More is different: a path from integrability to quantum chaos

Based on 2411.12806

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1/21/2025@YITP

Introduction

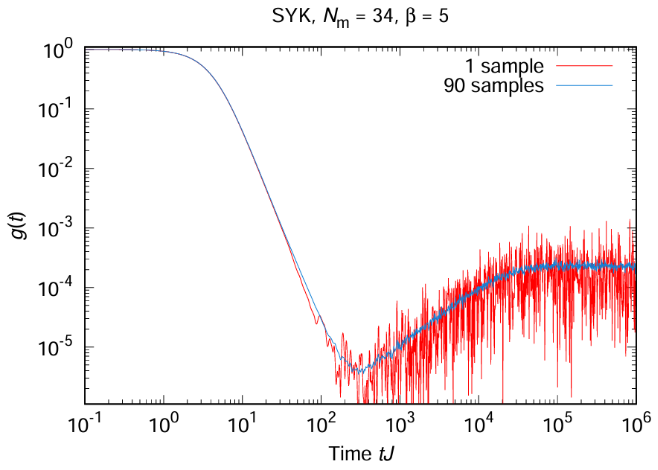
- Quantum chaos is a prevalent phenomenon in non-integrable many-body systems and has been widely studied in recent years.
- It has deep connections to quantum gravity, hydrodynamics, random matrix theory etc.
- However, we do not have a rigorous definition for quantum chaos though we usually characterize it in two different ways.

Introduction

- For a quantum chaotic system with N degrees of freedom, the first mean is to compute out-of-time-ordered correlator $\mathrm{F} = \langle W(t)V(0)W(t)V(0) \rangle$ (OTOC).
- Its typical behavior is $\mathrm{F}/\langle \mathrm{WW} \rangle \langle \mathrm{VV} \rangle = 1 - f(t)/N$, where f(t) is a growing function, which could be polynomial or exponential $\mathrm{e}^{\lambda \mathrm{t}}$.
- For the latter, we call it fast-scrambling, in which the d.o.f. are mixed up exponentially fast until the time scale $O(logN)$.
- λ is quantum Lyapunov exponent analogous to the classical counterpart.
- It has a chaos bound $\lambda \leq 2\pi/\beta$. [Maldacena-Stanford-Shenker 15]

Introduction

- The other probe is spectral form factor (SFF) $\mathbb{E}\left[Tre^{-(\beta+it)H}Tre^{-(\beta-it)H}\right]$ for an ensemble of models.
- SFF has three stages:
  1, exponential decay
  2, ramp
  3, plateau
- The last two are universally described by random matrices at time scales $O(e^{aN})$.
- In particular, the ramp is due to spectral correlation.



[Cotler et.al 16]

Introduction

- Due to the separate time scales, these two probes are dominated by different physics.
- In holographic models where the chaos bound is saturated $\lambda = 2\pi/\beta$, the exponential growth of OTOC is related to the boost symmetry near a black hole horizon with temperature $1/\beta$ in the dual spacetime.
- On the other hand, the ramp of SFF is related to non-perturbative effect of including Euclidean wormhole contributions. [Saad-Shenker-Stanford 19]

Introduction

- A prototype of model that exhibits both features is the SYK model.
- This quantum mechanic model consists of N Majorana fermions and all-to-all p-local random couplings.
- In low temperatures, it has a conformal limit, around which the dynamics is dominated by a Schwarzian derivative.
- Since Schwarzian derivative is the boundary graviton description of the AdS_2 JT gravity, SYK model is proposed as a 0+1 dimensional holographic model.

[Sachdev-Ye 93'; Kitaev 15']
[Maldacena-Stanford 16']
[Maldacena-Stanford-Yang 16']

Introduction

- The IR spectrum of SYK model scales as $\rho(\mathrm{E}) \sim \mathrm{e}^{S_0} \sinh C\sqrt{E}$.
- It has two features:
  1, a square root $\sqrt{E}$ edge bounded from below, which is also a feature of random matrices.
  2, exponential growth $\mathrm{e}^{C\sqrt{\mathrm{E}}}$ for large E, which is consistent with black hole state counting.
- Both features are closely related to the Schwarzian derivative dynamics.

A question

- Given the close relation between quantum chaos and quantum gravity, we do not have a complete understanding **why holography emerges**.
- To answer this question probably requires a thorough understanding of quantum gravity.
- Nevertheless, in this talk, I will introduce a small step towards this direction by explaining how quantum chaos could merge from integrability (non-chaotic for sure) in a constructive way.

Take-home message

- We start with an integrable model, the commuting SYK model, and use it as a building block to assemble multiple such models.
- While the commuting SYK model is non-chaotic, as we include more and more copies of such models, the **integrability is explicitly broken** and quantum chaotic features emerge.
- There is a **critical temperature** T_c, above which the model is chaotic.
- As we have more copies of the integrable building blocks, T_c **decreases monotonically**, which expands the chaotic regime: “More Is Different”.

Warm-up: commuting SYK

- Consider 2N Majorana fermions ψ_{i}. They obey $\{\psi_{\mathrm{i}}, \psi_{\mathrm{j}}\} = 2\delta_{\mathrm{ij}}$ and form a 2^N dimensional Hilbert space.
- Let us construct $\mathcal{X}_j = i\psi_{2j-1}\psi_{2j}$. Since each $\mathcal{X}_j$ contains two different Majorana fermions. Different $\mathcal{X}_j$ commutes with each other. They can be understood as a Pauli Z matrix at site j on a classical spin chain.

