One-dimensional force-free numerical simulations of Alfven waves around a spinning black hole

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Motivation of study of Alfven wave along magnetic field line around Kerr black hole

- Noda, Nanbu, Tsukamoto, Takahashi, PRD, 105, 064018 (2022): Superradiance with Alfven wave with calculation region between two light surface
  ⇐ Verification with numerical simulation → △
  ⇐ Calculation region from horizon to infinity → ○

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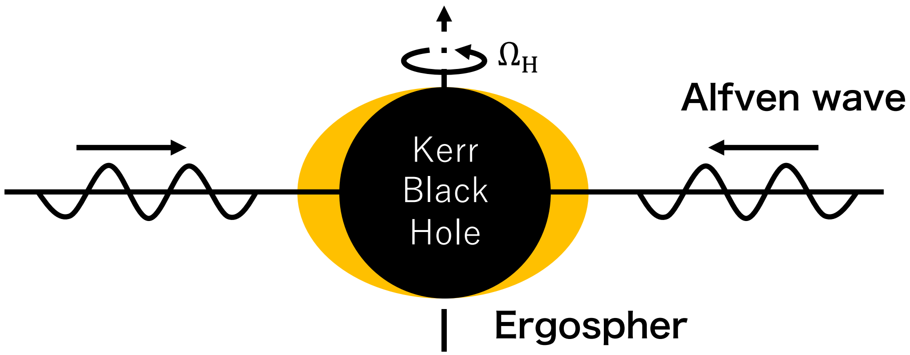
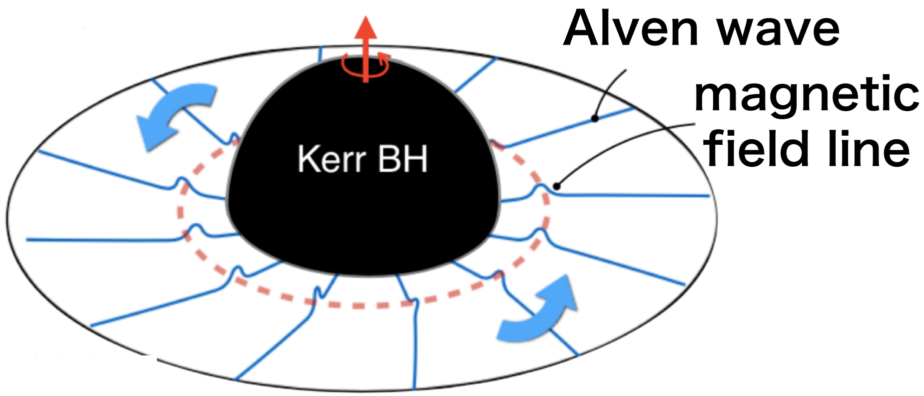
- Peculiar phenomena of Alfven wave around Kerr black hole → ◎



Common aspects of Noda, et al. (2022) and this study

Alfven wave along field line of force-free field around Kerr black hole



(Noda et al. 2022)

Natural unit system：

$G = c = 1, M_{\rm BH} = 1,$

$\epsilon_0 = \mu_0 = 1$

Boyer-Lindquist coordinates： $x^{\underline{\mu}} = (t, r, \theta, \varphi)$

$$ds^2 = g_{\underline{\mu}\underline{\nu}} dx^{\underline{\mu}}\, dx^{\underline{\nu}} = -\alpha^2\, dt^2 + \frac{\Sigma}{\Delta}\, dr^2 + \Sigma\, d\theta^2 + R^2\,(d\,\varphi - \Omega\, dt)^2$$

$$\alpha^2 = \frac{\Sigma\Delta}{\mathrm{A}},\ \ \Omega = \frac{2\,M\,ar}{A},\ \Delta = r^2 - 2Mr + a^2,\ \ \Sigma = r^2 + a^2\cos^2\theta,\ \ A = (r^2 + a^2)^2 - a^2\,\Delta\,\sin^2\theta$$

$\left(a_* = \frac{a}{M}\right.$ **: Nondimensional spin parameter of BH** $\left.\right)$

Electromagntic field tensor

- **Maxwell equation：**

$$\nabla_\lambda F^{\lambda\mu} = \frac{1}{\sqrt{-g}}\partial_\lambda\left(\sqrt{-g}F^{\lambda\mu}\right) = -J^\mu$$

$$\nabla_\lambda {}^*F^{\mu\nu} = 0、\quad {}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

- **Force-free condition：** $J^\mu F_{\mu\nu} = 0$

Force-Free Magnetodynamics (FFMD)

Field tensor of force-free field can be expressed by Euler potential: $F_{\mu\nu} = \partial_\mu \phi_1 \partial_\nu \phi_2 - \partial_\nu \phi_1 \partial_\mu \phi_2$

Maxwell equation (inhomogeneous) + Force-free condition

Maxwell equation + force-free condition

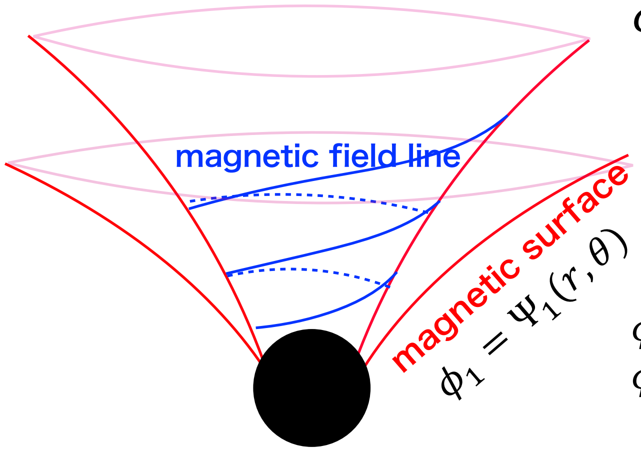
$$\partial_\lambda \phi_i \partial_\nu [\sqrt{-g} W^{\lambda\alpha\nu\beta} \partial_\alpha \phi_1 \partial_\beta \phi_2] = 0 \quad (i = 1,2)$$

$$W^{\lambda\alpha\nu\beta} = g^{\lambda\alpha} g^{\nu\beta} - g^{\lambda\beta} g^{\alpha\nu}$$

Solution of the stationary magnetic field around a spinning black hole

$$\bar{\phi}_1 = \Psi_1(r,\theta), \quad \bar{\phi}_2 = \varphi - \Omega_{\mathrm{F}}(\Psi_1) t + \Psi_2(r,\theta)$$

Ω_{F} : angular velocity of magnetic field line



To obtain global magnetosphere in the Kerr spacetime is very difficult (Grad-Shafranov eq.)

ϕ_1 = const. defines magnetic surface
ϕ_2 = const. magnetic field line on a magnetic surface

Solution of the stationary magnetic field on the equatorial plane around a spinning black hole

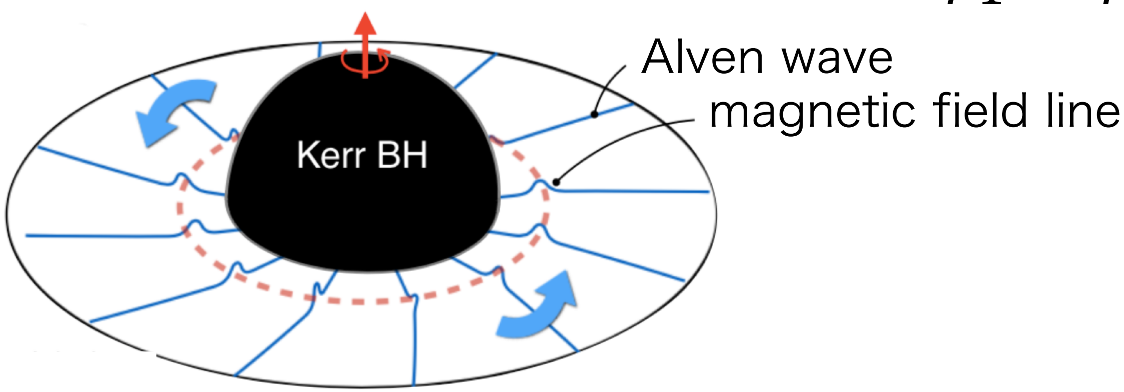
$$\bar{\phi}_1 = \cos\theta, \quad \bar{\phi}_2 = \varphi - \Omega_F t + \int \kappa dr, \quad \text{where } \kappa = \frac{I X^2}{\Delta}$$

(Constants) Ω_F : **angular velocity of magnetic field line**、

I : **current**

Alfven wave

Perturbations of Alfven wave : $\phi_1 = \bar{\phi}_1 + \delta\phi_1, \ \phi_2 = \bar{\phi}_2$



Brief review of Noda, et al. (2022)

- Corotating natural coordinates, $(T',\ X',\ \Psi',\rho')$

$$T' = t + \int \iota dr = T + \int \iota dr, \qquad X' = r = X,$$
$$\Psi' = -\cos\theta = \Psi, \qquad \rho' = \varphi + \int \kappa dr - \Omega_{\mathrm{F}} T = \rho - \Omega_{\mathrm{F}} T,$$

Where $\iota = \frac{\mathrm{I}\Sigma \mathrm{R}^2}{\Gamma\Delta}(\Omega - \Omega_{\mathrm{F}})$, $\Gamma = -\alpha^2 + \mathrm{R}^2(\Omega - \Omega_{\mathrm{F}})^2$. $\quad \Gamma = 0 \quad \rightarrow$ Light surface

- $\partial_\lambda \phi_i \partial_\nu [\sqrt{-g} W^{\lambda\alpha\nu\beta} \partial_\alpha \phi_1 \partial_\beta \phi_2] = 0 \quad (i = 1{,}2) \rightarrow \qquad r = r_{\mathrm{in}}, r_{\mathrm{out}}$

$$\left[-\frac{X^2}{\Delta}H + \frac{J_B^2 X^4 g_{\varphi\varphi}^2}{\Gamma\Delta^2}(\Omega - \Omega_F)^2\right]\partial_T^2 \delta\phi_1 + \partial_X(-\Gamma\partial_X \delta\phi_1) - |\partial\phi_2|^2 \delta\phi_1 = 0.$$

Corotating natural coordinates has a singularity at light surface.

