Operator Size Dynamics: Theory & Applications

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Outline

1. Introduction:

Why operator size?   Relation to various topics

2. Theoretical Analysis

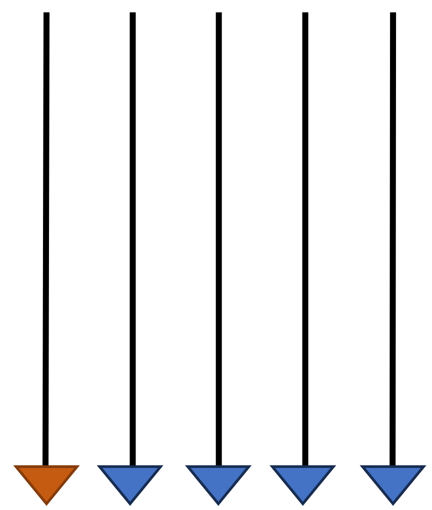
Closed / Open systems with all-to-all interactions

3. Application in Shadow Tomography

Information Scrambling

- Quantum information never disappears.

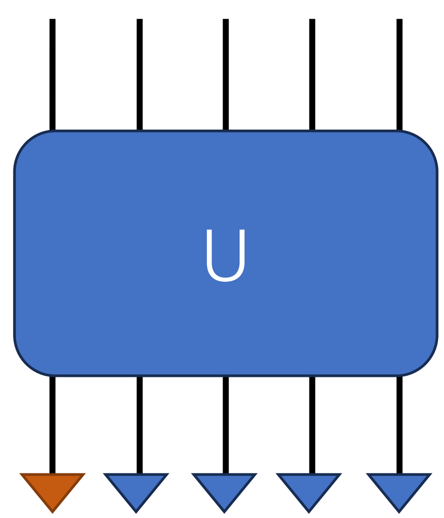
Initial state



$$|\psi_0\rangle = |\psi\rangle \otimes |0\rangle \otimes |0\rangle \ldots$$

Manipulate $|\psi\rangle$: $\{\sigma_1^x, \sigma_1^y, \sigma_1^z\}$

After Evolution



$$|\psi_t\rangle = U|\psi_0\rangle$$

Manipulate $|\psi\rangle$:

$$\{U\sigma_1^x U^\dagger, U\sigma_1^y U^\dagger, U\sigma_1^z U^\dagger\}$$

Becomes complex

Operator Size Distribution

- The scrambling can be quantified by size

$$\sigma_1^x(t) = \sum_n \sum_{\{i_1,\dots,i_n\}} \sum_{\{\alpha_1,\dots,\alpha_n\}} C_{i_1\dots i_n}^{\alpha_1,\dots,\alpha_n} \sigma_{i_1}^{\alpha_1} \sigma_{i_2}^{\alpha_2} \dots \sigma_{i_n}^{\alpha_n}.$$

- The size of a basis $\sigma_{i_1}^{\alpha_1} \sigma_{i_2}^{\alpha_2} \dots \sigma_{i_n}^{\alpha_n}$ is n.

- The size distribution of $\sigma_1^x(t)$ is

$$P(n,t) = \sum_{\{i_1,\dots,i_n\}} \sum_{\{\alpha_1,\dots,\alpha_n\}} \left| C_{i_1\dots i_n}^{\alpha_1,\dots,\alpha_n} \right|^2 \geq 0.$$

  - Normalization: $\sum_n P(n,t) = 1$.

A classical distribution

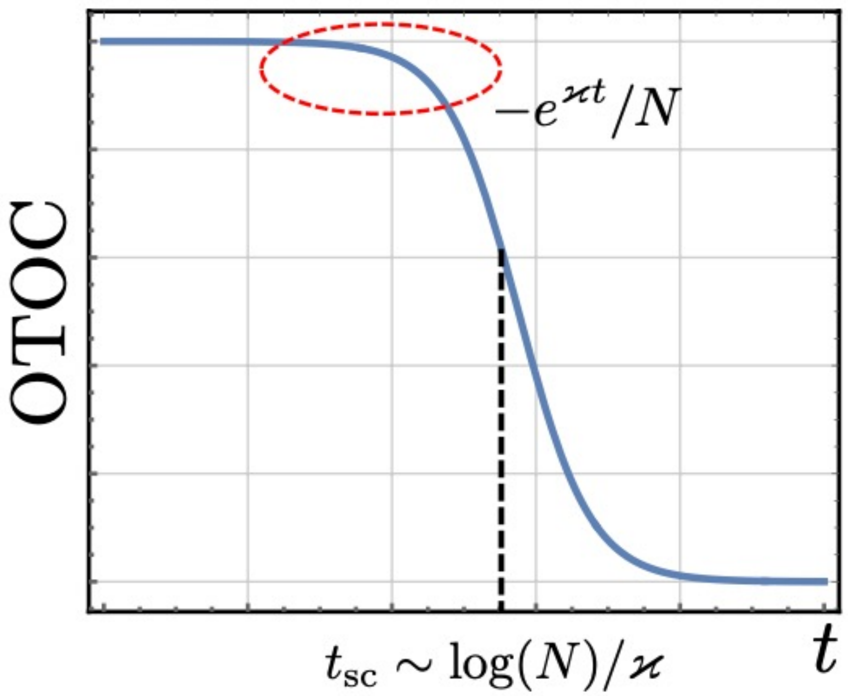
D. A. Roberts, D. Stanford, and L. Susskind, JHEP 2015, 51 (2015).

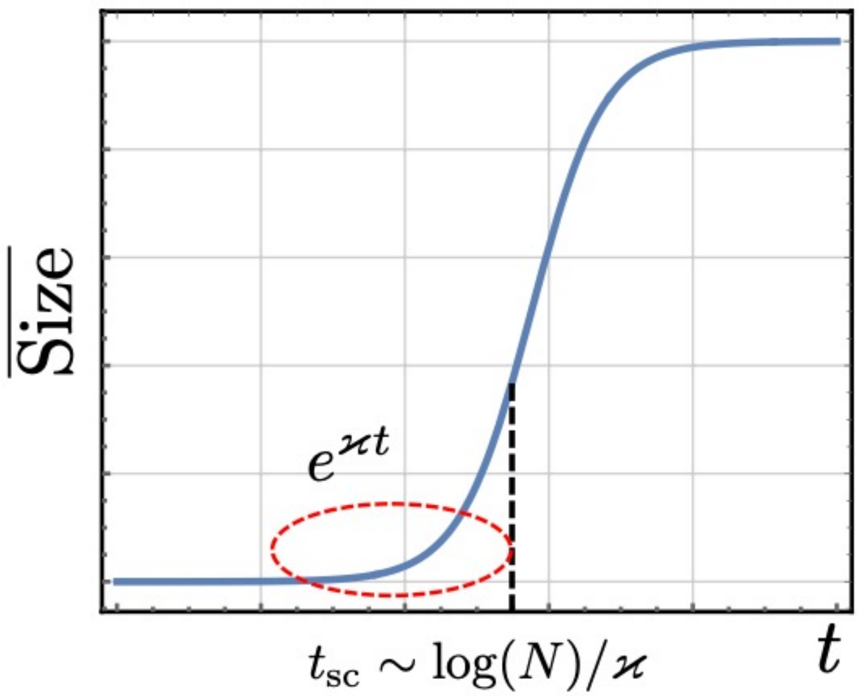
Moments and OTOC

- The average size is the first moment

$$\sum_n n\,P(n,t) = \frac{3N}{4} - \frac{1}{4}\sum_{i,\alpha} \langle \sigma_1^x(t)\sigma_i^\alpha \sigma_1^x(t)\sigma_i^\alpha \rangle.$$

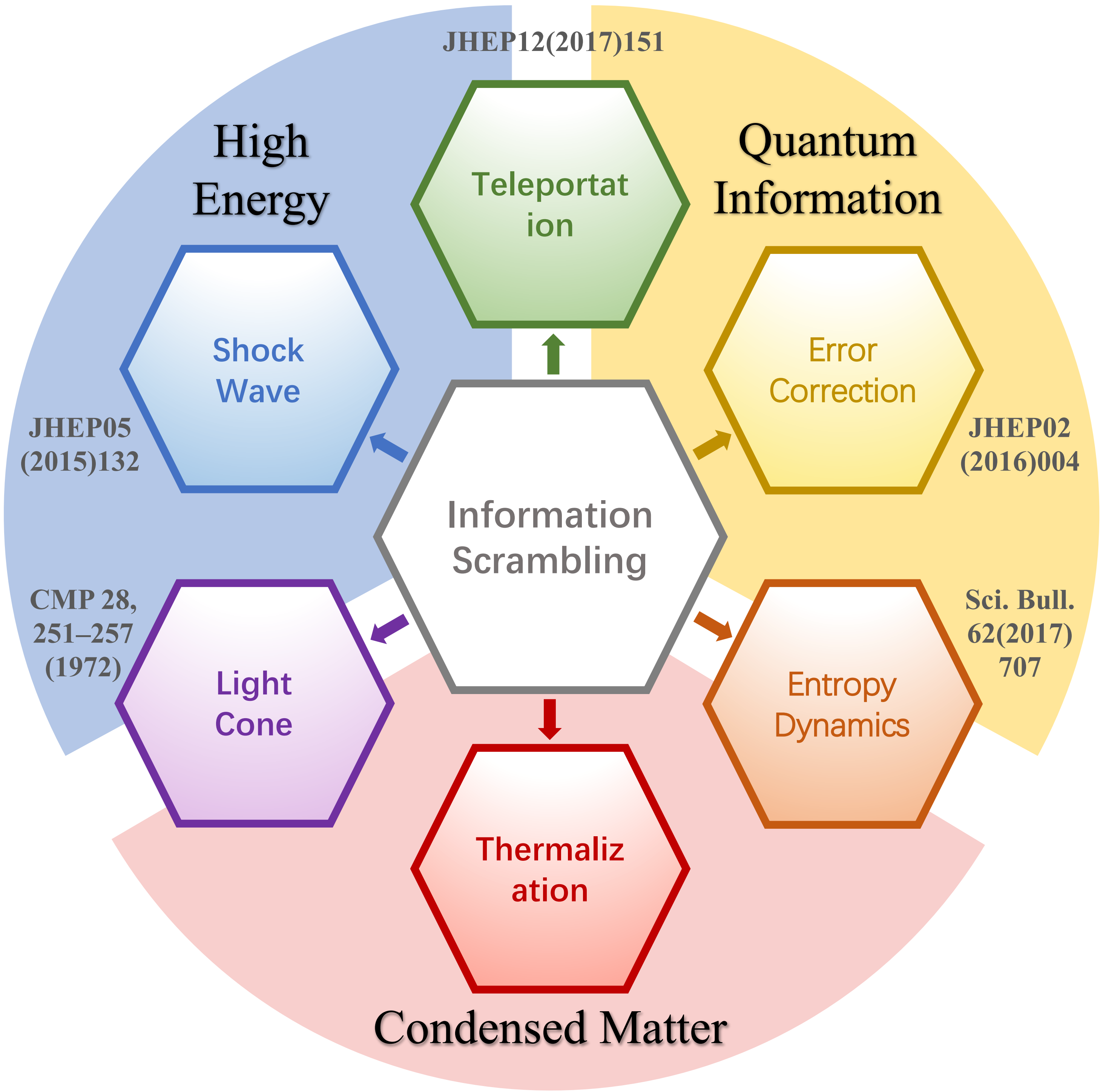
OTOC





- Higher Moments: $\overline{n^k} = \sum_n n^k\, P(n,t)$.

Interplay Between CMP, HEP, and QI

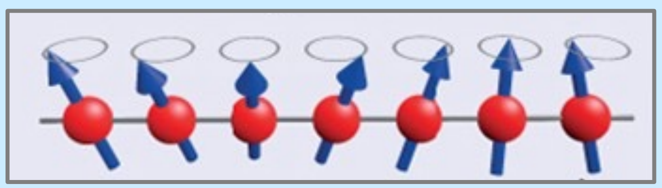


Theoretical Analysis in Many-body Systems

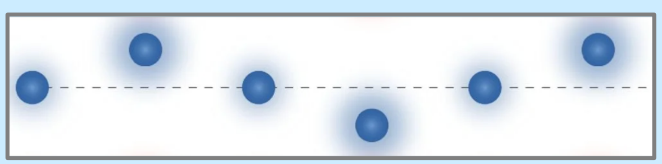
Transports in Many-body Systems

- Collective modes are important in transports.

Spin/Charge Transports

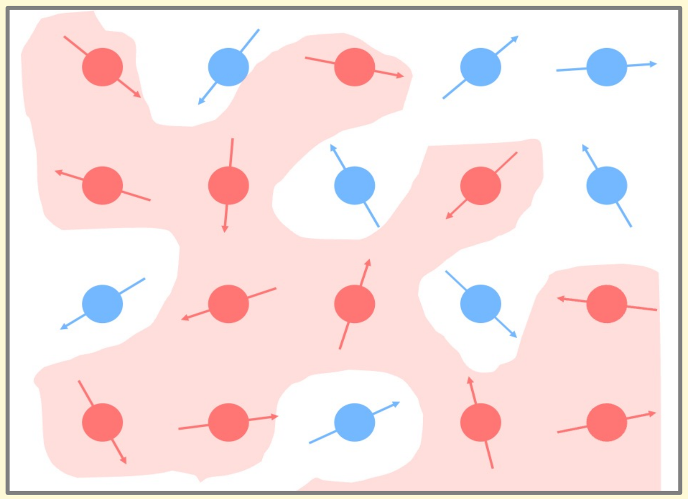


Magnon



Phonon

How about Information?