[PG 23']

- Consider a random Hamiltonian constructed by $\mathcal{X}_j$

$$H = \sum_{i_1 \cdots i_p} J_{i_1,\cdots,i_p} \mathcal{X}_{i_1} \cdots \mathcal{X}_{i_p}$$

where each coupling constant is an independent Gaussian random variable.

Warm-up: commuting SYK

- All terms commute and H can be simultaneously diagonalized term by term. It is an integrable model.
- This commuting SYK model is indeed equivalent to the p-local version of the SK model, which is the well-known spin glass model that has nontrivial replica symmetry breaking phase at low temperatures.
- Above the spin-glass critical temperature, we can study the quantum chaotic dynamics of this model. It turns out that OTOC grows quadratically, which is consistent with the fact that this model is integrable.
- This is the **building block** for our later construction.

The d-commuting SYK

- For the 2N Majorana fermions ψ_i. Consider a new Hamiltonian

$$\tilde{H} = \frac{1}{\sqrt{d}} \sum_{a=1}^{d} \tilde{H}_a, \quad \tilde{H}_a = \sum_{i_1 < \cdots < i_p} J^a_{i_1 \cdots i_p} \mathcal{X}^a_{i_1} \cdots \mathcal{X}^a_{i_p}$$

where $\mathcal{X}_j^a$ are bilinear in Majorana fermions

$$\mathcal{X}^a_j \equiv i\psi_{2j-1}\psi_{[(2j-4+2a) \mod (2N)]+2}, \quad j = 1, \cdots, N$$

$$\begin{cases} \mathcal{X}_j^1 = i\psi_{2j-1}\psi_{2j} \\ \mathcal{X}_j^2 = i\psi_{2j-1}\psi_{2j+2} \\ \mathcal{X}_j^3 = i\psi_{2j-1}\psi_{2j+4} \\ \cdots\cdots \end{cases}$$

- Each a is a commuting SYK model because each term commutes with each other
- The full Hamiltonian is the sum over **d different commuting SYK models**.

The d-commuting SYK

- The couplings $J^a_{i_1\cdots i_n}$ are all Gaussian random variables

$$\mathbb{E}[J^a_{i_1\cdots i_p}] = 0, \quad \mathbb{E}[(J^a_{i_1\cdots i_p})^2] = \sigma^2 \equiv J^2 / \binom{N}{p}$$

- For d=1, this model is the integrable commuting SYK model.
- For d>1, $\mathcal{X}^a_j$ with **different a** may not commute with each other when they share one common Majorana fermion.
- This explicitly breaks the integrability.

The d-commuting SYK

- An analogous understanding is that we can view each $\mathcal{X}_j^a$ as a Pauli matrix on N sites but written in different basis labeled by a.
- For d=1, it is the well-known spin-glass model, the SK model, in which the Hamiltonian is just p-local σ_j^z random couplings.
- For d>1, this model is analogous to the quantum spin-glass model with p-local σ_j^z random couplings, plus σ_j^x random couplings, and σ_j^y random couplings etc.

[Erdos-Schroder 14']

The double-scaled limit

- Due to the non-integrability, this model is hard to solve explicitly. However, there is a **double-scaled limit**, in which the formalism can be largely simplified.
- We will consider large N and p but keeping $p \sim O(\sqrt{N})$, namely

$$N \to \infty, \qquad \lambda \equiv 4p^2/N \quad fixed$$

- Under this limit, this model can be formulated as the d-color chord diagram algebra. This is a generalization of the chord diagram algebra of the double-scaled SYK model.

d-color chord diagrams

- Let us consider the partition function expanded in Taylor series

$$Z(\beta) = \mathbb{E}[\mathrm{tr} e^{-\beta \tilde{H}}] = \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} M_n \qquad M_n \equiv \frac{1}{d^{n/2}} \sum_{a_i} \mathbb{E}[\mathrm{tr} \tilde{H}_{a_1} \cdots \tilde{H}_{a_n}]$$

- For n-moment M_n, we expand out $(I = \{i_1, \cdots, i_p\}, X_I^a = \chi_{i_1}^a \cdots \chi_{i_p}^a)$

$$\mathbb{E}[\mathrm{tr} \tilde{H}_{a_1} \cdots \tilde{H}_{a_n}] = \sum_{I_i} \mathbb{E}[J_{I_1}^{a_1} \cdots J_{I_n}^{a_n}] \mathrm{tr} X_{I_1}^{a_1} \cdots X_{I_n}^{a_n}$$

- Since J_I^a are Gaussian random variables, the expectation value is just the sum over all Wick contractions, which gives **pairwise identical $\{I, a\}$ indices** for X_I^a's in the trace.

- $M_n = 0$ for odd n; M_n with even n becomes a counting problem of chord diagrams with n/2 chords.

d-color chord diagrams

- Examples: $M_2 = \frac{\sigma^2}{d} \sum_{a,I} \mathrm{tr} X_I^a X_I^a = 2^N$ $\quad (X_I^a X_I^a = \mathbb{I})$

$$M_4 = \frac{1}{d^2} \sum_{a,b} (\mathbb{E}[\mathrm{tr}\tilde{H}_a \tilde{H}_a \tilde{H}_b \tilde{H}_b] + \mathbb{E}[\mathrm{tr}\tilde{H}_a \tilde{H}_b \tilde{H}_b \tilde{H}_a] + \mathbb{E}[\mathrm{tr}\tilde{H}_a \tilde{H}_b \tilde{H}_a \tilde{H}_b])$$

$$= 2 \cdot 2^N + \frac{\sigma^4}{d^2} \sum_{a,b,I,I'} \mathrm{tr} X_I^a X_{I'}^b X_I^a X_{I'}^b$$

- To evaluate the trace of four X's, we need to swap their orders using

$$X_I^a X_{I'}^b = (-)^{|I \cap I'|} X_{I'}^b X_I^a, \quad (a \neq b)$$

where $|I \cap I'|$ means the # of overlapping Majorana fermions in two X's.