- Tortoise coordinate, x : $\quad \frac{dx}{dX'} = -\frac{1}{\Gamma}$

r	r_{in}	……	r_{out}
x	$-\infty$	……	$+\infty$

$$\delta\phi_1 = e^{-i\omega T} R(X)$$

$$\frac{d^2R}{dx^2} - V_{\mathrm{eff}} R = 0, \quad V_{\mathrm{eff}} = -\Gamma|\partial\phi_2|^2 + \frac{\omega^2 X^2}{\Delta}\left[H\Gamma - \frac{J_B^2 X^2 g_{\varphi\varphi}^2}{\Delta}(\Omega - \Omega_F)^2\right].$$

Noda et al. (2022)

Force-free & Alfvén wave

$$\partial_\lambda \phi_i \partial_\nu [\sqrt{-g} W^{\lambda\alpha\nu\beta} \partial_\alpha \phi_1 \partial_\beta \phi_2] = 0$$

Alfvén wave

$$\longrightarrow \quad \frac{d^2 R}{dx^2} - V_{\rm eff} R = 0$$

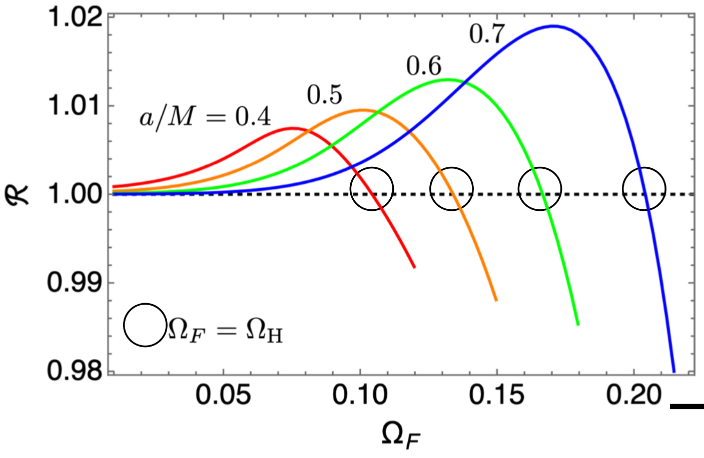
Asymptotic sols.

$$R = \begin{cases} \exp\left[-i\omega \int \frac{dx}{\Delta} X^2 g_{\varphi\varphi} |J_B| (\Omega_F - \Omega)\right] & \text{for} \quad x \to -\infty, \\ A_{\rm in} \exp\left[-i\omega \int \frac{dx}{\Delta} X^2 g_{\varphi\varphi} |J_B| (\Omega_F - \Omega)\right] + A_{\rm out} \exp\left[i\omega \int \frac{dx}{\Delta} X^2 g_{\varphi\varphi} |J_B| (\Omega_F - \Omega)\right] & \text{for} \quad x \to +\infty. \end{cases}$$

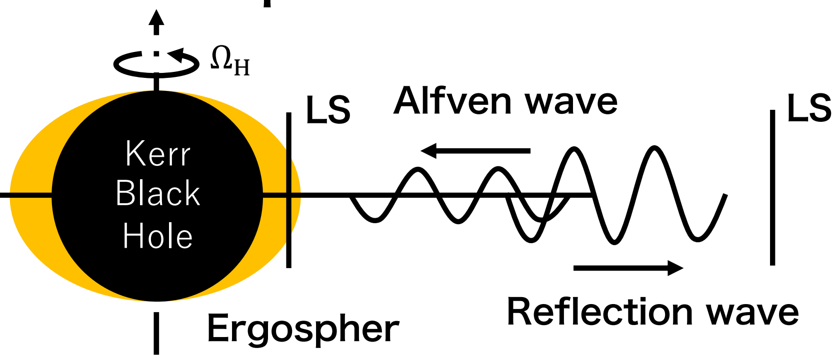
Conservation of Wronskian

$$\mathcal{R} = \left|\frac{A_{\rm out}}{A_{\rm in}}\right|^2 = 1 - \frac{f_{\rm in}}{f_{\rm out}} \frac{\Omega|_{r_{\rm in}} - \Omega_F}{\Omega|_{r_{\rm out}} - \Omega_F} \frac{1}{|A_{\rm in}|^2}$$

Alfvénic superradiance



Superradiance with Alfven wave



Motivation of study of Alfven wave along magnetic field line around Kerr black hole

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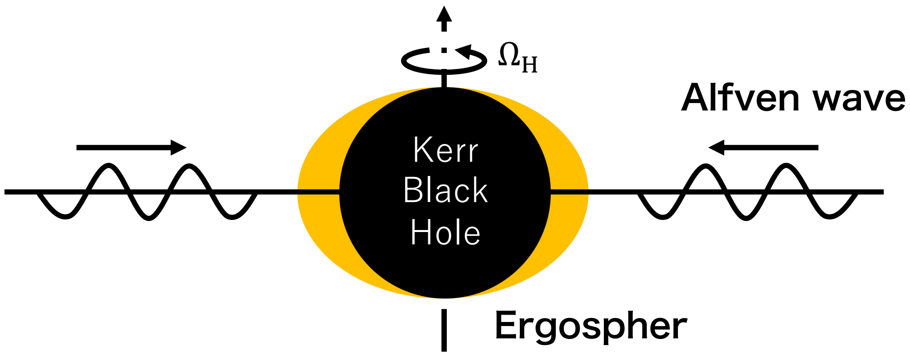
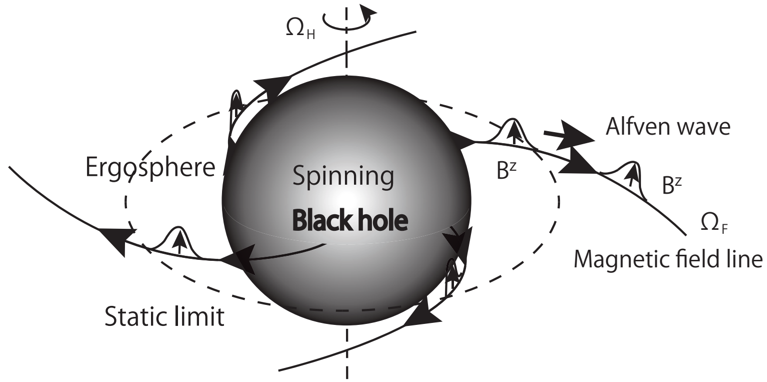


Background of numerical calculation of Alfven wave around Kerr black hole

- Nonrotating natural coordinates：$(t,\ X,\ \Psi, \rho)$

$$t = t, \qquad X = r,$$
$$\Psi = -\cos\theta, \qquad \rho = \varphi + \int \kappa dr,$$

where $\kappa = \frac{I\mathrm{X}^2}{\Delta}$.



- Metric of nonrotating natural coordinates:

$$g^{\mu\nu} = \begin{pmatrix} g^{tt} & 0 & 0 & g^{\varphi t} \\ 0 & g^{rr} & \frac{I'}{\mathrm{X}} & \kappa g^{rr} \\ 0 & 0 & g^{\theta\theta}\sin^2\theta & 0 \\ g^{\varphi t} & \kappa g^{rr} & 0 & g^{\varphi\varphi} + \kappa^2 g^{rr} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha^2} & 0 & 0 & -\frac{\Omega}{\alpha^2} \\ 0 & \frac{\Delta}{\Sigma} & 0 & \frac{IX^2}{\Sigma} \\ 0 & 0 & \frac{\sin^2\theta}{\Sigma} & 0 \\ -\frac{\Omega}{\alpha^2} & \frac{IX^2}{\Sigma} & 0 & \frac{\alpha^2 - R^2\Omega^2 + \frac{I^2\mathrm{X}^4}{\Sigma}\sin^2\theta}{\alpha^2 R^2} \end{pmatrix}$$

Background field and perturbational field of Alfven wave on nonrotating natural coordinates

- Force-free equation of Euler potentials with natural coordinates:

  $$\partial_\lambda \left(\sqrt{-g}\, W^{\lambda\alpha\mu\beta} \partial_\alpha \phi_1 \, \partial_\beta \phi_2\right) \partial_\mu \phi_i = 0 \quad (i = 1,2),$$

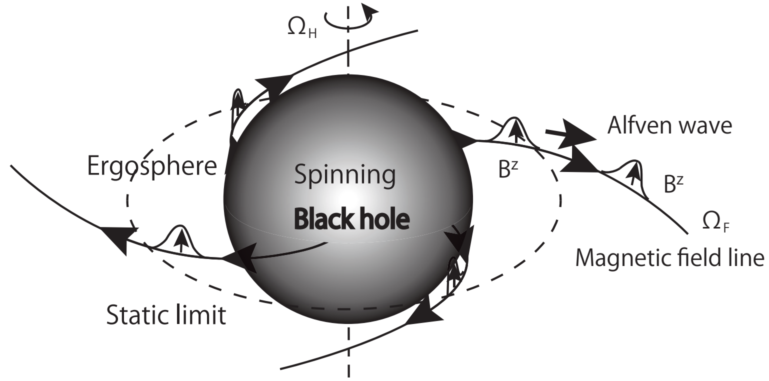
  where $W^{\lambda\alpha\mu\beta} \equiv g^{\lambda\alpha} g^{\mu\beta} - g^{\lambda\beta} g^{\mu\alpha}$.

