Black holes and trace relations

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Recent Develoments in BHs and QG
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Talk based on collaborations with:

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2209.12696, 2304.10155, 2312.16443, 2412.08695

Closely related works:

- Chang, Lin: 2209.06728, 2402.10129
- Chang, Feng, Lin, Tao: 2306.04673
- Budzik, Murali, Vieira: 2306.04693
- ……

Introduction

Grand goal: microstate structures of BPS black holes in AdS/CFT

- Black hole microstates in $AdS_5 \times S^5 \rightarrow$ Requires strong coupling & large N.
- Much of the recent progress happened at weak-coupling & low N.

- Qualitative/semi-quantitative lessons
- Hope for non-renormalization in coupling (in “cohomology” spectrum) [Minwalla] [Chang, Lin]
- Black holes in “quantum” gravity: finite $G_N \sim 1/N^2$

Important features to understand the new BPS operators:

- gravitons vs. black holes
- “monotone” vs. “fortuitous”
- Role of trace relations

Today’s (modest) goal:

- illustrate with examples how to “construct” new operators (at low N=2,3).
- Simple applications: features reminiscent of “hairy” BH cohomologies

The problem

1/16-BPS states in 4d maximal SYM with SU(N) gauge group

3 chiral multiplets: ϕ_m , $\bar{\phi}^m$ and $\psi_{m\alpha}$, $\bar{\psi}^m_{\dot{\alpha}}$ $(m = 1,2,3)$

vector multiplet: $A_\mu \sim A_{\alpha\dot{\beta}}$ and λ_α , $\bar{\lambda}_{\dot{\alpha}}$ $(\mu = 1, \cdots, 4)$ $(\alpha = \pm, \dot{\alpha} = \dot{\pm})$

- Among $16Q + 16S$, pick a pair Q & $S = Q^\dagger$ that the BPS states preserve.

$$2\{Q, Q^\dagger\} = E - E_{BPS} = E - (R_1 + R_2 + R_3 + J_1 + J_2)$$

- Nilpotency $Q^2 = 0$: BPS states ~ harmonic forms $\overset{1\text{ to }1}{\longleftrightarrow}$ Q-cohomology classes.

Gauge-invariant local BPS operators: (at $x^\mu = 0$ on R^4)

- Free ($g_{YM} \to 0$): All gauge invariants of the invariant fields under Q & $Q^\dagger$:

$\bar{\phi}^m$, ψ_{m+}, $\bar{\lambda}_{\dot{\alpha}}$, $f_{++} \equiv F_{1+i2\,,\,3+i4}$ & holomorphic derivatives $\partial_1 - i\partial_2 \equiv \partial_{+\dot{+}}$, $\partial_3 - i\partial_4 \equiv \partial_{+\dot{-}}$

- Many of them are non-BPS at $g_{YM} \neq 0$: At small $g_{YM} \ll 1$,

$$Q\,\bar{\phi}^m = 0\,,\; Q\psi_{m+} \sim g_{YM}\epsilon_{mnp}[\bar{\phi}^n, \bar{\phi}^p]\,,\; Qf_{++} \sim g_{YM}[\psi_{m+}, \bar{\phi}^m]\,,\; Q\,\bar{\lambda}_{\dot{\alpha}} = 0\,,\; [Q, D_{+\dot{\alpha}}] \sim g_{YM}[\,\bar{\lambda}_{\dot{\alpha}}\,,\;\;]$$

- Classical Q & $Q^\dagger$ → Anomalous dimension $QQ^\dagger + Q^\dagger Q \sim E - E_{BPS}$ at 1-loop, $O(g_{YM}^2)$.

Question: Spectrum of 1-loop BPS operators? Their cohomology classes?

(Index of these cohomologies studied in depth (2018~), accounting for S_{BH})

Gravitons vs. black holes

We shall spend some time with "graviton" and then move on to BH's.

Constructing the representatives of graviton cohomologies:

- (Partly) anti-symmetrized scalars in trace are Q-exact. E.g.

$$\mathrm{tr}\left(\bar\phi\cdots[\bar\phi^m,\bar\phi^n]\cdots\bar\phi\right)\sim Q\mathrm{tr}\left(\bar\phi\cdots\epsilon^{mnp}\psi_{p+}\cdots\bar\phi\right)$$

- Symmetrized scalars → scalar chiral primaries of single trace cohomologies

$$\mathrm{tr}[\bar\phi^{(m_1}\cdots\bar\phi^{m_n)}]$$

- From these, the following procedures generate all graviton cohomologies:

PSU(1,2|3) superconformal descendants: All single-trace cohomologies

E.g.