- The exact counting of this factor is hard but is largely simplified in the double-scaled limit.

d-color chord diagrams

- Since $p \sim O(\sqrt{N}) \ll N$, the typical overlap is **sparse**. Given a X_I^a with **s** spots overlap, there are $\binom{2p}{s}$ choices for these spots, and $\binom{2N-2p}{p-s}$ ways for another $X_{I'}^b$ to overlap these spots.
- Therefore, the probability of s spots overlapping between two typical X_I^a's is

$$P(s) = \binom{2p}{s}\binom{N-2p}{p-s} / \binom{N}{p} \to \frac{(\lambda/2)^s}{s!} e^{-\lambda/2}$$

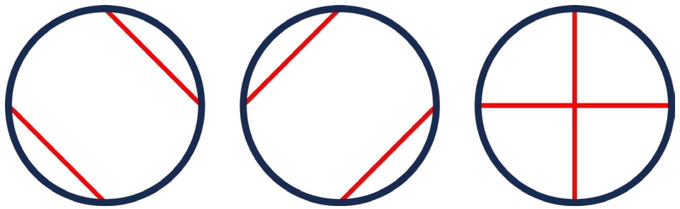
- The swap of two typical X_I^a's (with different a) gives $\sum_I (-)^s P(s) = e^{-\lambda}$

d-color chord diagrams

- In the M_4 example, we have

$$\frac{\sigma^4}{d^2}\sum_{a,b,I,I'}\mathrm{tr}X_I^aX_{I'}^bX_I^aX_{I'}^b \longrightarrow \frac{\sigma^4(d^2-d)2^N}{d^2}\sum_{I_a,I_b}(-)^sP(s)=(1-1/d)2^Ne^{-\lambda}$$

- Pictorially, we write M_4 as the sum over three chord diagrams



- Each chord represents two identical $\widetilde{\mathrm{H}}_{\mathrm{a}}$. We can color each chord by d colors. Different color chords with crossing gives a factor $\mathbf{q}=e^{-\lambda}$, non-crossing gives 1. M_n is the sum over all possible d-color chord diagrams.

Recap: Double-scaled SYK

- The SYK model has a p-local random Hamiltonian directly constructed by Majorana fermions:

$$H = \sum_{i_1 \cdots i_p} J_{i_1,\cdots,i_p} \psi_{i_1} \cdots \psi_{i_p} = \sum_I J_I \Psi_I$$

- In the double scaling limit, to compute n-th moment of H, $\mathrm{M_n} = \mathbb{E}[Tr(H^n)]$, we need to sum over all chord diagrams with n/2 chords, where each chord is a Wick contraction of Ψ_I.
- Now the rule becomes that each crossing of the chord equals a weight $\mathbf{q} = \boldsymbol{e^{-\lambda}}$ with $\lambda = 2p^2/N$.

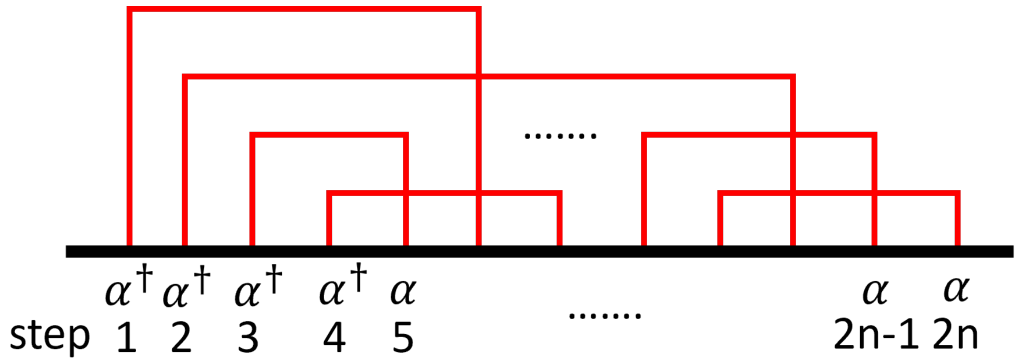
[Berkooz, Jia, Narayan, Simon, Lin, Stanfrod, Garcia-Garcia, Verbaarschot, Narovlansky, Isachenkov, Torrents......]

