Chaos in deformed SYK models

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"Recent Developments in Black Holes and Quantum Gravity"

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Motivation

cond-matt {
- * Better understand quantum many body systems
- * signatures of quantum chaos (Lyapunov, Krylov)

QG {
- * Microscopic models of spacetime
- * potential for holography in dS

The SYK model (and its deformations)

$$H_q = i^{q/2} \sum J_{i_1 \ldots i_q} \underbrace{\psi_{i_1} \psi_{i_2} \ldots \psi_{i_q}}$$

$J_{i_1 \ldots i_q}$ → random

$\{\psi_i, \psi_j\} = \delta_{ij}$

↑ N majorana fermions

$$\langle J^2_{i_1 \ldots i_q} \rangle = \frac{2^{q-1}}{q} \frac{\mathcal{J}^2 (q-1)!}{N^{q-1}}$$

($q=2$ $H \sim \sum J \psi \psi$ integrable!)

- at low temp - max chaotic
- used to model strange matals

properties of the syk model

* when $N \to \infty$ effective description in terms of bilinear fields $G(\tau_1,\tau_2) = \frac{1}{N}\sum_{i=1}^{N} \langle T\psi_i(\tau_1)\psi_i(\tau_2)\rangle$; $\Sigma(\tau_1,\tau_2) \equiv$ self energy

$$\langle Z_{syk}[\beta]\rangle = \int DG\, D\Sigma\, e^{-N S_{eff}[G,\Sigma]}$$

$$S_{eff} = -\tfrac{1}{2}\log\det(\partial_\tau - \Sigma) + \tfrac{1}{2}\int_0^\beta\int_0^\beta d\tau_1 d\tau_2 \left(\Sigma G - J^2 \frac{2^{q-1}}{q^2} G^q\right)$$

(EOM – Schwinger–Dyson)

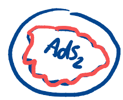
* $q \to \infty$ large $\Rightarrow$ further simplification. $[\lambda = \frac{q^2}{N}$ fixed $\lambda \to 0]$ DSSYK
* at low temp syk nearly conformal $\Delta_\psi = \frac{1}{q}$
* $S_{eff} = S_{CFT} + S_{Schwartzian} + \ldots$

Dual gravitational description

* JT gravity (+Dilaton)

$U(\phi) = 2\phi \to$ Dilaton potential

$$I = \# - \frac{1}{16\pi G}\int_{\mathcal{M}} d^2x\, \phi\sqrt{g}\,(R+2) + 2\int_{\partial\mathcal{M}} \phi_b K$$



* AdS_2 + dynamics which reduces to the boundary
* Maximal chaos $\lambda_L = \frac{2\pi}{\beta}$
* Can deformations of syk lead to different gravity backgrounds (different IR)?

Deformations of the SYK model

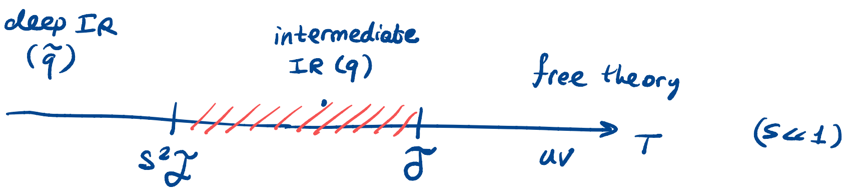
* Relevant deformations of SYK (important in IR)

* $H_q = \# \sum J_{i_1 \dots i_q} \psi_{i_1} \dots \psi_{i_q} \to \Delta_\psi = 1/q$

* e.g.: $\psi_i \, \partial_\tau^{2n+1} \psi_i \to 2n+1 + \frac{2}{q} > 1$ (irrelevant)