$$\mathrm{tr}(\bar\phi^{(m}\bar\phi^{n)})\ ,\ \mathrm{tr}(\bar\phi^m\bar\lambda_{\dot\alpha})\ ,\ \mathrm{tr}(\bar\lambda_{\dot+}\bar\lambda_{\dot-}),$$
$$\mathrm{tr}(\bar\phi^m\psi_{n+})-\tfrac{1}{3}\delta^m_n\mathrm{tr}(\bar\phi^l\psi_{l+})\ ,\ \mathrm{tr}(\bar\lambda_{\dot\alpha}\psi_{m+}-\epsilon_{mnp}\bar\phi^nD_{+\dot\alpha}\bar\phi^p),$$
$$\mathrm{tr}(\bar\phi^mf_{++}-\tfrac{1}{4}\epsilon^{mnp}\psi_{n+}\psi_{p+})\ ,\ \mathrm{tr}(\bar\lambda_{\dot\alpha}f_{++}-\tfrac{2}{3}\psi_{m+}D_{+\dot\alpha}\bar\phi^m+\tfrac{1}{3}\bar\phi^mD_{+\dot\alpha}\psi_{m+})$$

& the derivatives $\partial_{+\dot\alpha}$ acting on them (conformal descendants)

Products of all these single-traces: multi-trace (~multi-particle) cohomologies

- Large N & low E: #(trace) = #(particle). This construction has concrete physical meanings.

Monotone vs. fortuitous

Even at finite N, or $E > N$, the operators listed above do represent cohomologies, which we shall keep calling "graviton" cohomologies.

- They continue to be Q-closed no matter what the matrices sizes are: no use of trace relations of finite N matrices.
- The actual BPS states have mixed trace numbers, but the trace basis of the previous page still provides representatives of cohomologies.

This leads to the following characterizations of two types of cohomologies: [Chang, Lin]

- gravitons ("monotone")

  Fix the "shape" of operator, like $tr(XY)$ or $tr(Z\psi_1)tr(Xf - 1/2\,\psi_2\psi_3)$: Q-closed at all N
- black holes ("fortuitous"):

  Basically, all the remainders: those not cohomologous to gravitons

  A necessary condition → Q-closed by trace relations. Q-closed only for $N \leq N_{max}$.

"Gravitons" at finite N have good gravity picture: giant gravitons

Calling "fortuitous → BH" may be a bit overstating, but I shall often do so.

Gravitons & trace relations

No trace relations for Q-closedness:

- First constructed all single trace cohomologies, and then multiplied them.
- Trace relations never used for Q-closedness, so the construction applies to all N.

Trace relations for Q-exactness

- However, as N reduces at fixed operator shape, some of them may be redundant due to trace relations: E.g., an SU(2) trace relation relates $tr(Z^2)tr(Z^2) = 2\, tr(Z^4)$

"Gravitons" at finite N ~ giant gravitons

- Multi-trace cohomologies I explained are overcomplete, due to trace relations.
- Typical trace relations in cohomologies: "polynomials of single trace gravitons" = Q(…)

  E.g. SU(2):

$$\mathrm{tr}(X^2)\mathrm{tr}(Y^2) - [\mathrm{tr}(XY)]^2 \sim \mathrm{tr}([X,Y][X,Y]) \sim Q\mathrm{tr}(\psi_3[X,Y])$$

$$\mathrm{tr}(X^2)\mathrm{tr}(YZ) - \mathrm{tr}(XY)\mathrm{tr}(XZ) \sim \mathrm{tr}([X,Y][X,Z]) \sim Q\mathrm{tr}(\psi_3[X,Z] - [X,Y]\psi_2)$$

- Relatively easy to find SU(2) relations, since SU(2) adjoint ~ SO(3) vector.

  E.g. $(a \times b)\cdot(c \times d) = (a\cdot c)(b\cdot d) - (a\cdot d)(b\cdot c)$, etc. : identities changing #(inner products)

SU(3) graviton trace relations: examples

Some gravitons
$$u^{ij} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\right) \ , \quad u^{ijk} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j}\phi^{k)}\right) \ ,$$
$$v^{i}{}_{j} \equiv \mathrm{tr}\left(\phi^{i}\psi_{j}\right) - \tfrac{1}{3}\delta^{i}_{j}\mathrm{tr}\left(\phi^{a}\psi_{a}\right) \ , \quad v^{ij}{}_{k} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{k}\right) - \tfrac{1}{4}\delta^{i}_{k}\mathrm{tr}\left(\phi^{(j}\phi^{a)}\psi_{a}\right) - \tfrac{1}{4}\delta^{j}_{k}\mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_{a}\right)$$
$$w^{i} \equiv \mathrm{tr}\left(f\phi^{i} + \tfrac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right) \ , \quad w^{ij} \equiv \mathrm{tr}\left(f\phi^{(i}\phi^{j)} + \epsilon^{a_1a_2(i}\phi^{j)}\psi_{a_1}\psi_{a_2}\right) \ ,$$