Recap: Double-scaled SYK

- Double-scaled SYK is the d=1 version with **q factor for all crossings**.
- It is equivalent to the q-deformed harmonic oscillators

$$[\alpha, \alpha^\dagger]_q \equiv \alpha\alpha^\dagger - q\alpha^\dagger\alpha = 1$$

where $\alpha^\dagger$ and α creates/annihilates a chord for a chord diagram



Recap: Double-scaled SYK

- This algebra can be exactly solved by q-hermite polynomials.
- The Hamiltonian is equivalent to $h = \alpha + \alpha^\dagger$, which can be diagonalized by

$$h|\theta\rangle = E(\theta)|\theta\rangle = \frac{2\cos\theta}{\sqrt{1-q}}|\theta\rangle\,, \quad \theta \in [0,\pi]$$

$$\langle\theta|\theta'\rangle = \frac{\delta(\theta-\theta')}{\mu(\theta)}, \quad \mu(\theta) = \frac{(e^{2i\theta}, e^{-2i\theta}, q; q)_\infty}{2\pi}$$

$$\langle\theta|n\rangle = \psi_n(\theta) = \frac{H_n(\cos\theta|q)}{(q;q)_n}, \quad \psi_0(\theta) = 1$$

$$Z(\beta) = 2^N \langle 0|e^{-\beta h}|0\rangle = 2^N \int_0^\pi d\theta \mu(\theta) e^{-\beta E(\theta)}$$

$$\rho(E) = \mu(E(\theta))|d\theta/dE|, \quad E \in [-|E_0|, |E_0|]$$

$$E_0 = -2/\sqrt{1-q}$$

- $\mu(\theta)$ gives a **compact spectrum** $\rho(E)$ with $E = 2\cos\theta/\sqrt{1-q}$
- For $q \to 1$, near E_0, $\rho(E) \sim \sinh 2\pi\sqrt{(E-E_0)/\lambda^{3/2}}$ **(Schwarzian limit)**

d-color chord algebra

- The generalization to d-color case is straightforward

$$[\alpha_a, \alpha_a^\dagger] = 1, \quad [\alpha_a, \alpha_b^\dagger]_q = 0 \quad (a \neq b)$$

- The Hamiltonian is $H = \frac{1}{\sqrt{d}} \sum_{a=1}^{d} H_a \equiv \frac{1}{\sqrt{d}} \sum_{a=1}^{d} (\alpha_a + \alpha_a^\dagger)$
- This algebra has a similar Fock space representation of d colors.
- However, we do not know how to diagonalize H and solve the spectrum for generic cases.
- Instead, we will consider three limit cases.

Case I: Infinite d

- If d is infinite, we can approximately regard the crossing of chords of different colors by **independent probability**.
- For any two crossing chords, the probability of same color is 1/d, and the probability of different color is 1-1/d.
- This amounts to regard all chords as a single color with a change the q factor (in the double-scaled SYK) by

$$q \to \bar{q} = 1/d + (1 - 1/d)q$$

- The model reduces a double-scaled SYK model.

Case I: Infinite d

- This gives a **new perspective** for the double-scaled SYK model.
- Let us recall the Hamiltonian of SYK model

$$\tilde{H}_{SYK} = i^p \sum_{i_1<\cdots<i_{2p}} J_{i_1\cdots i_{2p}} \psi_{i_1} \cdots \psi_{i_{2p}} \qquad \mathbb{E}[J_{i_1\cdots i_{2p}}] = 0, \quad \mathbb{E}[(J_{i_1\cdots i_{2p}})^2] = J^2 / \binom{2N}{2p}$$

- It includes all p-local random couplings.
- Imagine we **decompose the Hamiltonian into d groups**. In each group, all terms are commutative to each other, but different groups may not.
- The upper bound of terms in each group is $\binom{N}{p}$, the # of terms in each commuting SYK.

Case I: Infinite d

- Suppose most groups in this decomposition roughly at the same order as the upper bound in the double-scaled limit.
- The terms in each group definitely **does not** look like our specific construction, but we could expect the non-commutativity between groups **on average follows the same statistics**.
- The number of groups is $d \approx \binom{2N}{2p} / \binom{N}{p} \sim N^p \to \infty$
- $\bar{q} = 1/d + (1 - 1/d)q \to q$ perfectly matches with the q factor of the double-scaled SYK (from an independent single-color chord analysis).
- d-commuting SYK is a **sparse** equivalent version of SYK.

Case II: q=0 with finite d

- Quantum chaos or holography emerges at **infinite d**.
- How about **finite d**? Any intermediate regime with **partial quantum chaos or partial holography**?
- The second solvable case is **q=0**, namely **different color chords do not intersect**.
- In this case, we can regard each commuting SYK component Hamiltonian $H_a = \alpha_a + \alpha_a^\dagger$ as a **free random variable**.

Case II: q=0 with finite d

- Using free probability theory, we can show the following results.
- **[Theorem]** If the spectrum of one free random variable is non-compact, then the spectrum of the sum over finite numbers of free random variables is also non-compact.
- Since commuting SYK model has Gaussian spectrum (because the random coupling is Gaussian and all terms can be diagonalized simultaneously), the spectrum of d-commuting SYK model is **non-compact**, especially **unbounded from below**.

[Nica-Speicher 06']

Case II: q=0 with finite d

- To solve the spectrum, define the resolvent

$$R(z) = Tr(z - H)^{-1}$$

- Using the technique of free cumulant, $R(z)$ obeys the following self-constrainting equation

$$R(z) = \sqrt{\frac{d}{2\pi}} \int_{\mathbb{R}} \frac{dx}{z/d + (1 - 1/d)/R(z) - x} e^{-dx^2/2}$$

- Consider **large but finite d**. One can solve this equation either **perturbatively** or **non-perturbatively**.

Case II: q=0 with finite d

- Perturbative solution. Expanding RHS leads to

$$\int \frac{R(z)e^{-x^2/2}}{\sqrt{2\pi}}\left(1+\frac{1-zR(z)+x^2R(z)^2}{d}+O(d^{-2})\right)$$

- Requiring the second term vanish leads to Wigner's semi-circle

$$1-zR(z)+R(z)^2=0 \implies R(x)=R_\infty(x)\equiv(x-\sqrt{x^2-4})/2$$

  which has **compact spectrum** with ground energy at -2.

