(Beyond) effective field theory with homotopy transfer

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Based on

- The L_∞-algebra of the S-matrix, ASA, [arXiv:1903.05643]
- Homotopy Transfer and Effective Field Theory I: Tree-level, ASA, Olaf Hohm, Chris Hull, Victor Lekeu [arXiv:2007.07942], ...II, ...III [to appear]

I will outline a dictionary from QFTs to L_∞-algebras:

n-point 1PI correlator	$\leftrightarrow$	$\kappa(\bullet, \underbrace{[\bullet, \bullet, \cdots \bullet]}_{\mathbf{n}-\mathbf{1}\text{ arguments}})$
antibracket	$\leftrightarrow$	inner product κ
S-matrix elements	$\leftrightarrow$	minimal model
vacuum moduli space	$\leftrightarrow$	Maurer-Cartan locus
effective theory	$\leftrightarrow$	homotopy transfer

It is based on the [Zinn–Justin 1974] antifield formalism, that assigns a 1PI (1-particle irreducible) generating functional Γ to a solution S of the quantum BV master equation.

How to recognise an L_∞-algebra

If C^c_{ab} are structure constants C^c_{ab} of a Lie algebra $\mathfrak{g}$:

$$Q^2 = 0 \quad \text{for} \quad Q = \tfrac{1}{2} C^c_{ab} c^a c^b \frac{\partial}{\partial c^c} \iff C^d_{[ab} C^e_{c]d} = 0\,.$$

c^a has *ghost number* 1 (aka *degree*), Q increases degree by 1.

[AKSZ '95]

Any degree-1 differential Q with $Q|_{z=0} = 0$:

$$Q = (\underbrace{C^a_b}_{\partial = [\bullet]} z^b + \tfrac{1}{2} \underbrace{C^a_{bc}}_{[\bullet,\bullet]} z^b z^c + \tfrac{1}{3!} \underbrace{C^a_{bcd}}_{[\bullet,\bullet,\bullet]} z^b z^c z^d + \dots) \frac{\partial}{\partial z^a}$$

and $Q^2 = 0$ — **the Jacobi identities** — defines an L_∞-algebra.

One n-ary bracket $\overbrace{[\bullet, \bullet, \cdots \bullet]}^{n \text{ arguments}}$ for each $n = 1, 2 \dots$
"Ghosts" z^a bosonic or fermionic depending on degree mod 2.

WHY L_∞-ALGEBRAS, MORALLY?

"Meta-theorem[1]": every deformation problem has an L_∞-algebra X; deformations solve

Maurer-Cartan equation:

$$\partial v + \frac{1}{2}[v,v] + \frac{1}{3!}[v,v,v] + \cdots = 0\,, \quad v = v^a T_a \in X\,, \quad \deg v = 0\,.$$

Solutions $v \mod$ gauge form the **MC locus** (or **moduli space**).

These define translations in ghost space

$$z^a \to z^a + v^a = e^v(z^a)$$

such that

$$e^v Q e^{-v}$$

defines an L_∞-algebra whenever Q does.
Every perturbation expansion involves an L_∞-algebra.

[1] Now an actual theorem [Pridham, Lurie].

L_∞-ALGEBRA FOR A SCALAR QFT

Vacuum correlator generating functional for a scalar $\phi(x)$:

$$Z[J] = \langle 0|\mathrm{T}\exp\left(\int d^4x\, J(x)\phi(x)\right)|0\rangle\,.$$

Form the 1PI functional Γ in the usual way (Legendre):

$$\Gamma[\Phi] \equiv \log Z[J] + \int d^4x\, J(x)\Phi(x)\,, \quad J = J[\Phi] = \delta\Gamma/\delta\Phi\,.$$

Γ is a formal power series in the **classical field** $\Phi(x)$.
Its Taylor coefficients around $\Phi = 0$ give "1PI VEVs":

$$\left.\frac{\delta^n\Gamma}{\delta\Phi(x_1)\cdots\delta\Phi(x_n)}\right|_{\Phi=0} \propto \langle\phi(x_1)\cdots\phi(x_n)\rangle_{\mathrm{1PI}}$$

In perturbation theory, Γ = (kinetic term) + (all 1PI graphs).
Γ is the classical action S at tree level.

