OPERADS & SFT HOMOTOPY ALGEBRAS

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based on collaborations with
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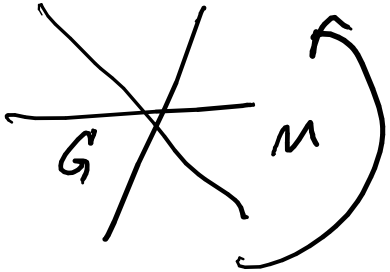
Cor

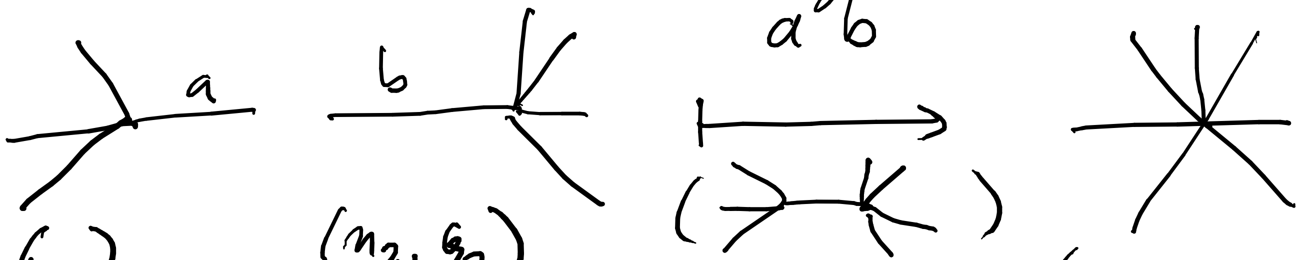
$2(G-1)+n>2$



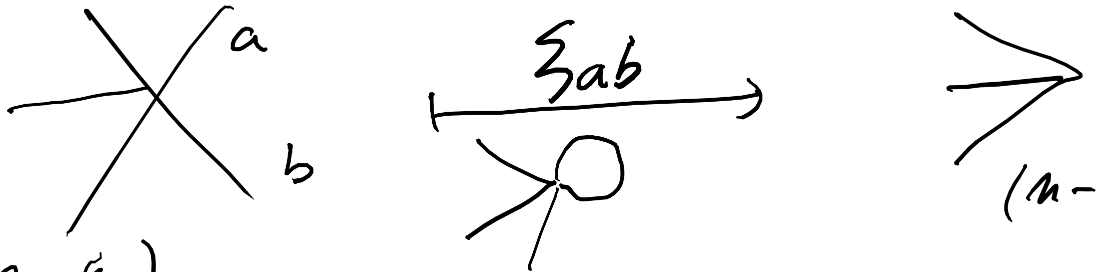
$\rho \in \Sigma_n$

skeletal version
(non-skeletal
$\{1, \dots, n\} \to C$ set
$|C| = n$)

operations



(n_1, g_1) (n_2, g_2) (n_1+n_2-2, g_1+g_2)



(n, G) $(n-2, G+1)$

Modular operad

$\mathcal{P}(\ast_{G}\,n)$ – collection of d.g.v.s

+ morphisms deg = 0

$$\mathcal{P}(\ast) \xrightarrow{\mathcal{P}(\rho)} \mathcal{P}(\rho(\ast)) \qquad \Sigma\text{-action}$$

$$\mathcal{P}(\ast\, a) \otimes \mathcal{P}({}_b\,\ast) \xrightarrow{{}_a\circ_b} \mathcal{P}(\ast)$$

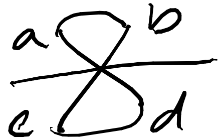
$$\mathcal{P}(\ast^{a\,b}) \xrightarrow{\xi_{ab}} \mathcal{P}(\ast)$$

such that:

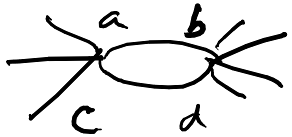
1. $a \circ b$, ξ_{ab} are Σ-equivariant