- Q-exact polynomials

  (only partial list)

$$t^{10}[1,2](u_2u_3) : (R^{(0,0)}_{10})^{i}_{jk} = \epsilon_{a_1a_2(j}\epsilon_{k)b_1b_2}u^{a_1b_1}u^{ia_2b_2}$$
$$t^{12}[0,0](u_2u_2u_2) : R^{(0,0)}_{12} = \epsilon_{a_1a_2a_3}\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}u^{a_3b_3}$$
$$t^{12}[2,2](u_2u_2u_2, u_3u_3) : (R^{(0,0)}_{12})^{ij}_{kl} = \epsilon_{a_1a_2(k}\epsilon_{l)b_1b_2}\left(u^{a_1b_1}u^{a_2b_2}u^{ij} + 6u^{a_1b_1(i}u^{j)a_2b_2}\right)$$
$$t^{12}[0,3](u_2v_3) : (R^{(0,1)}_{12})_{ijk} = \epsilon_{(i|a_1a_2}\epsilon_{|j|b_1b_2}u^{a_1b_1}v^{a_2b_2}{}_{|k)}$$
$$t^{12}[1,1](u_2v_3, u_3v_2) : (R^{(0,1)}_{12})^{i}_{j} = \epsilon_{ja_1a_2}\left(4u^{a_1b}v^{ia_2}{}_{b} + 3u^{ia_1b}v^{a_2}{}_{b}\right)$$
$$t^{12}[2,2](u_2v_3, u_3v_2) : (R^{(0,1)}_{12})^{ij}_{kl} = \epsilon_{a_1a_2(k}\left(u^{a_1(i}v^{j)a_2}{}_{l)} + u^{ija_1}v^{a_2}{}_{l)}\right)$$
$$t^{14}[1,0](u_2u_2v_2) : (R^{(0,1)}_{14})^{i} = \epsilon_{a_1a_2a_3}u^{ia_1}u^{ba_2}v^{a_3}{}_{b}$$
$$t^{14}[0,2](u_2u_2v_2, u_3v_3) : (R^{(0,1)}_{14})_{ij} = \epsilon_{a_1a_2(i|}\left(\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}v^{b_3}{}_{|j)} - 2\epsilon_{|j)b_1b_2}u^{a_1b_1c}v^{a_2b_2}{}_{c}\right)$$
$$t^{14}[2,1](u_2u_2v_2, u_3v_3) : (R^{(0,1)}_{14})^{ij}_{k} = \epsilon_{ka_1a_2}\left(3u^{(a_1b}u^{ij)}v^{a_2}{}_{b} + 4u^{a_1b}u^{a_2(i}v^{j)}{}_{b} + 24u^{a_1b(i}v^{j)a_2}{}_{b}\right)$$
$$t^{14}[1,3](u_2u_2v_2, u_3v_3) : (R^{(0,1)}_{14})^{i}_{jkl} = \epsilon_{(j|a_1a_2}\epsilon_{|k|b_1b_2}\left(u^{a_1b_1}u^{a_2b_2}v^{i}{}_{|l)} + 6u^{ia_1b_1}v^{a_2b_2}{}_{|l)}\right)$$
$$t^{14}[3,2](u_2u_2v_2, u_3v_3) : (R^{(0,1)}_{14})^{ijk}_{lm} = \epsilon_{a_1a_2(l}\left(u^{(a_1i}u^{jk)}v^{a_2}{}_{m)} + 6u^{a_1(ij}v^{k)a_2}{}_{m)}\right)$$
$$t^{14}[1,3](v_2v_3) : (R^{(0,2)}_{14})^{i}_{jkl} = \epsilon_{a_1a_2(j}v^{a_1}{}_{k}v^{ia_2}{}_{l)}$$
$$t^{16}[0,1](u_2v_2v_2, v_3v_3) : (R^{(0,2)}_{16})_{i} = \epsilon_{ia_1a_2}\left(12u^{bc}v^{a_1}{}_{b}v^{a_2}{}_{c} + 13u^{a_1b}v^{a_2}{}_{c}v^{c}{}_{b} + 12v^{a_1b}{}_{c}v^{a_2c}{}_{b}\right)$$
$$t^{16}[1,2](u_2v_2v_2, v_3v_3) : (R^{(0,2)}_{16})^{i}_{jk} = \epsilon_{a_1a_2(j}\left(3u^{ib}v^{a_1}{}_{k)}v^{a_2}{}_{b} - 7u^{ia_1}v^{b}{}_{k)}v^{a_2}{}_{b} + 6u^{a_1b}v^{i}{}_{k)}v^{a_2}{}_{b} + 24v^{a_1b}{}_{k)}v^{ia_2}{}_{b}\right)$$
$$t^{16}[2,3](u_2v_2v_2, v_3v_3) : (R^{(0,2)}_{16})^{ij}_{klm} = \epsilon_{a_1a_2(k}\left(u^{a_1(i}v^{j)}{}_{l}v^{a_2}{}_{m)} + 3v^{a_1(i}{}_{l}v^{j)a_2}{}_{m)}\right)$$
$$t^{18}[0,0](u_3v_2v_2) : R^{(0,2)}_{18} = \epsilon_{a_1a_2a_3}u^{a_1bc}v^{a_2}{}_{b}v^{a_3}{}_{c}$$
$$t^{20}[1,0](v_2v_2v_3) : (R^{(0,3)}_{20})^{i} = 2v^{a}{}_{c}v^{b}{}_{a}v^{ic}{}_{b} - 3v^{i}{}_{a}v^{c}{}_{b}v^{ab}{}_{c}$$
$$t^{22}[2,0](u_2v_2v_2v_2) : (R^{(0,3)}_{22})^{ij} = u^{ij}v^{a}{}_{b}v^{b}{}_{c}v^{c}{}_{a} - 3u^{a(i}v^{j)}{}_{b}v^{b}{}_{c}v^{c}{}_{a} + 3u^{ab}v^{(i}{}_{a}v^{j)}{}_{c}v^{c}{}_{b}$$
$$t^{24}[0,0](u_2v_2v_2v_3) : R^{(0,3)}_{24} = \epsilon_{a_1a_2a_3}u^{a_1b}v^{a_2}{}_{b}v^{a_3c}{}_{d}v^{d}{}_{c}$$
$$t^{26}[1,0](v_2v_2v_2v_3) : (R^{(0,4)}_{26})^{i} = v^{i}{}_{a}v^{a}{}_{b}v^{d}{}_{c}v^{bc}{}_{d}$$
$$t^{30}[0,0](v_2v_2v_2v_2v_2) : R^{(0,5)}_{30} = v^{a}{}_{b}v^{b}{}_{c}v^{c}{}_{d}v^{d}{}_{e}v^{e}{}_{a}$$
$$t^{30}[3,0](v_2v_2v_2v_2v_2) : (R^{(0,5)}_{30})^{ijk} = \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k)}{}_{a_2}v^{b}{}_{c}v^{c}{}_{d}v^{d}{}_{b} \ .$$