Attempt to define L_∞-algebra over the graded vector space

$$X = X_0 \oplus X_1\,, \quad X_0 = X_1 = \{\text{scalar fields on } \mathbb{R}^4\},$$

with brackets $[\bullet, \bullet \cdots \bullet] : (X_0)^n \to X_1$, else zero:

$$[\phi_1, \cdots \phi_n](x) \propto \int \frac{\delta^{n+1}\Gamma}{\delta\Phi(x_1)\cdots\delta\Phi(x_n)\delta\Phi(x)}\Big|_{\Phi=0} \phi_1(x_1)\cdots\phi_n(x_n)\,.$$

Jacobi identities are automatic. Easy to see with **classical antifield** $\overset{\star}{\Phi}(x)$ and antibracket

$$(\Phi(x), \overset{\star}{\Phi}(y)) = \delta^4(x-y)\,, \quad (\deg \overset{\star}{\Phi}(x) = -1)$$

so that

$$Q = (\Gamma, \bullet) = -\int \frac{\delta\Gamma}{\delta\Phi(x)} \frac{\delta}{\delta\overset{\star}{\Phi}(x)} \implies Q^2 = 0\,.$$

We have an L_∞-algebra if the scalar field has vanishing VEV:

$$Q|_{\Phi=0} = 0 \iff \frac{\delta\Gamma}{\delta\Phi(x)}|_{\Phi=0} = 0 \iff \langle 0|\phi(x)|0\rangle = 0\,.$$

Why 1PI instead of connected or general correlators?
Consider a v in the MC locus. $v \in X_0$ is a scalar field solving

$$\sum_{n=1}^{\infty} \int \frac{1}{n!} \frac{\delta^n\Gamma}{\delta\Phi(x_1)\cdots\delta\Phi(x_n)}\Big|_{\Phi=0} v(x_1)\cdots v(x_n) \equiv \frac{\delta\Gamma[v]}{\delta\Phi} = 0\,.$$

$v =$ (const.) thus extremises the [Coleman–Weinberg ’73] potential!

Therefore we identify [ASA ’19]

vacuum moduli space $\quad\leftrightarrow\quad$ Maurer-Cartan locus

(C.f. solutions of the EOM in string field theory determining the conformal manifold of the worldsheet CFT [Sen ’90].)

The LSZ formula for the S-matrix also has an L_∞-interpretation — as the *minimal model* — when expressed via Γ: [ASA '19]

Let $\mathcal{A}[\varphi]$ be the generating function for non-trivial connected S-matrix elements. Can then prove [Jevicki–Lee '88, ASA '19]

$$\mathcal{A}[\varphi] = \Gamma[\Phi_\varphi]\,, \qquad \Phi_\varphi \propto \varphi + \mathcal{O}(\varphi^2) \quad \text{solves} \quad \delta\Gamma/\delta\Phi = 0\,. \quad (*)$$

Here φ is an on-shell 1-particle state of *renormalised mass*, so

$$(\delta^2\Gamma/\delta\Phi^2)\varphi = 0.$$

$(*)$ is the **geometric interpretation** of the minimal model of L_∞/A_∞-algebras due to [Kajiura '01, '03].

$(*)$ is naturally solved recursively in Φ as a power series in φ. At tree level this leads to practical recursion relations: Berends-Giele & perturbiner methods [Macrelli Sämann Wolf '19, Lopez-Arcos Vélez '19]

THE ZINN-JUSTIN [1974] 1PI FUNCTIONAL

For (perturbative) gauge theory, we need antifields. Let $S[\phi, \overset{\star}{\phi}]$ be the BV master action, J classical source, $\overset{\star}{\Phi}$ classical antifield,

$$Z[J, \overset{\star}{\Phi}] \equiv \int \mathcal{D}\phi \ \exp\left(iS[\phi, \overset{\star}{\Phi}] + \int dx\, J(x)\phi(x)\right)$$

Define $\Gamma[\Phi, \overset{\star}{\Phi}]$ again via Legendre with respect to J; $\overset{\star}{\Phi}$ is fixed:

$$\Gamma[\Phi, \overset{\star}{\Phi}] \equiv \log Z[J, \overset{\star}{\Phi}] + \int d^4x\, J(x)\Phi(x)\,, \quad J = J[\Phi, \overset{\star}{\Phi}] = \delta\Gamma/\delta\Phi\,.$$

Taking $\delta/\delta\overset{\star}{\Phi}$ produces terms $Q_{\mathrm{BRST}}\phi(x)$ inside correlators. The

Zinn-Justin Γ thus encodes expressions of the form

$$\langle \phi(x_1)\cdots\phi(x_n) Q_{\mathrm{BRST}}\phi(x_{n+1}) Q_{\mathrm{BRST}}\phi(x_{n+2})\cdots\rangle_{1\mathrm{PI}}$$

in its Taylor expansion.
(This is reviewed in e.g. [Henneaux Teitelboim] or [Gomis Paris Samuel])