- Stationary solution on equatorial plane around Kerr black hole:

  $$\phi_1 = \bar{\phi}_1 = \Psi, \; \phi_2 = \bar{\phi}_2 = \rho - \Omega_{\mathrm{F}} t$$

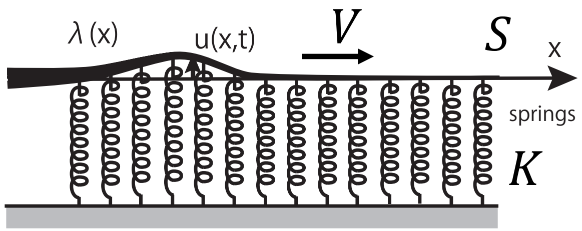
- Perturbation for Alfven wave:

  $$\phi_1 = \bar{\phi}_1 + \delta\phi_1 = \Psi + \psi,$$
  $$\phi_2 = \bar{\phi}_2 = \rho - \Omega_{\mathrm{F}} t$$



Linearized force-free equation of Alfven wave on nonrotating natural coordinates

Equation of peculiar string



- Force-free equation of $i = 1 \rightarrow$ Trivial equation
- Force-free equation of $i = 2 \rightarrow \partial_\lambda \left(\sqrt{-g}\, Z^{\lambda\alpha} \partial_\alpha \psi\right) = 0,$

where $Z^{\lambda\alpha} = W^{\lambda\alpha\sigma\beta}\partial_\sigma \bar{\phi}_2 \partial_\beta \bar{\phi}_2 = W^{\lambda\alpha\rho\rho} - \Omega_{\mathrm{F}}(W^{\lambda\alpha\rho t} + W^{\lambda\alpha t\rho}) + \Omega_{\mathrm{F}}^2 W^{\lambda\alpha tt}$

$$\Rightarrow \quad \lambda \frac{\partial^2 \psi}{\partial t^2} - \frac{\partial}{\partial X}\left(S \frac{\partial \psi}{\partial X}\right) + K\psi + \frac{\partial}{\partial t}\left(V \frac{\partial \psi}{\partial X}\right) + \frac{\partial}{\partial X}\left(V \frac{\partial \psi}{\partial t}\right) = 0,$$

where $\lambda = -\sqrt{-g} Z^{tt} = \frac{1}{\alpha^2}\left(\frac{\Sigma}{R^2} + \frac{I^2\Sigma^2}{\Delta}\right),$

$S = \sqrt{-g} Z^{XX} = \frac{1}{\sin^2\theta}\left[\alpha^2 - R^2(\Omega_{\mathrm{F}} - \Omega)^2\right] = \frac{-\Gamma}{\sin^2\theta},$

$K = \sqrt{-g} Z^{\Psi\Psi} = \frac{1}{\Delta}\left[\alpha^2 - R^2(\Omega_{\mathrm{F}} - \Omega)^2 + I^2\Sigma\sin^2\theta\right] = \frac{I^2\Sigma\sin^2\theta - \Gamma}{\Delta},$

$V = -\sqrt{-g} Z^{tX} = \frac{I\Sigma}{\alpha^2}(\Omega_{\mathrm{F}} - \Omega)$

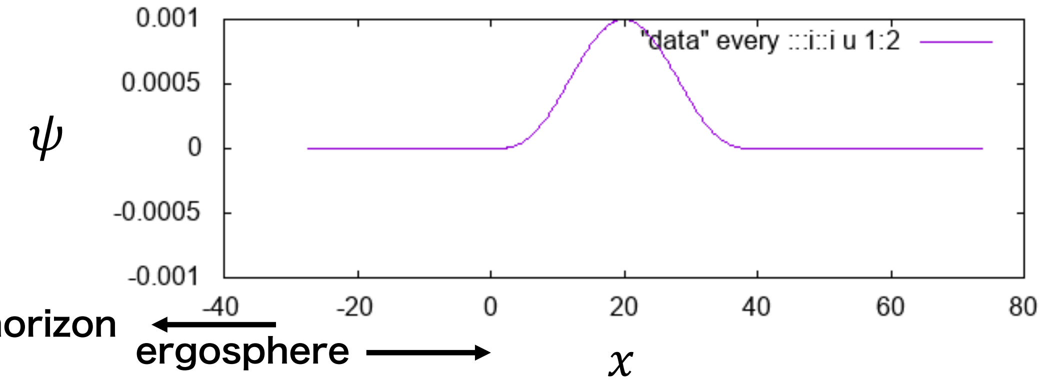
$\Gamma = -\alpha^2 + R^2(\Omega - \Omega_{\mathrm{F}})^2$

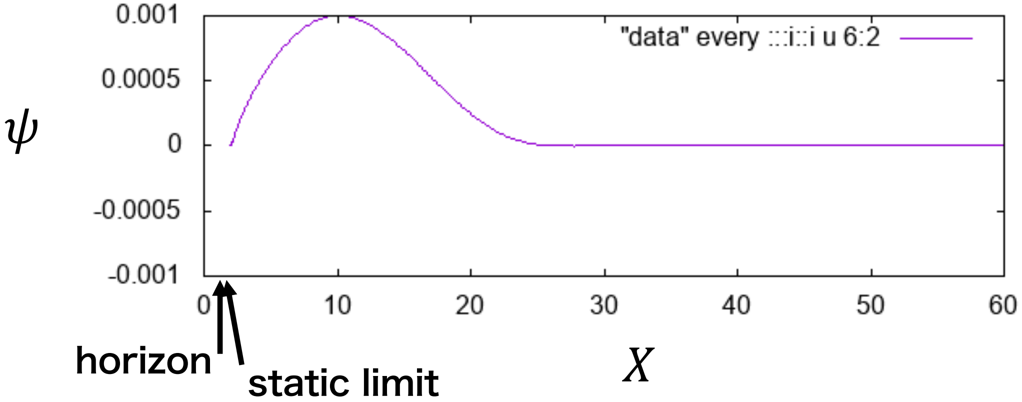
$\Gamma = 0 \rightarrow$ **light surface**

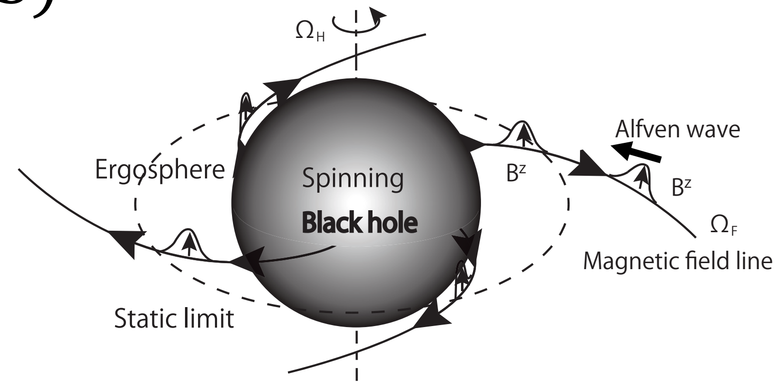
Numerical calculation：
2 step Lax-Wendroff method

Numerical results: Refection of initially inward pulse

$$a_* = 0.2,\ \Omega_F = 0.027\ (\Omega_H = 0.0505)$$







$$x = X - 2M + r_H \log\left[\frac{X - r_H}{2M - r_H}\right]$$

(Tortoise coordinate)

$x = 0$: Static limit ($X = 2M$)

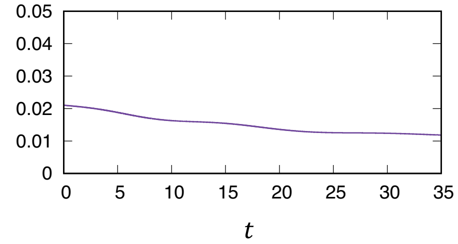
Reflection is found, but not superradiance

Energy balance of Alfven wave in case of initially inward pulse

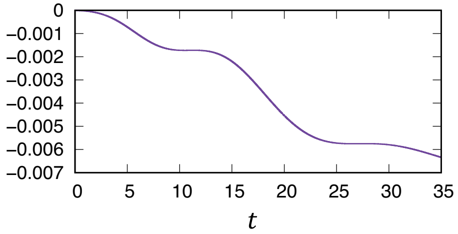
$a_* = 0.2,\ \Omega_F = 0.027\ (\Omega_H = 0.0505)$

Alfven wave only

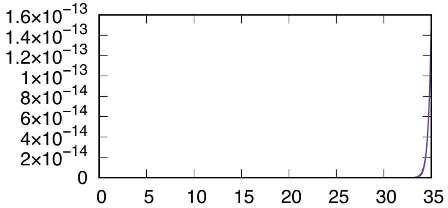
$\delta^2 E(t)$



$\delta^2 F_1(t)$

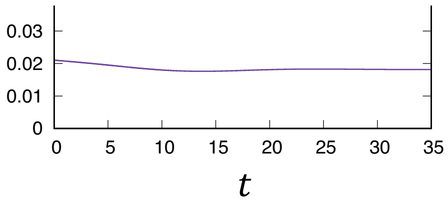


$\delta^2 F_2(t)$



Energy is not conserved!