[Algebraic method: Groebner basis]

BH cohomology: an ansatz

Operators which are Q-closed by trace relations:

- There are systematic ways of finding trace relations (Cayley-Hamilton identity, restricted Schur polynomials), but at least so far, inconvenient to use them to find useful relations.

We know many trace relations of graviton cohomologies

- Some linear combinations of the relations $\{R_a(g_I) = Qr_a\}$ exactly vanish:

$$\sum_a f_a(g)\, R_a(g) = 0$$

- "Relations of relations." (Hierarchy: relations → r of r → r of r of r, … → "syzygy")
- To us now, "ansatze" for the new cohomologies via trace relations:

$$Q[\, \sum_a f_a(g)\, r_a\,] = 0$$

- This may look like a limited ansatz. But once we construct new BH operators, we can include them to the set $\{g_I\}$ and construct new relations & relations of relations.
- One can repeat this step and have a hierarchy of ansatze.

- Of course, the ansatze may fail to be nontrivial cohomologies, by being Q-exact.

SU(2) example

Consider the following SU(2) graviton trace relations:

[Recall $(v_2)^m{}_n = tr(\bar{\phi}^m\psi_n - 1/3\ \delta^m_n\ \bar{\phi}^p\psi_p)$]

$$Q\text{tr}(\psi_{(c}\psi_m\psi_{n)}) \propto \epsilon_{ab(c}(v_2)^a{}_m(v_2)^b{}_{n)} \equiv R(v_2)_{cmn}$$

terms $\propto \delta^a_m$ or $\propto \delta^b_n$ vanish by contractions/symmetrizations

- It can be best understood by writing SU(2) adjoints as 3-vectors:

$$Q[\psi_{(c}\cdot(\psi_m\times\psi_{n)})] \sim \epsilon_{ab(c}(\phi^a\times\phi^b)\cdot(\psi_m\times\psi_n) = 2\epsilon_{ab(c}\ (\phi^a\cdot\psi_m)(\phi^b\cdot\psi_{n)})$$