* $H_{def} = H_q + s H_{\tilde q} \qquad s > 0 \qquad q \geq \tilde q$

naive scaling dimension $\Delta = \frac{\tilde q}{q}$ relevant IR



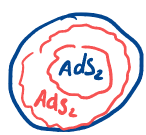
* large N and large q techniques generalize.
* effective theory $N \to \infty$ $\quad G(\tau_1, \tau_2) = \frac{1}{N} \sum_{i=1}^{N} \langle T \psi_i(\tau_1) \psi_i(\tau_2) \rangle$; $\Sigma(\tau_1, \tau_2) \equiv$ self energy

integro-differential system
$$\begin{cases} G^{-1}(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2) \partial_{\tau_2} - \Sigma(\tau_1, \tau_2) \\ \Sigma(\tau_1, \tau_2) = \mathcal{J}^2 \left(\frac{2^{q-1}}{q} G(\tau_1, \tau_2)^{q-1} + s^2 \frac{2^{\tilde q - 1}}{\tilde q} G(\tau_1, \tau_2)^{\tilde q - 1} \right) \end{cases}$$

* $N \to \infty \quad q \to \infty \quad (q, \tilde q \to \infty \,;\, n \equiv \frac{q}{\tilde q}$ fixed$) \quad \lambda = \frac{q^2}{N} \to 0$

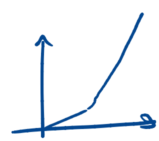
$$G(\tau) = \frac{\operatorname{sgn}(\tau)}{2} \left(1 + \frac{g(\tau)}{q} + \dots \right)$$

$$\partial_\tau^2 g(\tau) = 2 \mathcal{J}^2 e^{g(\tau)} + 2 n s^2 \mathcal{J}^2 e^{g(\tau)/n}$$



closed form for the 2pt function!

* Dual description $\qquad U(\phi) = \begin{cases} 2\phi & \phi \ll \phi_0 \\ \# + \alpha \phi & \phi \gg \phi_0 \end{cases}$



[intermediate IR $\quad \frac{S_{thermal}}{N} = \# + \frac{\pi^2}{q^2}\frac{1}{\beta J}$ (relation to dS ?)

deep IR $\quad \frac{S_{thermal}}{N} = \# + \frac{\sqrt{1+4s^2}}{2s^2}\frac{\pi^2}{q^2}\frac{1}{\beta J}$ } $s \to is$

$\mathcal{H} = \mathcal{H}_q + is\,\mathcal{H}_{\tilde{q}} \Rightarrow$???

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signatures of quantum chaos

* classical chaos - hypersensitivity on initial conditions

$$\frac{\partial q(t)}{\partial q(0)} = \{q(t), p(0)\} \sim e^{\lambda t} \leftarrow \text{Lyapunov exponent}$$

* If semiclassical limit exists $\frac{1}{i\hbar}[q(t), p(0)]$
* proposed generalization $\quad c(t) = -\langle [W(t), V(0)]^2 \rangle_\beta$

Lyapunov Exponent

* $OTOC(t) = \frac{1}{N^2}\sum_{i,j=1}^{N} Tr\left(\rho^{\frac{1}{4}}\psi_i(t)\rho^{1/4}\psi_j(0)\rho^{1/4}\psi_i(t)\rho^{1/4}\psi_j(0)\right)$

$$\rho = \frac{1}{Z(\beta)}e^{-\beta H}$$

$$OTOC = f_0 - \frac{f_1}{N}\exp(\lambda_L t)$$

* $\lambda_L \leq \frac{2\pi}{\beta}$ (good only at large N)

krylov exponent

* $O(t) = \underbrace{e^{iHt}\, O\, e^{-iHt}}$ (how fast does this spread?)

Span $\{O, [H,O], [H[H,O]] \ldots\} \xrightarrow{\text{krylov basis}} \{O_1, O_2, O_3 \ldots\}$

$O(t) = \sum \varphi_n(t) O_n$ $\qquad$ $\langle n \rangle_t = \sum |\varphi_n(t)|^2 \cdot n \sim e^{\lambda_K \cdot t}$

exponential in quantum chaotic

* krylov / lyapunov exponents - different measures of chaos

$$\boxed{\lambda_L \leq \lambda_K \leq \frac{2\pi}{\beta}}$$

tight

* in certain cases $\lambda_L = \lambda_K$ same

Questions

Q1: what are λ_L and λ_K in deformed syk models (✓)

Q2: are they equal? (not equal!)