Energy balance



$$\delta^2 E(t) = \int_{X_1}^{X_2} \sqrt{-g}\, \delta^2 S^t(X,t)\, dX$$

(Energy in calculation region)

$$\delta^2 F_1(t) = \int_{T_1}^{T} \sqrt{-g}\, \delta^2 S^r(X_1,t')\, dt'$$

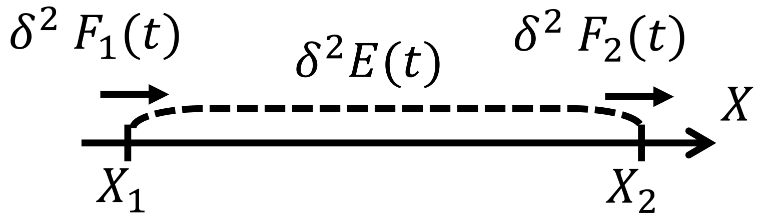
(Energy transported at left boundary)

$$\delta^2 F_2(t) = \int_{T_1}^{T} \sqrt{-g}\, \delta^2 S^r(X_2,t')\, dt'$$

(Energy transported at right boundary)

Energy balance

$$\delta^2 E(t) - \delta^2 F_1(t) + \delta^2 F_2(t)$$



Calculation region

Energy conservation law on nonrotating natural coordinates

$S^{\mu} = -\xi_{\nu(t)} T^{\mu\nu}$: Energy flux density 4-vector

$\xi_{\nu(t)}$=(1,0,0,0): Killing vector,
$T^{\mu\nu}$: Energy-momentum tensor of electromagnetic field.

Energy conservation law of Alfven wave

$$\nabla_{\mu}\, \delta^2 S^{\mu} = \frac{1}{\sqrt{-g}}\, \partial_{\mu}(\sqrt{-g}\delta^2 S^{\mu}) = \xi_T^{\nu} \delta^2 f_{\nu}^{\mathrm{L}} = \delta^2 f_t^{\mathrm{L}}$$

where$f_{\mu}^{\mathrm{L}} = J^{\nu} F_{\mu\nu}$: Lorentz force density.

$$\frac{\partial}{\partial T} \alpha \delta^2 S^t + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial X^i} \left(\alpha\sqrt{\gamma}\, \delta^2 S^i\right) = -\alpha \delta^2 f_t^{\mathrm{L}} = \alpha \Omega_{\mathrm{F}} \delta J^{\mu} \partial_{\mu} \psi \neq 0、$$

where $\sqrt{-g} = \alpha\sqrt{\gamma}$

Only with Alfven wave, energy is not conserved

2^{nd} perturbation of $\boldsymbol{\phi_2}$: $\chi\,(X, t)$

2nd peruturbation of ϕ_2

$$\phi_2 = \bar{\phi}_2 + \delta^2\phi_2 = \rho - \Omega_{\rm F} t + \chi$$

- Force-free equation of $i = 2$ → Trivial equation
- Force-free equation of $i = 1$

$$\rightarrow \partial_\lambda\left(\sqrt{-g}\, g^{\Psi\Psi} g^{\lambda\alpha}\partial_\alpha\chi\right) = \partial_\lambda\left(\sqrt{-g}\, Y^{\lambda\alpha\mu}\partial_\alpha\psi\partial_\mu\psi\right) \equiv \sqrt{-g}s,$$

where $Y^{\lambda\alpha\mu} = W^{\lambda\alpha\mu\sigma}\partial_\sigma\bar{\phi}_2 = W^{\lambda\alpha\mu\rho} - \Omega_{\rm F} W^{\lambda\alpha\mu t}$

$$\Rightarrow \boxed{\frac{\partial^2\chi}{\partial t^2} + \frac{1}{g^{tt}}\frac{\partial}{\partial X}\left(g^{XX}\frac{\partial\chi}{\partial X}\right) = \frac{\sqrt{-g}}{g^{tt}} s}$$

where $s \equiv \frac{1}{\sqrt{-g}}\partial_\lambda\left(\sqrt{-g}\, Y^{\lambda\alpha\mu}\partial_\alpha\psi\partial_\mu\psi\right) = \frac{1}{\sqrt{-g}}(\partial_t\,\Theta\,\partial_X\,\psi - \partial_X\,\Theta\,\partial_t\,\psi)$,

$$\Theta \equiv -\frac{I\,X^2}{\alpha^2}\partial_t\psi - R^2(\Omega_{\rm F} - \Omega)\partial_X\,\psi$$

Numerical calculation： 2 step Lax-Wendroff method

Energy conservation law of Alfven wave and fast wave

Energy conservation law of Alfven wave and fast wave

$$\nabla_\mu \delta^2 S^\mu = \frac{1}{\sqrt{-g}} \partial_\mu\left(\sqrt{-g}\delta^2 S^\mu\right) = \delta^2 f_t^{\mathrm{L}}$$

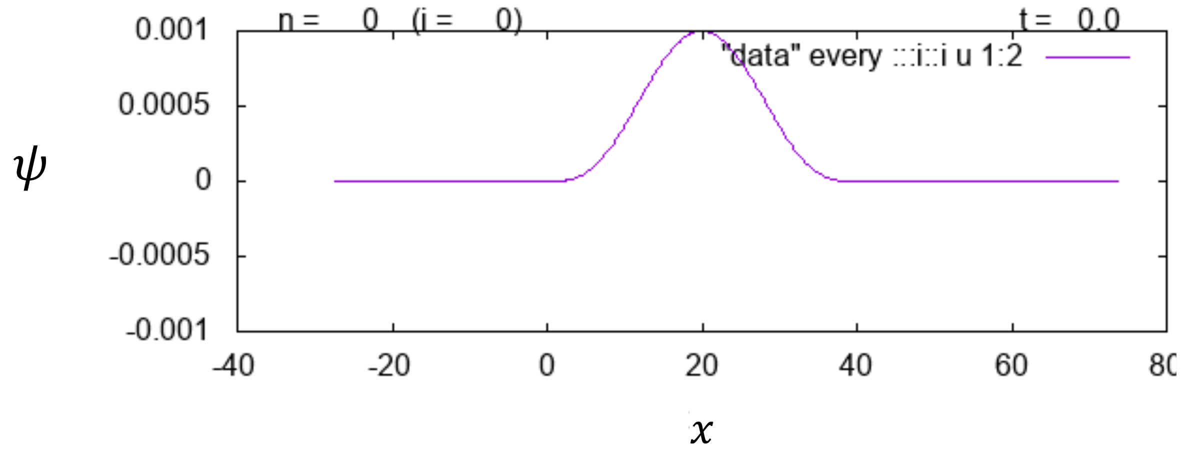
$$= \frac{1}{\sqrt{-g}}\Omega_{\mathrm{F}}\left[\partial_\lambda\left(\sqrt{-g}\, g^{\Psi\Psi} g^{\lambda\alpha}\partial_\alpha\chi\right) - \partial_\lambda\left(\sqrt{-g}\, Y^{\lambda\alpha\mu}\partial_\alpha\psi\partial_\mu\psi\right)\right] = 0$$

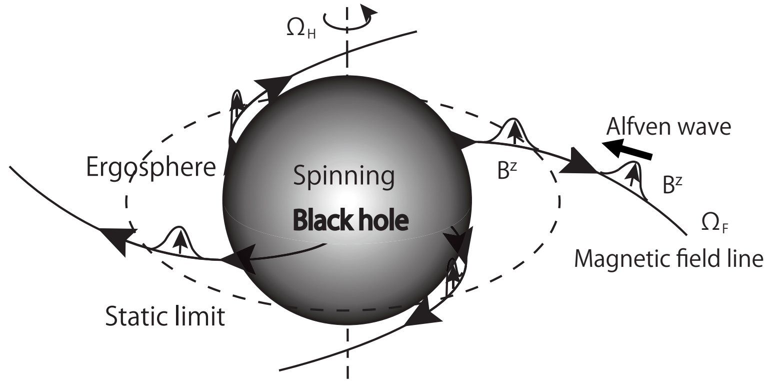
Energy is conserved!

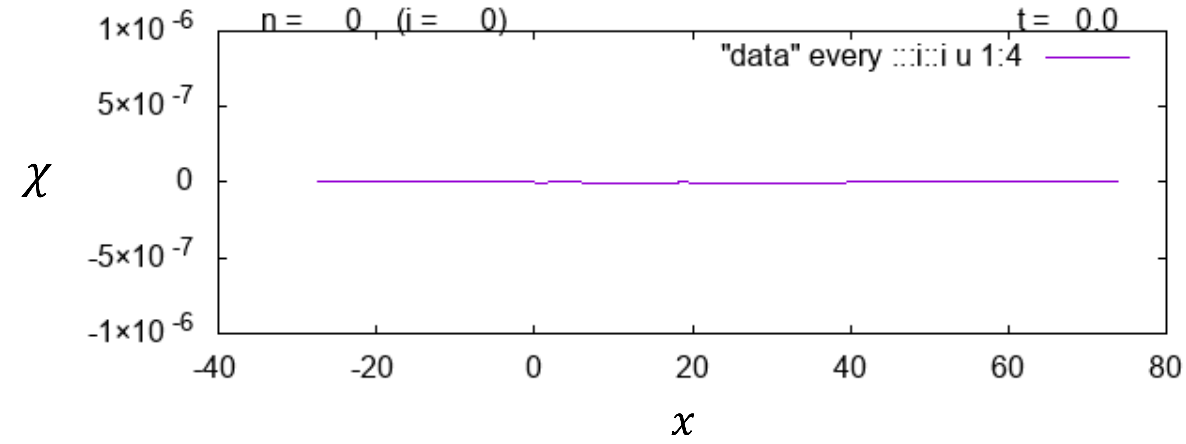
$$\frac{\partial}{\partial t}\alpha\delta^2 S^t + \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial X^i}\left(\alpha\sqrt{\gamma}\,\delta^2 S^i\right) = 0$$

Numerical results of Alfven wave and induced fast wave: initially inward pulse

$$a_* = 0.2,\ \Omega_F = 0.027\ (\Omega_H = 0.0505)$$







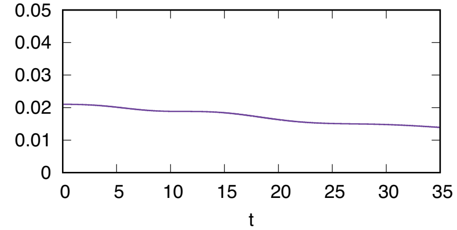
Reflection is found, but not superradiance