- Perturbative in higher orders give corrections but all have a **bounded spectrum from below**!

Case II: q=0 with finite d

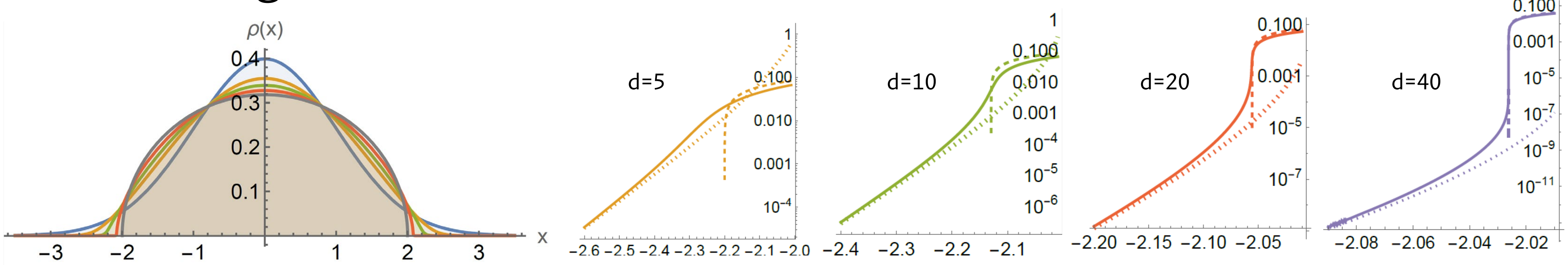
- To reconcile with the **[Theorem]** about non-compact spectrum, one has to solve this equation **non-perturbatively**.
- For the spectrum away and below the edge around -2, define y=z+2, we find

$$\rho(z) \sim \frac{d^{3/2} e^{-d-1}}{\sqrt{2\pi}} e^{-2d(-y)-dy^2/2}$$

- For small |y|, keeping the linear term, we can identify the exponential decay tail e^{E/T_c} with a **critical temperature** $T_c = 1/(2d)$.

Case II: q=0 with finite d

- The physical interpretation of T_c is: if the system temperature $T > T_c$, the relevant spectrum is the same as RMT and scales as $\sqrt{E+2}$, so we may regard it as chaotic; if $T < T_c$, the relevant spectrum is the exponential tail and the dynamics should be non-chaotic.
- As $T_c = 1/(2d)$, increasing d will expand the chaotic regime over the non-chaotic regime.



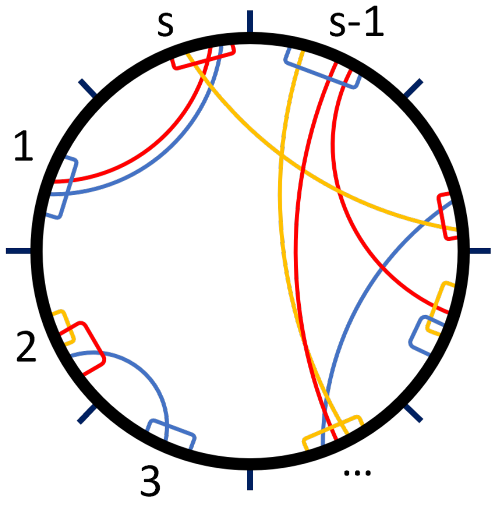
Case III: $q \to 1$ with finite d

- Another approximately solvable case is q close to 1, for which a recent development formulates the chord diagram in a **coarse-grained** way.
- We slightly improved this coarse-grained method and solved the IR part of the spectrum in the Schwarzian limit.
- We find again an exponential tail of the spectrum for small $d<9$. This time the tail is the **1-loop effect** rather than non-perturbative effect as the $q=0$ case.

[Berkooz-Brukner-Jia-Mamroud 24']

Case III: q→1 with finite d

- The idea is to consider chord diagrams with **large numbers of chords** and put the ends of all chords into **s segments**.
- For **a**-color chords emanating from **i-th** segment, define n_{ij}^{a} as the number of chords ending on **j-th** segment. Clearly, $n_{ij}^{a} = n_{ji}^{a}$.
- The intersection of chords belongs to **two types**
  1) intersections within the same segment
  2) intersections between different segments

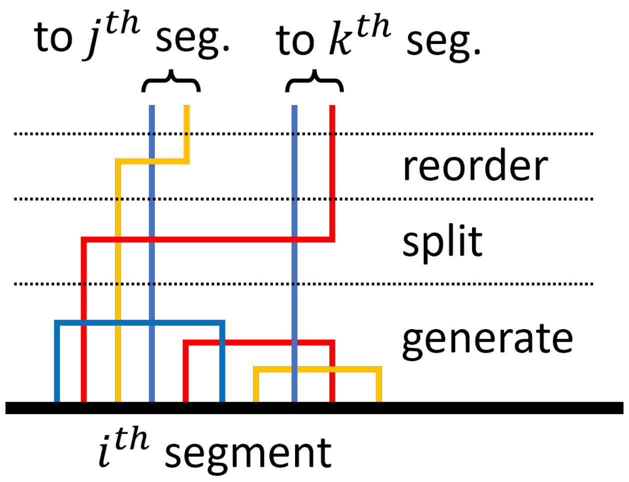


Case III: $q \to 1$ with finite d

- The intersections within the same segment has three steps:
  a. generate **b. split** **c. reorder**
- These intersections are short-distance and relevant to **UV** physics. So, we simply set **q=1** for their contributions.
- The intersections between different segments contribute as

$$q^{\sum_{a\neq b}\sum_{i>k>j>l} n_{ij}^{a} n_{kl}^{b}}$$

- Keeping **q≠1** gives the interesting **IR** physics.