Quantum master eq. for S is *classical* master eq. for Γ:

$$(\Gamma, \Gamma) \equiv \int dx \, \frac{\delta\Gamma}{\delta\Phi(x)} \frac{\delta\Gamma}{\delta\overset{\star}{\Phi}(x)} = 0 \iff \Delta \exp(iS[\phi, \overset{\star}{\phi}]) = 0 \,.$$

$Q \equiv (\Gamma, \bullet)$ has $Q^2 = 0$ iff $(\Gamma, \Gamma) = 0$ (*Zinn-Justin equation*). This is the absence of perturbative gauge anomalies.

We thus formally associate an L_∞-algebra over $\mathbb{R}[[\hbar]]$ (defined by Γ) to a *loop* i.e. *quantum* L_∞-algebra over $\mathbb{R}$ (defined by S).

For finite-dimensional algebras this is a precise statement under certain conditions [ASA Hull Hohm Lekeu '21 (?)].

HOMOTOPY TRANSFER

Homotopy transfer is the operation of constructing a L_∞-algebra on a subspace $\bar{X} < X$ of an L_∞-algebra X.

The 1-ary bracket ∂ makes X a cochain complex (due to Jacobi). Homotopy transfer works if $\iota : \bar{X} \hookrightarrow X$ is an *isomorphism in cohomology* (under $\bar{\partial}$ and ∂), + some other conds.

There is a convenient **geometric interpretation** again: [ASA Hull Hohm Lekeu '20]
if $Q \equiv Q^a(z)\partial/\partial z^a$ defines the L_∞-algebra X, write

$$z = (\bar{z}, z^\perp)$$

where $\bar{z}^{\bar{a}}$ are "ghosts" dual to generators of $\bar{X}$. Then solve

$$Q(z^\perp) = 0 \implies z^\perp = z^\perp(\bar{z})\,.$$

This defines a formal power series extending $\iota : \bar{X} \hookrightarrow X$ to a morphism of L_∞-algebras. This is a *homotopy equivalence* or *quasi-isomorphism*: their minimal models are isomorphic.

Effective field theory & homotopy transfer

Tree level. Here $\Gamma = S$. The geometric interpretation says

$$\bar{\Gamma}[\bar{z}] = \Gamma[\bar{z}, z^\perp], \quad \text{where} \quad z^\perp \quad \text{solves} \quad \delta\Gamma/\delta z^\perp = 0\,;$$

integrating out means solving EOMs.

Loop level. $\Gamma \neq S$. The Γ story is the same (less trivial due to $\mathbb{R}[[\hbar]]$-related complications). S defining a loop L_∞-algebra means homotopy transfer is more subtle; proposals generally formalise the *Losev trick*: ['04, later Mnev, Cattaneo, many others]

$$\exp(\hbar^{-1}\bar{S}[\bar{z}]) = \int d\phi^\perp \exp(\hbar^{-1}S[\bar{z}, \phi^\perp, \overset{\star}{\phi}{}^\perp = 0])$$

which is morally a morphism of loop L_∞-algebras. (Rigorous proposals: e.g. [Merkulov '09, Münster & Sachs '12, Doubek Jurčo Pulmann '17])

The point is that integrating over a lagrangian subspace of $z^\perp$s implies (assuming the path integral plays nice)

$$\Delta \exp(\hbar^{-1}S[z]) \implies \bar{\Delta}\exp(\hbar^{-1}\bar{S}[\bar{z}]) = 0\,.$$

We have the following relation involving original (Γ, S) and effective $(\bar{\Gamma}, \bar{S})$: [ASA Hull Hohm Lekeu '21 (?)]

$$\begin{array}{ccc} S & \xrightarrow{\text{1PI}} & \Gamma \\ {\scriptstyle \text{path integral}}\downarrow & & \downarrow{\scriptstyle \text{homotopy transfer}} \\ \bar{S} & \xrightarrow{\text{1PI}} & \bar{\Gamma} \end{array}$$

For finite-dimensional algebras satisfying conditions (most crucially *contractibility*) this is again a precise statement.

Integrating out is a quasi-isomorphism between cyclic (i.e. with inner product/antibracket) L_∞-algebras $\Gamma, \bar{\Gamma}$.

NB there is no locality restriction; L_∞-algebras can accommodate nonlocalities (as in e.g. closed SFT).

We could go further *beyond* EFT via *arbitrary* quasi-isos! The cyclicity assumption seems unwarranted, c.f. [Saberi's talk].