- Look for the dominant mode for $\sum_{i,\alpha}\langle \sigma_1^x(t)\sigma_i^\alpha\sigma_1^x(t)\sigma_i^\alpha\rangle.$

Sachdev-Ye-Kitaev Model

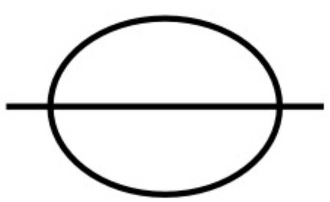
- The SYK model

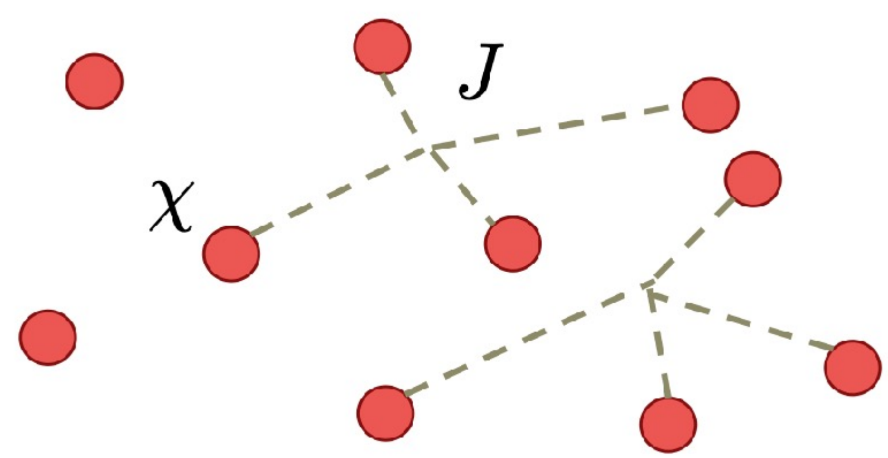
$$H_{\text{SYK}} = \sum_{1\le i_1<i_2<\ldots<i_q\le N} J_{i_1 i_2 \ldots i_q} \chi_{i_1} \chi_{i_2} \cdots \chi_{i_q},$$

The random variables

$$\overline{J_{i_1 i_2 \ldots i_q}} = 0, \qquad \overline{J^2_{i_1 i_2 \ldots i_q}} = \frac{(q-1)! J^2}{N^{q-1}} = \frac{(q-1)! 2^{q-1} \mathcal{J}^2}{q N^{q-1}}$$

Self-energy are melons:

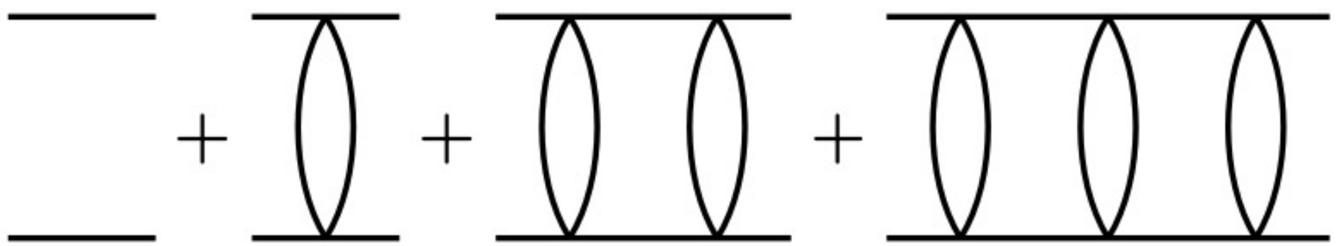
$\Sigma(\tau) =$  $= J^2 G^{q-1}(\tau)$

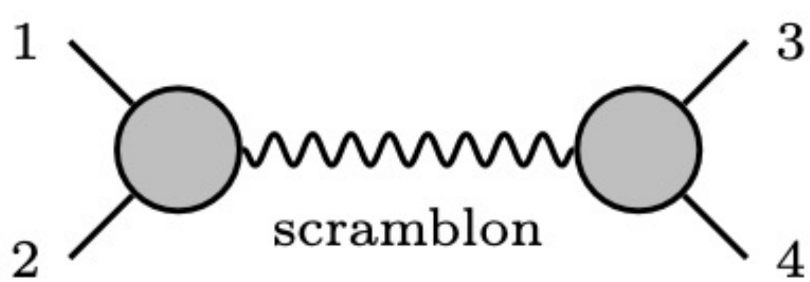


S. Sachdev and J. Ye, PRL 70, 3339 (1993); Kitaev talk @2015.

OTOC Ladder Diagram

- The early-time OTOC are ladders

$\mathrm{OTOC}_c =$  $+ \ldots,$

$=$ 

$= \left(-\frac{e^{\varkappa t}}{Nc}\right) \Upsilon^{\mathrm{R},1}(t_{12}) \Upsilon^{\mathrm{A},1}(t_{34}).$

- Only valid at

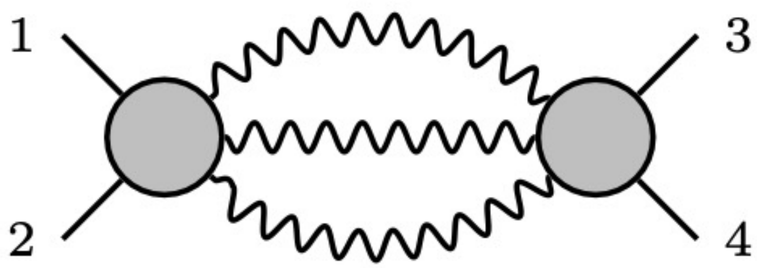
$$t \ll \ln N \qquad \text{(Early-time limit)}$$

S. Sachdev and J. Ye, PRL 70, 3339 (1993); Kitaev talk @2015.

Scramblon Effective Theory

- Scrambling are dominated by scramblons

$$\langle W^\dagger(t_1)V^\dagger(t_3)W(t_2)V(t_4)\rangle =$$

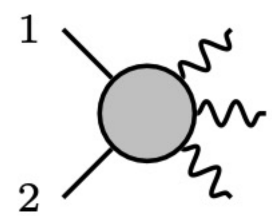


$$= \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!} \Upsilon_W^{\mathrm{R},m}(t_{12})\, \Upsilon_V^{\mathrm{A},m}(t_{34}).$$

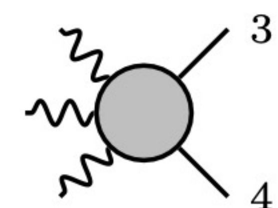
Propagator

Scattering amplitude

$$= \lambda = \frac{e^{\varkappa \frac{t_1+t_2-t_3-t_4}{2}}}{C}$$



$$= \Upsilon_W^{\mathrm{R},m}(t_{12}),$$

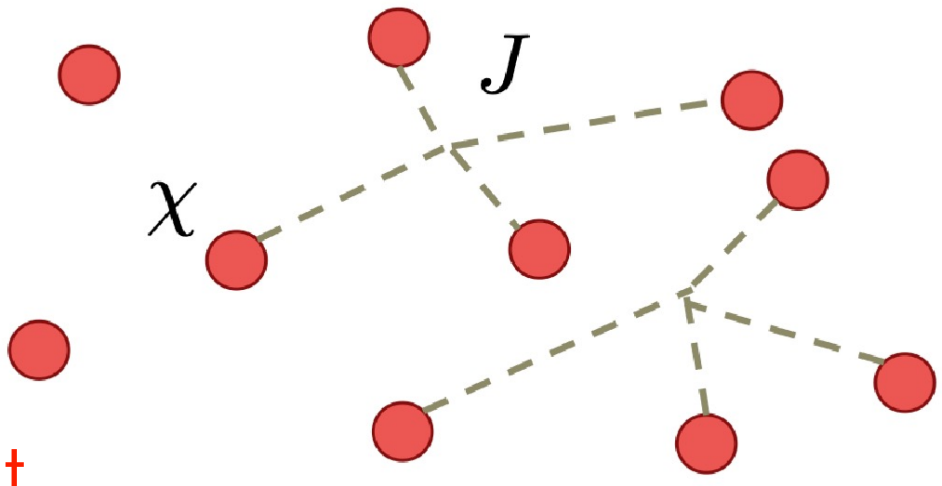


$$= \Upsilon_V^{\mathrm{A},m}(t_{34}).$$

D. Stanford, Z. Yang, S. Yao, JHEP 2022, 200 (2022); Y. Gu, A. Kitaev, and PZ†, JHEP 2022,133 (2022).

The Brownian SYK Model

- The SYK is easier when Brownian



$$H_{\mathrm{BSYK}} = \sum_{1\le i_1<i_2<\ldots<i_q\le N} J_{i_1 i_2 \ldots i_q}(t)\chi_{i_1}\chi_{i_2}\cdots\chi_{i_q},$$

Time-dependent

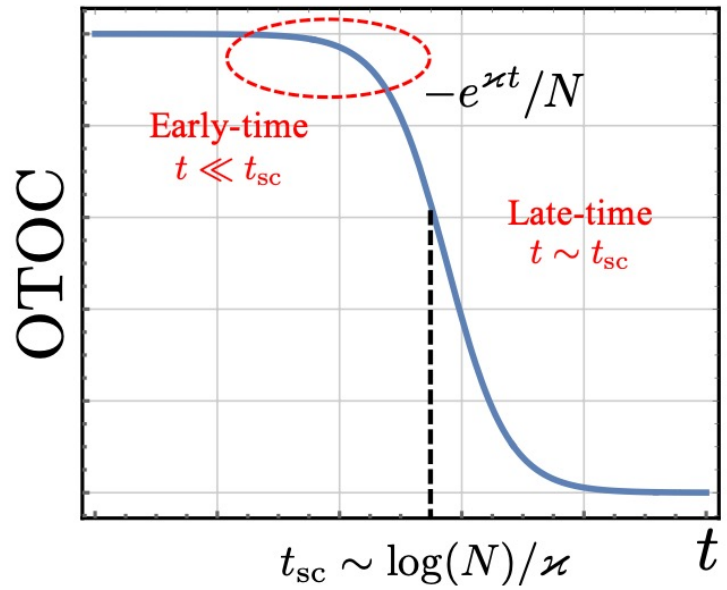
- All scattering amplitudes are known:

$$2\Delta = 1/(q-2)$$

$$f(x,t) = \sum_n \frac{(-x)^n}{n!}\Upsilon^n(t) = \left(\frac{1}{e^{\varkappa|t_{12}|/2}+x}\right)^{2\Delta},$$

- Result:

$$\mathrm{OTOC}(\{t_i\}) = \frac{1}{\lambda^{2\Delta}} U\left(2\Delta, 1, \frac{e^{\varkappa|t_{12}|/2}e^{\varkappa|t_{34}|/2}}{\lambda}\right)$$

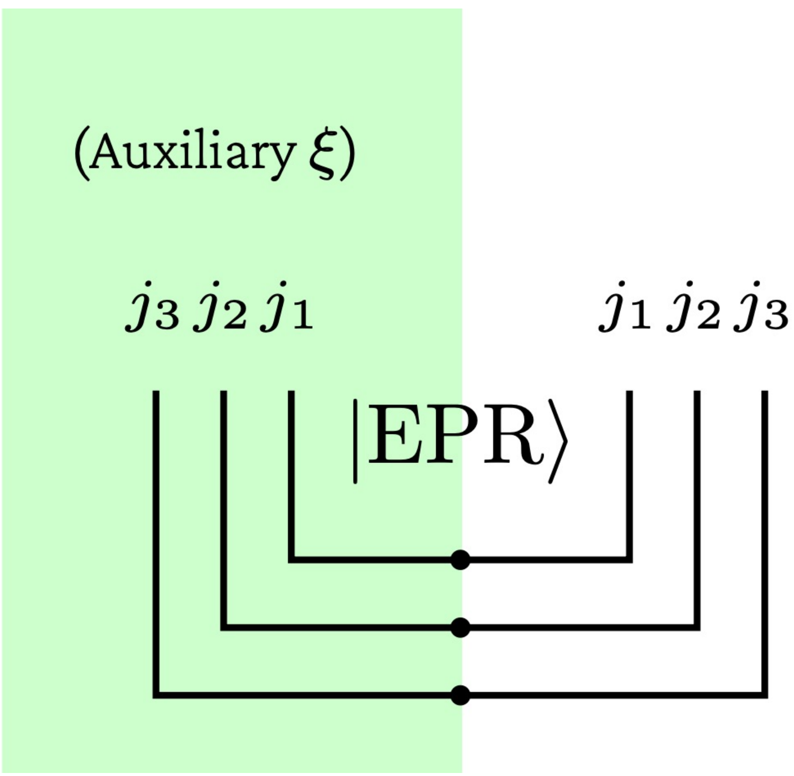


Y. Gu, A. Kitaev, and PZ†, JHEP 2022,133 (2022).

Size Distribution & Total Spin

- Theoretical Trick: states are easier than operators!

operator-state mapping: $O \to |O\rangle = (O \otimes I)|\mathrm{EPR}\rangle$



1. Introduce auxiliary spin system τ_i^α
2. Prepare σ_i and τ_i in singlets

$$|\mathrm{EPR}\rangle_i = |s\rangle_i = (|01\rangle - |10\rangle)/\sqrt{2}$$

3. Since $\sigma_i^\alpha |s\rangle_i = |t\rangle_i$

$$P(n,t) = \langle O(t)|\delta_{n,\sum s_{tot,j}^2}|O(t)\rangle$$

Z. Liu and PZ[†], PRL 132, 060201 (2024).

➡ Exp. protocol

Relation between Early-time & Late-time

- Using the scramblon effective theory:

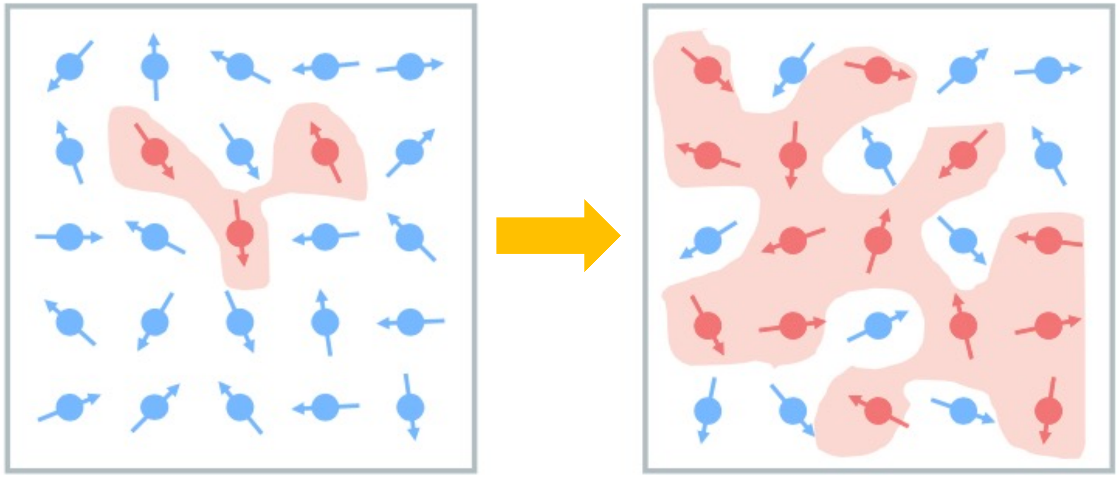
$$\mathcal{P}(s,t) = N P(sN,t) = \int_0^\infty dy\, \bar{h}(y)\delta(s - s_{\rm sc}(1-\bar{f}(\lambda y)))$$

with $f(x) = \sum_m (-x)^m \Upsilon^m / m!, \qquad f(x) = \int dy e^{-xy} h(y)$

Early-time limit $\lambda \ll 1$

$$\overline{\mathcal{P}}(s,t) = (s_{\rm sc}\lambda\overline{\Upsilon}^1)^{-1}\overline{h}\left(s(s_{\rm sc}\lambda\overline{\Upsilon}^1)^{-1}\right)$$

$h(y)$ contains the Full information!