Case III: q→1 with finite d

- Putting two parts together, the full partition function is ($n = \sum_{a=1}^{d} \sum_{i>j} n_{ij}^{a}$)

$$Z(\beta) = \sum_{\{n_{ij}^a\}} q^{\sum_{a\neq b} \sum_{i>k>j>l} n_{ij}^a n_{kl}^b} \underbrace{\frac{e^{\beta^2/(2s)}}{\prod_{i>j}\prod_a (n_{ij}^a)!} \frac{\beta^{2n}}{(ds^2)^n}}_{\text{q=1 intra-segment intersections}}$$

- We need to sum over all n_{ij}^{a}.

Define $\displaystyle\sum_{a\neq b} \sum_{i>k>j>l} n_{ij}^a n_{kl}^b = \frac{1}{2}\mathbf{n}\cdot M \cdot \mathbf{n}$

$$\Rightarrow \quad q^{\sum_{a\neq b} \sum_{i>k>j>l} n_{ij}^a n_{kl}^b} = e^{-\frac{1}{2}\lambda \mathbf{n}\cdot M\cdot \mathbf{n}} = \frac{1}{\sqrt{\det(-2\pi\lambda M)}} \int d\mathbf{J} e^{\mathbf{n}\cdot\mathbf{J} + \frac{1}{2\lambda}\mathbf{J}\cdot M^{-1}\cdot \mathbf{J}}$$

$\xrightarrow{\text{sum over } n_{ij}^a}$

$$Z(\beta) = \int d\mathbf{J} \frac{e^{\frac{1}{2\lambda}\mathbf{J}\cdot M^{-1}\cdot\mathbf{J}}}{\sqrt{\det(-2\pi\lambda M)}} e^{\left(\frac{1}{s} + \frac{2}{s^2 d}\sum_{i>j;a} e^{J_{ij}^a}\right)\beta^2/2}$$

Case III: q→1 with finite d

- For q→1, $\lambda \to 0$, consider low temperature $\beta = \beta_r/\sqrt{\lambda}$.
- Consider **large s limit**, and J^a_{ij} become continuous bilocal functions

$$\tau_i = \beta_r\, i/s \qquad\qquad J^a_{ij} = J^a(\tau_i, \tau_j)$$

- M^{-1} becomes a **double derivative** acting on two τ variables

$$M^{-1} = \frac{\beta_r^2}{2s^2} K^{-1} \partial_1 \partial_2 \qquad K_{ab} = 1 - \delta_{ab}$$

- Eventually, we derive a **Liouville-like action**

$$S = \int_0^{\beta_r} d\tau \int_0^{\tau} d\tau' \, \frac{1}{4} J^a(\tau,\tau') K^{-1}_{ab} \partial_\tau \partial_{\tau'} J^b(\tau,\tau') + \frac{1}{d} \sum_a e^{J^a(\tau,\tau')} \qquad Z(\beta) \sim \int d\mathbf{J}\, e^{S/\lambda}$$

Case III: q→1 with finite d

- Since λ is small, we can use saddle approximation and compute the partition function in orders of loops.
- The saddle is **homogeneous** $J^a = J$ (SYK solution)

$$e^J = \frac{\cos^2 \omega\beta_r/2}{\cos^2 \omega(\tau_{12} - \beta_r/2)} \qquad \omega = \sqrt{1-1/d}\cos\omega\beta_r/2$$

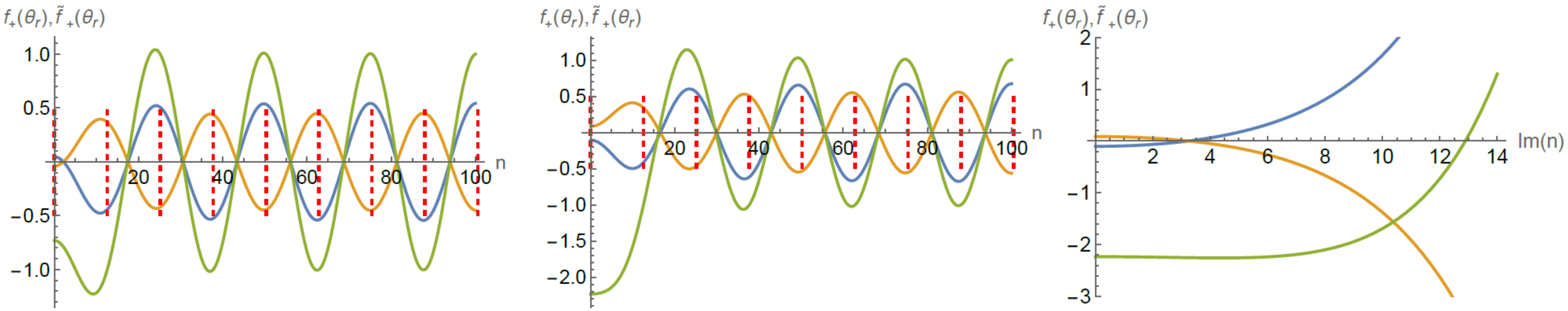
- Around the saddle, consider fluctuations of J^a diagonalizing the K matrix. There is **one** fluctuation δJ with K-eigenvalue d-1, and **d-1** fluctuations δJ^i with K-eigenvalue -1.