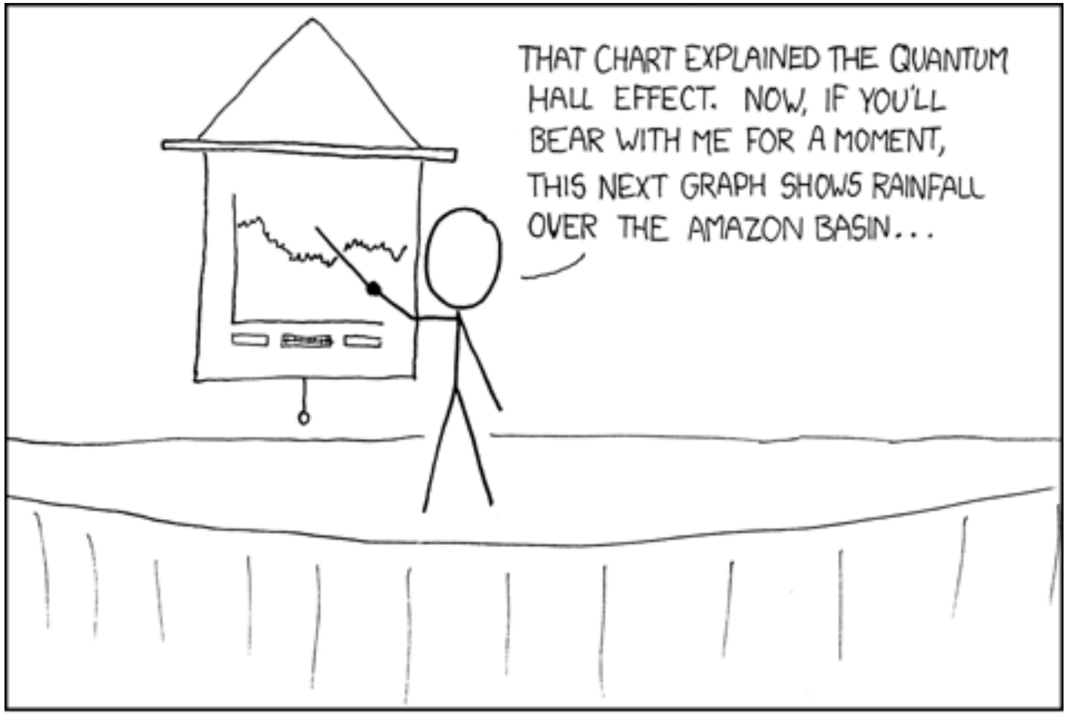
Q3: can these exponents diagnose chaos transitions between two syk Hamiltonian. (only Lyapunov!)

$N \to \infty$ (large q, finite q)

Q:

* Relation of flows in grav and syk - beyond thermo flows (e.g. Dil pot?) ⇒ in last paper of Damian/sam/Dio some match...

Results


THAT CHART EXPLAINED THE QUANTUM HALL EFFECT. NOW, IF YOU'LL BEAR WITH ME FOR A MOMENT, THIS NEXT GRAPH SHOWS RAINFALL OVER THE AMAZON BASIN...

Questions to answer

- What are λ_L and λ_K in deformed SYK models?
- Are they equal?
- Can they both diagnose chaos/chaos transitions?