Self-consistent relation

Z. Liu and PZ[†], PRL 132, 060201 (2024).

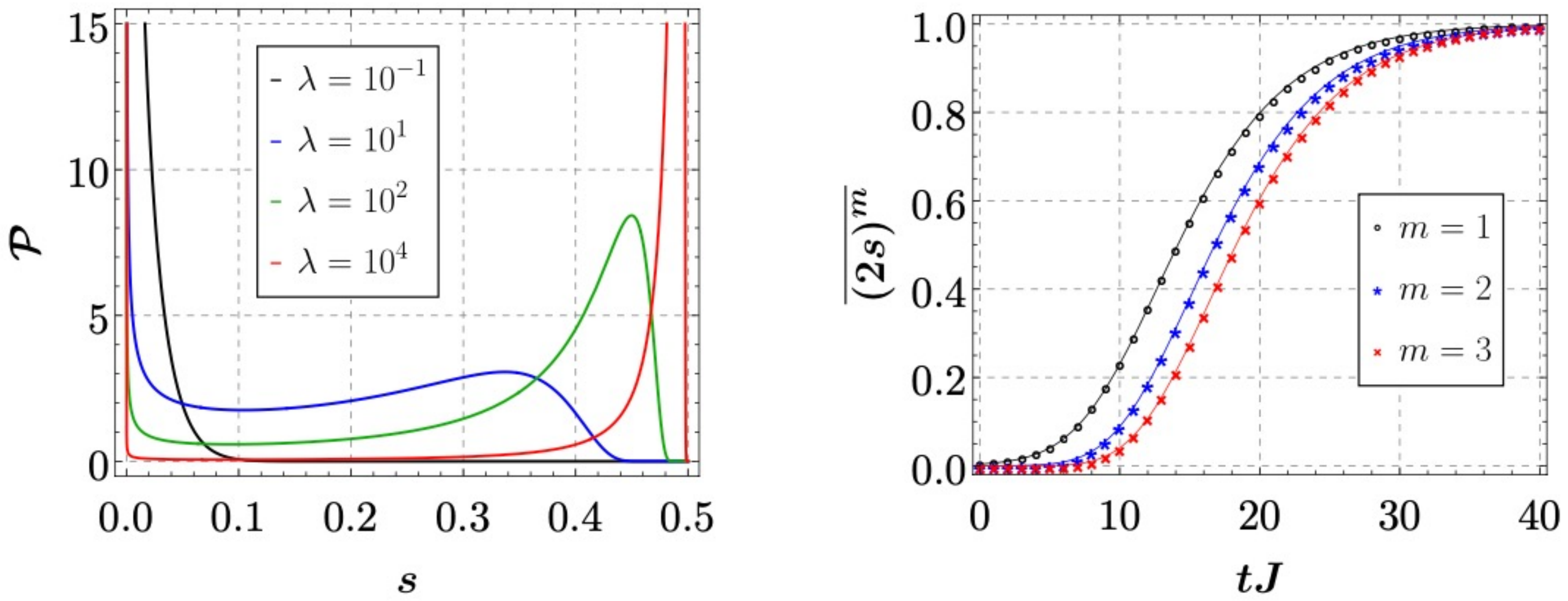
Late-time Size Distribution

- In the long-time limit, assuming $f(\infty) = 0$:

$$\mathcal{P}(s,\infty) = \delta(s - s_{\text{sat}}), \qquad \text{“Maximally Scrambled”}\ \delta s \sim N^{-1/2} \to 0.$$

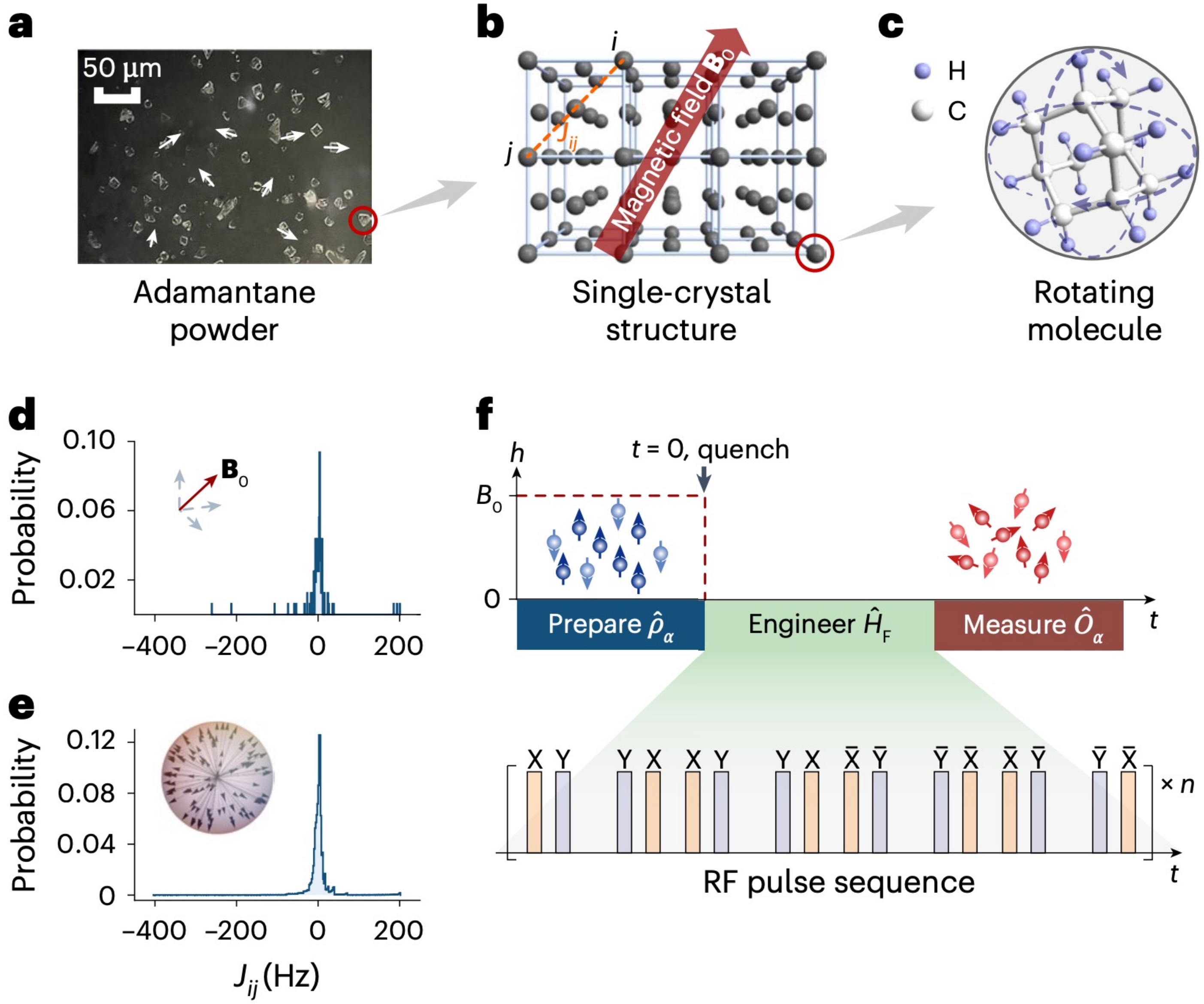
Example: Brownian SYK

$$\mathcal{P}(s,t) = \frac{2p^{2\Delta-1}e^{-p}}{\lambda\Gamma(2\Delta+1)(1-2s)^{\frac{2\Delta+1}{2\Delta}}}, \qquad p = \frac{(1-2s)^{-\frac{1}{2\Delta}} - 1}{\lambda},$$



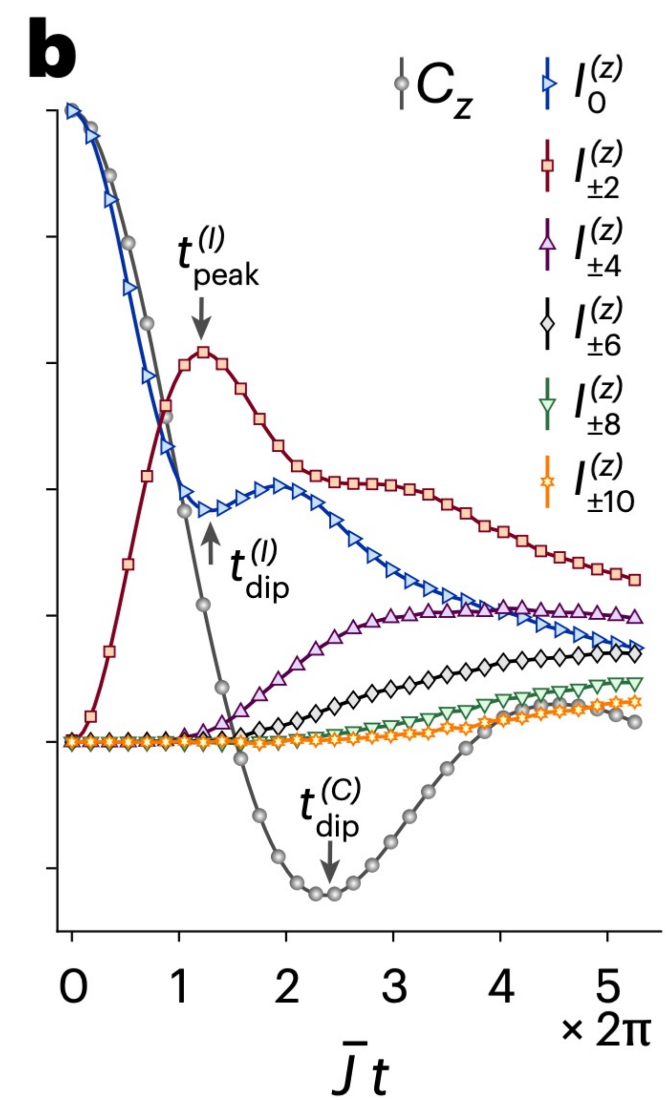
Z. Liu and PZ[†], PRL 132, 060201 (2024).

Experimental Progress



$$F(\phi,t) = \frac{1}{c_O}\,\mathrm{Tr}[e^{-i\hat{O}_\alpha\phi}\,\hat{O}_\alpha(t)\,e^{i\hat{O}_\alpha\phi}\,\hat{O}_\alpha(t)]$$

Fourier Transform → I_m



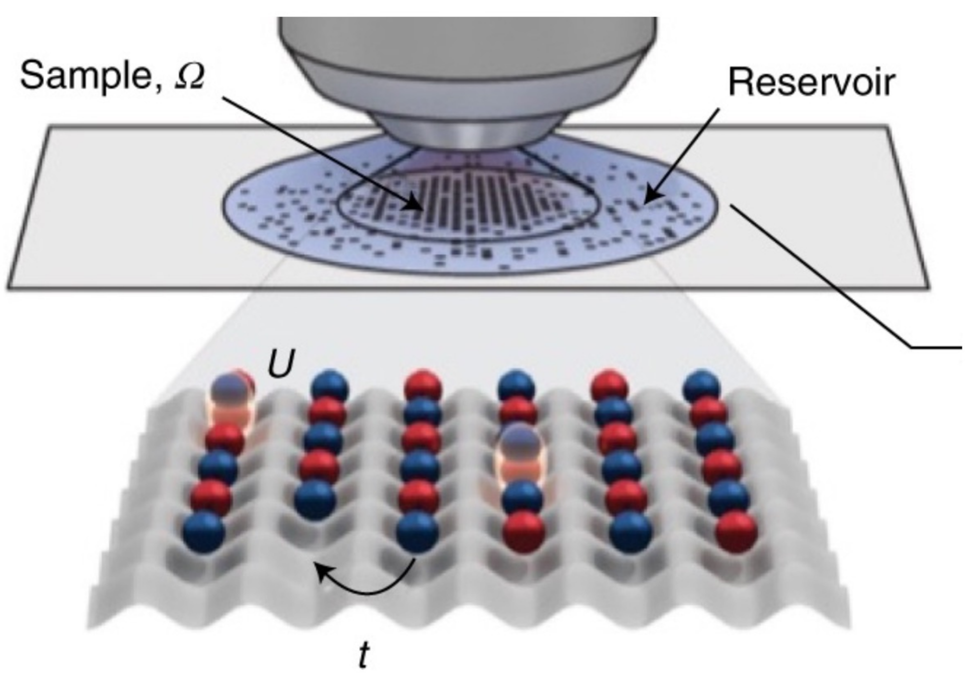
Verify scramblon theory?

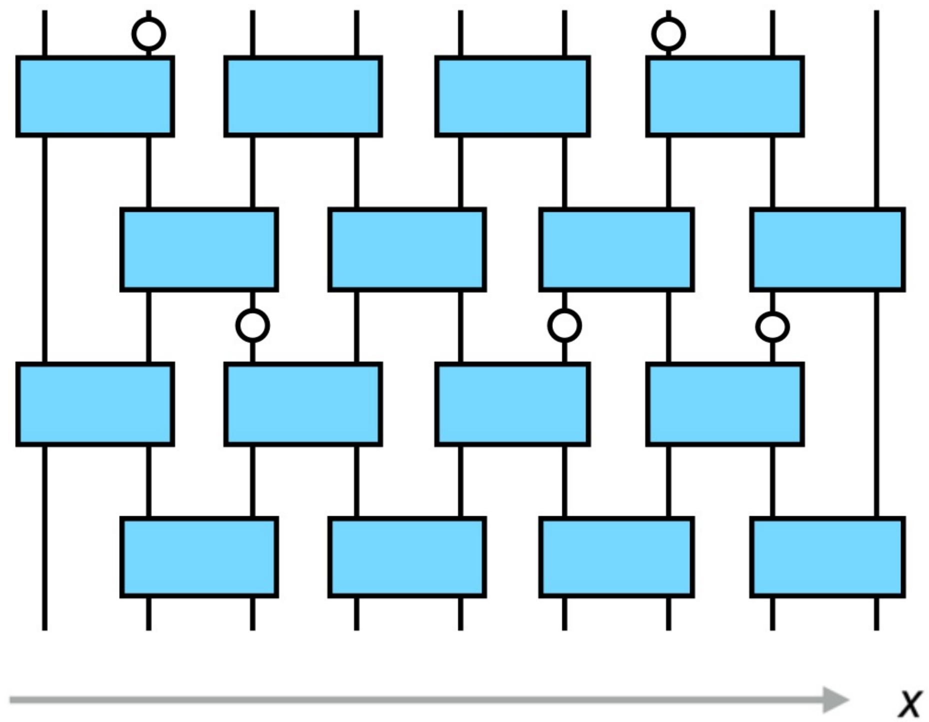
Collaboration with Hui Zhai and Xinhua Peng' s groups, Nature Physics (2024)

From Closed to Open

Why open systems?

