

$\mathtt{q}=3$ example $\qquad |\psi\rangle =$



We study Rényi multi-entropy $S_n^{(\mathtt{q})}$ of $|\psi\rangle$.

This is a natural generalization of Page's work from $\mathtt{q}=2$ to general $\mathtt{q}$.

Recent researches on entanglement measures in random tensor models

Reflected entropy in random tensor models

[C. Akers, T. Faulkner, S. Lin, P. Rath, 2021]

[C. Akers, T. Faulkner, S. Lin, P. Rath, 2022]

[C. Akers, T. Faulkner, S. Lin, P. Rath, 2024]

Entanglement of purification in random tensor models

[C. Akers, T. Faulkner, S. Lin, P. Rath, 2023]

Entanglement negativity in random tensor models

[H. Shapourian, S. Liu, J. Kudler-Flam, A. Vishwanath, 2020]

[J. Kudler-Flam, V. Narovlansky, S. Ryu, 2021]

2. Review of single random tensor

[P. Hayden, S. Nezami, X.-L. Qi, N. Thomas, M. Walter, Z. Yang, 2016]
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- Single random tensor is a good model to study BH evaporation as in Page's work.

- Random average can be represented by Wick contraction due to Gaussian property.

- By counting the number of loops in graphical representation, we can compute $S_n^{(\mathtt{q})}$.

3. Black hole Rényi multi-entropy curves

[N. Iizuka, S. Lin, MN, 2024]

- Black hole Rényi multi-entropy curve is a natural generalization of Page curve for multi-partite systems.
- Multi-entropy curves change their behavior from multi-entropy time, which is later than Page time.
- Multi-entropy curves are nonzero when BH evaporates because of correlations between Hawking radiation.

Setup for $\mathrm{q}=3$ black hole Rényi multi-entropy curve

$$\mathcal{H}=\mathcal{H}_{\mathrm{R1}}\otimes\mathcal{H}_{\mathrm{R2}}\otimes\mathcal{H}_{\mathrm{BH}}$$

$$\dim\mathcal{H}_{\mathrm{R1}}=d_{\mathrm{R1}},\dim\mathcal{H}_{\mathrm{R2}}=d_{\mathrm{R2}},\dim\mathcal{H}_{\mathrm{BH}}=d_{\mathrm{BH}}$$

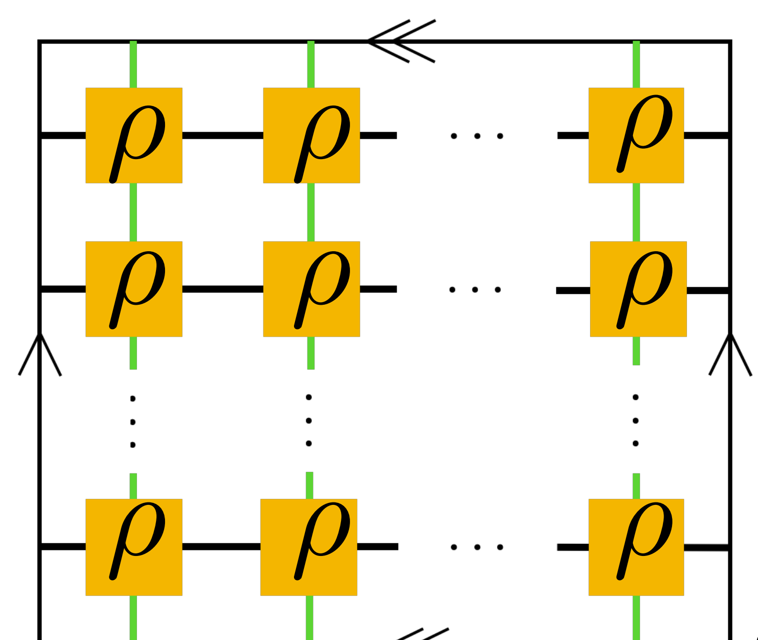
Single random tensor state

$$|\psi\rangle=\sum_{i=1}^{d_{\mathrm{R1}}}\sum_{j=1}^{d_{\mathrm{R2}}}\sum_{k=1}^{d_{\mathrm{BH}}}c_{ijk}|\mathrm{R1}_i\rangle\otimes|\mathrm{R2}_j\rangle\otimes|\mathrm{BH}_k\rangle$$

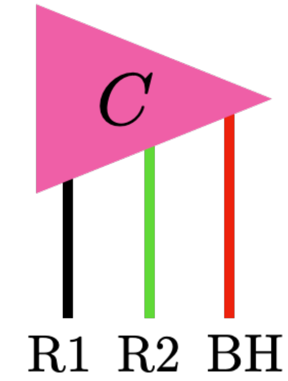
We compute $S_2^{(3)}$ of this random state $|\psi\rangle$.

$$S_2^{(3)}=-\overline{\log\frac{Z_2^{(3)}}{(Z_1^{(3)})^4}}\approx-\log\frac{\overline{Z_2^{(3)}}}{\overline{(Z_1^{(3)})^4}}$$