Useful dificulty map

- **Finite *N*,** large matrices - hard
- **Large *N***, integro-differential equations
- **Large *N*, large *q*** – ODE – analytic solution

Hard

Easy(er)

Large q expansion – a single SYK

- An expansion to the next-to-leading order in 1/q can be performed, at large temperatures:

$$\lambda_K = \frac{2\pi}{\beta}\left(1 - \left(2 - \frac{7\pi^2+12}{9q}\right)\frac{1}{\beta\mathcal{J}} + \cdots\right) \qquad \lambda_L = \frac{2\pi}{\beta}\left(1 - \left(2 + \frac{5\pi^2-12}{9q}\right)\frac{1}{\beta\mathcal{J}} + \cdots\right)$$

- Already at this level the exponents do not exactly coincide!

$$\frac{\beta}{2\pi}(\lambda_K - \lambda_L) = \frac{4\pi^2}{3q\beta\mathcal{J}} + \cdots .$$

Large N and finite q – Numeric strategy

Outline of the numerical procedure:

- Rewrite the the Schwinger-Dyson equation in Fourier space

$$G^{-1}(\tau_1,\tau_2) = \delta(\tau_1-\tau_2)\partial_{\tau_2} - \Sigma(\tau_1,\tau_2)\ ,$$
$$\Sigma(\tau_1,\tau_2) = \mathcal{J}^2\left(\frac{2^{q-1}}{q}G(\tau_1,\tau_2)^{q-1} + s^2\frac{2^{\tilde{q}-1}}{\tilde{q}}G(\tau_1,\tau_2)^{\tilde{q}-1}\right)\ .$$

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega)\ ,$$

$$G(\tau+\beta/2) = \int d\omega\ e^{-\omega(\tau+\frac{\beta}{2})}\frac{\rho(\omega)}{1+e^{-\beta\omega}}\ .$$

- Re-express them in terms of the spectral density
- Solve them with an iterative procedure in Fourier space to obtain the fermion 2pt function and various analytic continuations in Lorentzian time.
- From the moments of the 2pt function we can already obtain the Krylov exponent.

Large N and finite q – Numeric strategy

- Solve 4pt function from Kernel equation:

$$K\left(t_1,t_2,t_3,t_4\right)=G^R(t_{13})G^R(t_{24})\mathcal{J}^2\left(\frac{2^{q-1}}{q}(q-1)G^W(t_{34})^{q-2}+s^2\frac{2^{\tilde{q}-1}}{\tilde{q}}(\tilde{q}-1)G^W(t_{34})^{\tilde{q}-2}\right)$$

$$F\left(t_1,t_2\right)=\int dt_3dt_4\,K\left(t_1,t_2,t_3,t_4\right)F\left(t_3,t_4\right)$$

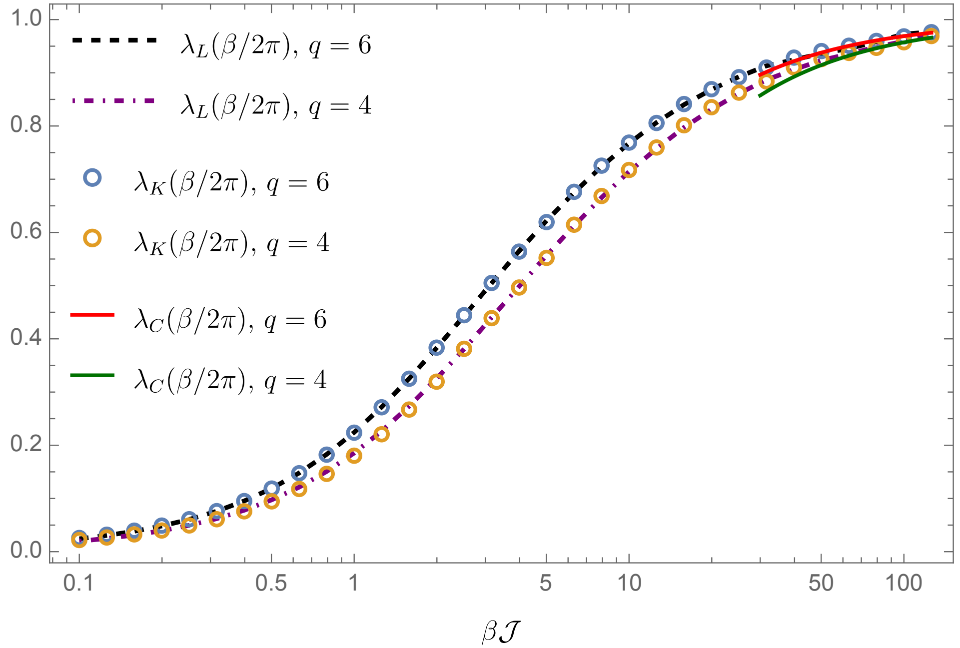
$$\mathrm{OTOC}(t_1,t_2)=F_0(t_1,t_2)+\frac{1}{N}F(t_1,t_2)+\cdots\ .$$