- Experimentally, any realistic system is open.
- Theoretically, Novel dynamical phases.





Operator Size in Open Systems

- We define size in open systems by only count system operator

$$\text{size}\left(\sigma_{i_1}^{\alpha_1}\sigma_{i_2}^{\alpha_2}\ldots\sigma_{i_n}^{\alpha_n}O_B\right)=n$$

- An expansion of operator evolution:

$$\sigma_1^x(t)=\sum_n\sum_{\{i_1,\ldots,i_n\}}\sum_{\{\alpha_1,\ldots,\alpha_n\}}C_{i_1\ldots i_n}^{\alpha_1,\ldots,\alpha_n}\sigma_{i_1}^{\alpha_1}\sigma_{i_2}^{\alpha_2}\ldots\sigma_{i_n}^{\alpha_n}O_{B,\{i,\alpha\}}.$$

$$P(n,t)=\sum_{\{i_1,\ldots,i_n\}}\sum_{\{\alpha_1,\ldots,\alpha_n\}}\left|C_{i_1\ldots i_n}^{\alpha_1,\ldots,\alpha_n}\right|^2\langle O_{B,\{i,\alpha\}}^2\rangle.$$

PZ and Z. Yu, PRL 130, 250401 (2023); Also see N. Y. Yao et al, P. Nandy et al ...

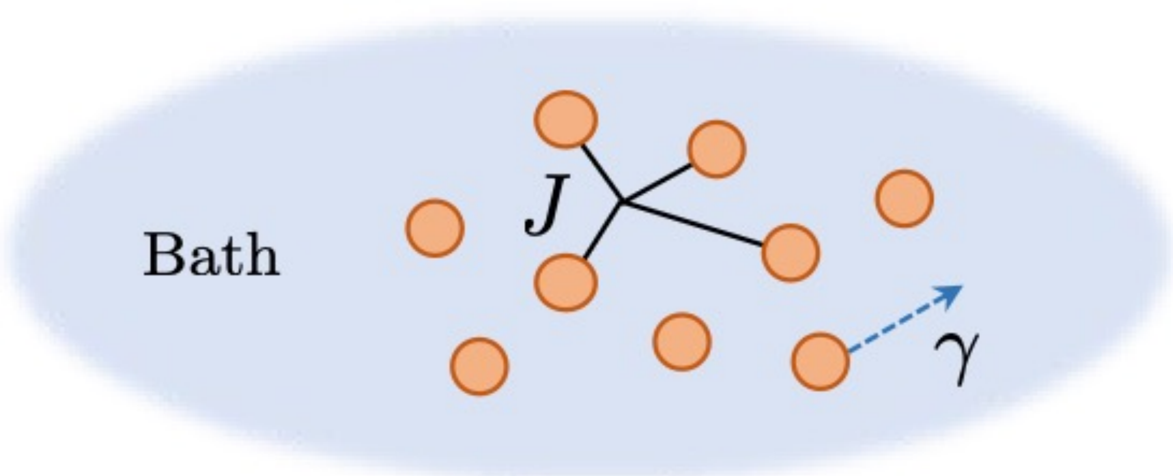
Scrambling Transition

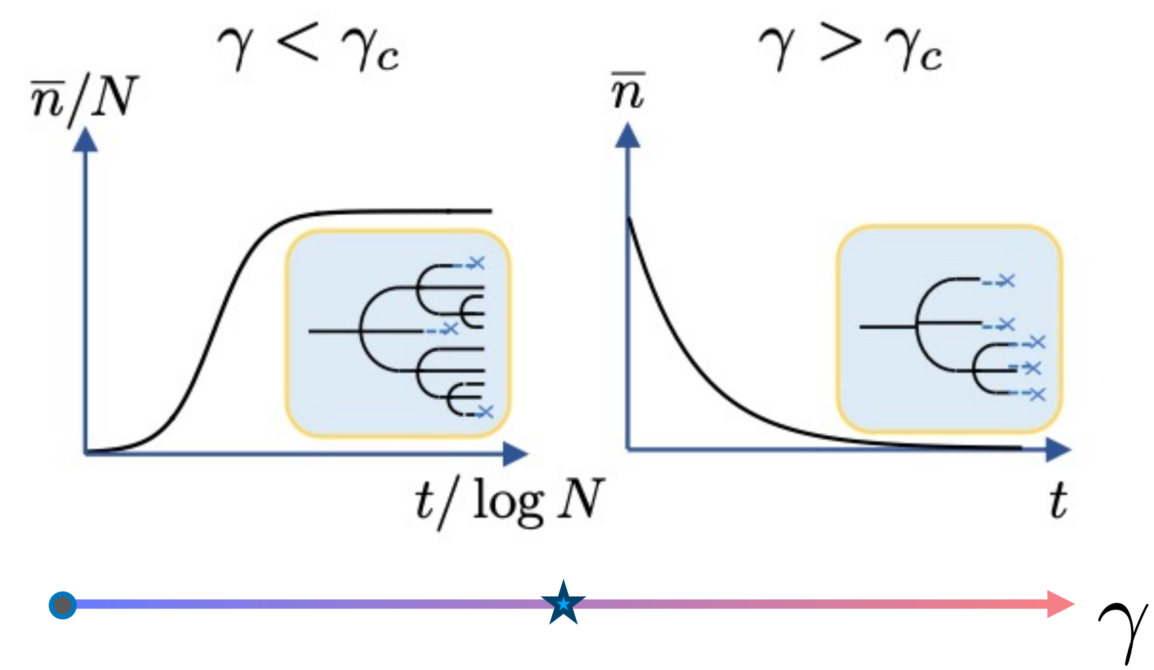
- Competition:

  A. System interaction $i\,[J\sigma_i\sigma_j\,,\sigma_1] \sim \sigma_1\sigma_j$

  B. System-both coupling $\mathrm{SWAP}_{1B}\;\sigma_1\;\mathrm{SWAP}_{1B} \sim \sigma_B$

- There is a scrambling transition





PZ and Z. Yu, PRL 130, 250401 (2023), E. Altman, et. al , PRL 131, 220404 (2023)

SYK Solvable Model

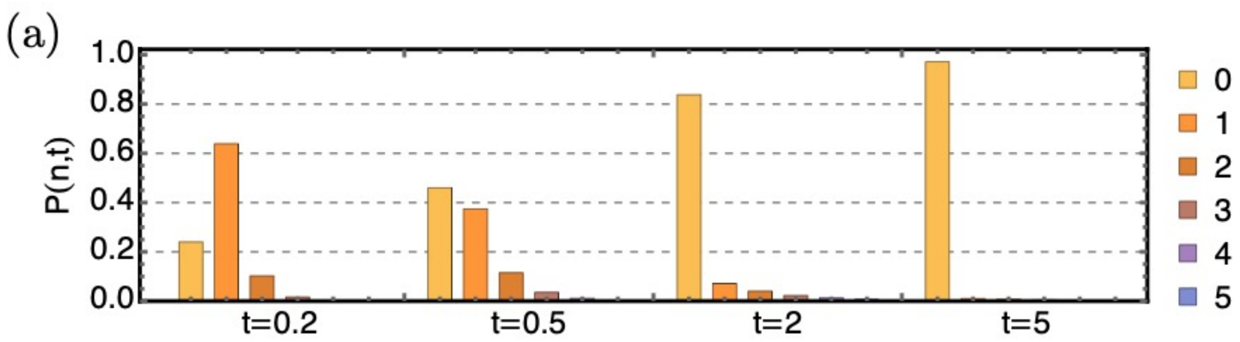
- The transition can be verified in

$$H(t) = \sum_{i_1<\ldots<i_q} i^{\frac{q}{2}} J_{\{i_p\}}(t) \chi_{i_1}\chi_{i_2}\ldots\chi_{i_q} + \sum_{j,a} iV_{ja}(t)\chi_j\psi_a$$

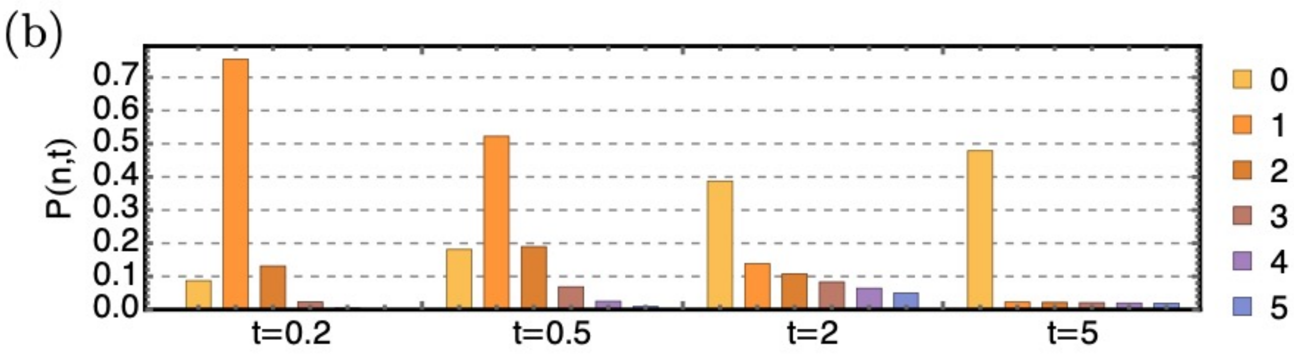
System fermion χ, bath fermion ψ

- Scrambling transition occurs at $V/J = 1$

dissipative



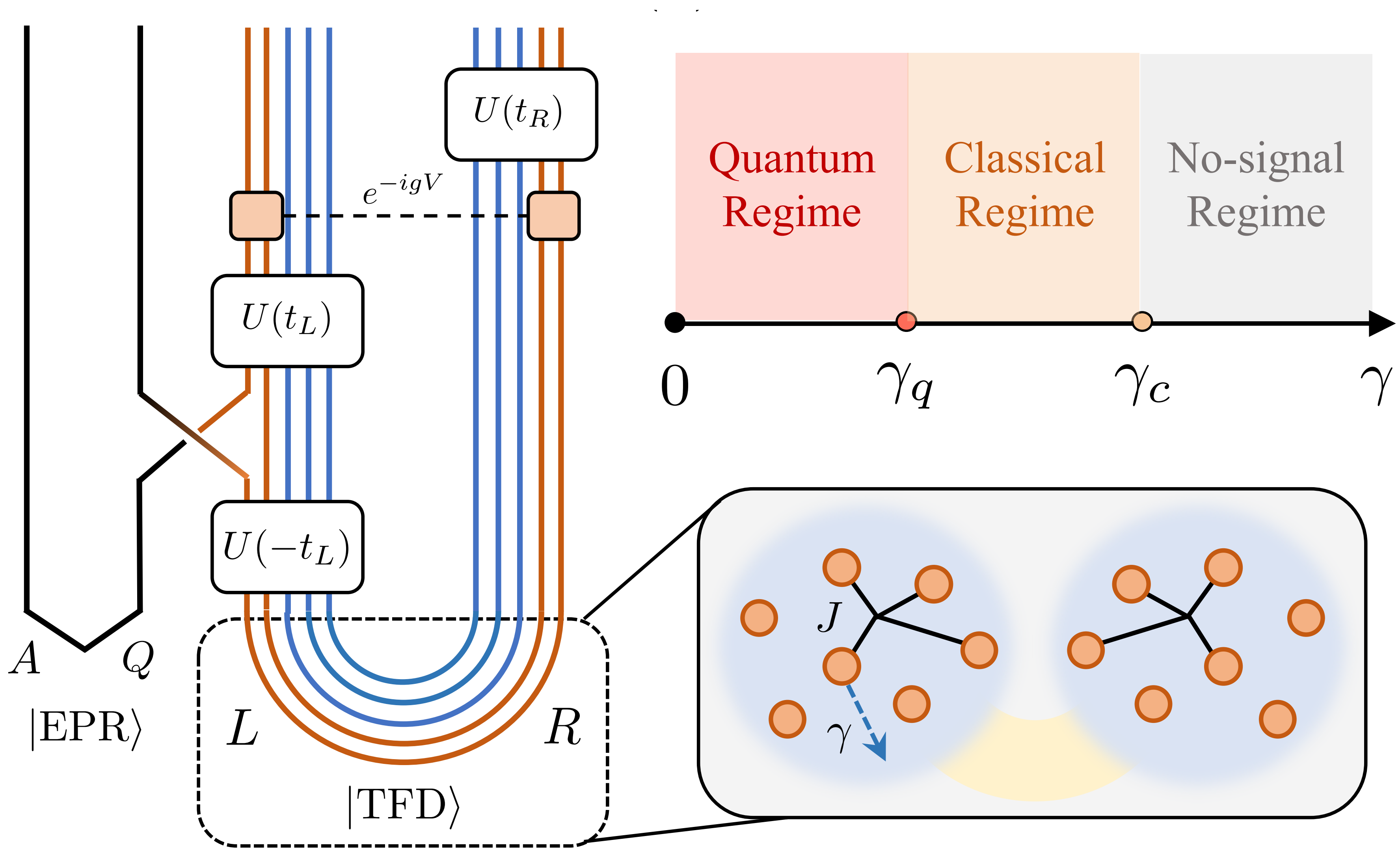
scrambling



PZ and Z. Yu, PRL 130, 250401 (2023).