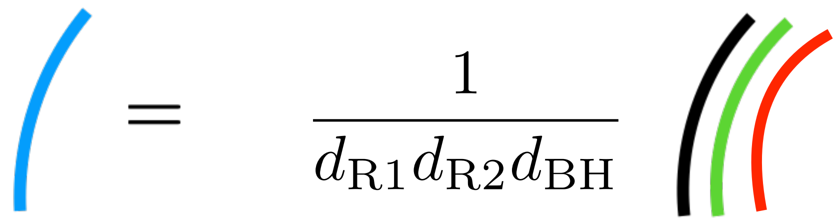
$$Z_n^{(3)}=$$

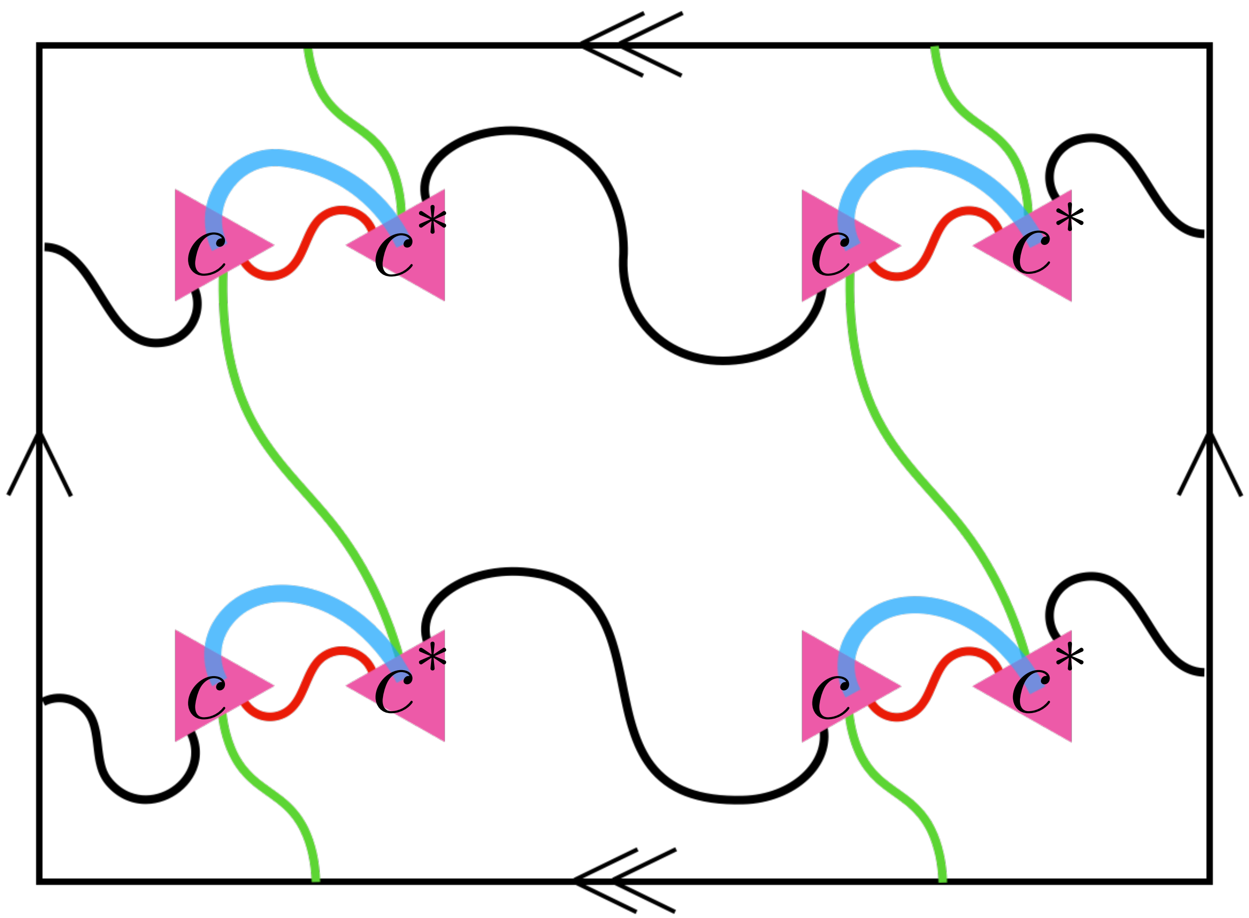


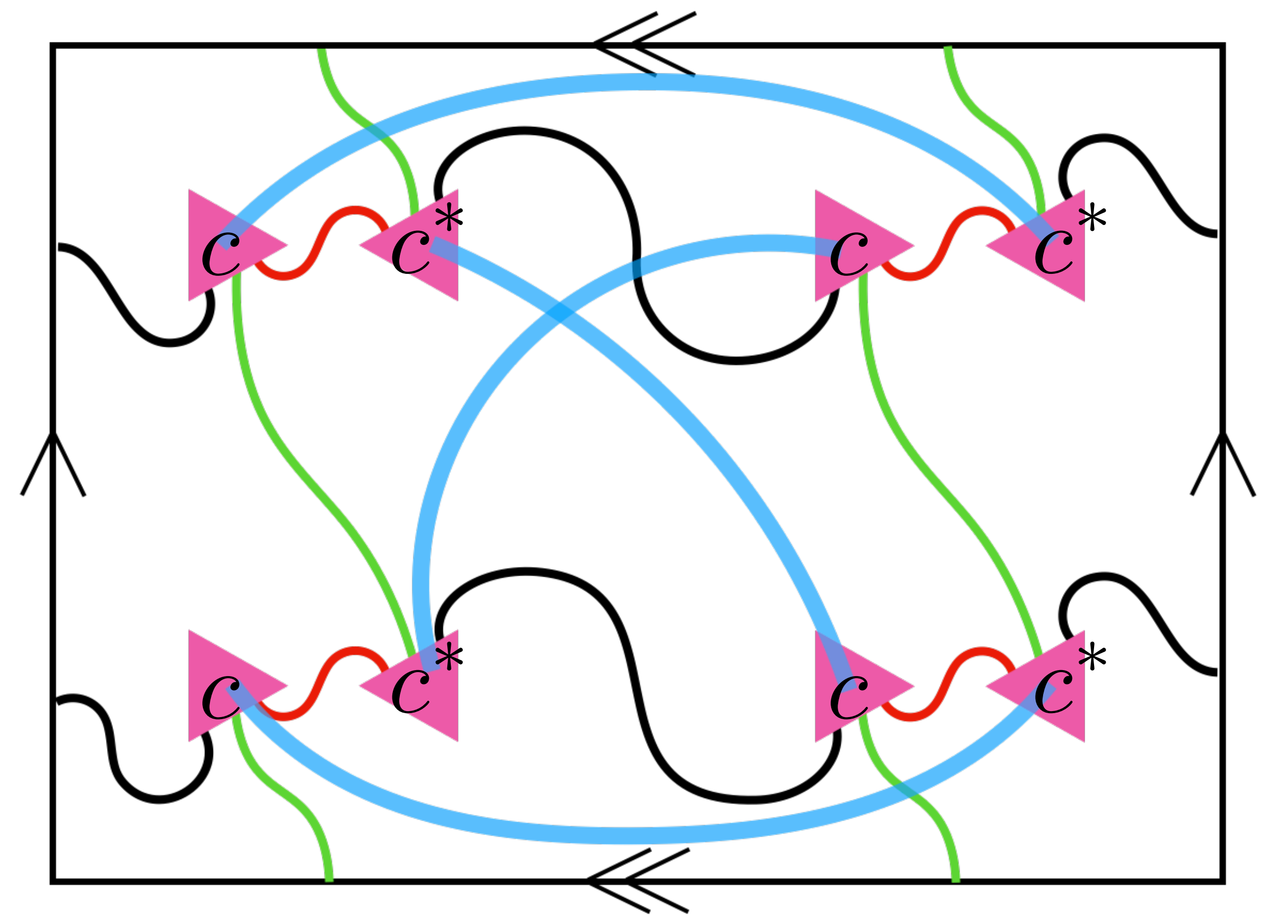
Counting the number of loops, we can compute $\overline{Z_2^{(3)}}$.



Two examples of Wick contraction by







$$\frac{d_{\rm R1}^2 d_{\rm R2}^2 d_{\rm BH}^4}{d_{\rm R1}^4 d_{\rm R2}^4 d_{\rm BH}^4}$$

$$\frac{d_{\rm R1}^3 d_{\rm R2}^1 d_{\rm BH}^1}{d_{\rm R1}^4 d_{\rm R2}^4 d_{\rm BH}^4}$$

$(\mathtt{q}=3, n=2)$ **Rényi multi-entropy** $S_2^{(3)}$

$$S_2^{(3)} = -\log\left[\frac{d_{\mathrm{R1}}d_{\mathrm{R2}}d_{\mathrm{BH}}(9+d_{\mathrm{R1}}^2+d_{\mathrm{R2}}^2+d_{\mathrm{BH}}^2)+2(d_{\mathrm{R1}}^2+d_{\mathrm{R2}}^2+d_{\mathrm{BH}}^2+d_{\mathrm{R1}}^2d_{\mathrm{R2}}^2+d_{\mathrm{R2}}^2d_{\mathrm{BH}}^2+d_{\mathrm{BH}}^2d_{\mathrm{R1}}^2)}{(d_{\mathrm{R1}}d_{\mathrm{R2}}d_{\mathrm{BH}}+1)(d_{\mathrm{R1}}d_{\mathrm{R2}}d_{\mathrm{BH}}+2)(d_{\mathrm{R1}}d_{\mathrm{R2}}d_{\mathrm{BH}}+3)}\right]$$

which is totally symmetric.

At large d_{BH}

$$S_2^{(3)} = \underline{2\log d_{\mathrm{R1}} + 2\log d_{\mathrm{R2}}} - \underline{\frac{2\left(d_{\mathrm{R1}}^2+d_{\mathrm{R2}}^2-2\right)}{d_{\mathrm{R1}}d_{\mathrm{R2}}}}\frac{1}{d_{\mathrm{BH}}} + \mathcal{O}\left(\frac{1}{d_{\mathrm{BH}}^2}\right)$$

no correlation between $\mathcal{H}_{\mathrm{R1}}, \mathcal{H}_{\mathrm{R2}}$

correlation between $\mathcal{H}_{\mathrm{R1}}, \mathcal{H}_{\mathrm{R2}}$

Hawking limit

$$S_{2\ \mathrm{Hawking}}^{(3)} := \lim_{d_{\mathrm{BH}}\to\infty} S_2^{(3)} = 2\log d_{\mathrm{R1}}d_{\mathrm{R2}}$$

Setup for black hole Rényi multi-entropy curve

Divide Hawking radiation into $\mathtt{q}-1$ subsystems equally.

$$d_{\mathrm{R}1} = \cdots = d_{\mathrm{R}\mathtt{q}-1} \equiv d_{\mathrm{R}}$$

We fix the dimension of total system.

$$d_{\mathrm{Total}} = d_{\mathrm{R}}^{\mathtt{q}-1} d_{\mathrm{BH}} = \mathrm{fixed}$$

Rényi multi-entropy $S_n^{(\mathtt{q})}$ becomes a function of d_{R} .