- Discretize and treat as matrix diagonalization
- Numerical difficulties associated with low temperatures (many coding hours of our talented students & postdocs)

Large N Finite q - single SYK

q = 4/6 of fermions in each interaction term

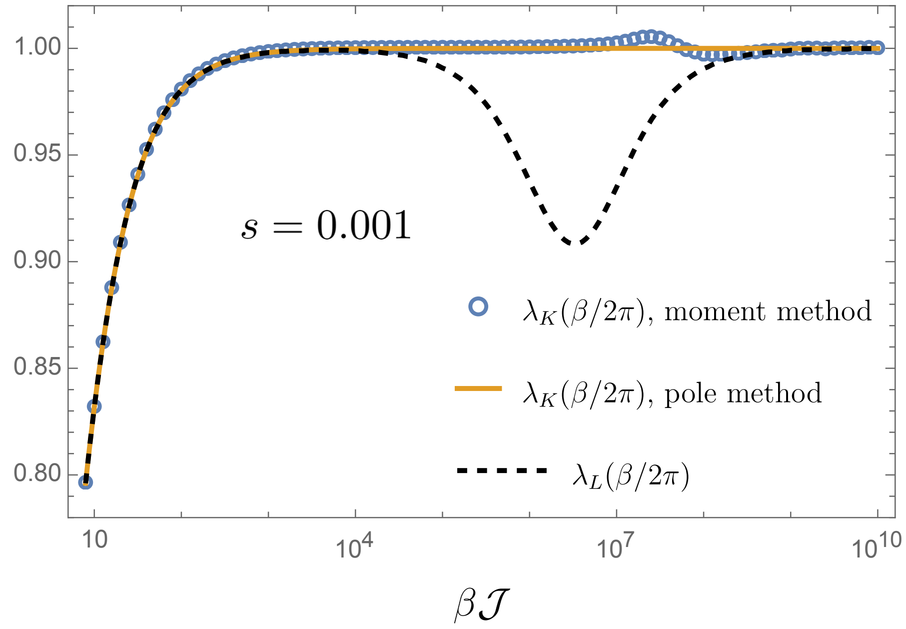
- Maximal chaos at low temperatures.
- Krylov – good approximation to the Lyapunov exponent.
- But not exactly equal (differences fall within numerical accuracy, but we already know from expansion)



[**SC,** Demulder, Galante, Sheorey, Shoval (2024)]

Flows - Chaos to Chaos $H_{\text{def}} = H_q + sH_{q/2}, \quad q \to \infty$

- **Two regimes of nearly maximal chaos.**
- Chaos transition at $\beta\mathcal{J} \sim 1/s^2$
- Krylov complexity correctly (but poorly!) bounds (above) the Lyapunov exponent.
- Krylov does **not** diagnose chaos transitions!
- Analytic expansions near maximally chaotic regimes available.