Case III: $q \to 1$ with finite d

- These modes all obey the following eigen differential equation
$$\left[\partial_1\partial_2 + \frac{h(h-1)\omega^2}{\cos^2\omega(\tau_{12}-\beta_r/2)}\right] f(\tau_1,\tau_2) = \eta f(\tau_1,\tau_2)$$
- For $f = \delta J$, h=2, which is exactly the **Schwarzian mode**.
- For $f = \delta J^i$, we have a **nontrivial h**
$$h = \frac{1}{2}\left(1 \pm \sqrt{\frac{d-9}{d-1}}\right)$$
- For d>9, $h \in [0,1]$; For 1<d<9, $h \in 1/2 + i\mathbb{R}$.

Case III: q→1 with finite d

- Let us focus on small d<9.
- One can show that there is **critical** β_c, such that a pair opposite eigenvalues η and $-\eta$ collide at **zero** when $\beta_r/\sqrt{\lambda} = \beta_c$.



$T > T_c$ $T < T_c$

Case III: q→1 with finite d

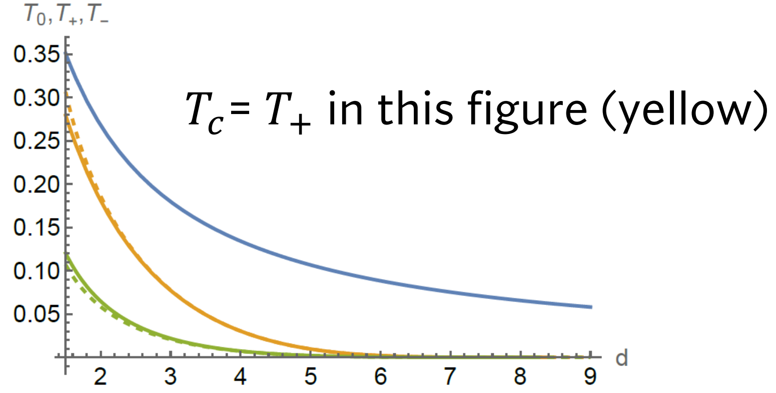
- Since the 1-loop contribution is $Z_{1-\mathrm{loop}} \sim 1/\sqrt{\prod \eta}$, this leads to a singularity scaling as (recall d-1 δJ^i modes in total)

$$Z_{1-\mathrm{loop},\, h\neq 2} \sim \frac{1}{(\beta_c - \beta)^{(d-1)/2}}$$

- This singularity gives a tail of spectrum as (E<0)

$$\rho(E) \sim (-E)^{(d-3)/2} e^{E/T_c}$$

- The critical temperature T_c **drops monotonically as d increases**.

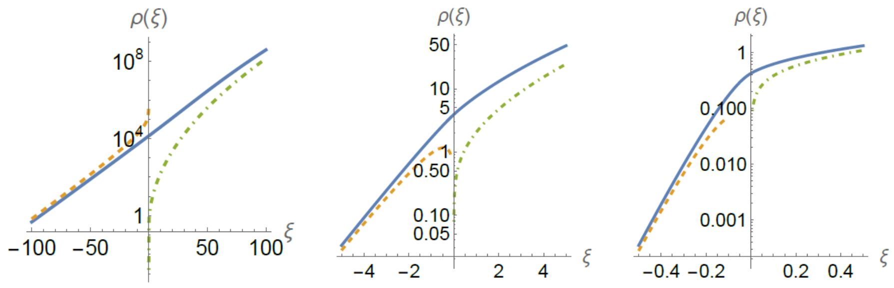


Case III: q→1 with finite d

- Putting the saddle and all 1-loop contributions together, we have

$$Z(\beta) \sim \underbrace{\frac{1}{(1/T_c - \beta)^{(d-1)/2}}}_{h \neq 2} \underbrace{\frac{1}{\beta^{3/2}}}_{h=2} \underbrace{e^{-\beta E_0 + c/\beta}}_{saddle}$$

- The full spectrum is an interpolation between the **holographic spectrum** $\sim \sinh 2\pi\sqrt{(E - E_0)/\lambda^{3/2}}$ **in UV** and the **(most likely) non-holographic exponential tail** $(E_0 - E)^{(d-3)/2} e^{E/T_c}$ **in IR**.



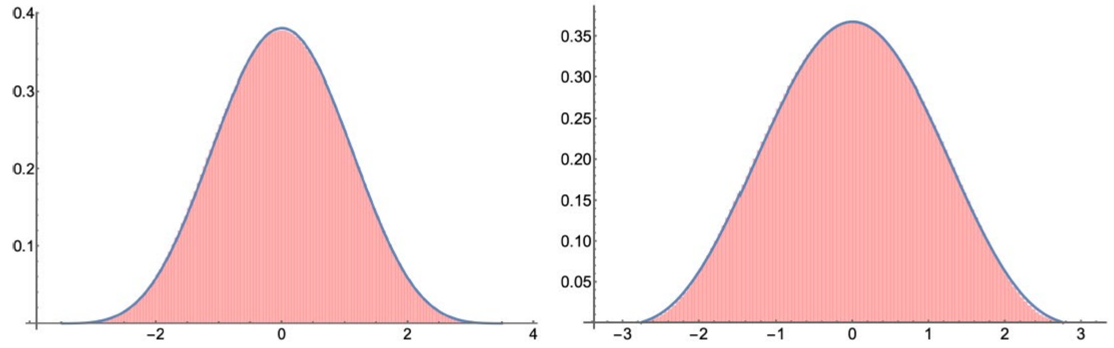
(a) $d = 2$ (b) $d = 4$ (c) $d = 6$

Case III: $q \to 1$ with finite d

- How about d>9?
- Unfortunately, there is no null eigenvalue for all temperatures. We suspect higher loop contributions may lead to an exponential tail.
- Why is spectrum related to quantum chaos?
- To check this claim, we need to compute OTOC or other dynamical observables in a future work. However, our preliminary numerics support this claim.