2. $a \circ b\,(x \otimes y) = (-1)^{|x||y|}\, b \circ a\,(y \otimes x)$

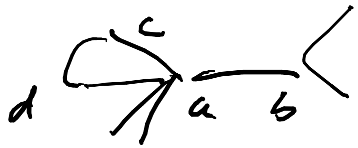
3. $\xi_{ab}\,\xi_{cd} = \xi_{cd}\,\xi_{ab}$



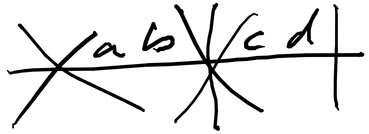
4. $\xi_{ab}\; c \circ d = \xi_{cd}\; a \circ b$



5. $a \circ b\,(\xi_{cd} \otimes 1) = \xi_{cd}\; a \circ b$



6. $a \circ b\,(1 \otimes c \circ d) = c \circ d\,(a \circ b \otimes 1)$

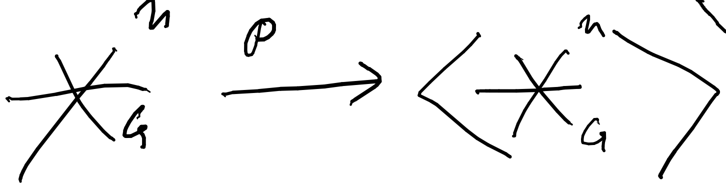


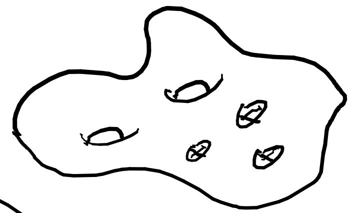
$G = 0$ + forgetting ξ_{ab}

cyclic operad

Adjoint functor   modular envelope

Ex. 1) $\mathrm{Mod}(\mathcal{C}om^c)$





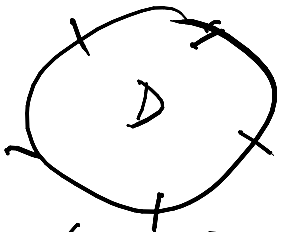
$G = g \qquad \left(G = 2g + \frac{n}{2} - 1 \right)$

1-dim v.s. in degree 0

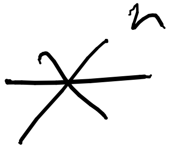
differential $d = 0$

Σ-action trivial $\qquad a \circ b\,,\ \xi_{ab}$ – obvious

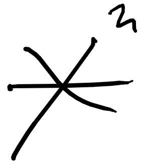
2) $\mathcal{A}ss^c$



$G = 0$



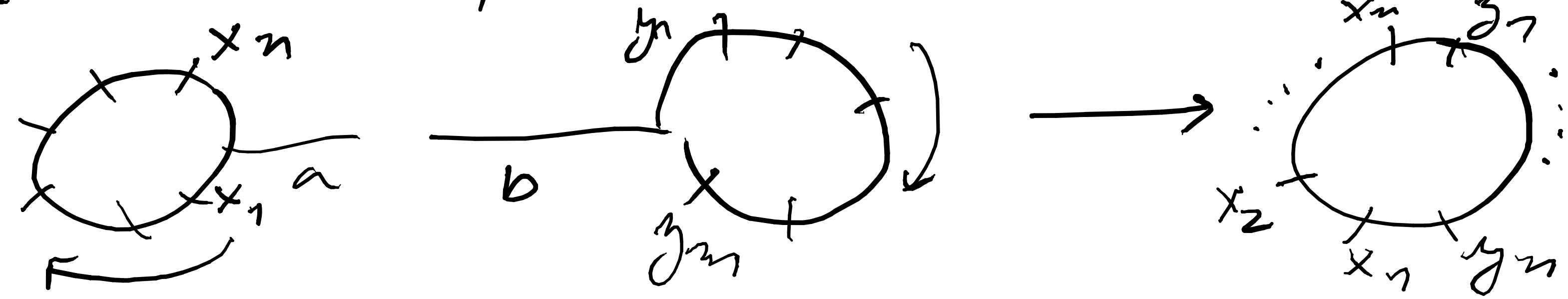
$\longrightarrow$ $\langle$ cyclic orderings on