Scrambling & Teleportation

- The scrambling transition is a teleportation transition

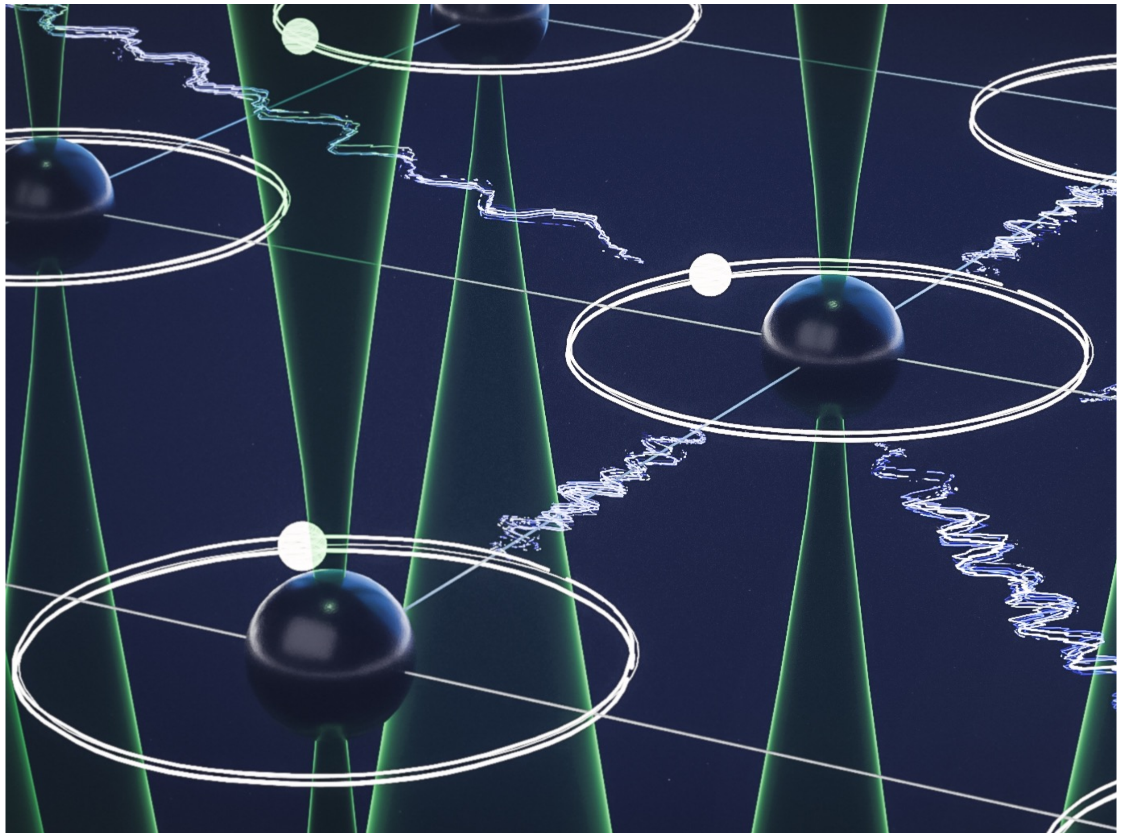


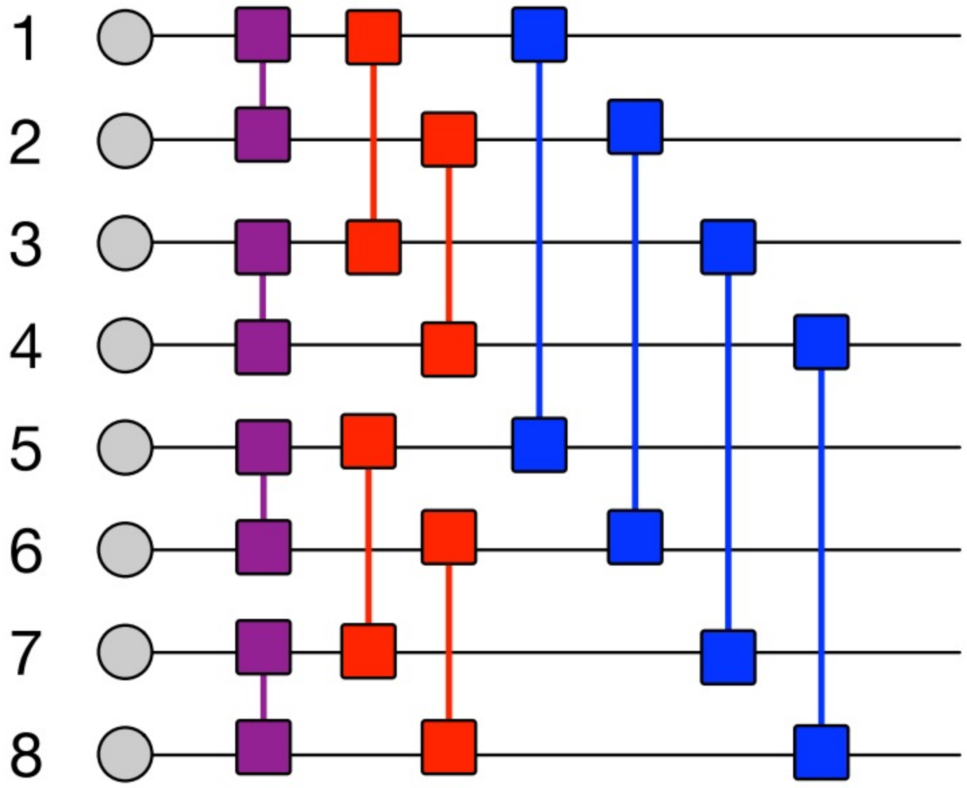
S. Zhou, PZ[†] and Z. Yu[†], arXiv:2406.02277.

Applications in Shadow Tomography

New Quantum Platforms

- Rydberg atom array in optical tweezers.





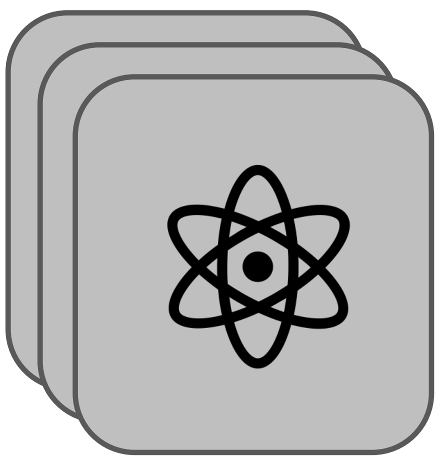
No Hamiltonian, No symmetry, No locality.

Observables → Quantum Information

Quantum State Tomography

- How to storage a generic quantum state?

Naïve answer: store the full density matrix:



$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots \\ \rho_{21} & \rho_{22} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

Pauli basis:

$$\rho = \sum_{P_i \in \{I,X,Y,Z\}} c_{\{P_i\}} P_1 P_2 \ldots P_N / 2^N$$

$$c_{\{P_i\}} = \mathrm{tr}\left(\rho P_1 P_2 \ldots P_N\right)$$

4^N elements!

$N = 384,\ 4^N \approx 10^{115}$

Requires 10^3TB to store!

Classical Shadow Tomography

- Only certain properties are of interests:

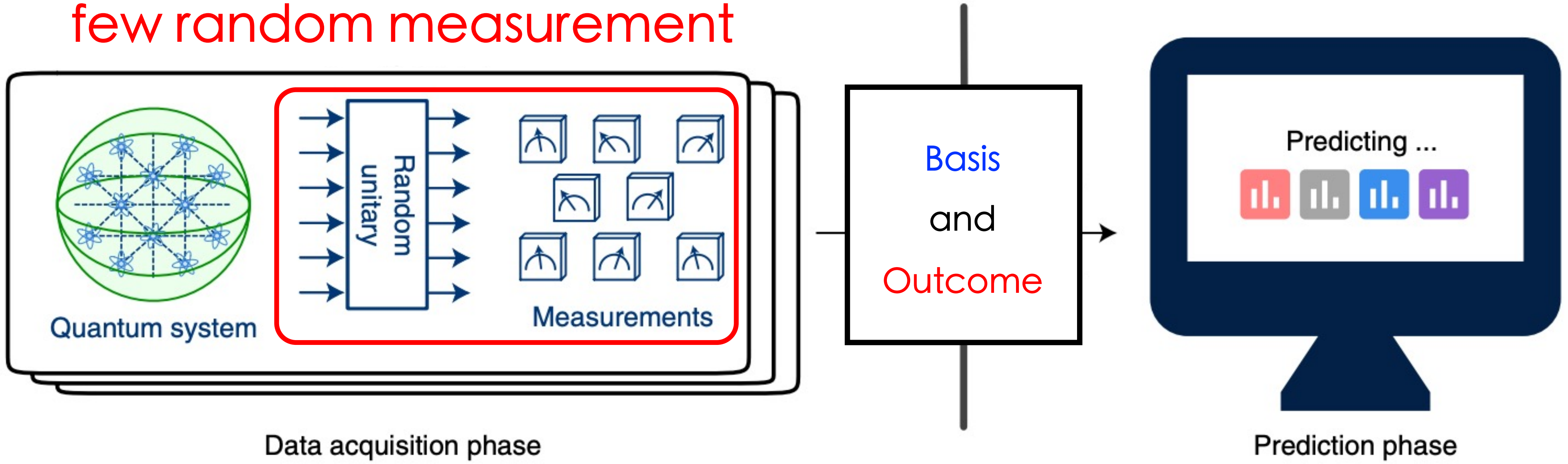
  ✓ Local Observable $\mathrm{tr}(\rho P)$

  ✓ Entropy $\mathrm{tr}(\rho^2)$

  ✓ Fidelity $\langle\psi|\rho|\psi\rangle$

- Shadow Tomography with Randomness

few random measurement

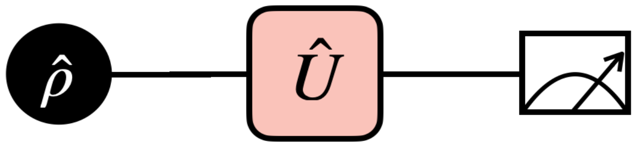


[1]. H.-Y. Huang, R. Kueng, and J. Preskill, Nature Physics volume 16, p1050–1057 (2020).

A Single-qubit Example

$$\rho = I/2 + (c_X X + c_Y Y + c_Z Z)/2$$

- The shadow tomography protocol:

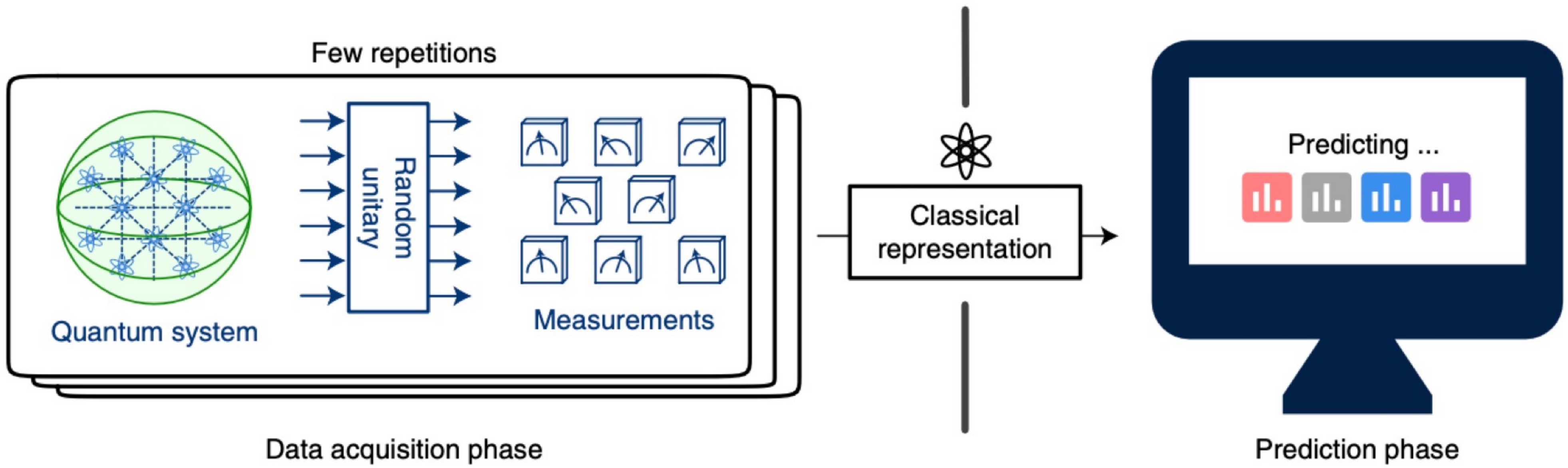


= Randomly measure in x/y/z direction:

Direction	Outcome $\|o\rangle$ (Shadow)	Probability	Partial Average	Full Average
x	$\|+x\rangle$	$(1+c_X)/2$	$(I + c_X X)/2$	$\bar{\sigma} = I/2 + (c_X X + c_Y Y + c_Z Z)/6$
	$\|-x\rangle$	$(1-c_X)/2$		
y	$\|+y\rangle$	$(1+c_Y)/2$	$(I + c_Y Y)/2$	
	$\|-y\rangle$	$(1-c_Y)/2$		
z	$\|+z\rangle$	$(1+c_Z)/2$	$(I + c_Z Z)/2$	
	$\|-z\rangle$	$(1-c_Z)/2$		

Quantum Experiment | Data collection

General Strategy



Random unitary $U \in \mathcal{E}(U)$

$$\rho \to U\rho U^\dagger \to |z^a\rangle\langle z^a|$$

Classical snapshot

$$|o^a\rangle = U^\dagger |z^a\rangle, \quad p^a = \langle o^a|\rho|o^a\rangle$$

Measurement channel

$$\overline{\sigma} = \mathcal{M}[\rho]$$

Rebalance $\rho = \mathcal{M}^{-1}[\overline{\sigma}]$

Make prediction

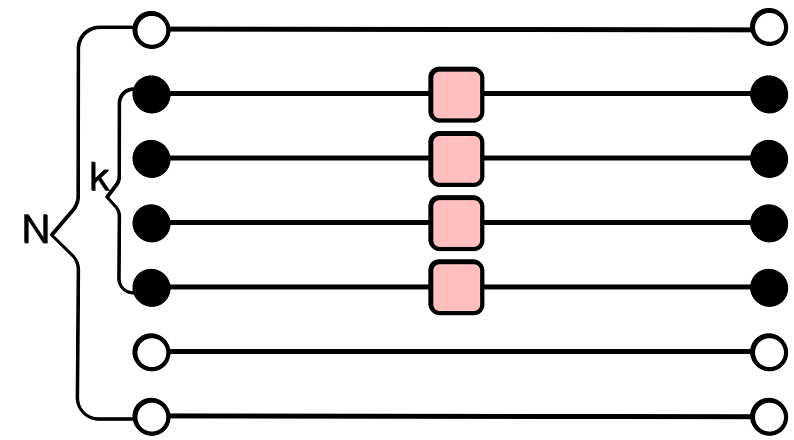
$$\langle P\rangle = \mathrm{tr}\left(P\mathcal{M}^{-1}[\overline{\sigma}]\right)$$

Sample Complexity

- How many samples are needed?