Now we find the following relation of relations: $\epsilon^{abc}(v_2)^m{}_a(v_2)^n{}_b R(v_2)_{cmn} = 0$

- An $SU(3)_R$ invariant of four 3 x 3 traceless matrices, involving 2 ϵ's ~ no ϵ's
- Possible terms $tr(v^4)$, $tr(v^2)tr(v^2)$ all vanish from cyclicity & fermion statistics.
- Leads to an operator which become Q-closed by using trace relations:

$$\epsilon^{abc}(v_2)^m{}_a(v_2)^n{}_b\text{tr}(\psi_{(c}\psi_m\psi_{n)})$$

Easy to show it is not Q-exact. (I will skip this, unless asked. → Eunwoo Lee's simple proof)

- The first fortuitous/"black hole" operator found in the maximal SYM, in 2022...!

SU(3) example

More complicated BH/fortuitous operators from this ansatz:

- Lightest BH operator in SU(3): Looks complicated, but takes the form $\sum_a f_a(g)\, r_a$

$$
\begin{aligned}
&288 v^j{}_a v^{ka}{}_i \epsilon_{c_1c_2(j} \mathrm{tr}\left(\phi^{c_1}\phi^{c_2}\phi^i\psi_{k)}\right) - 72 v^a{}_b v^{bk}{}_a \epsilon_{c_1c_2(k} \mathrm{tr}\left(\phi^{c_1}\phi^{c_2}\phi^d\psi_{d)}\right) \\
&+36\epsilon_{a_1a_2i} u^{a_1k} v^{a_2}{}_j \Big[2\mathrm{tr}\left(\phi^{(i}\phi^c\phi^{j)}\psi_{(c}\psi_{k)}\right) + 2\mathrm{tr}\left(\phi^{(i|}\phi^c\phi^{|j)}\psi_{(c}\psi_{k)}\right) \\
&\qquad +9\mathrm{tr}\left(\phi^{(i}\phi^j\psi_{(c}\phi^{c)}\psi_{k)}\right) - 6\mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(c}\phi^c\psi_{k)}\right)\Big] \\
&-9\epsilon_{a_1a_2j} u^{a_1b} v^{a_2}{}_b \Big[2\mathrm{tr}\left(\phi^{(j}\phi^c\phi^{d)}\psi_{(c}\psi_{d)}\right) + 2\mathrm{tr}\left(\phi^{(j|}\phi^c\phi^{|d)}\psi_{(c}\psi_{d)}\right) \\
&\qquad +9\mathrm{tr}\left(\phi^{(j}\phi^d\psi_{(c}\phi^{c)}\psi_{d)}\right) - 6\mathrm{tr}\left(\phi^{(j}\phi^{d)}\psi_{(c}\phi^c\psi_{d)}\right)\Big] \\
&-20 u^{ai} v^j{}_a \epsilon_{b_1b_2b_3} \Big[2\mathrm{tr}\left(\psi_{(i}\psi_{j)}\phi^{b_1}\phi^{b_2}\phi^{b_3}\right) + \mathrm{tr}\left(\psi_{(i}\phi^{b_1}\psi_{j)}\phi^{b_2}\phi^{b_3}\right)\Big] \\
&-36 u^{ai} v^j{}_a \epsilon_{b_1b_2(i} \Big[\mathrm{tr}\left(\psi_{j)}\psi_c\phi^{b_1}\phi^{b_2}\phi^c\right) + \mathrm{tr}\left(\psi_{j)}\psi_c\phi^{b_1}\phi^c\phi^{b_2}\right) + \mathrm{tr}\left(\psi_{j)}\psi_c\phi^c\phi^{b_1}\phi^{b_2}\right)\Big] \\
&-36 u^{ai} v^j{}_a \epsilon_{b_1b_2(i} \Big[\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\psi_c\phi^{b_2}\phi^c\right) + \mathrm{tr}\left(\psi_{j)}\phi^{b_1}\psi_c\phi^c\phi^{b_2}\right) + \mathrm{tr}\left(\psi_{j)}\phi^c\psi_c\phi^{b_1}\phi^{b_2}\right)\Big] \\
&-36 u^{ai} v^j{}_a \epsilon_{b_1b_2(i} \Big[\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^{b_2}\psi_c\phi^c\right) + \mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^c\psi_c\phi^{b_2}\right) + \mathrm{tr}\left(\psi_{j)}\phi^c\phi^{b_1}\psi_c\phi^{b_2}\right)\Big] \\
&-36 u^{ai} v^j{}_a \epsilon_{b_1b_2(i} \Big[\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^{b_2}\phi^c\psi_c\right) + \mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^c\phi^{b_2}\psi_c\right) + \mathrm{tr}\left(\psi_{j)}\phi^c\phi^{b_1}\phi^{b_2}\psi_c\right)\Big] \\
&+12 u^{ai} v^j{}_a \epsilon_{b_1b_2(i} \Big[5\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^{b_2}\right)\mathrm{tr}\left(\psi_c\phi^c\right) + 2\mathrm{tr}\left(\psi_{j)}\phi^{(b_1}\phi^{c)}\right)\mathrm{tr}\left(\psi_c\phi^{b_2}\right) - 2\mathrm{tr}\left(\psi_{j)}\phi^{b_2}\right)\mathrm{tr}\left(\psi_c\phi^{(b_1}\phi^{c)}\right)\Big]
\end{aligned}
$$