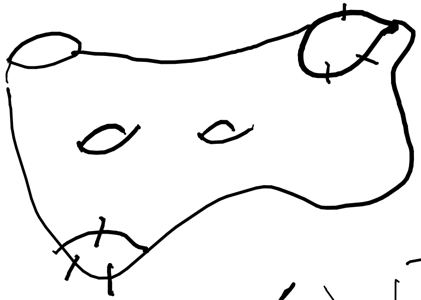
$\rangle$

Σ-action – permutations of points on ∂D

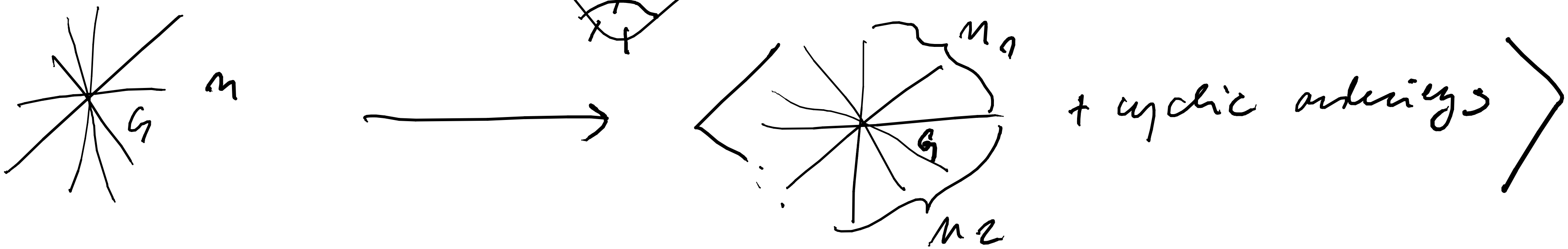
$a \circ b$



3. $\mathrm{Mod}(\mathrm{Ass}^c)$



$G = 2g + b - 1$



$a \circ_b$ sewing two surfaces along open punctures

ξ_{ab} (2 kinds) - self-sewing of surface within the same boundary component

- self-sewing using points on different boundary components

4. Quantum open-closed

2-colored, combination of 1. and 3.

$$G = 2g + b + \frac{n}{2} - 1$$

5. super versions

e.g. Mod (com^c $N=1$) type II

4-closed NS-NS, NS-R, R-NS, R-R

Odd modular operads

$a \circ b \qquad \xi_{ab}$ – now degree $\underline{\underline{1}}$

$\Downarrow$

modification of axioms

$$\xi_{ab}\,\xi_{cd} = \ominus\,\xi_{cd}\,\xi_{ab}$$

$$\xi_{ab}\;{}_c\!\circ_d = \ominus\,\xi_{cd}\;{}_a\!\circ_b$$

$$\vdots$$

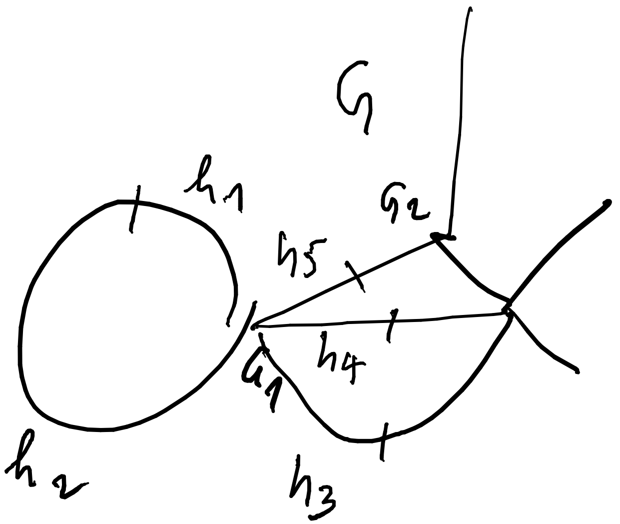
Ex. 1) $\mathrm{End}\,V \qquad V$ – d.g. (super) v.s.