We interpret increasing d_{R} while fixing d_{Total}
as the time evolution of an evaporating BH.

Initial state

$$d_{\mathrm{R}}^{\mathtt{q}-1} = 1, d_{\mathrm{BH}} = d_{\mathrm{Total}}$$

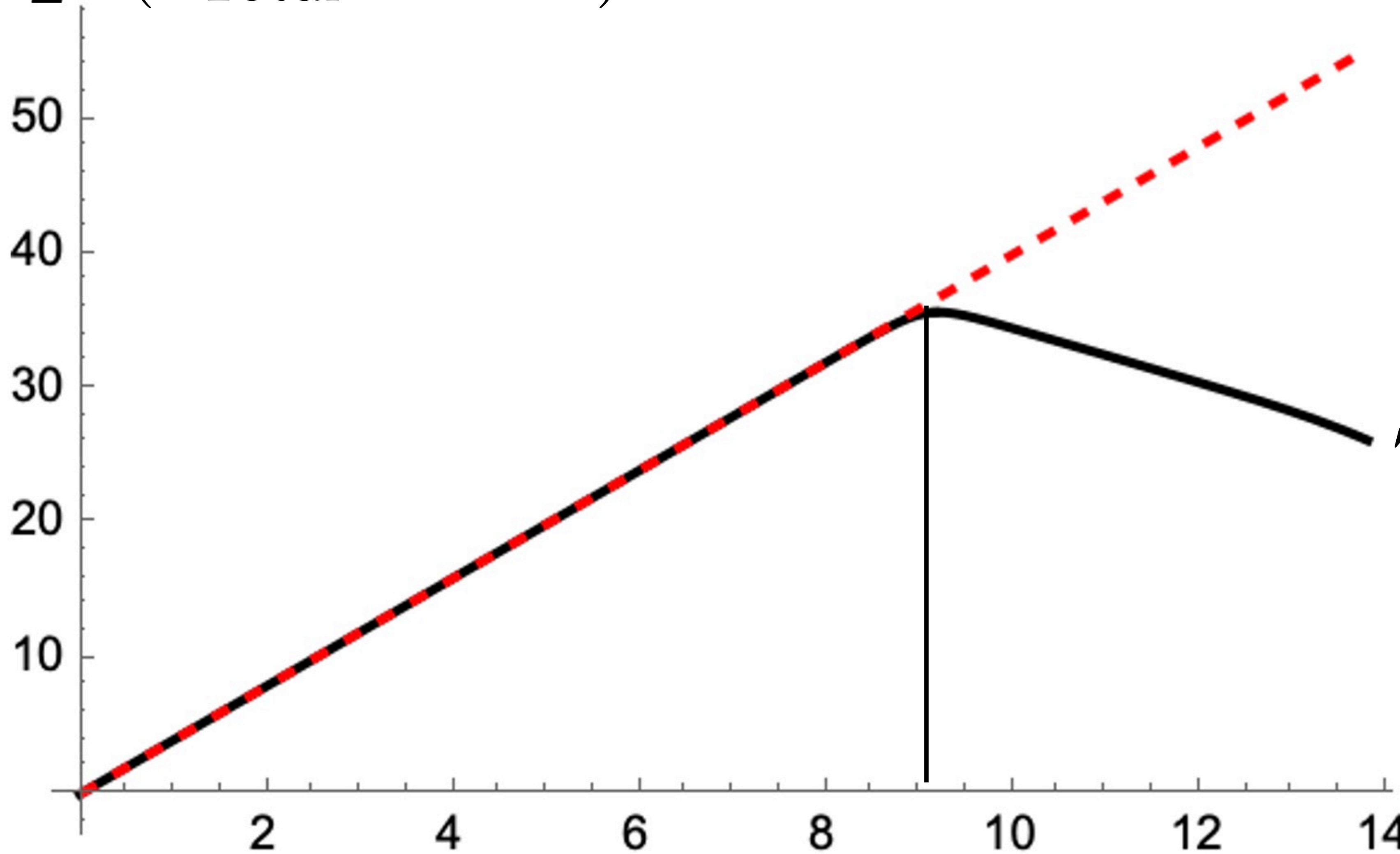
Final state

$$d_{\mathrm{R}}^{\mathtt{q}-1} = d_{\mathrm{Total}}, d_{\mathrm{BH}} = 1$$

Black hole Rényi multi-entropy curve

$$S^{(3)}_{2\ \mathrm{Hawking}} := \lim_{d_{\mathrm{BH}}\to\infty} S_2^{(3)} = 4\log d_{\mathrm{R}}$$

$S_2^{(3)}(d_{\mathrm{Total}} = 10^{12})$



Hawking limit

$S_2^{(3)}$ is nonzero when BH evaporates.

$\log d_{\mathrm{R}}$

Multi-entropy time

$$d_{\mathrm{R}} = d_{\mathrm{BH}} = \frac{d_{\mathrm{Total}}}{(d_{\mathrm{R}})^{\mathtt{q}-1}} \quad \Leftrightarrow \quad d_{\mathrm{R}} = (d_{\mathrm{Total}})^{1/\mathtt{q}} \quad (\text{Multi-entropy time})$$