- Wasn't easy to prove that this is not Q-exact (heavy use of computer).

- The ansatz is surprisingly useful to generate illustrating examples. (E.g. all discussed today)

Product cohomologies

At this moment, note that if O_1 and O_2 represent cohomologies, so does $O_1 O_2$:

- Leibniz rule of Q ensures its Q-closedness.
- In fact this allowed an easy construction of multi-graviton cohomologies beyond large N.
- It is in principle unclear if $O_1 O_2$ is nontrivial or not (i.e. Q-exact).

How does the lightest SU(2) BH cohomology react to the multiplications of gravitons?

$$O_0 \equiv \epsilon^{abc}(v_2)^m{}_a (v_2)^n{}_b \mathrm{tr}(\psi_{(c}\psi_m\psi_{n)})$$

- E.g. multiply smallest scalar primaries: becomes Q-exact by SU(2) trace relations

$$\begin{aligned}
O_0(\bar\phi^{(m}\cdot\bar\phi^{n)}) = -\frac{1}{14}Q[&20\epsilon^{rs(m}(\bar\phi^{n)}\cdot\psi_{p+})(\bar\phi^p\cdot\psi_{r+})(\bar\phi^q\cdot\psi_{q+})(f_{++}\cdot\psi_{s+}) \\
&-20\epsilon^{prs}(\bar\phi^{(m}\cdot\psi_{p+})(\bar\phi^{n)}\cdot\psi_{r+})(\bar\phi^q\cdot\psi_{q+})(f_{++}\cdot\psi_{s+}) \\
&+30\epsilon^{prs}(\bar\phi^{(m}\cdot\psi_{p+})(\bar\phi^{n)}\cdot\psi_{r+})(\bar\phi^q\cdot\psi_{s+})(f_{++}\cdot\psi_{q+}) \\
&-7\epsilon^{a_1a_2p}\epsilon^{b_1b_2(m}(\bar\phi^{n)}\cdot\psi_{p+})(\bar\phi^q\cdot\psi_{q+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+}) \\
&+18\epsilon^{a_1a_2p}\epsilon^{b_1b_2(m}(\bar\phi^{n)}\cdot\psi_{q+})(\bar\phi^q\cdot\psi_{p+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})]
\end{aligned}$$

- Multiplying many other conformal primaries (w.o. derivatives) yield Q-exact operators.
- So O_0 abhors to be dressed by many types of gravitons. If this holds for all gravitons, it is tempting to interpret this as the finite N version of black hole no-hair theorem.

Spinning graviton hairs

The product can be nontrivial if the graviton is a large enough conformal descendant

$$O_{\rm BH}(\partial_{+\dot{+}})^{j_1}(\partial_{+\dot{-}})^{j_2}(\text{primary graviton}) \quad \ldots (\bigstar)$$

Currently, we know two evidences for this statement:

- SU(2) index exhibits higher order terms above O_0 accounted for by these products

$$Z - Z_{\rm grav} \;=\; \chi_D\left[-t^{24} - \chi_{(1,3)}t^{32} - (\chi_{1,\bar{3}} + \chi_{(3,\bar{6})})t^{34} - \chi_{(2,3)}t^{35} + (\chi_{(3,1)} + \chi_{(3,8)})t^{36} - (\chi_{2,\bar{3}} + \chi_{4,6})t^{37} + \cdots\right]$$

$PSU(1,2|3)$ character

- For SU(N), we showed that they aren't Q-exact by trace relations for infinitely many j_1, j_2.