Energy balance in case of initially inward pulse

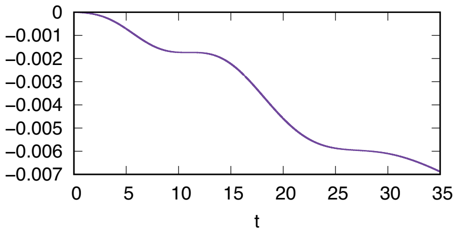
$$a_* = 0.2,\ \Omega_{\mathrm{F}} = 0.027\ (\Omega_{\mathrm{H}} = 0.0505)$$

Alfven wave and fast wave

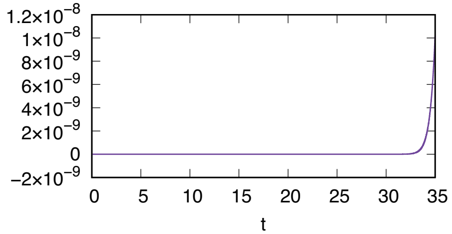
$\delta^2 E(t)$



$\delta^2 F_1(t)$

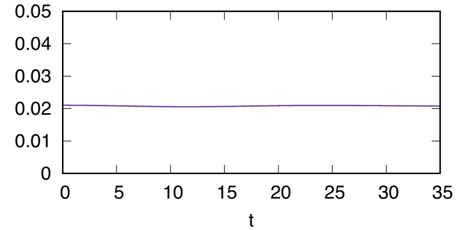


$\delta^2 F_2(t)$



Energy is conserved!

Energy balance



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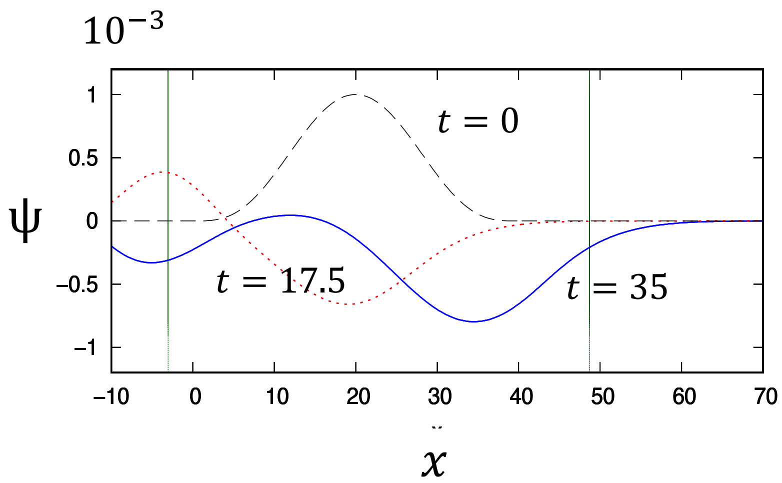
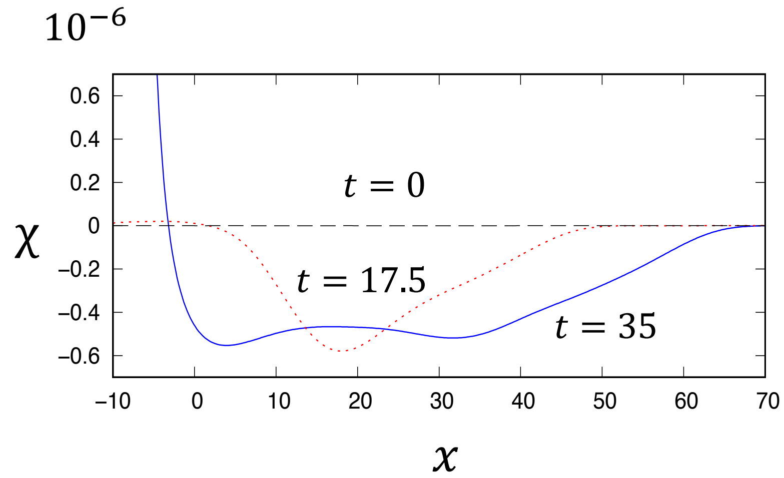
Numerical results: initially inward pulse

$$a_* = 0.2,\ \Omega_F = 0.027\ (\Omega_H = 0.0505)$$





Angular momentum conservation law only with Alfven wave on nonrotating natural coordinates

$M^{\mu} = -\eta_{\nu(\rho)} T^{\mu\nu}$: Angular momentum flux density 4-vector

$\eta_{\nu(\rho)}$=(0,0,0,1): axial Killing vector,

$T^{\mu\nu}$: Energy-momentum tensor of electromagnetic field.

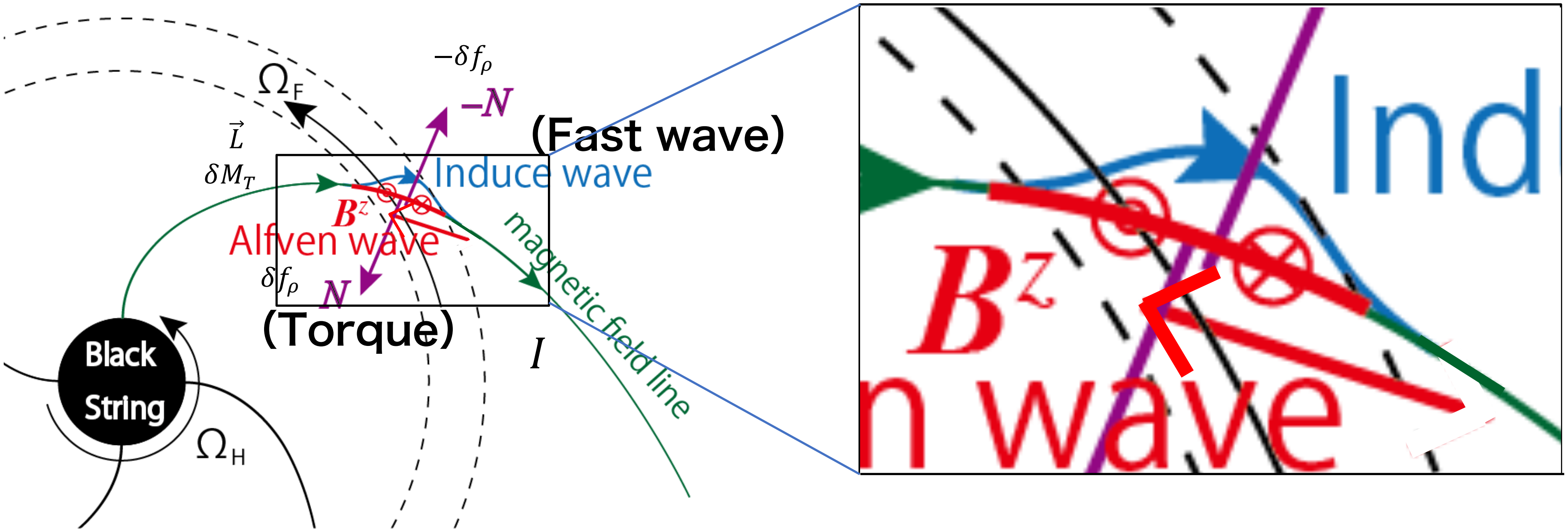
Angular momentum conservation law of Alfven wave

$$\nabla_\mu \, \delta^2 M^\mu = \frac{1}{\sqrt{-g}} \, \partial_\mu\left(\sqrt{-g}\delta^2 M^\mu\right) = \eta^{\nu}_{(\rho)} \delta^2 f^{\mathrm{L}}_{\nu} = \delta^2 f^{\mathrm{L}}_{\rho}.$$

$$\frac{\partial}{\partial t}\alpha\delta^2 M^t + \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial X^i}\left(\alpha\sqrt{\gamma}\,\delta^2 M^i\right) = -\alpha\delta^2 f^{\mathrm{L}}_{\rho} = \alpha\delta J^\mu \partial_\mu \psi \neq 0、$$

Only with Alfven wave, angular momentum is not conserved

Fast wave induced by Alfven wave: Relativistic effect



- Perturbation： $\phi_1 = \bar{\phi}_1 + \delta\phi_1 = \Psi + \psi,\ \phi_2 = \bar{\phi}_2 + \delta^2\phi_2 = \rho - \Omega_F + \chi$

  Alfven wave (ψ) fast wave (χ)

Angular momentum conservation law with Alfven wave and induced fast wave on nonrotating natural coordinates

Angular momentum conservation law with Alfven wave and induce fast wave

$$\nabla_\mu \delta^2 M^\mu = \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\delta^2 M^\mu\right) = \eta^\nu_{(\rho)}\delta^2 f^{\mathrm{L}}_\nu = \delta^2 f^{\mathrm{L}}_\rho.$$