Decoupled radiation state

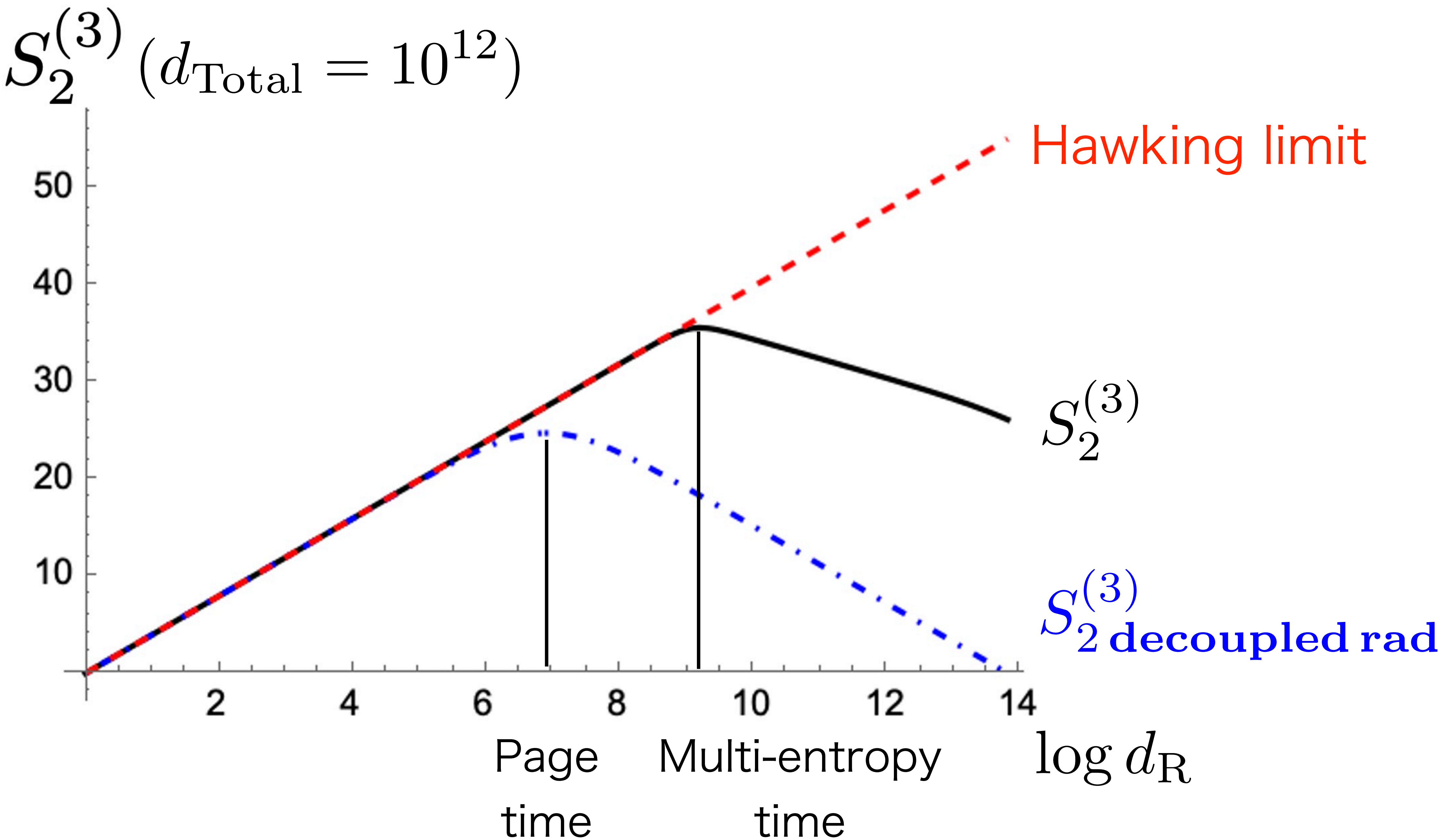
$$|\psi\rangle_{\textbf{decoupled rad}} = \left(\sum_{i_1=1}^{d_{\mathrm{R1}}}\sum_{j_1=1}^{d_{\mathrm{BH1}}} c_{i_1 j_1}|\mathrm{R1}_{i_1}\rangle \otimes |\mathrm{BH1}_{j_1}\rangle\right) \otimes \left(\sum_{i_2=1}^{d_{\mathrm{R2}}}\sum_{j_2=1}^{d_{\mathrm{BH2}}} d_{i_2 j_2}|\mathrm{R2}_{i_2}\rangle \otimes |\mathrm{BH2}_{j_2}\rangle\right)$$

$$d_{\mathrm{R1}} = d_{\mathrm{R2}} = d_R \qquad d_{\mathrm{BH1}} = d_{\mathrm{BH2}} = \sqrt{d_{\mathrm{BH}}}$$

$$S^{(3)}_{2\,\textbf{decoupled rad}} = -2\log\left[\frac{d_R\sqrt{d_{\mathrm{BH}}}(10 + 2d_R\sqrt{d_{\mathrm{BH}}} + d_R^2 + d_{\mathrm{BH}}) + 2(1 + 2d_R^2 + 2d_{\mathrm{BH}})}{(d_R\sqrt{d_{\mathrm{BH}}}+1)(d_R\sqrt{d_{\mathrm{BH}}}+2)(d_R\sqrt{d_{\mathrm{BH}}}+3)}\right]$$

Black hole Rényi multi-entropy curve of $|\psi\rangle_{\textbf{decoupled rad}}$ is essentially same as Page curve for $\mathtt{q} = 2$.

Multi-entropy time is later than Page time.

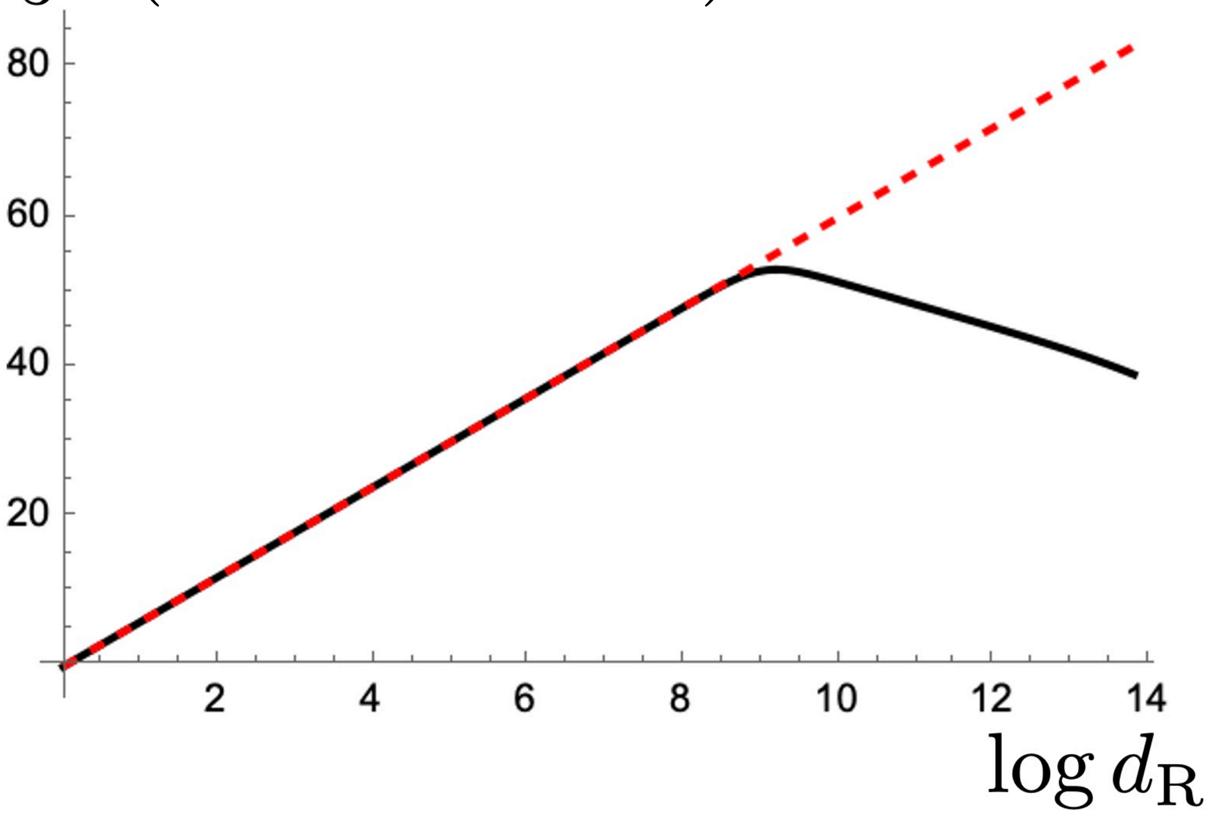


$$d_{\mathrm{R}} = d_{\mathrm{BH}} = \frac{d_{\mathrm{Total}}}{(d_{\mathrm{R}})^{\mathtt{q}-1}} \quad \Leftrightarrow \quad d_{\mathrm{R}} = (d_{\mathrm{Total}})^{1/\mathtt{q}} \quad \text{(Multi-entropy time)}$$