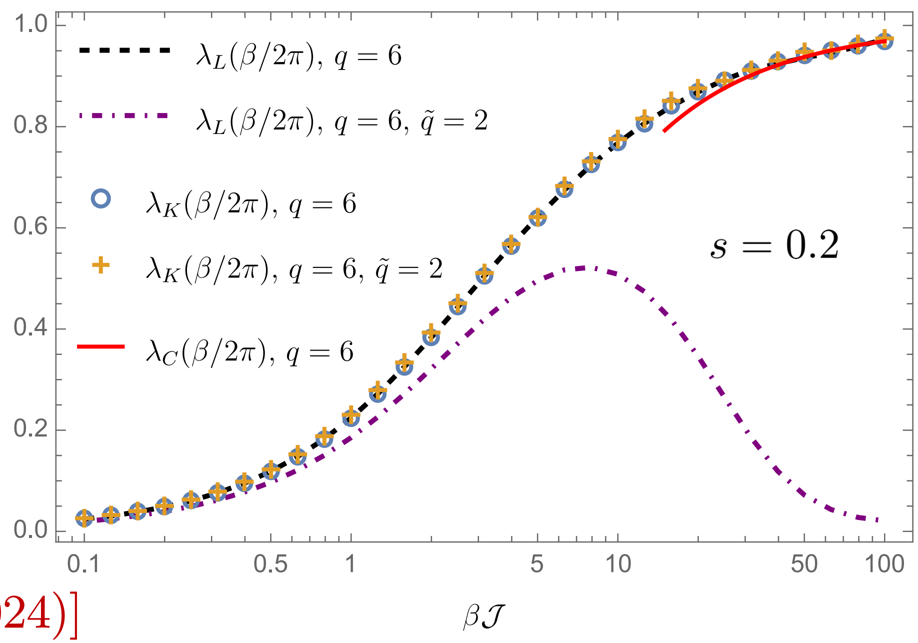


[**SC,** Demulder, Galante, Sheorey, Shoval (2024)]

Flows - Chaos to integrable

$$H_{\text{def}} = H_6 + sH_2$$

- Lyapunov exponent sees the transition
- Krylov seems unable to diagnose the chaotic properties of the flows

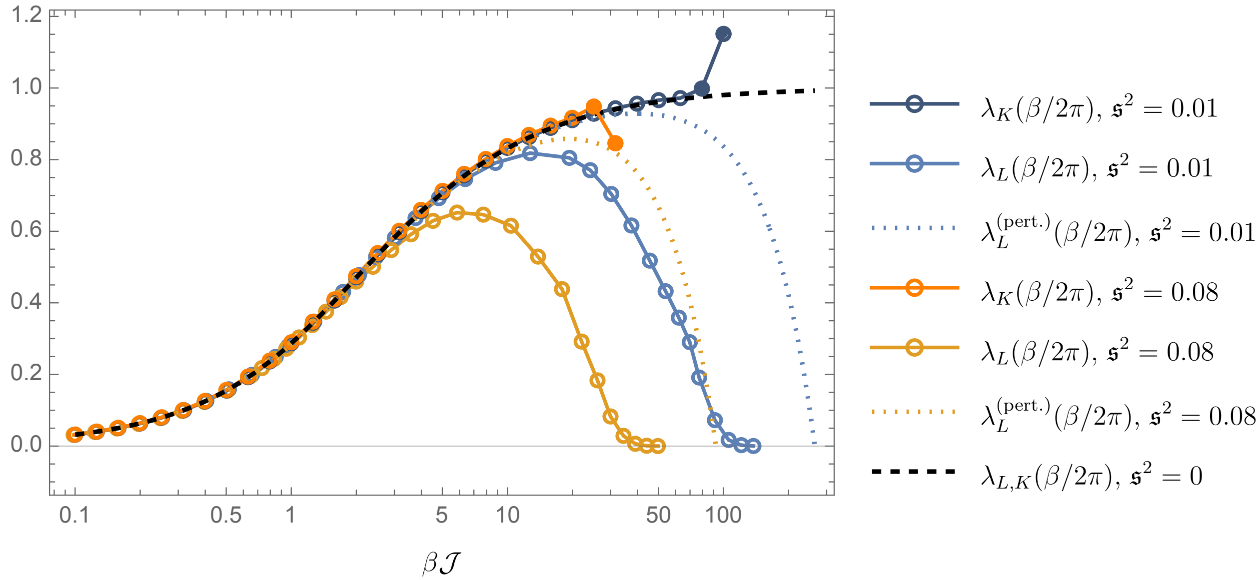


[**SC,** Demulder, Galante, Sheorey, Shoval (2024)]