1. Random single-qubit rotations

$$U = u_1 u_2 \ldots u_N$$



Example: Pauli observable $P = X_3 Y_4 Z_5$ (size $k = 3$)

- Each sample measures $P_3 P_4 P_5$ with $P_i \in \{X, Y, Z\}$.

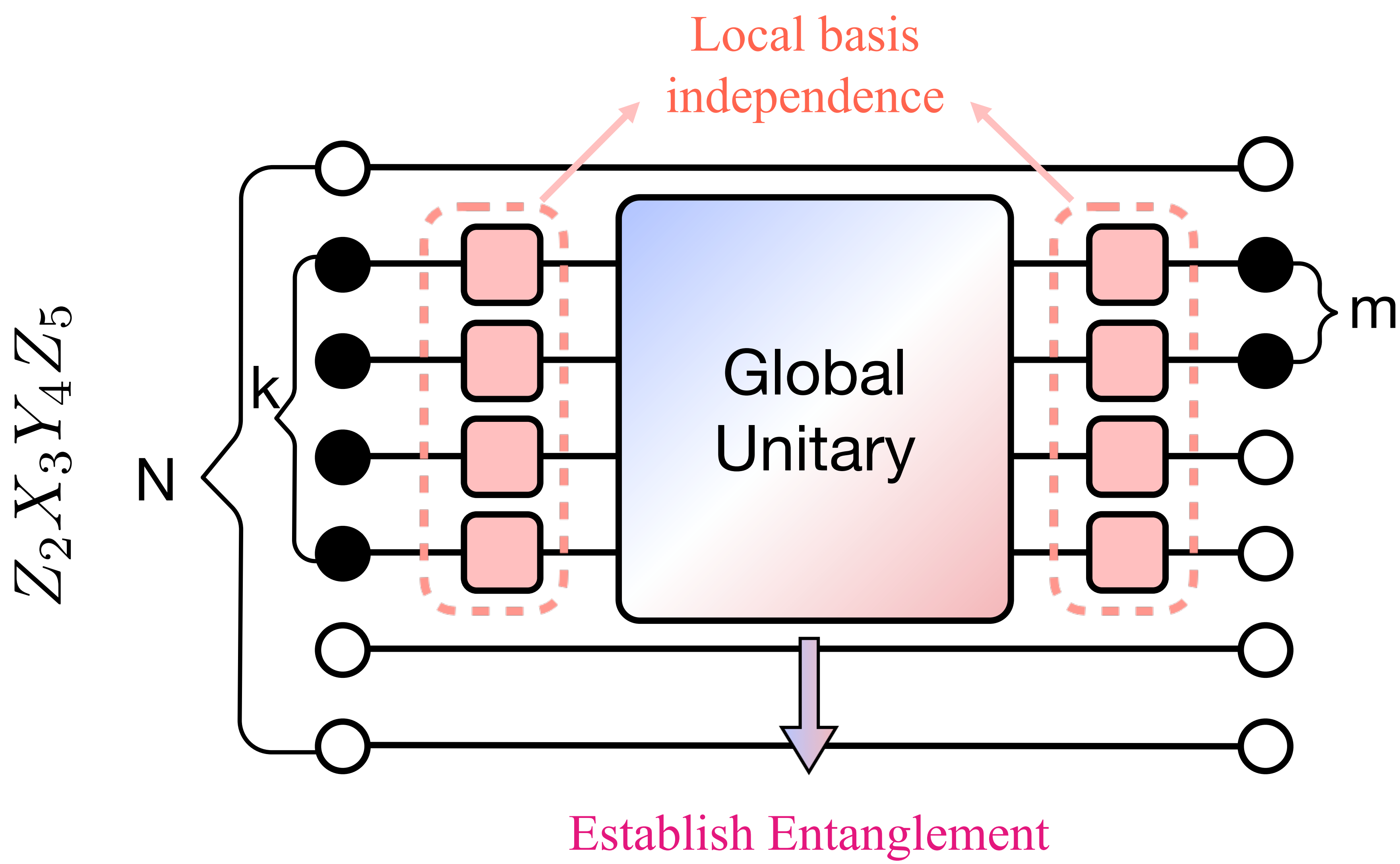
Sample complexity $\sim 3^k$

Independent of N!

Sample Complexity

- How many samples are needed?

  2. Adding Clifford entanglement unitary



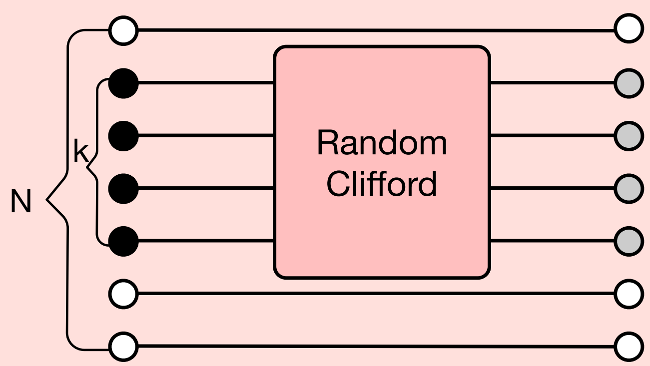
Sample Complexity
3^m

Low complexity

Small operator size

Barrier for Sample Complexity?

Nature Physics 16, p1050 (2020).



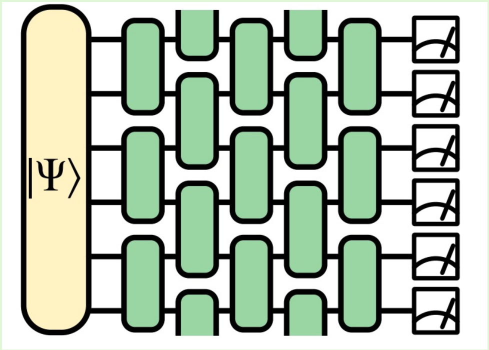
k-qubit Clifford unitary

$\pi(m)$ is binomial,

with $||O||_{sh} = 2^k + 1$

with Prior Knowledge

PRR 5, 023027 (2023); PRL 130, 230403 (2023), and others

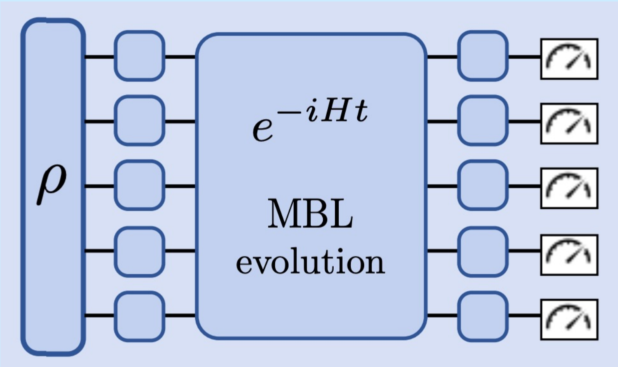


Shallow circuits with circuit depth d$\sim \ln k$

has $||O||_{sh} > 2^k$

no Prior Knowledge

Collaboration with Tian-Gang Zhou, Quantum 8, 1467 (2024)



Hamiltonian dynamics with MBL systems

has $||O||_{sh} \sim 2.25^k$

no Prior Knowledge

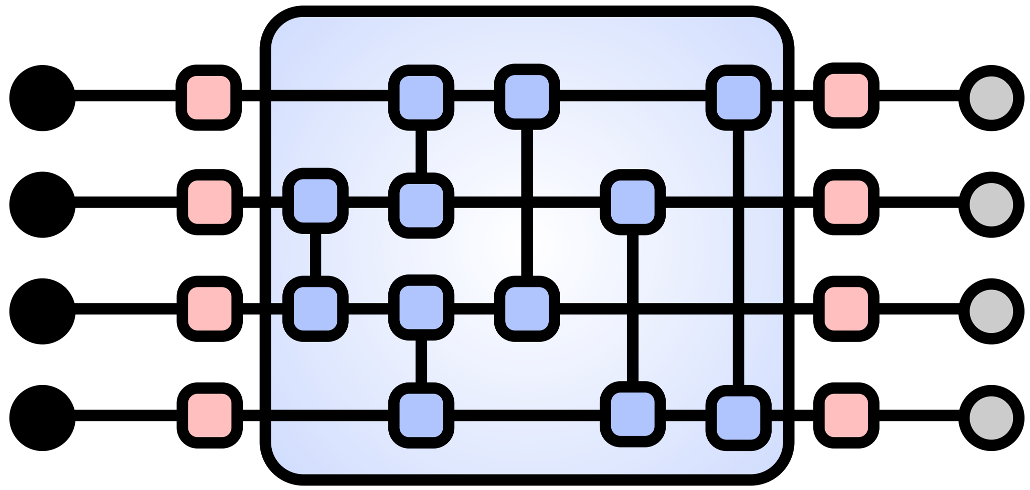
Local thermalization

Easier for simulation

- Can the sample complexity be lower than 2^k ?

Contractive Unitary

- Main result: the target-driven design of a unitary operation that rapidly contract operator size:



$$U_{ct} = \prod_{i<j} U_{ij}^{ZZ},$$

$$U_{ij}^{ZZ} = \exp\left(-\frac{i\pi}{4} Z_i Z_j\right)$$

Best two-qubit gate

Information of the precise location of $\hat{O}$	Contractive Unitary	Random Clifford	Shallow Circuit
Known	1.8^k	2^k	$> 2^k$
Unknown	$k \times 1.8^k$	$k \times 2^k$	

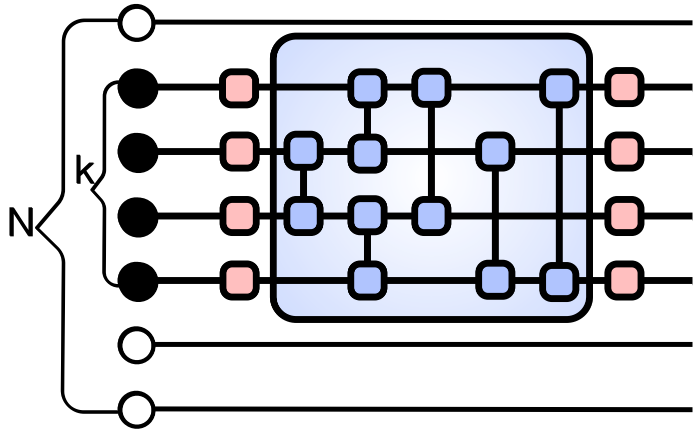
Sliding trick

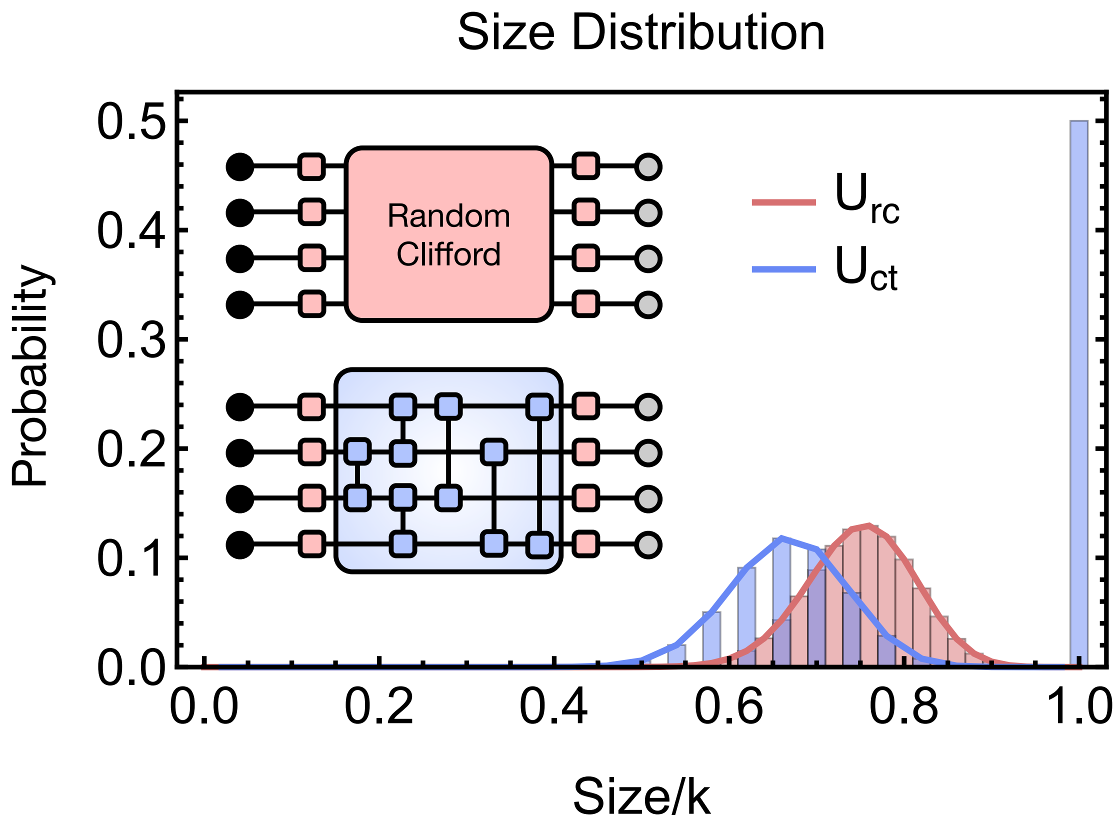
Y. Wu, C. Wang, J. Yao, H. Zhai, Y.-Z. You, and P. Zhang †, 2412.01850.