  Package letters dressed by (covariant) derivatives to holomorphic fields

$$X(z) \equiv \sum_{n=0}^{\infty}\frac{1}{n!}(z_1D_1 + z_2D_2)^n X(0) = \sum_{n=0}^{\infty}\frac{1}{n!}(z_\alpha D_\alpha)^n X(0) \quad \text{and so on}$$

  Q-action is local in holomorphic fields [Grant, Grassi, SK, Minwalla] [Chang, Yin]

$$Q\phi^m(z) = [\lambda(z), \phi^m(z)]\ , \qquad Q\psi_m(z) = \epsilon_{mnp}[\phi^n(z), \phi^p(z)] + \{\lambda(z), \psi_m(z)\}$$
$$Qf(z) = [\phi^m(z), \psi_m(z)] + [\lambda(z), f(z)]\ , \qquad Q\lambda(z) = \{\lambda(z), \lambda(z)\}$$

  Can show $O_{BH}(0)\, O_{grav}(z)$ cannot be Q-exact by trace relation: some of (★) nontrivial

  Then, can slightly generalize it to prove that: Infinitely many of (★) are nontrivial.

Product hairs & gravity

Conformal primary gravitons: mostly disallowed hairs

Conformal descendant gravitons: infinitely many of them are allowed hairs

$$O_{\mathrm{BH}}(\partial_{+\dot{+}})^{j_1}(\partial_{+\dot{-}})^{j_2}(\text{primary graviton})$$

Gravity dual interpretation: Consider a probe BPS particle around BPS black hole.

- The orbit stays outside event horizon if $j_1 + j_2 >$ (threshold set by BH size).
- We interpret that product cohomologies represent hairy BH microstates.
- "Gray galaxy" type hairy black holes have this type of graviton hairs at large j.

[SK, Kundu, E. Lee, J. Lee, Minwalla, Patel] (2023) [Bajaj, Kumar, Minwalla, Mukherjee, Rahaman] (2024)

- For a given O_{BH}, if one can compute the minimal value of j's which makes the product nontrivial, one would be able to probe the "size" of the BH state from QFT.

Note: Anomalous dimensions of $O_1 \partial^j O_2$ are generally suppressed as $\sim \frac{A}{j^{\#}}$ for large j.

[Alday, Maldacena] (2007) [Komargodski, Zhiboedov] (2012)

Dual giant hairs & fortuity

Do all hairy BH cohomologies admit product representatives? The answer is no.

- D3-branes wrapping AdS_5 black holes (called "dual giant gravitons") provide important charged hairs. [Choi, Jain, SK, Krishna, E. Lee, Minwalla, Patel] (2024)
- We consider the microstates of their BPS limit.

"Dual giant physics + fortuity of BH cohomologies" forbid product representatives.

- Dual giant reduces RR 5-form flux inside, making the objects inside described morally by the $SU(N-1)$ theory: core BH microstate ~ $SU(N-1)$ cohomology?
- However, if one promotes a BH cohomology O_{N-1} of SU(N-1) theory to an operator of the same shape in the SU(N) theory, it generally fails to be Q-closed due to fortuity.

$$Q\left[(O_{N-1} \text{ promoted to } SU(N))(\text{dual giant operator})\right] \neq 0$$

- It turns out, morally keeping the "shape" information of O_{N-1}, the two operators are suitably "fused" (gauge orientations entangled) for the product to be Q-closed.

Cohomologies with dual giant hairs

Illustration at low N:

- $SU(3)$ cohomologies representing a "dual giant" operator (scalar primary) wrapping the $SU(2)$ core BH microstate $O_0 \equiv \epsilon^{abc}(v_2)^m{}_a(v_2)^n{}_b\mathrm{tr}(\psi_{(c}\psi_m\psi_{n)})$
- Again used ansatz: O_0 wrapped by a 3-scalar dual giant (r's defined by $R_a = Qr_a$)