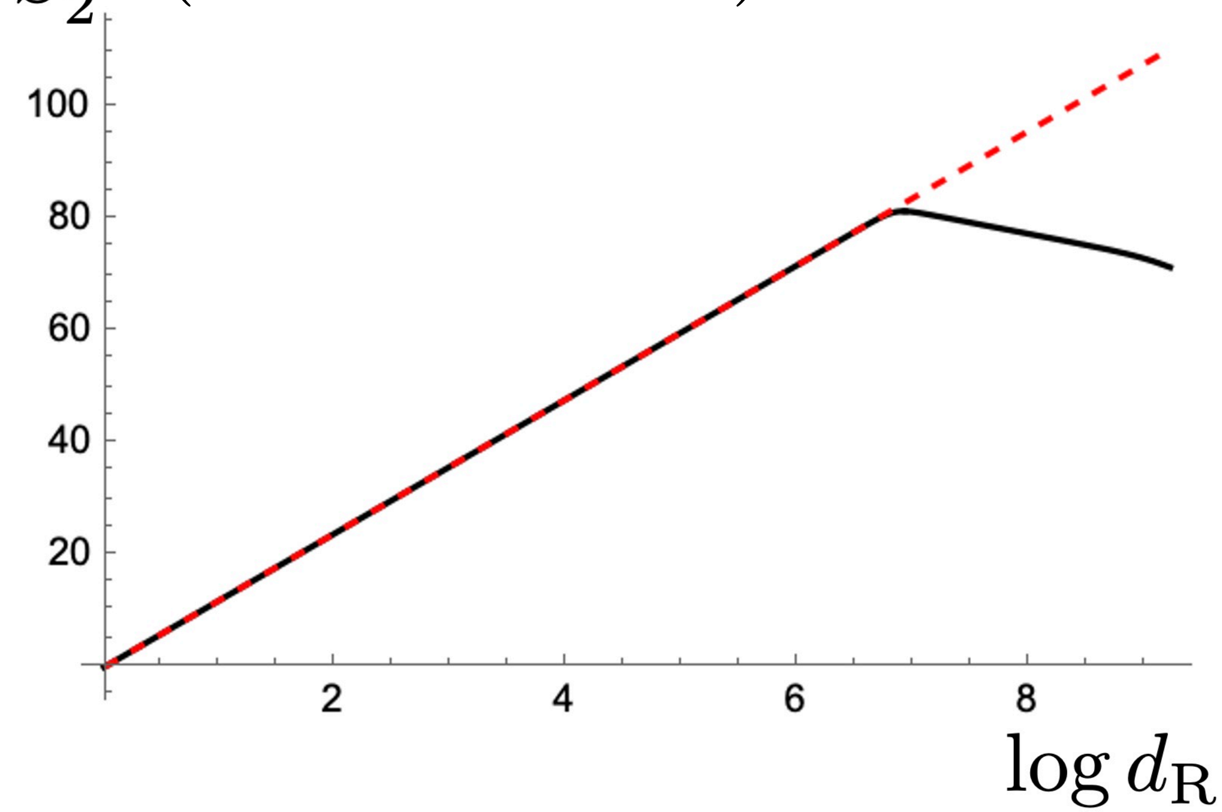
$$(d_{\mathrm{R}})^{\mathtt{q}-1} = d_{\mathrm{BH}} = \frac{d_{\mathrm{Total}}}{(d_{\mathrm{R}})^{\mathtt{q}-1}} \quad \Leftrightarrow \quad d_{\mathrm{R}} = (d_{\mathrm{Total}})^{\frac{1}{2(\mathtt{q}-1)}} \quad \text{(Page time)}$$

Other black hole Rényi multi-entropy curves

$S_3^{(3)}\,(d_{\mathrm{Total}} = 10^{12})$



$S_2^{(4)}\,(d_{\mathrm{Total}} = 10^{12})$



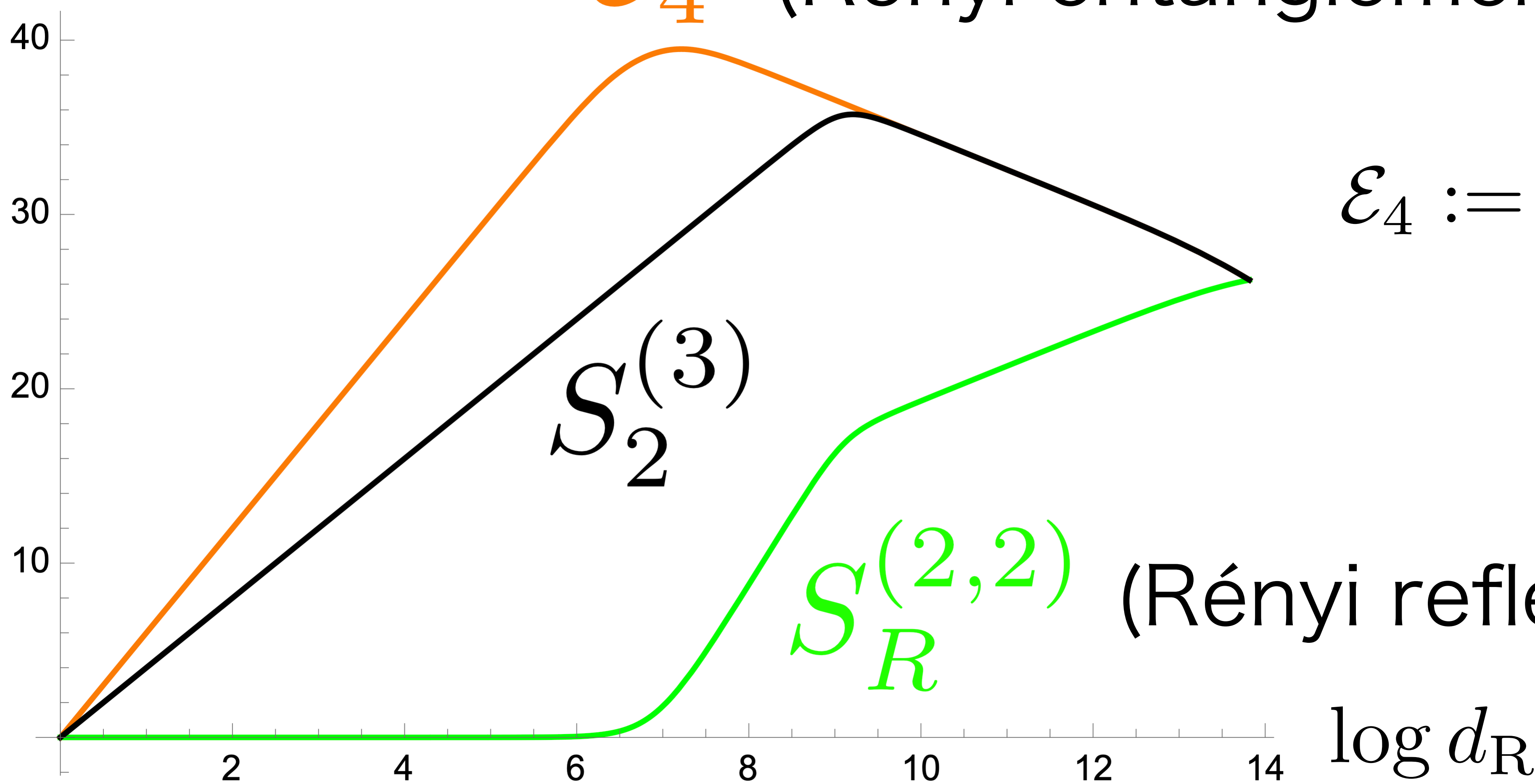
Multi-entropy time depends on $\mathtt{q}$, but not on n .

$$d_{\mathrm{R}} = d_{\mathrm{BH}} = \frac{d_{\mathrm{Total}}}{(d_{\mathrm{R}})^{\mathtt{q}-1}} \quad \Leftrightarrow \quad d_{\mathrm{R}} = (d_{\mathrm{Total}})^{1/\mathtt{q}} \quad \text{(Multi-entropy time)}$$