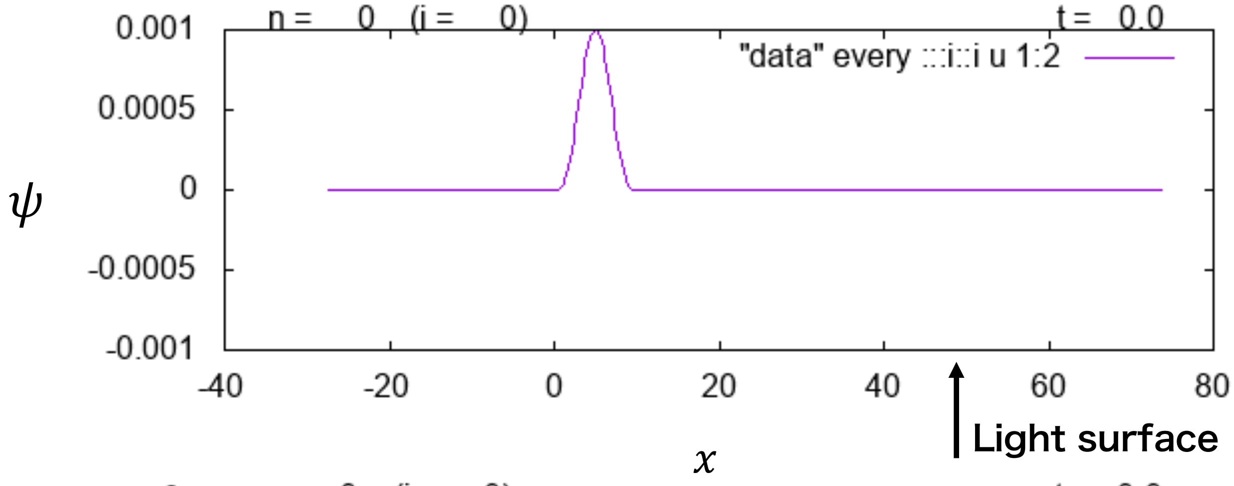
$$\frac{\partial}{\partial t}\alpha\delta^2 M^t + \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial X^i}\left(\alpha\sqrt{\gamma}\,\delta^2 M^i\right) = -\,\alpha\delta^2 f^{\mathrm{L}}_\rho$$

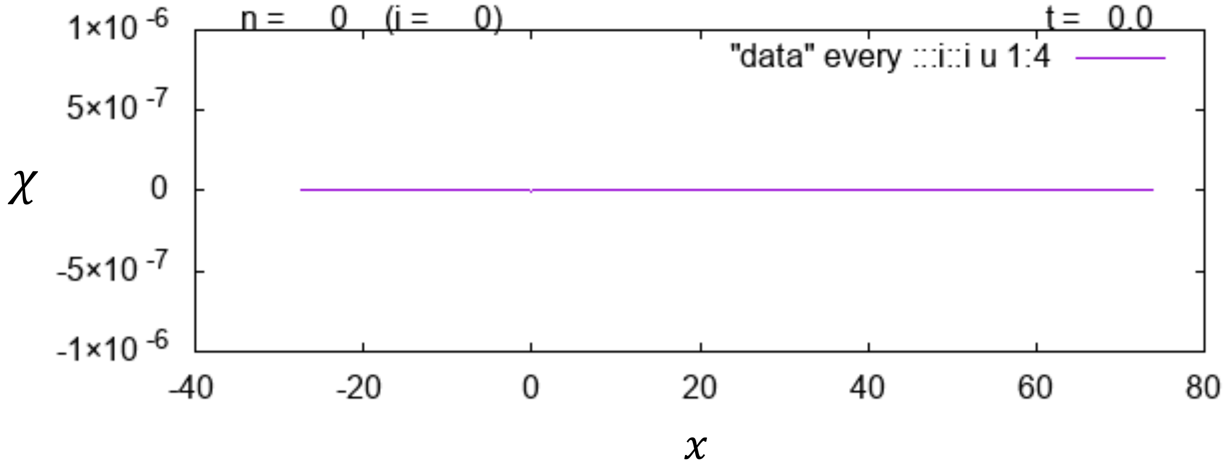
$$= \frac{\alpha}{\sqrt{-g}}\left[\partial_\lambda\left(\sqrt{-g}\,g^{\Psi\Psi}g^{\lambda\alpha}\partial_\alpha\chi\right) - \partial_\lambda\left(\sqrt{-g}\,Y^{\lambda\alpha\mu}\partial_\alpha\psi\partial_\mu\psi\right)\right] = 0、$$

Angular momentum is conserved when the induced fast wave is considered.

Outward pulse in case of $0 < \Omega_F < \Omega_H$

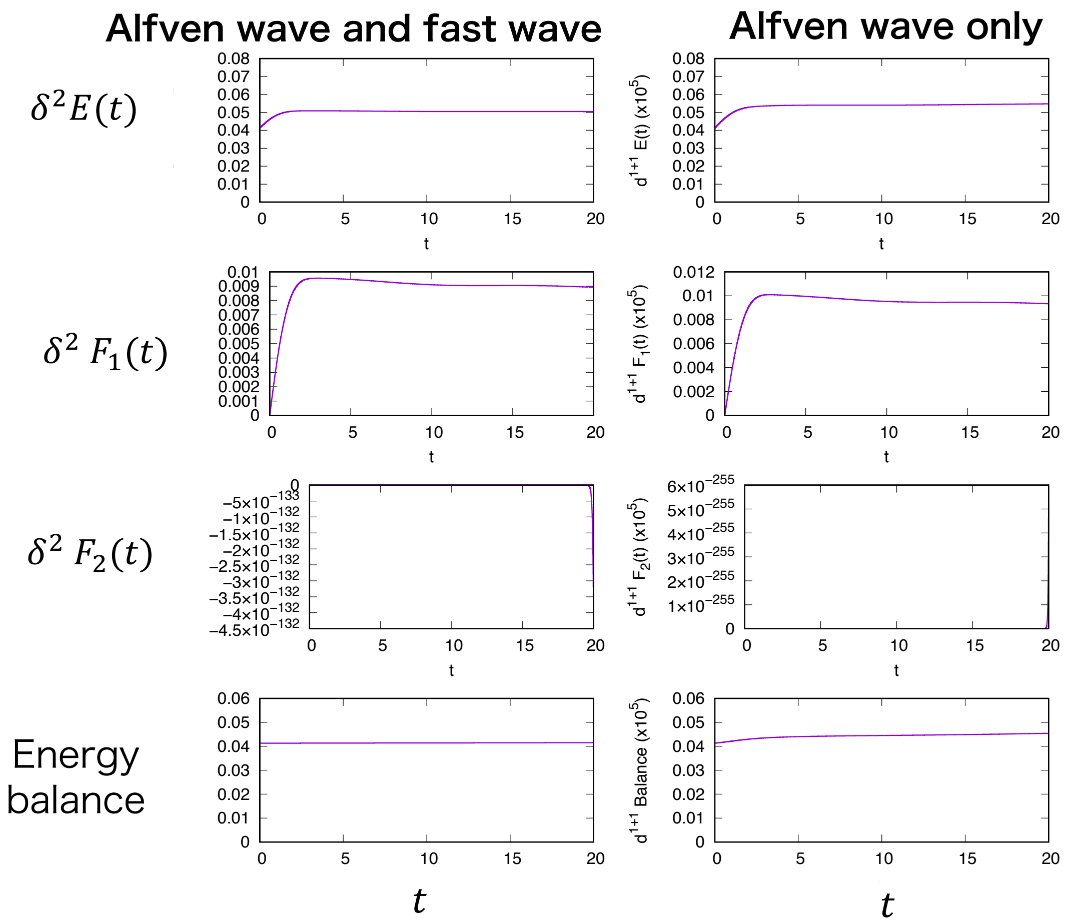
$a_* = 0.2,\ \Omega_F = 0.027\ (\Omega_H = 0.0505)$





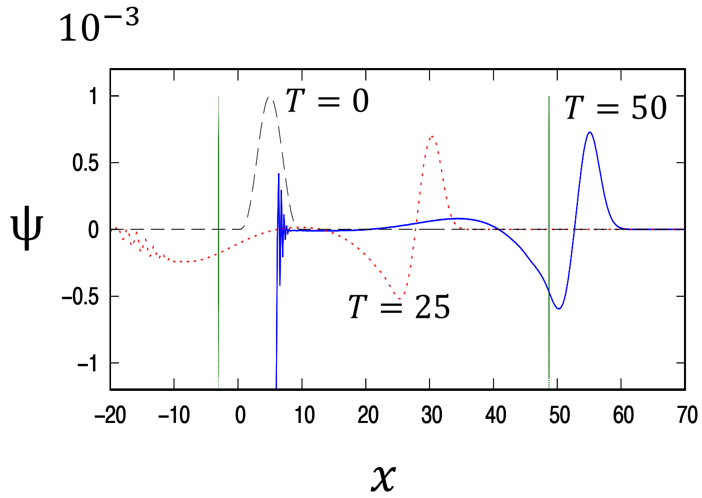
Energy balance of outward pulse in case of $\mathbf{0 < \Omega_F < \Omega_H}$

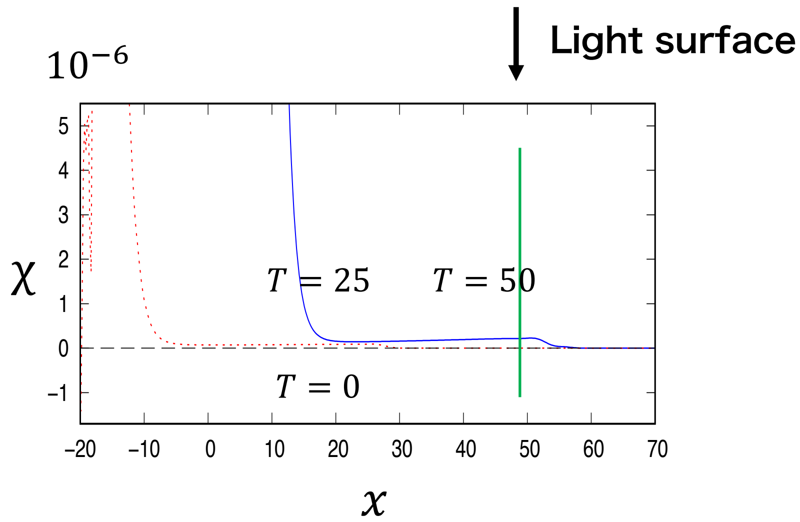
$$a_* = 0.2,\ \Omega_F = 0.027\ (\Omega_H = 0.0505)$$