$$
\begin{aligned}
(\mathcal{O}^3_{30})^{ijk} = \; & 24\epsilon^{a_1a_2(i}\epsilon^{j|b_1b_2}v^c{}_dv^{|k)}{}_{a_1}v^d{}_{b_1}(r^{(0,2)}_{12})_{a_2b_2c} - \frac{16}{15}\epsilon^{a_1a_2(i}v^{j|}{}_bv^b{}_cv^c{}_{a_1}(r^{(0,2)}_{12})^{|k)}_{a_2} \\
& + \frac{4}{15}\epsilon^{a_1a_2a_3}v^b{}_{a_1}v^{(i}{}_{a_2}v^j{}_{a_3}(r^{(0,2)}_{12})^{k)}_b - \frac{32}{15}\epsilon^{a_1a_2(i}v^j{}_bv^{k)}{}_{a_1}v^b{}_c(r^{(0,2)}_{12})^c_{a_2} + 16\epsilon^{a_1a_2(i}v^{j|}{}_dv^b{}_{a_1}v^d{}_c(r^{(0,2)}_{12})^{|k)c}_{a_2b} \\
& + 85\epsilon^{a_1a_2(i|}u^{bc}v^d{}_{a_1}v^{|j}{}_b(r^{(0,3)}_{14})^{k)}_{a_2cd} + 5\epsilon^{a_1a_2(i|}u^{bc}v^d{}_bv^{|j}{}_{a_1}(r^{(0,3)}_{14})^{k)}_{a_2cd} - 40\epsilon^{a_1a_2(i}u^{j|b}v^c{}_{a_1}v^d{}_b(r^{(0,3)}_{14})^{|k)}_{a_2cd} \\
& - \frac{5}{3}\epsilon^{a_1a_2(i|}u^{cd}v^b{}_{a_1}v^{|j}{}_b(r^{(0,3)}_{14})^{k)}_{a_2cd} + \frac{5}{3}\epsilon^{a_1a_2(i}u^{j|c}v^b{}_{a_1}v^d{}_b(r^{(0,3)}_{14})^{|k)}_{a_2cd} - 40\epsilon^{a_1a_2(i}u^{j|b}v^c{}_{a_1}v^{|k)}{}_d(r^{(0,3)}_{14})^d_{a_2bc} \\
& + \frac{125}{7}\epsilon^{a_1a_2(i}v^{j|b}{}_{a_1}v^{cd}{}_b(r^{(0,3)}_{14})^{|k)}_{a_2cd} + 31\epsilon^{a_1a_2(i}v^{bc}{}_{a_1}v^{d|j}{}_b(r^{(0,3)}_{14})^{k)}_{a_2cd} + \frac{24}{7}\epsilon^{a_1a_2(i}v^{jk)}{}_bv^{cd}{}_{a_1}(r^{(0,3)}_{14})^b_{a_2cd} \\
& + \frac{543}{7}\epsilon^{a_1a_2a_3}v^{b(i}{}_{a_1}v^{j|c}{}_{a_2}(r^{(0,3)}_{14})^{|k)}_{a_3bc} + \frac{8}{5}\epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{|jk)}{}_b(r^{(0,2)}_{16})_{a_2} + \frac{16}{15}\epsilon^{a_1a_2(i}v^j{}_{a_1}v^{k)b}{}_{a_2}(r^{(0,3)}_{16})_b \\
& + \frac{251}{7}\epsilon^{a_1a_2a_3}v^{b_1}{}_{a_1}v^{b_2(i}{}_{a_2}(r^{(0,3)}_{16})^{jk)}_{a_3b_1b_2} - \epsilon^{a_1a_2a_3}v^{(i|}{}_{a_1}v^{b_1b_2}{}_{a_2}(r^{(0,3)}_{16})^{|jk)}_{a_3b_1b_2} - 9\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{bc}{}_d(r^{(0,3)}_{16})^{d|k)}_{a_2bc} \\
& + \frac{174}{7}\epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{c|j}{}_d(r^{(0,3)}_{16})^{k)d}_{a_2bc} - \frac{17}{2}\epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{cd}{}_{a_2}(r^{(0,3)}_{16})^{|jk)}_{bcd}
\end{aligned}
$$

E.g. contains terms of the form
$tr(\phi\phi\psi)tr(\phi\phi\psi)tr(\phi\psi\psi\psi)$,
$tr(\phi\psi)tr(\phi\phi\psi)tr(\phi\phi\psi\psi\psi)$,
...

- How to see a BH wrapped by a dual giant? Don't know the general rule, but…

  Restrict 3 x 3 matrices to SU(2) x U(1) blocks: ∃ term of the form $O_0(SU(2))\ \phi(U(1))^3$

  This term actually obstructs the full SU(3) operator from being Q-exact.

- Similar cohomologies constructed for O_0 wrapped by dual giants made of $n \geq 3$ scalars
- "$n = 1{,}2$ branes" are too small to wrap BH: probes the size of the core BH operator

Concluding remarks

Constructions of BPS BH microstates in limited sense: weak-coupling, mostly at low N.

Ansatz: trace relations & relations of relations → some black hole cohomologies

- More trace relations, beyond those from gravitons? (E.g. hierarchical constructions)

Physics of BPS black hole microstates?

- Aspects of BH: complexity, BPS chaos [Chen, Lin, Shenker], … → more studies needed
- Computational complication at large N & charges: Simpler toy models?

  (SYK [Chang, Chen, Sia, Yang]; 3d vector models ↔ AdS4 higher spin gravity, …)

Hairy black hole microstates:

- Product representatives for hairs with large angular momenta
- Fused (orientation-entangled) brane hairs w/ charges: subtleties of fortuity/trace relations

These cohomologies are intrinsically new observables in SUSY QFT in general:

- Beyond chiral rings, barons/mesons in SQCD, … → "semi-chiral rings" [Budzik, Gaiotto, …]