Contractive Unitary Protocol

- This leads to the size at the output:

$$m = \text{Size}\left(\hat{U}_{\text{ct}}\hat{O}\hat{U}^{\dagger}_{\text{ct}}\right) = \begin{cases} N_{XY} & \text{if} \quad N_{XY} \in \text{odd}, \\ k & \text{if} \quad N_{XY} \in \text{even}. \end{cases}$$





$$w(\hat{O})_{\text{ct}} = \sum_m \frac{\pi_{\text{ct}}(m)}{3^m}$$

$$= \sum_{N_{XY} \in \text{even}} \binom{k}{N_{XY}} \frac{2^{N_{XY}}}{3^k} \frac{1}{3^k} \rightarrow \sim \frac{1}{2}\frac{1}{3^k}$$

$$+ \sum_{N_{XY} \in \text{odd}} \binom{k}{N_{XY}} \frac{2^{N_{XY}}}{3^k} \frac{1}{3^{N_{XY}}} \rightarrow \sim \frac{1}{2}\left(\frac{5}{9}\right)^k$$

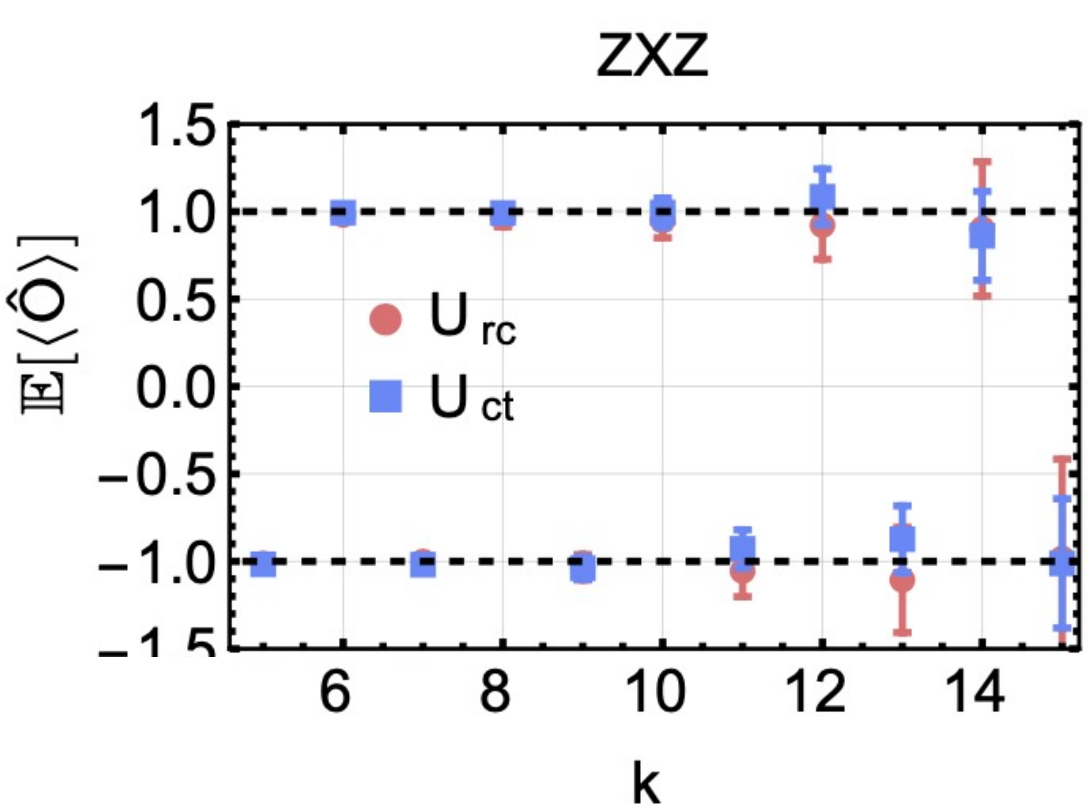
$$\|\hat{O}\|^2_{\text{ct}} \sim 2 \times 1.8^k < \|\hat{O}\|^2_{\text{rc}} = 2^k + 1$$

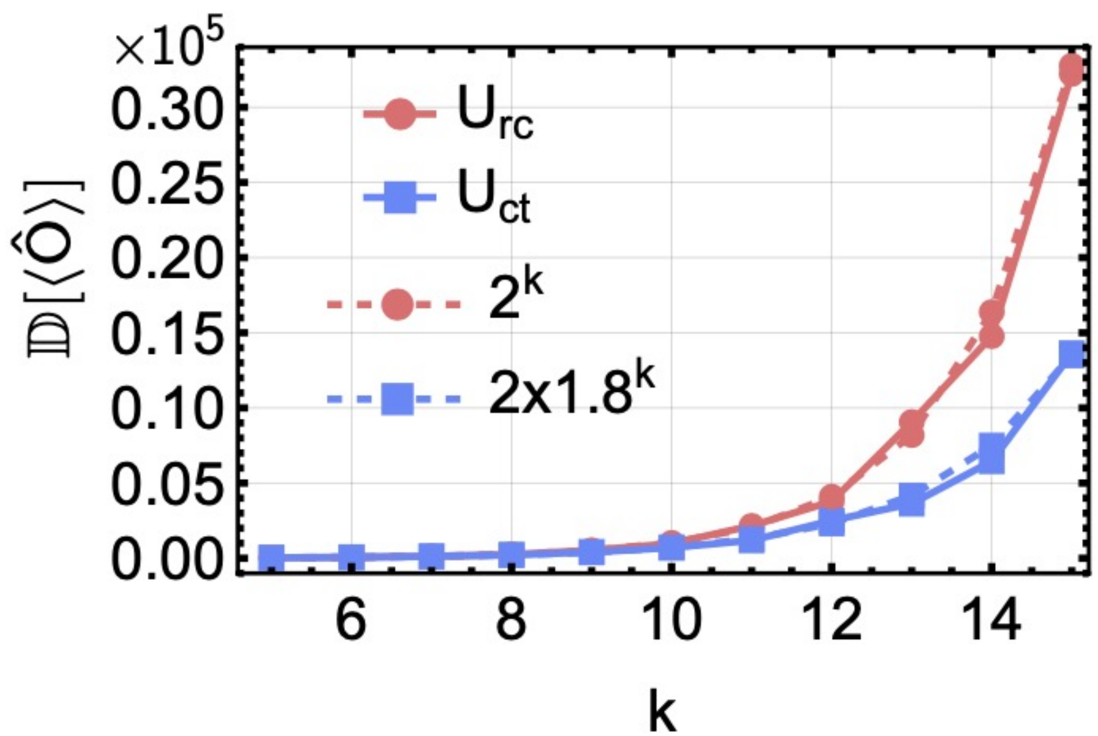
Numerical Demonstration

- 1D cluster states (PBC)

$$S = \{Z_i X_{i+1} Z_{i+2}\}$$

Observable $O = Z_1 Y_2 X_3 \dots X_{k-2} Y_{k-1} Z_k \quad \langle O \rangle = (-1)^k$





Take Home Message

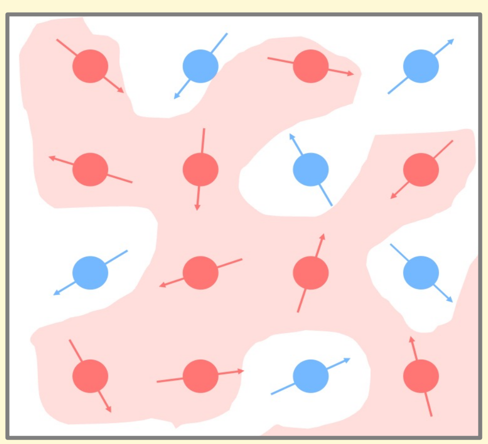
Summary

Information Scrambling

Fundamental Theory

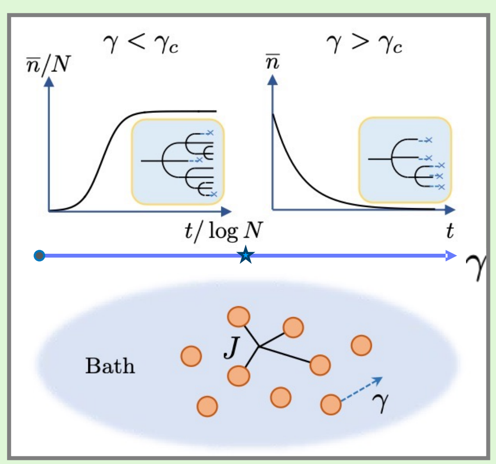
Verification & Application

Scramblon Theory



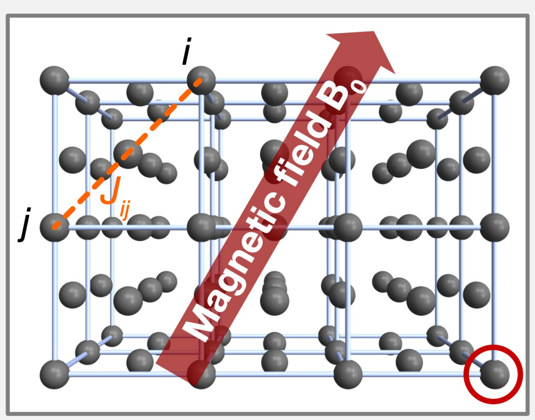
PRL 2024, JHEP 2022

Scrambling Transition



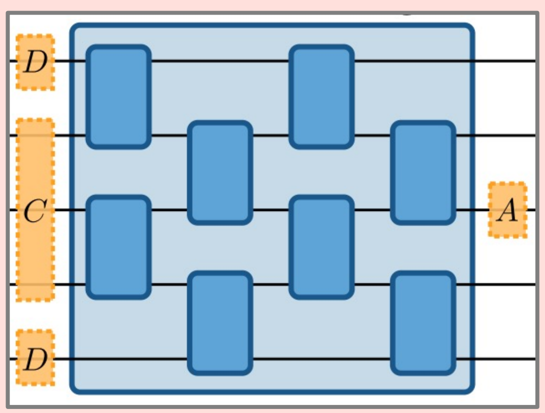
PRL 2023, AAPPS 2024

NMR Dynamics



Nature Physics 2024

Quantum Algorithm

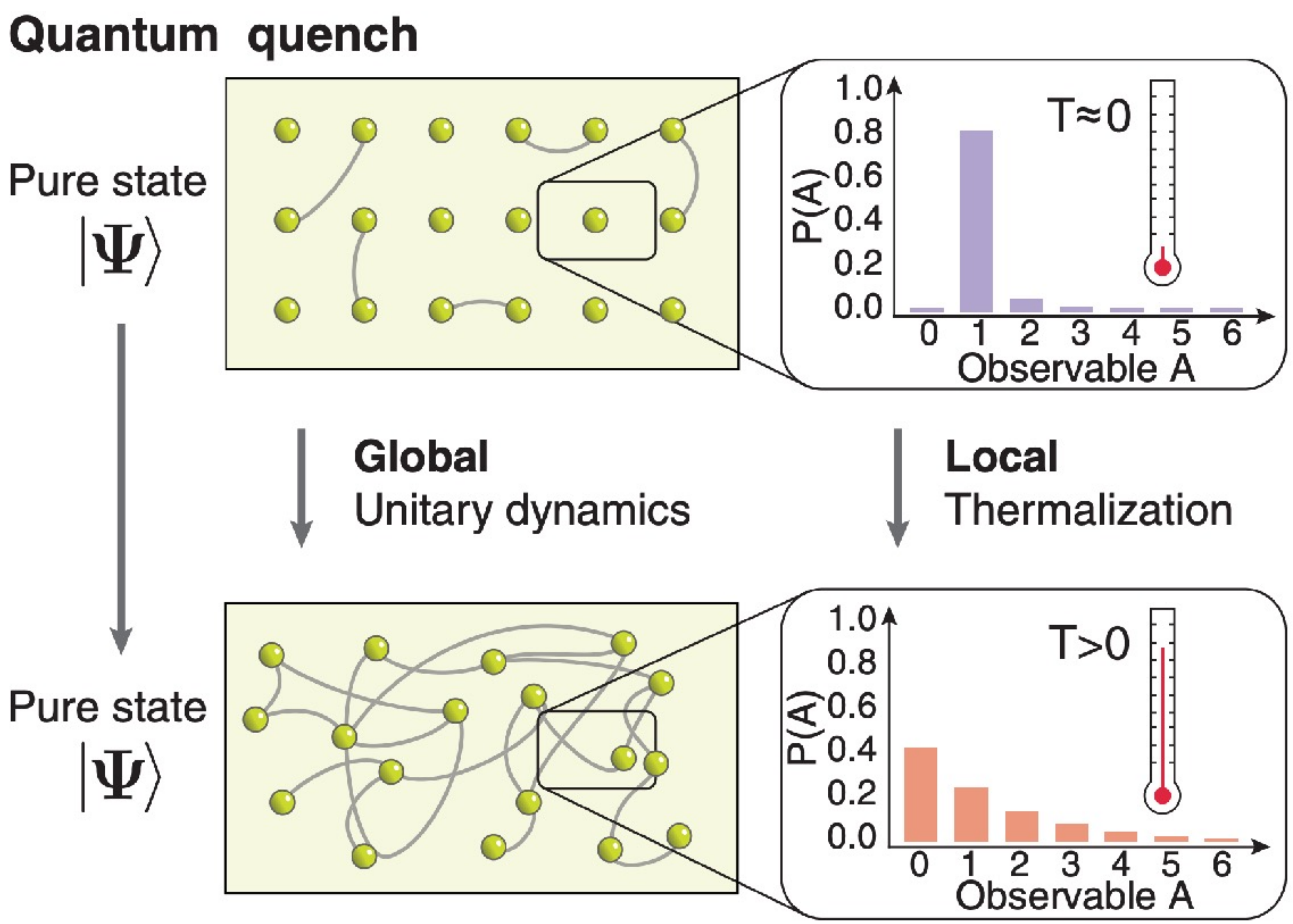


PRL 2024, PRL 2020

Thanks for you attention!

Scrambling & Thermalization

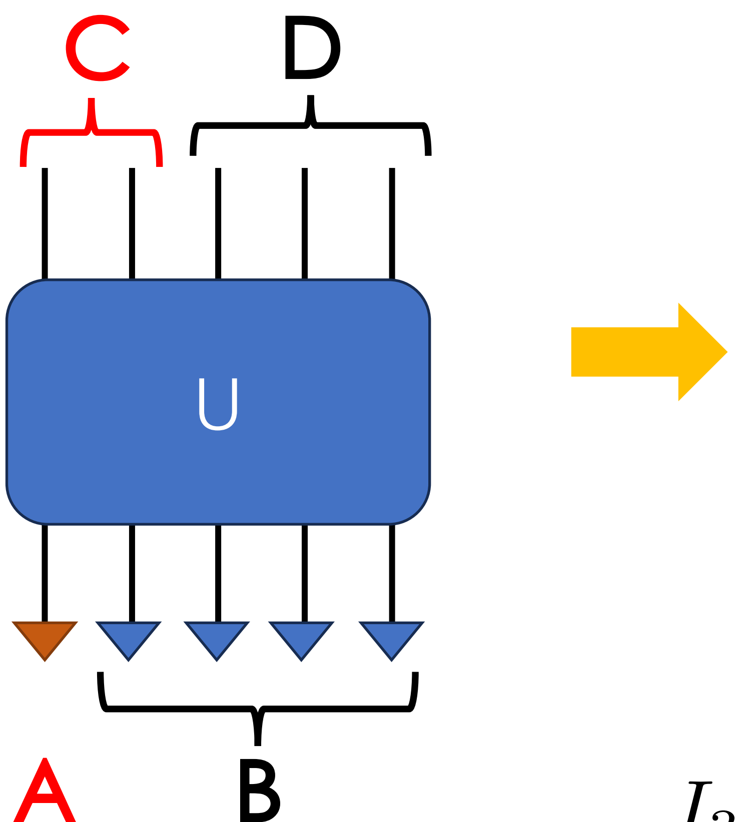
- Closed quantum systems thermalize!