Outward pulse in case of $0 < \Omega_F < \Omega_H$

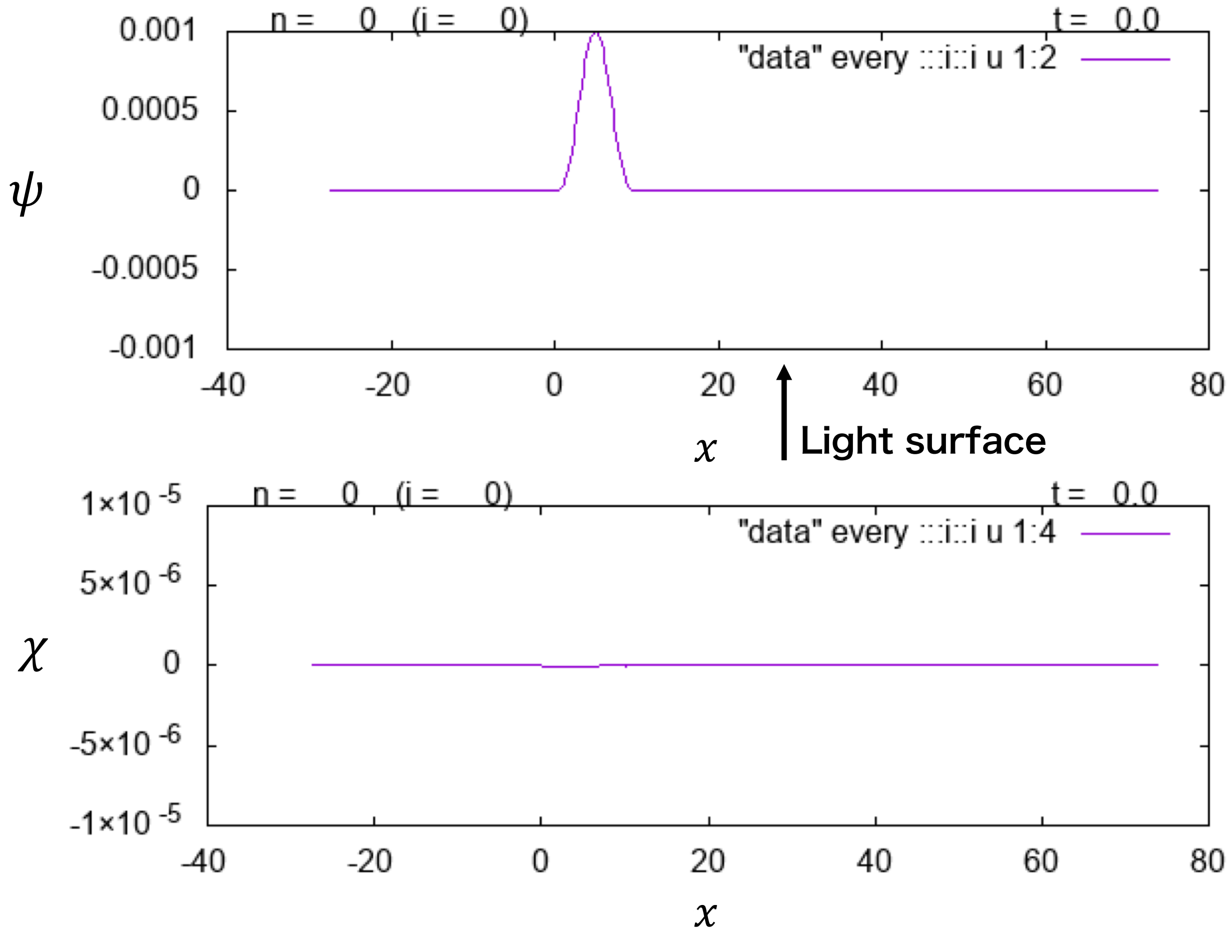
$a_* = 0.2,\ \Omega_F = 0.027\ (\Omega_H = 0.0505)$





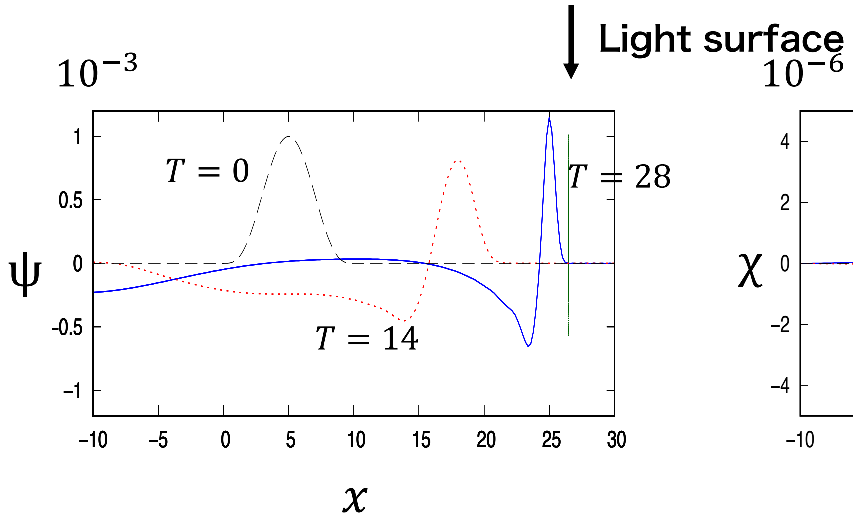
Outward pulse in case of $\Omega_F > \Omega_H$

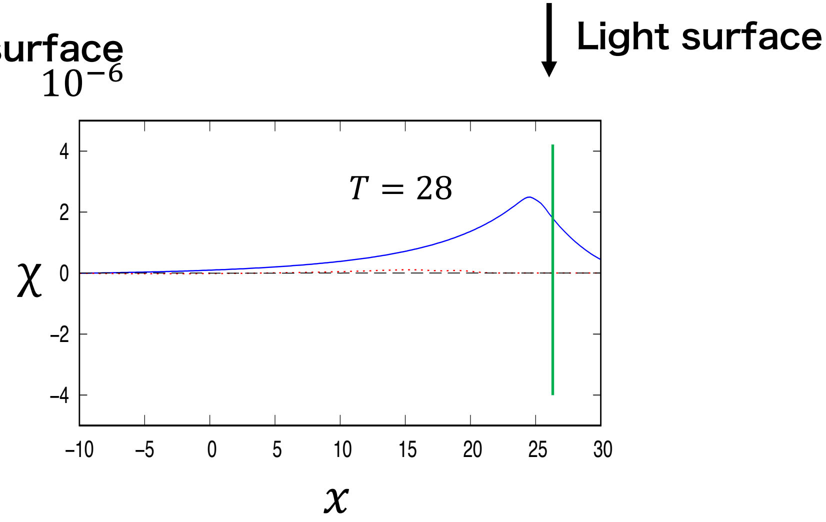
$$a_* = 0.2,\ \Omega_F = 0.06\ (\Omega_H = 0.0505)$$



Outward pulse in case of $\Omega_F > \Omega_H$

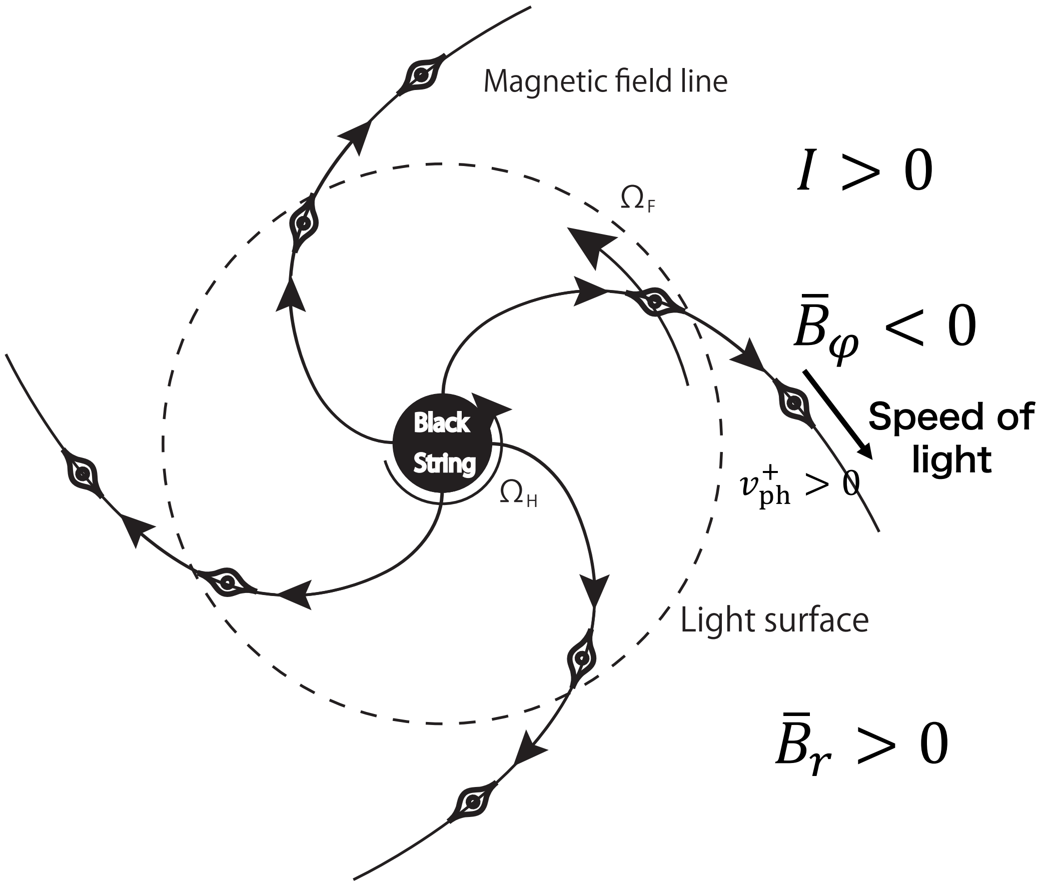
$$a_* = 0.2,\ \Omega_F = 0.06\ (\Omega_H = 0.0505)$$