Comparison with other measures

$$|\psi\rangle = \sum_{i=1}^{d_{\mathrm{R1}}} \sum_{j=1}^{d_{\mathrm{R2}}} \sum_{k=1}^{d_{\mathrm{BH}}} c_{ijk} |\mathrm{R1}_i\rangle \otimes |\mathrm{R2}_j\rangle \otimes |\mathrm{BH}_k\rangle \quad (d_{\mathrm{Total}} = 10^{12})$$

$\mathcal{E}_4$ (Rényi entanglement negativity)



(Rényi reflected entropy)

$$\mathcal{E}_4 := -\log \frac{\mathrm{Tr}\left(\rho_{AB}^{T_B}\right)^4}{(\mathrm{Tr}\rho_{AB})^4}$$

$$S_2^{(3)} = -\log \frac{Z_2^{(3)}}{(Z_1^{(3)})^4}$$

$$S_R^{(2,2)} := -\log \frac{Z_{2,2}}{(Z_{2,1})^2},$$

$$Z_{m,n} := \mathrm{Tr}_{AA^*}\left(\mathrm{Tr}_{BB^*} |\rho_{AB}^{m/2}\rangle\langle\rho_{AB}^{m/2}|\right)^n$$

3. Black hole Rényi multi-entropy curves

[N. Iizuka, S. Lin, MN, 2024]

- Black hole Rényi multi-entropy curve is a natural generalization of Page curve for multi-partite systems.

- Multi-entropy curves change their behavior from multi-entropy time, which is later than Page time.

- Multi-entropy curves are nonzero when BH evaporates because of correlations between Hawking radiation.

4. Early time and late time behaviors

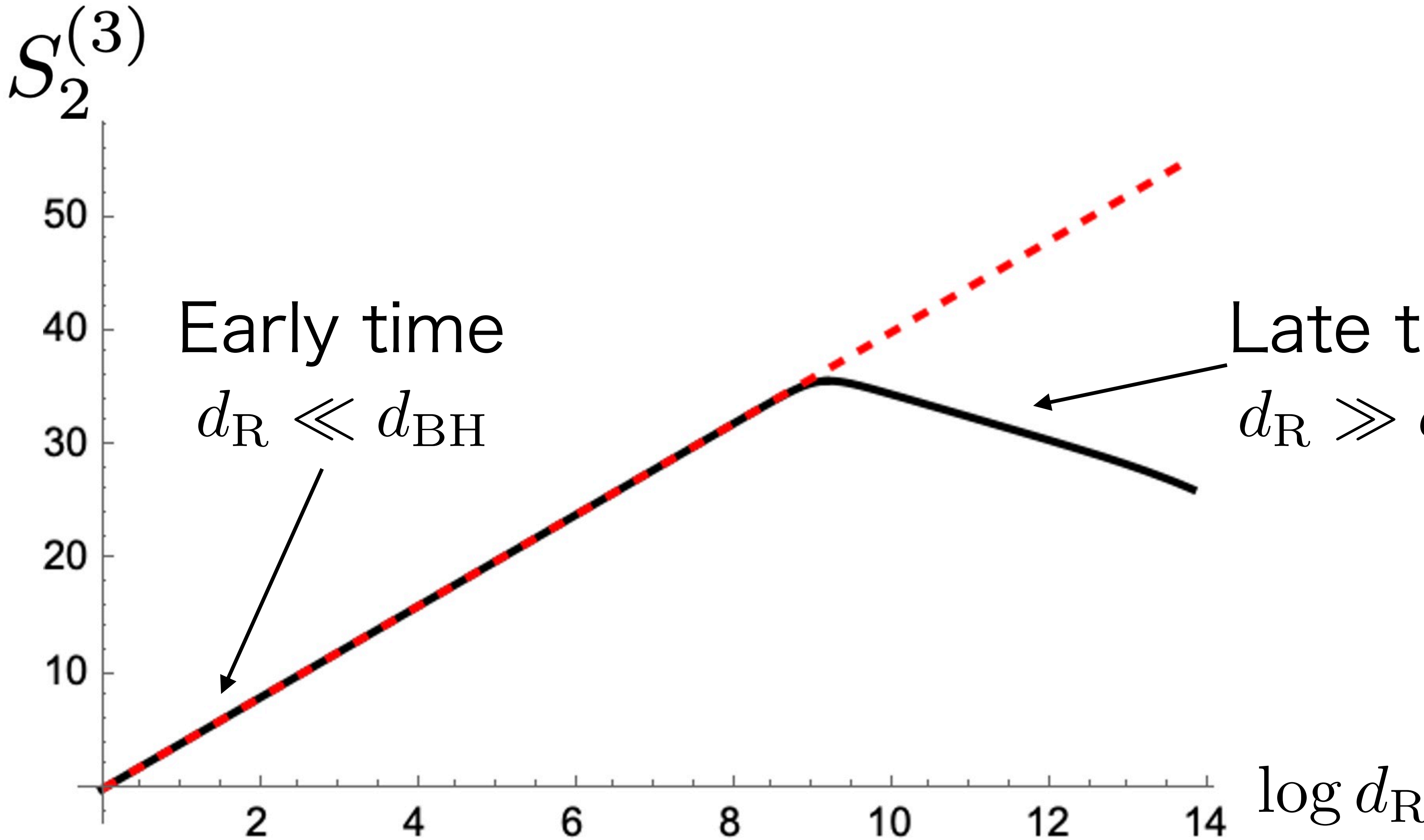
[N. Iizuka, S. Lin, MN, 2024]

- We obtain early time $(d_{\rm R} \ll d_{\rm BH})$ behavior of $S_n^{(\tt q)}$ by counting the leading and subleading contributions.

- We also conjecture late time $(d_{\rm R} \gg d_{\rm BH})$ behavior.

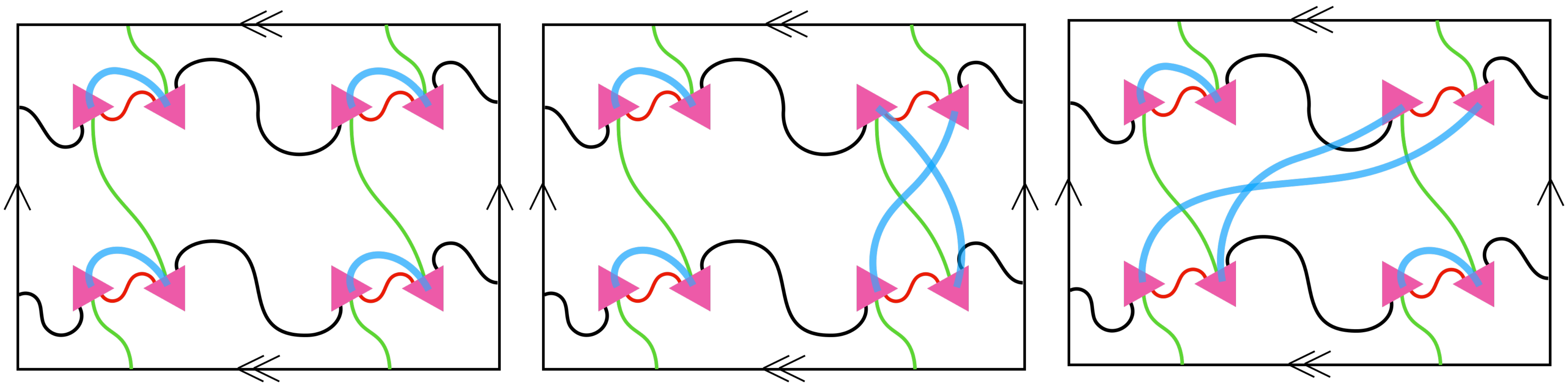
- The cross-over time between the early and late time behaviors is multi-entropy time.