Numerics

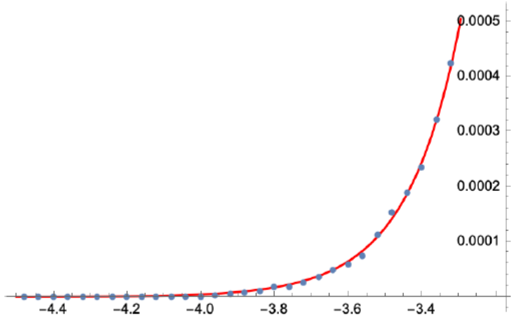
- We first checked the spectrum of d-commuting SYK model for N=15, p=2 by exact diagonalization and average ensembles. It globally matches well with double-scaled SYK with $q \to \bar{q} = 1/d + (1 - 1/d)q$.



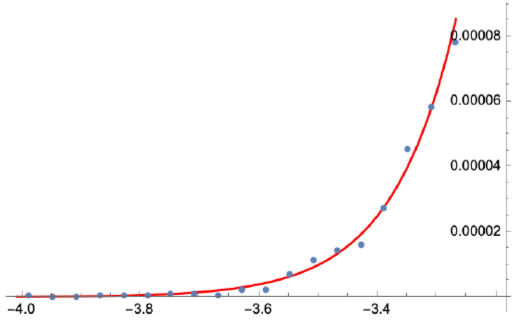
(a) $N = 15\ d = 2$ (b) $N = 15\ d = 5$

Numerics

- Then we checked the tail of the spectrum. The 1-loop result works well for small d=2,3 but becomes worse for larger d.
- This is not too surprising, given N=15 not much greater than d, and $\lambda = 16/15$ not close to 0.



(a) $d = 2$

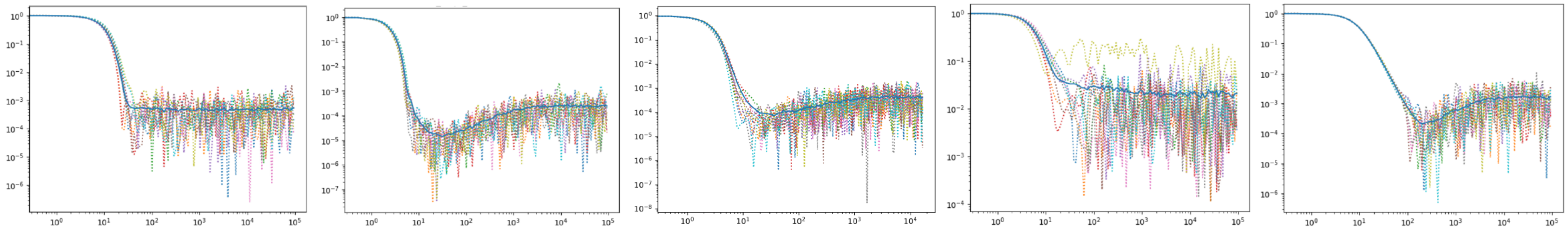


(b) $d = 3$

$T_c/\lambda^{1/2}$	$d = 2$	$d = 3$	$d = 4$
Theory	0.1809	0.07717	0.03025
Numerics	0.1787	0.07667	0.03291
Error	1.2%	0.6%	8.8%

Numerics

- To justify **a critical temperature around T_c** that separates chaotic and non-chaotic physics, we numerically checked the spectral form factor.
- **Ramp disappears for too low temperature and d=1**.



(a) $d = 1, \beta = 0,$ $h = 4.9 \times 10^{-4}$

(b) $d = 2, \beta = 0,$ $k = 0.77, h = 2.4 \times 10^{-4}$

(c) $d = 2, \beta = 2,$ $k = 0.45, h = 4.5 \times 10^{-4}$

(d) $d = 2, \beta = 10,$ $h = 2.1 \times 10^{-2}$

(e) $d \to \infty, \beta = 10,$ $k = 0.73, h = 1.6 \times 10^{-3}$

(theoretical β_c is 5.35)

An implication for quantum simulation

- It has an important operational implication for quantum simulations of holography. Given any Hamiltonian to be realized on a quantum simulator or quantum computer, we can **separate the terms into d intra-commuting groups** and **estimate the average non-commutativity parameter λ between groups**.
- Suppose d-commuting SYK has a **universality**, we can use (d, λ) to estimate a critical temperature for how quantum chaotic this model is.
- This could be a **benchmark** for distinguishing authentic holographic signals in future quantum simulations (e.g. traversable wormholes).

Conclusion

- We constructed a partially chaotic model **"d-commuting SYK"** using d copies of the integrable building blocks "commuting SYK". This model exhibits **quantum chaos above a critical temperature** $\mathrm{T_c}$ and is non-chaotic (and worth understanding further) below this temperature.
- This critical temperature is extracted from the spectral exponential tail e^{E/T_c} in IR.
- T_c monotonically decreases as d increases.
- This model may have universality for future quantum simulations of holography.

Thank you!