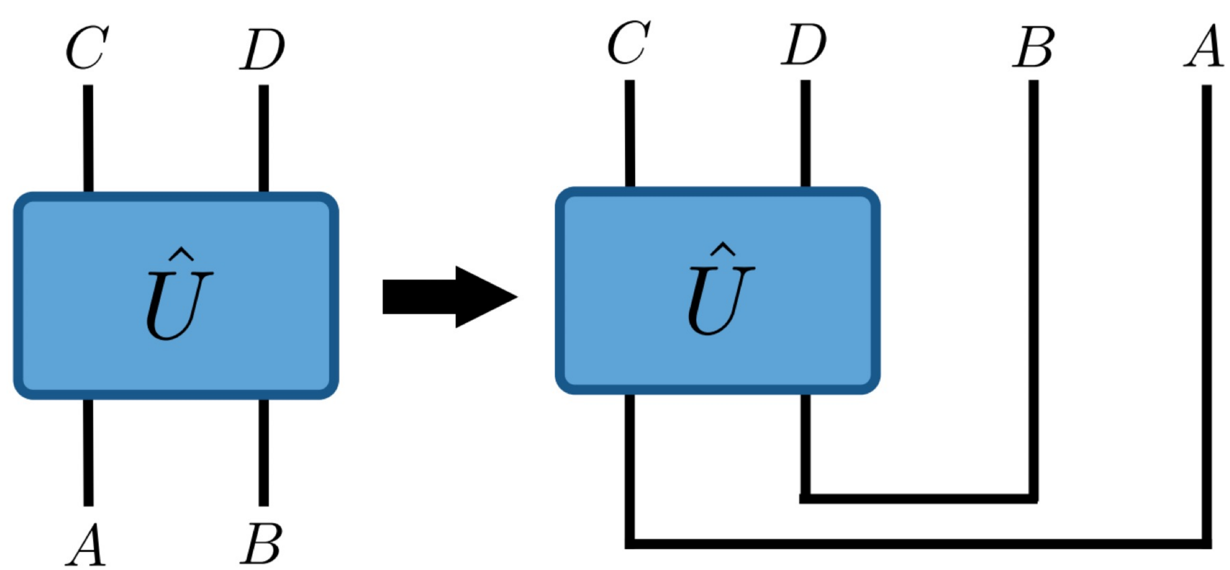
A. M. Kaufman et.al. Science, 353, 6301 (2016).

Scrambling & Entropy

- The scrambling is also described by tripartite information:



Operator-state Mapping



$$I_3(A;C,D) \equiv I(A,C) + I(A,D) - I(A,C \cup D)$$

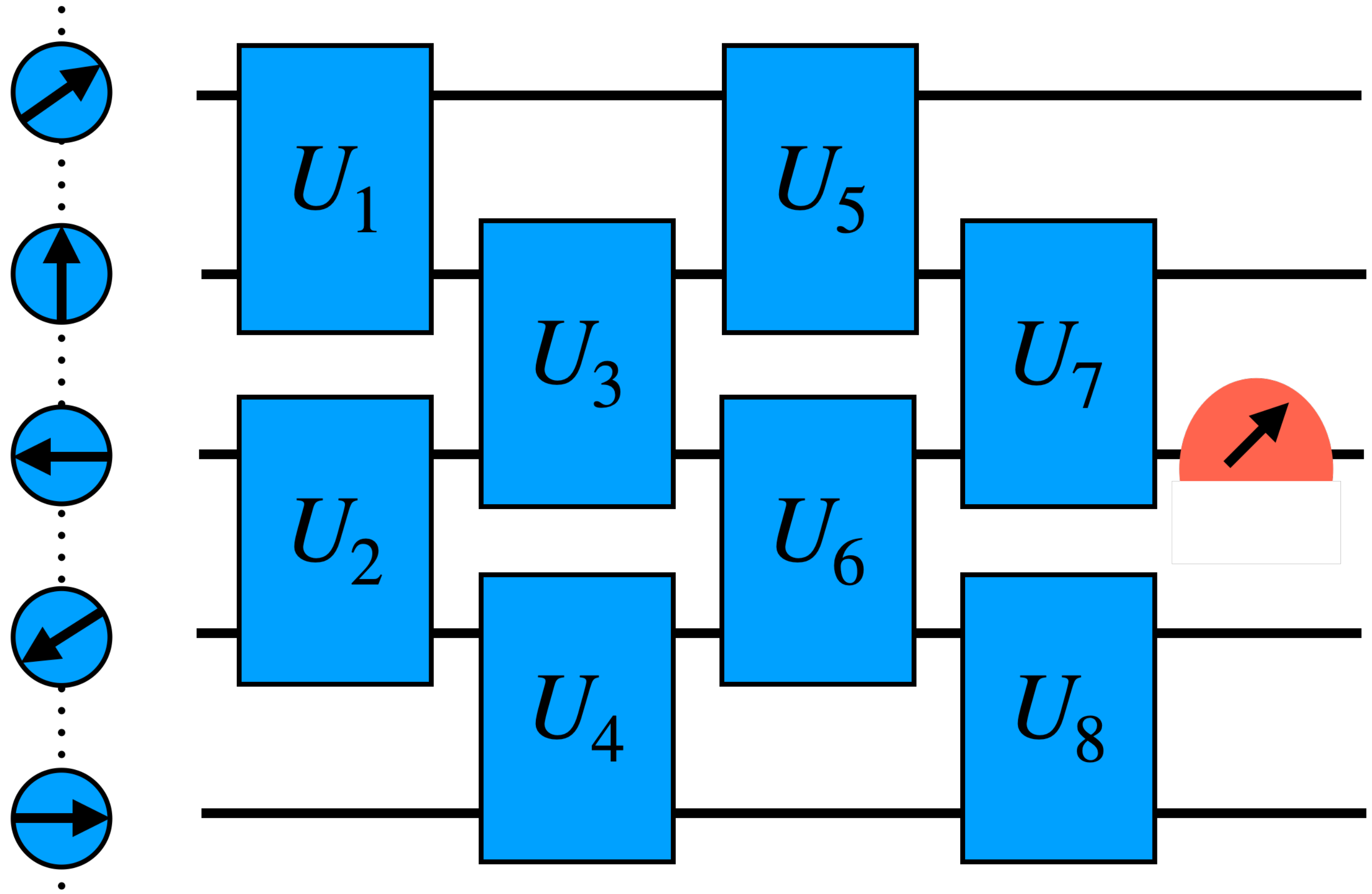
- Haar random unitary $I_3(A;C,D) = -2|A|$

Related to OTOC/Size

P. Hosur, X.-L. Qi, D. A. Roberts, B. Yoshida, JHEP 2016, 4 (2016).

Quantum Neural Network

- QNN is a parametrized quantum circuit:



$$M_z = \sum \langle \sigma_i^z \rangle / N$$

Output is:

$$\langle G | \prod_i U_i^\dagger \sigma_3^x \prod_i U_i | G \rangle$$

$$U_i = U_i(w_i^j)$$

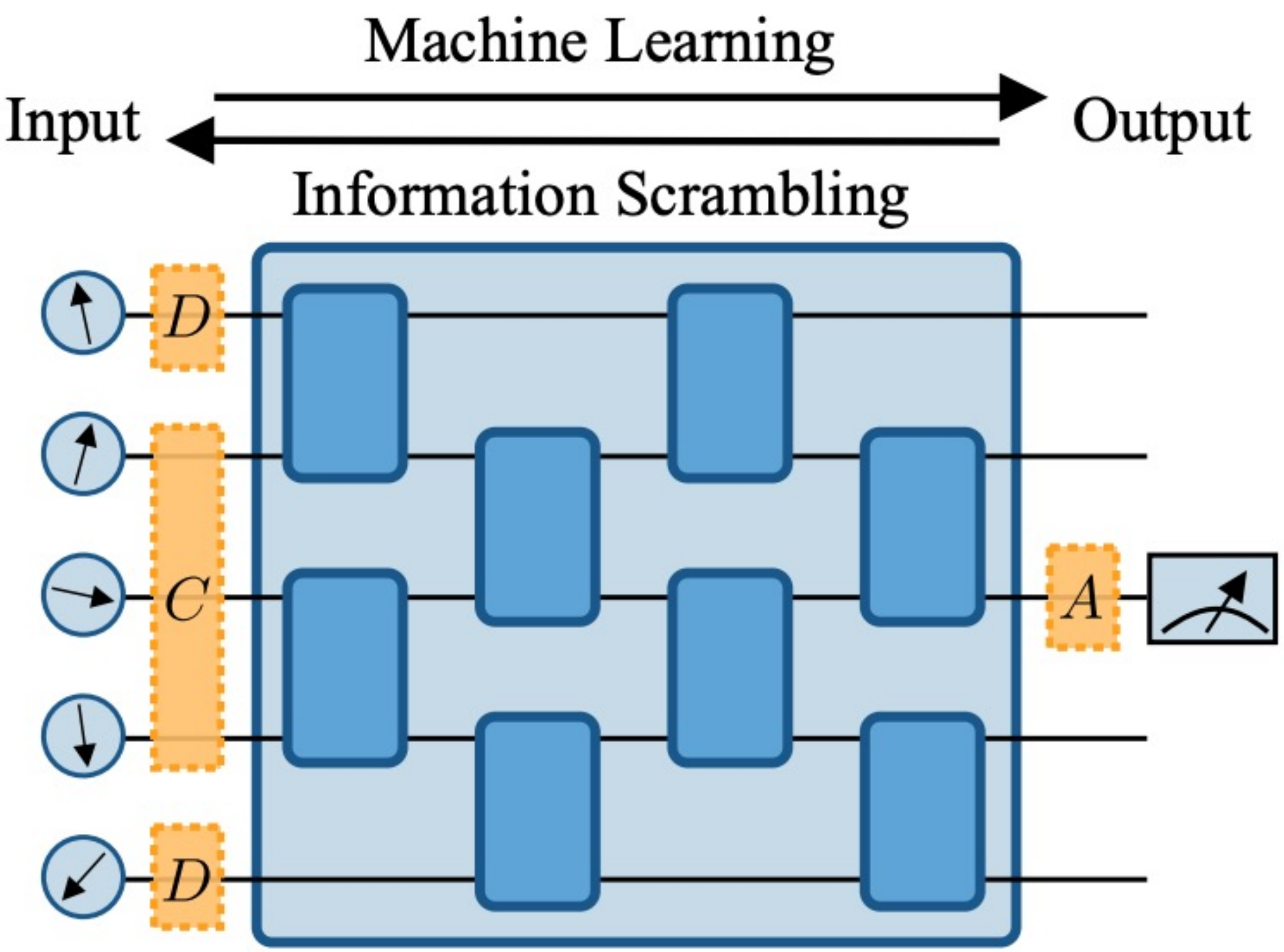
fitting parameters

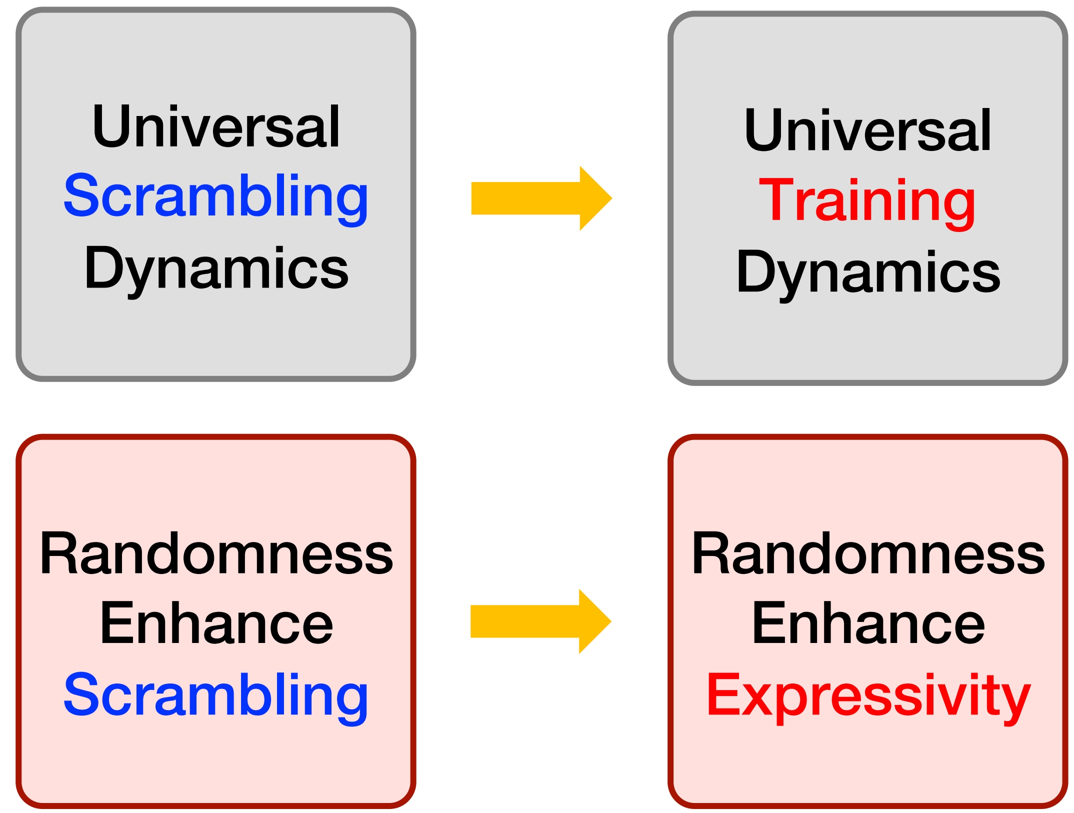
$$L = \sum_\alpha \left| \langle G^\alpha | \prod_i U_i^\dagger \sigma_3^x \prod_i U_i | G^\alpha \rangle - M_z^\alpha \right|$$

Other architectures?

QNN and Information Scrambling

- Learning is the reverse of scrambling:





Y. Wu, J. Yao, PZ†, and X. Li†, PRL 132, 010602 (2024).

H. Shen, PZ, Y.-Z. You, and H. Zhai, PRL 124, 200504 (2020).