Early time and late time behaviors of Black hole Rényi multi-entropy curves



We are interested in analytic expressions of the early and late time behaviors in $S_n^{(\mathtt{q})}$.

Early time $(d_{\mathrm{R}} \ll d_{\mathrm{BH}})$ behavior of $S_n^{(\mathtt{q})}$



By counting Wick contraction for the leading and subleadng contributions at large d_{BH}, we obtain

$$S_n^{(\mathtt{q})} = n^{\mathtt{q}-2} \log(d_{\mathrm{R}1} \cdots d_{\mathrm{R}\mathtt{q}-1}) - \frac{n^{\mathtt{q}-1}}{2} \frac{d_{\mathrm{R}1}^2 + d_{\mathrm{R}2}^2 + \cdots d_{\mathrm{R}\mathtt{q}-1}^2 - (\mathtt{q}-1)}{d_{\mathrm{R}1} \cdots d_{\mathrm{R}\mathtt{q}-1} d_{\mathrm{BH}}} + \mathcal{O}\left(d_{\mathrm{BH}}^{-2}\right)$$

$$S_1^{(\mathtt{q})} = \log(d_{\mathrm{R}1} \cdots d_{\mathrm{R}\mathtt{q}-1}) - \frac{1}{2} \frac{d_{\mathrm{R}1}^2 + d_{\mathrm{R}2}^2 + \cdots d_{\mathrm{R}\mathtt{q}-1}^2 - (\mathtt{q}-1)}{d_{\mathrm{R}1} \cdots d_{\mathrm{R}\mathtt{q}-1} d_{\mathrm{BH}}} + \mathcal{O}\left(d_{\mathrm{BH}}^{-2}\right)$$

Late time $(d_{\rm R} \gg d_{\rm BH})$ behavior of $S_n^{(\mathtt{q})}$

It is complicated to count the leading contribution at late time $d_1 = d_2 = \cdots = d_{\mathtt{q}-1} \equiv d_{\rm R} \gg d_{\rm BH}$

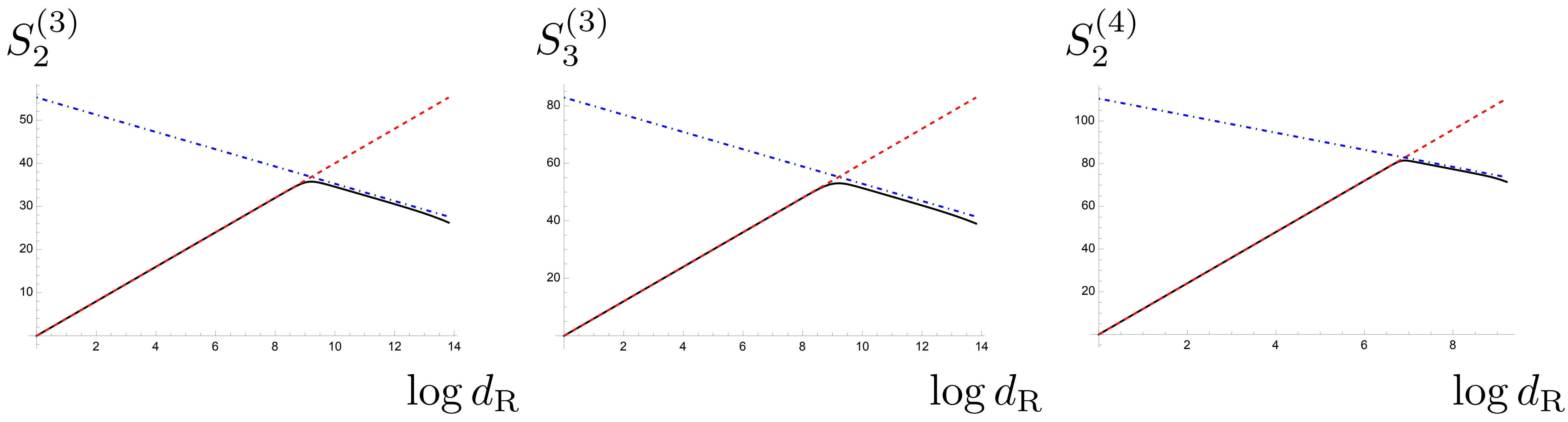
Based on our explicit results and the leading contribution at large d_1 , we conjecture

$$S^{(\mathtt{q})}_{n,\text{late time}} \approx n^{\mathtt{q}-2}\left((\mathtt{q}-2)\log d_{\rm R} + \log d_{\rm BH}\right) = n^{\mathtt{q}-2}\left(\log d_{\rm Total} - \log d_{\rm R}\right)$$

$$S^{(\mathtt{q})}_{1,\text{late time}} \approx (\mathtt{q}-2)\log d_{\rm R} + \log d_{\rm BH} = \log d_{\rm Total} - \log d_{\rm R}$$

$$d_{\rm Total} = d_{\rm R}^{\mathtt{q}-1} d_{\rm BH} = \text{fixed}$$

Comparison with explicit examples



Red: $S^{(\mathtt{q})}_{n,\text{early time}} \approx n^{\mathtt{q}-2}\left(\log d_{\text{Total}} - \log d_{\text{BH}}\right)$

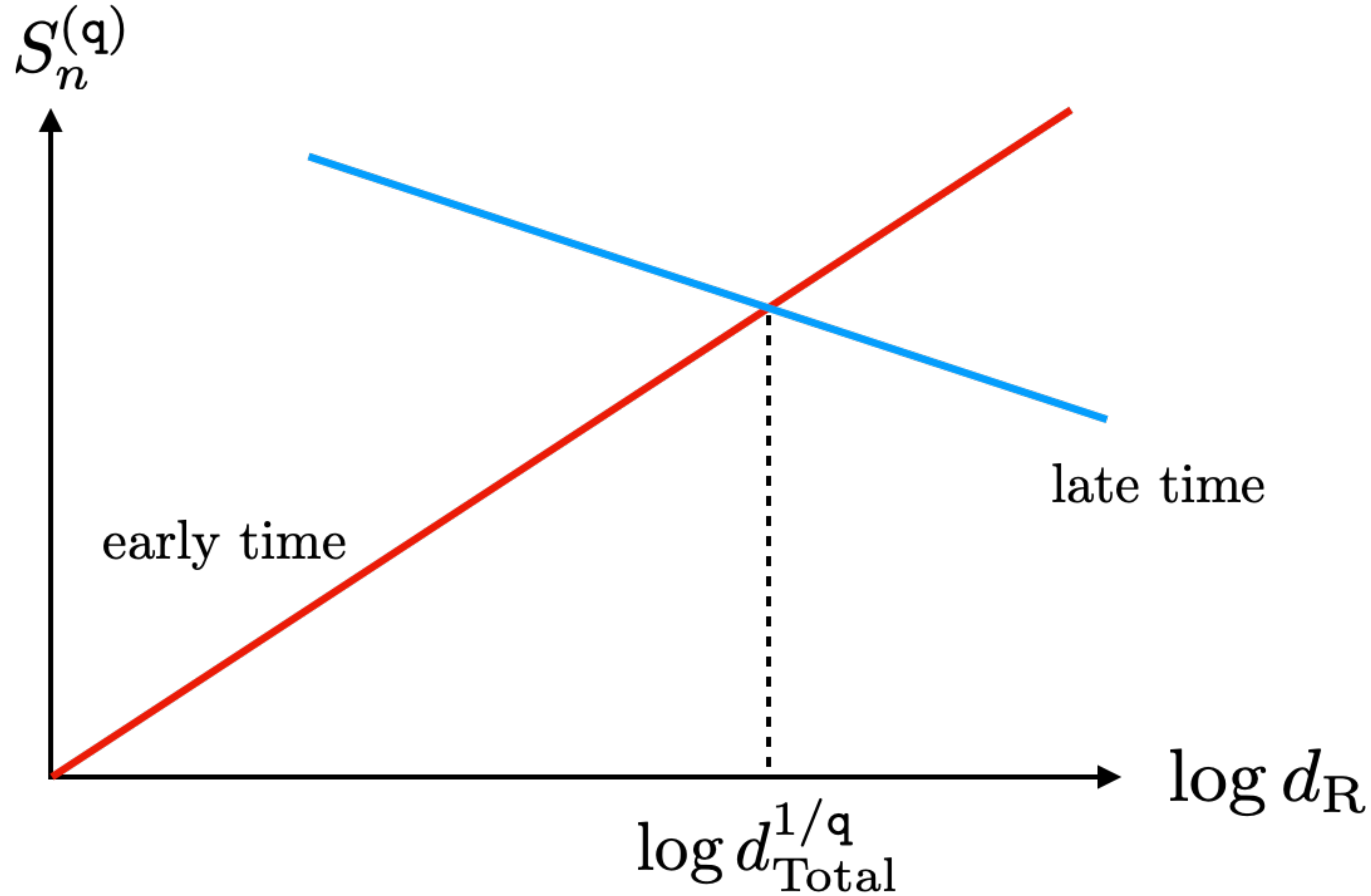
Blue: $S^{(\mathtt{q})}_{n,\text{late time}} \approx n^{\mathtt{q}-2}\left(\log d_{\text{Total}} - \log d_{\text{R}}\right)$