ω – deg -1 symplectic form, compatible with d

$\mathcal{P}(n,G) = \mathrm{Hom}(V^{\otimes n}, \mathbb{C}) =: \mathcal{E}_V(n,G)$ $\qquad \Sigma$ – permutation of inputs

${}_a\!\circ_b \mid \xi_{ab}$ contracting inputs with ω

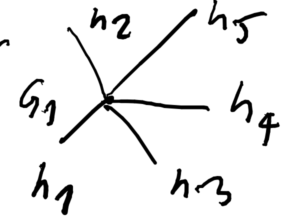
Ex. 2 Feynman transform $\mathcal{F}\mathcal{P}$ of a modular operad $\mathcal{P}$



$$G = \sum g_i + \#\,\text{loops}$$

decorated by

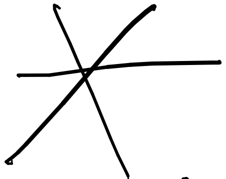
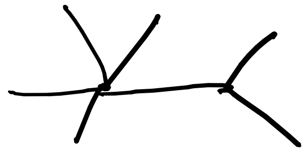
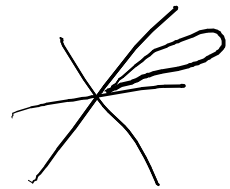
$$(p_1 \otimes p_2 \otimes p_3) \otimes (\uparrow e_1 \wedge \ldots \wedge \uparrow e_5)$$

$$p_1 \in \mathcal{P}\Big($$


$$\Big)^{\#}$$

$(a \circ b)_{\mathcal{F}\mathcal{P}}$ $\quad$ $(\xi_{ab})_{\mathcal{F}\mathcal{P}}$ $\quad$ grafting of graps and attaching edges, resp.

∂_{FP} – adding edge + modifying decorations using $(a^\circ b)^\#$ and $\xi_{ab}^\#$

∂_{FP}  $=$ $\displaystyle\sum_{\substack{n_1+n_2=n \\ G_1+G_2=G}}$  $+$ 

(n, G) $\qquad$ $(n, G-1)$

$G=0$, ξ_{ab} - trivial – cyclic cobar constr.

more accurately $\quad \partial_{FP}\, P(\ast) = d_P P(\ast) + \sum \cdots +$

Algebra over Feynman transform

Morphism of odd modular operads

$$\mathcal{F}\mathcal{P} \longrightarrow \mathrm{End}\, V$$

Baranikov:

$\Longleftrightarrow$ following data

$$\{ m(C,G) \in (\mathcal{P}(C,G) \otimes \mathcal{E}_V(C,G))^{\Sigma} \}$$

$$\underbrace{(d_{\mathcal{E}_V} - d_{\mathcal{P}})}_{d} m(C,G) = \underbrace{(\xi_{ab})_{\mathcal{P}} \otimes (\xi_{ab})_{\mathcal{E}_V}}_{\Delta} m(C \sqcup \{a,b\}, G-1) +$$

$$\underbrace{\frac{1}{2} \sum_{\substack{C_1 \sqcup C_2 = C \\ G_1 + G_2 = G}}}\ \underbrace{\left((a \circ b)_{\mathcal{P}} \otimes (a \circ b)_{\mathcal{E}_V} \right)}_{\{\cdot,\cdot\}} m(C_1 \sqcup \{a\}, G_1) \otimes m(C_2 \sqcup \{b\}, G_2)$$

(NC) BV-algebra on V

$$\left((P \otimes \mathcal{E}_V)^{\Sigma}, d, \Delta, \{\cdot,\cdot\} \right) \quad \text{d.g. Lie algebra}$$

$d+\Delta$ - differential

$\{\cdot,\cdot\}$ - Lie bracket

<u>Action</u> (Interaction part) $\qquad \deg S = 0$

$$S_{int} := \sum_{n,g} \hbar^g m(n,g) \quad \text{solution to the BV QME}$$

$$dS_{int} + \hbar \Delta S_{int} + \frac{1}{2}\{S_{int}, S_{int}\} = 0$$

<u>Theorem</u> (Barannikov)