Other studies of chaos-integrable transitions: [Berkooz, Brukner, Jia, Mamroud, 2024; Garcia-Garcia, Loureiro, Romero-Bermudez, Tezuka, 2018; Kim, Cao, 2021; Lunkin, Kitaev, Feigel'man, 2020; Nandy, Cadez, Dietz, Andreanov, Rosa 2022; Menzler, Jha 2024…]

Chaotic to integrable phase transitions?

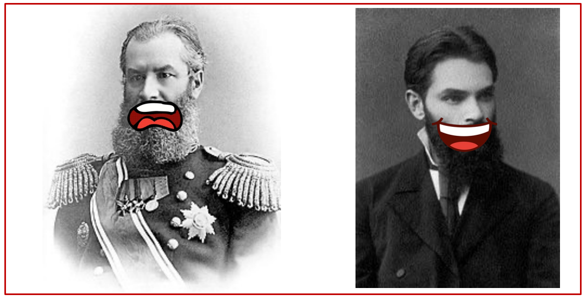
- One case where we could see hint of phase transition in Krylov.
- Linear slope of b_n stopped.
- Also, sharp phase transition of Lyapunov.
- At the same temperature.



$$H_{\text{def}} = H_\infty + sH_2$$

Recap of results

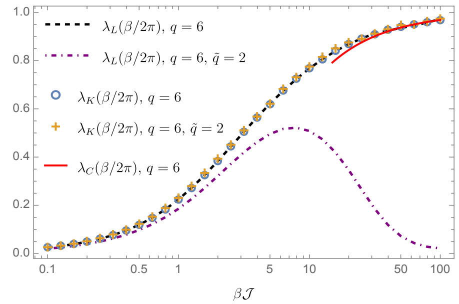
- We saw deviations between the two exponents already for the single SYK.
- For most cases of flow SYK Krylov did not provide a tool for diagnosic chaos transitions along RG flows



*Note that the late time volume in flow geometries **also** does not seem to see the flow [Rabinovici, Sanchez-Garrido, Shir, Sonner 2023; **SC**, Galante, Kramer 2022]

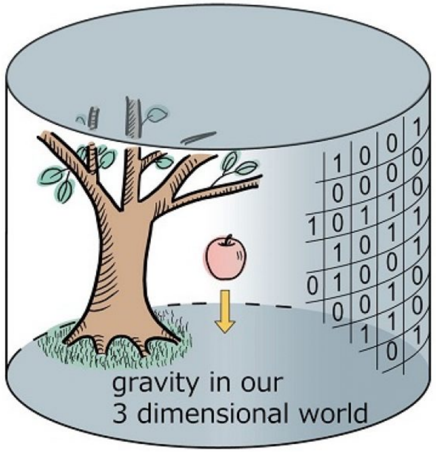
- In all examples, the Krylov exponent grew monotonically? (when it can be defined)
- Conjecture for Hermitian systems:

$$\beta\partial_\beta\left(\frac{\lambda_K\beta}{2\pi}\right)\geq 0$$



Potential for de Sitter holography?

- Holography is great, it helps us reinterpret conceptually hard questions about quantum gravity.
- We live in an expanding universe.
- A holographic approach could be very useful to understand quantum gravity in the expanding universe
- dS has a horizon and an entropy, what is it counting?
- But challenging – dS has no timelike boundary.
- Many attempts and approaches:
  - dS-CFT/Stretched horizon holography/ $\bar{T}T + \Lambda_2$/DSSYK-dS_3 ...



Can flow SYK help?