Our formulas approximate the black solid curves $S_n^{(\mathtt{q})}$ well at early and late times.

If we define $S_n^{(\mathtt{q})} := \dfrac{1}{1-n}\dfrac{1}{n^{\mathtt{q}-2}}\log\dfrac{Z_n^{(\mathtt{q})}}{\left(Z_1^{(\mathtt{q})}\right)^{n^{\mathtt{q}-1}}}$ as [A. Gadde, V. Krishna, T. Sharma, 2023] ,

$$S^{(\mathtt{q})}_{n,\text{early time}} \approx \log d_{\text{Total}} - \log d_{\text{BH}}, \quad S^{(\mathtt{q})}_{n,\text{late time}} \approx \log d_{\text{Total}} - \log d_{\text{R}}$$

Crossover time is multi-entropy time.



$$S^{(\mathtt{q})}_{n,\text{early time}} \approx n^{\mathtt{q}-2}\left(\log d_{\rm Total} - \log d_{\rm BH}\right)$$

$$S^{(\mathtt{q})}_{n,\text{late time}} \approx n^{\mathtt{q}-2}\left(\log d_{\rm Total} - \log d_{\rm R}\right)$$

Crossover time $d_{\rm BH} = d_{\rm R} \quad \Leftrightarrow \quad d_{\rm R} = \left(d_{\rm Total}\right)^{1/\mathtt{q}}$

is equal to multi-entropy time.

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- We also conjecture late time $(d_{\mathrm{R}} \gg d_{\mathrm{BH}})$ behavior.

- The cross-over time between the early and late time behaviors is multi-entropy time.

Summary

[N. Iizuka, S. Lin, MN, 2024]

- We study a special new measure called multi-entropy.

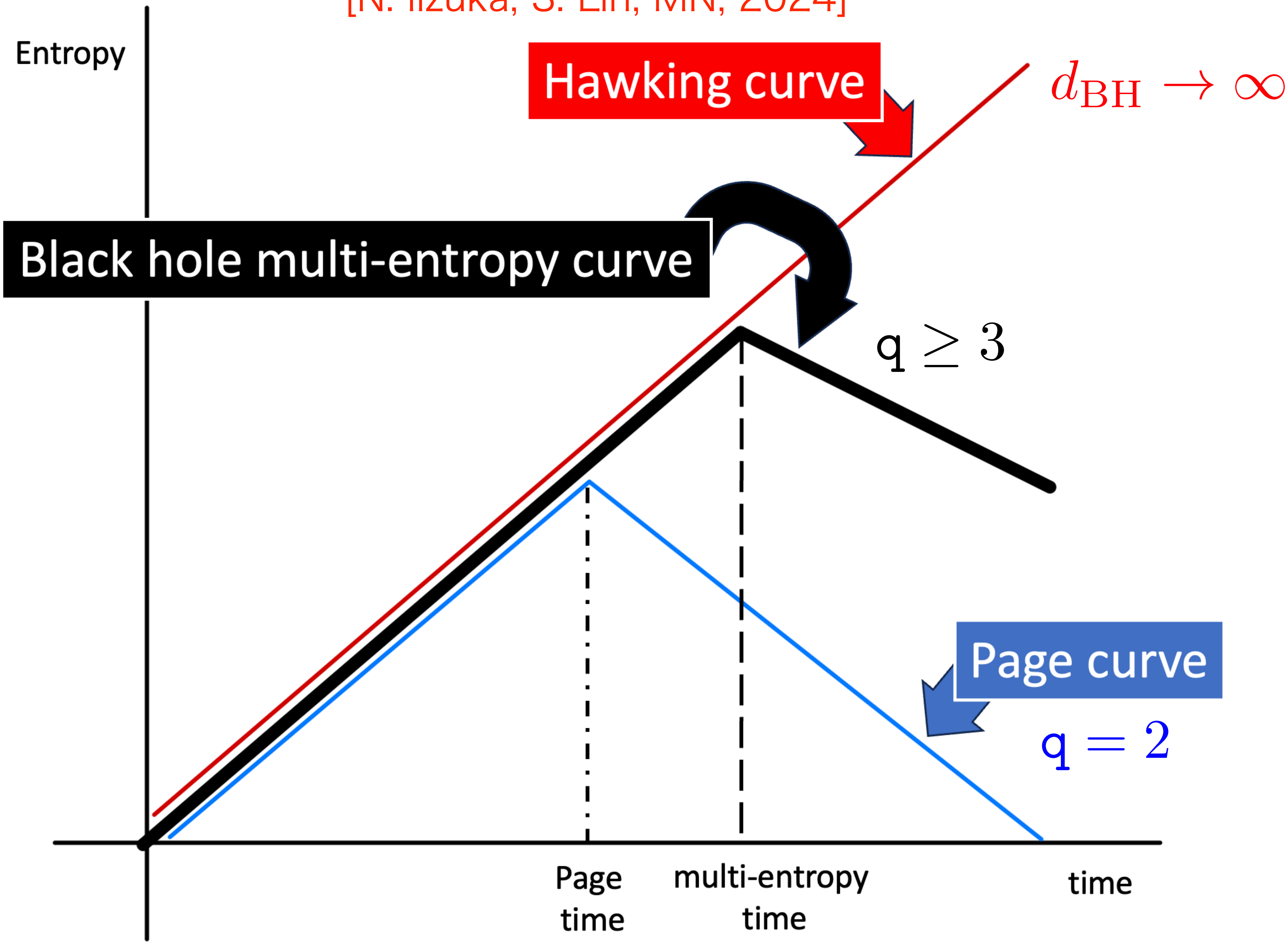
  [A. Gadde, V. Krishna, T. Sharma, 2022]

- We define a black hole multi-entropy curve, which describes how multi-entropy changes during BH evaporation.

- For $\mathtt{q}$ -partite black hole multi-entropy curves, we compute Rényi multi-entropy of a single random tensor with $\mathtt{q}$ indices.

Typical behavior of a black hole multi-entropy curve

[N. Iizuka, S. Lin, MN, 2024]



Future works

- ER=EPR and nonzero multi-entropy when BH evaporates
- Replica wormhole picture for multi-entropy of random states
  [G. Penington, S. H. Shenker, D. Stanford, Z. Yang, 2019]
- Analytic expressions in intermediate regime
- Multi-entropy curves for random tensor networks