Algebras over $\mathcal{F}P \Longleftrightarrow$ solutions to QME

- Generalization $\mathcal{E}_V \to$ any odd modular operad
- Action in physics terms

a_i - basis of V $\qquad a_I = a_{i_1} \otimes \dots \otimes a_{i_n}$

ϕ^i - dual basis $\qquad \phi_I = \phi^{i_1} \otimes \dots \otimes \phi^{i_n}$

$$\left(\mathcal{P}(n,G) \otimes \mathcal{E}_V(n,G) \right)^{\Sigma_n} \cong \mathcal{P}(n,G) \otimes_{\Sigma_n} V^{\# \otimes n}.$$

S_{int} - BV action ; Δ, $\{\cdot,\cdot\}$ - BV-operation , $\phi = a_i \otimes \phi^i$ string field

Ex. 1 $\mathrm{Mod}(\mathrm{Com}^c)$ $\qquad G = g$ (or $G = 2g + \frac{n}{2} - 1$)

$$\mathcal{P}(n,G) \otimes_{\Sigma_n} V^{\# n} \cong S^n(V^{\#}) \qquad S_{int} \in S(V^{\#})[[\hbar]]$$

$$S_{int} = \sum_{n,G} \frac{\hbar^G}{n!} \underbrace{f_n^G(a_I)}_{\text{graded sym.}} \phi^I$$

solutions QME $\Updownarrow$ Loop homotopy algebras $\qquad$ Zwiebach Markl

$G=0$ L_∞^c

Ex. 2 Ass^c $S_{int} = \sum_m \underbrace{f_m(a_I)}_{\text{cyclic sym.}} \phi^I$ A_∞^c

Stasheff
Zwiebach

Ex. 3 $\mathrm{Mod}(\mathrm{Ass}^c)$ $G = 2g + b - 1$

$$S_{int} = \sum_{g, b, \vec{m}} \frac{\hbar^G}{b!\, m_1! \cdots m_b!} f^{g,b}_{a_{I_1} \cdots a_{I_b}} \phi^{I_1} \cdots \phi^{I_b}$$

cyclic

Doubek, Münster, B.J.

Quantum A_∞^c

Ex. 4 Quantum open-closed 1+3 combination

Zwiebach $G = 2g + b + \frac{n_c}{2} - 1$

$G = 0$ classical open-closed
Stasheff – Kajiura

Ex. 5 type II
quantum (super) loop homotopy algebra

Münster, B.J.

Product?

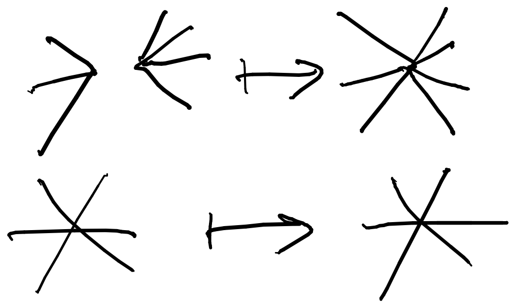
Doubek, Peksová, Pulmann, B.J.

Modular operads with <u>connected sum</u> #

deg 0

$$\#\quad P(C,g)\otimes P(C',g') \longrightarrow P(C\sqcup C', g+g'+1)$$

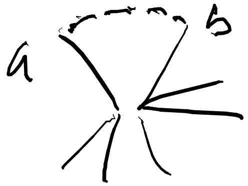
$$\#\quad P(C,g) \longrightarrow P(C, g+2)$$



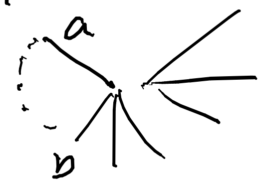
#– Σ-equivariant, associative
+ obvious compatibility conditions with ${}_a\circ_b$ and ξ_{ab}

e.g.

$$\xi_{ab}\# = \#\,{}_a\circ_b$$



$$\#\,\xi_{ab} = \xi_{ab}\#$$



$\vdots$

similarly for odd modular operads

Ex. 1 $\quad Mod(Com^c)$ $\qquad (G = 2g + \frac{n}{2} - 1)$

$(n, g) \# (n', g') \longrightarrow (n+n', g+g')$ i.e. connected sum of 2 surfaces