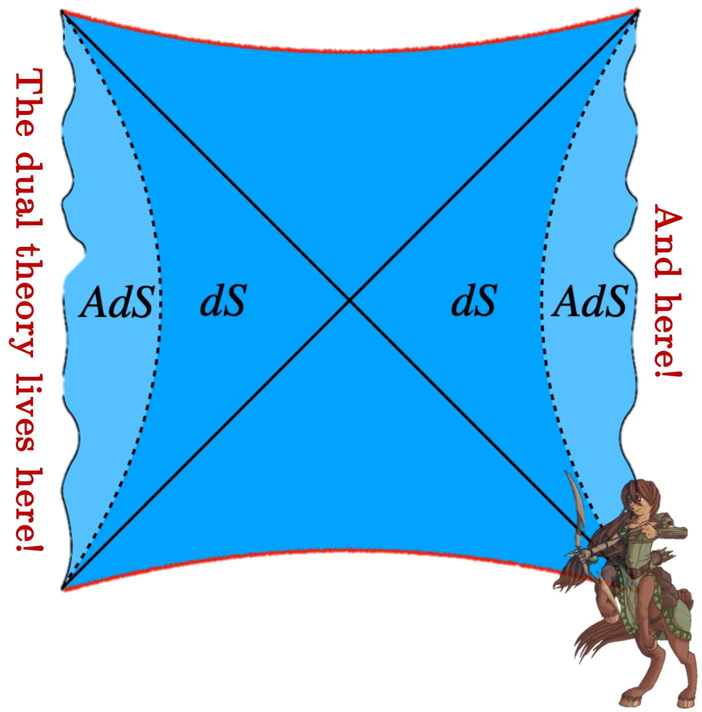
- Deformed SYK – dual IR deformations of nearly AdS_2 spacetimes.
- **Q1:** Can we put dS in the IR of AdS_2 ?
- **Q2:** If yes, can deformations of SYK be dual to this configuration?
- **Q1– answer:** In general dimensions – unclear, some no go theorems. Based on the idea that null geodesics leaving the AdS boundary cannot converge and then diverge. [Freivogel, Hubeny, Maloney, Myers, Rangamani, Shenker, '05]
- But in 2d there is no direction in which to converge and then diverge.

dS interpolating geometries in 2d

- Can embed a portion of dS inside AdS Space – **interpolating geometries** [Anninos, Hofman (2017)]
- JT gravity with Dilaton potential that changes sign

$$U(\phi) = \begin{cases} 2\phi & \phi \gg \epsilon \\ -2\phi & \phi \ll -\epsilon \end{cases}$$

- Now there is a timelike boundary where the dul theory can live!
- Can this be dual to some flow SYK?



Towards de Sitter Holography?

- Which SYK RG flow could lead to a dS interior? (hints from gravity behavior)
- dS flows has non-standard chaotic behavior (for instance the OTOC oscilates) [Anninos, Galante, Hofman 2019; **SC**, Galante, Kramer 2022, …]
- Thermodynamic behavior matches flow SYK when analytically continued to complex couplings [Anninos, Galante, Sheorey (2022)] $H_{\mathrm{def}} = H_q + sH_{\tilde{q}} \Rightarrow H_q + isH_{\tilde{q}}$
- **What does this really describe?** Non Hermitian Hamiltonian
- **Maybe dS is dual to an open quantum system**
  - Open boundary at future infinity
  - Perturbations make contact with a larger part of the system.
- Signatures of chaos in open quantum systems [Sá, Ribeiro, Prosen 2021; Bhattacharya, Nandy, Nath and Sahu, 2022; Liu, Tang , Zhai 2023; Bhattacharjee, Cao, Nandy, Pathak 2023; Srivatsa, Keyserlingk 2024…]
- Interesting to explore connection to dS!
- Explore relation to other propsals DSSYK/dS$_3$? [Narovlansky, Verlinde]

Lots to explore!


Thank you!
Any questions?