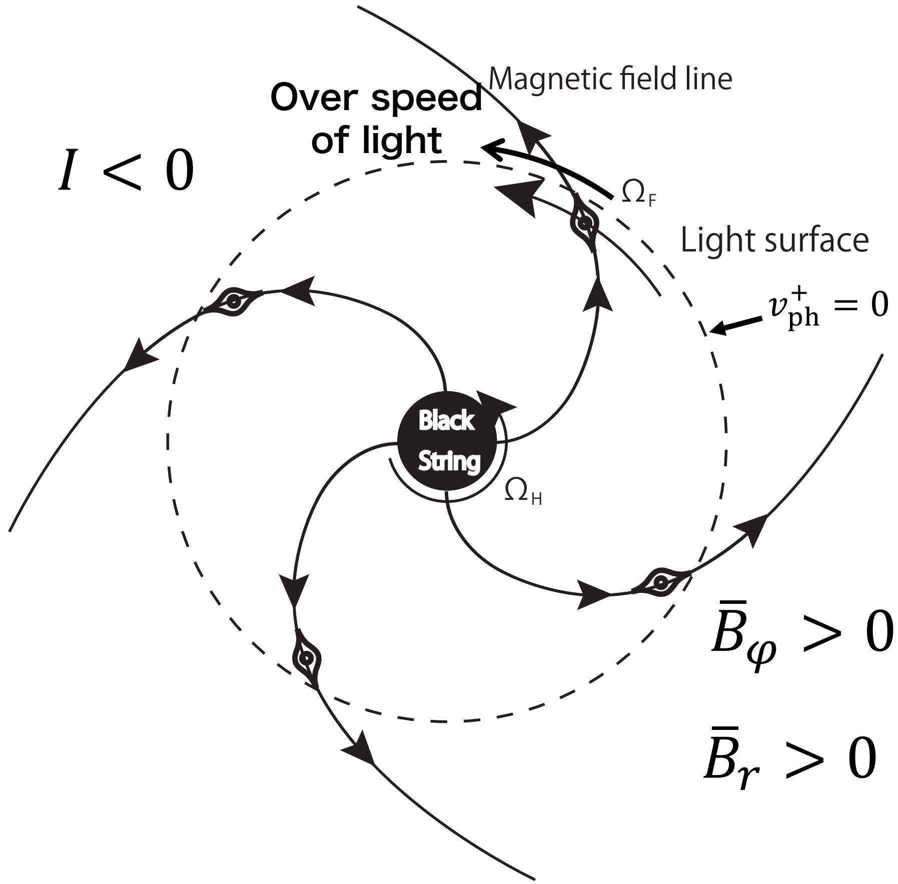


Alfven wave propagation region

$0 < \Omega_F < \Omega_H$



$\Omega_F > \Omega_H$



Dispersion relation of Alfven wave with small wavelength approximation

Sinusoidal perturbation: $\phi_1 = \bar{\phi}_1 + \delta\phi_1 = \bar{\phi}_1 + \psi,\ \psi = Ae^{-i\omega t + ikX}$

Dispersion relation: $\lambda\omega^2 - 2V\omega k + Sk^2 = 0$

Phase velocity: $v_{\mathrm{ph}}^{\pm} = \frac{\omega}{k} = \frac{V \pm \sqrt{V^2 + \lambda S}}{\lambda} = \frac{V \pm X\sqrt{K}}{\lambda}$

$V = \frac{I\Sigma}{\alpha^2}(\Omega_{\mathrm{F}} - \Omega)$

At outer light surface $(S = 0)$, $v_{\mathrm{ph}}^{+} = \frac{V + |V|}{\lambda}$

(i) Case of $0 < \Omega_{\mathrm{F}} < \Omega_{\mathrm{H}}$: $I > 0,\ \ V > 0, v_{\mathrm{ph}}^{+} = \frac{2V}{\lambda} > 0.$ → $v_{\mathrm{ph}}^{+} > 0$

(ii) Case of $\Omega_{\mathrm{F}} \geq \Omega_{\mathrm{H}}$: $I \leq 0,\ \ V \leq 0,\ v_{\mathrm{ph}}^{+} = 0.$

(iii) Case of $\Omega_{\mathrm{F}} \leq 0$: $I \geq 0,\ \ \Omega_{\mathrm{F}} - \Omega < 0, V \leq 0, v_{\mathrm{ph}}^{+} = 0.$

(ii), (iii) → $v_{\mathrm{ph}}^{+} = 0$ at light surface

Unstable region of Alfven wave

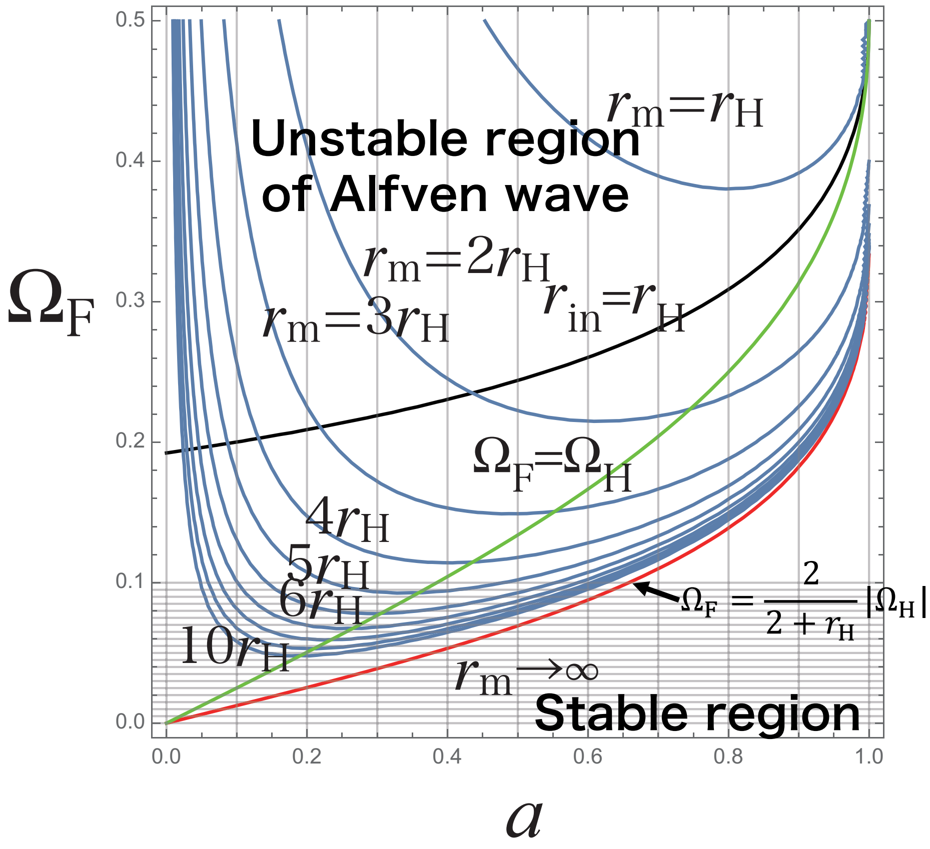
Sinusoidal perturbation: $\phi_1 = \bar{\phi}_1 + \delta\phi_1 = \bar{\phi}_1 + \psi,\ \psi = Ae^{-i\omega t + ikX}$

Dispersion relation: $\omega = \frac{V \pm X\sqrt{K}}{\lambda} k$

(i) When $K = V^2 + \lambda S < 0$, the Alfven wave is unstable.

(ii) Because $S < 0$ in the outer region of outer light surface, region of $K < 0$ is possible in the outer region.

(iii) In the case of $|\Omega_F| \leq \frac{2}{2+r_H}|\Omega_H|$, Alfven wave is stable over the whole region.



Summary

We have performed numerical simulations of force-free Alfven wave propagation along the stationary magnetic field line on the equatorial plane around a spinning black hole. We found the following interesting remarks with respect to the Alfven wave propagation around the spinning black hole.

- We observed a reflection of the Alfven wave near the ergosphere, while the reflection is not superradiance as suggested by Noda et al. (2022). The energy of the only Alfven wave is not conserved.
- Both in the cases of the inwardly and outwardly propagating Alfven wave, the Alfven wave induces the fast wave as the second order perturbation due to change in the angular momentum of Alfven wave in the case of $I \neq 0$ or $\Omega_{\mathrm{F}} - \Omega_{\mathrm{H}} \neq 0$. The total energy of the Alfven and fast waves is conserved.
- In the case of $0 < \Omega_{\mathrm{F}} < \Omega_{\mathrm{H}}$, the Alfven wave and the induced fast wave propagate through the outer light surface smoothly. However, in the case of $\Omega_{\mathrm{F}} > \Omega_{\mathrm{H}}$, the Alfven wave can not pass through the outer light surface and only the fast wave propagates through the outer light surface smoothly.
- In any case of $\frac{2}{2+r_{\mathrm{H}}}|\Omega_{\mathrm{H}}| < |\Omega_{\mathrm{F}}| < \frac{2}{2-r_{\mathrm{H}}}|\Omega_{\mathrm{H}}|$, in the outer enough region, the Alfven wave is unstable. We show the unstable region of Alfven wave in the last panel shown in this talk.