$\#(n, g) \rightarrow (n, g+1)$ i.e. adding handle

Ex. 2 $\quad Mod(Ass^c)$ – the same interpretation as above

$G = 2g + b - 1$

Ex. 3 $\quad End\, V \qquad \mathcal{E}_V(n, G) = Hom(V^{\otimes n}, \mathbb{C})$

$f \# g$ is induced by $\otimes$

$\# f = f \qquad f \in Hom(V^{\otimes n}, \mathbb{C})$ – doesn't depend on G

Algebra over $\mathcal{F}P$ gets a product $\cdot$ induced

by $\#_P$ and $\#_{\mathcal{E}_V}$

<u>Prop</u> $(P \otimes_{\mathbb{Z}} \mathcal{E}_V, \Delta, \cdot)$ is a BV-algebra (BD-algebra)

Algebras over $\mathcal{F}P \Longleftrightarrow \Delta e^S = 0$

<u>Remark</u> $\Delta(f \cdot g) = \Delta f \cdot g \pm f \Delta g \pm \{f, g\}$ $\quad$ $\{\cdot, \cdot\}$ the same as before

$\#$

$$\boxed{(d + \Delta) e^{\frac{S_{int}}{\hbar}} = 0}$$

or more accurately

$$k((\hbar)) \otimes P \otimes_{\mathbb{Z}} \mathcal{E}_V \Big/ \# f \sim \hbar f$$

Laurent

Minimal model (S-matrix)

- direct application of Homological Pert. Lemma

$$V = \underset{\text{physical}}{H} \oplus \underset{\text{trivial}}{\operatorname{Im} Q} \oplus \underset{\text{non-physical}}{C}$$

decomposition compatible with ω $\Rightarrow$ $\omega_H = \omega|_H$

$\Delta_H = \Delta|_H$

$$h \circlearrowright (V, Q) \underset{i}{\overset{p}{\rightleftarrows}} (H, 0)$$

$$hQ + Qh = ip - 1 \quad + \dots$$

h, Q, i, p extend to fields $\mathcal{F}(V)$ e.g. $Q = \{S_{\text{free}}, \cdot\}$

HPL $\overset{h'}{\circlearrowright}(\mathcal{F}(V), Q+\Delta) \underset{i'}{\overset{p'}{\rightleftarrows}} (\mathcal{F}(H), \Delta_H)$

- $e^{W} := p(1-\Delta h)^{-1} e^{S_{int}} = p' e^{S_{int}}$

$$\Delta_H e^{W} = 0$$

i.e. W defines a loop homotopy algebra on H

Theorem $e^{W} = \int_L e^{S} \quad \left(= p\, e^{\text{propagator}} e^{S_{int}}\right)$

$L \subset \mathrm{Im}\, Q \oplus C$ gauge fixing trivial fields $= 0$

propagator $= [\Delta, h_0]$

i.e. Minimal model $\Leftrightarrow$ effective

$\Delta e^{W} = 0$ Ward id.

$h_0 = \frac{1}{\#\,\text{triv.} + \#\,\text{nonph.}} h$, $S = S_{free} + S_{int}$

H. Kajiura

C. Albert

P. Mnev

A. Cattaneo

O. Gwilliam

;

- works for any algebra over Feynman transform of a general operad with connected sum. — Doubek, Peksová, Palmermann, B. J.

- HPL works also without connected sum (product)

perturbation $Q \to Q + \Delta + \{S_{int}, \cdot\} = Q + \delta$

gives on H $\quad \Delta_H + \{W, \cdot\}_H \qquad p'' = p(1 - h\delta)$

– with connected sum $\to$ relation

$$p''(f) = e^{-W} \int_L f e^S$$

– normalized path integral

– recursion relation for S-matrix (generalization of B.-G.) — Marcelli, Sä[illegible], Wolf, BJ

IBL_∞ (IBA_∞, ...) world

$$(\Delta + \{S, \cdot\})^2 = 0 \iff \text{BV QME}$$

More generally D — degree one differential operator on $\mathcal{F}(V)$

homological

$$D^2 = 0$$

$$\Updownarrow$$

Algebras over cobar construction over properads

- IBL_∞ point of view
  - nice interpretation of open-closed quantum SFT

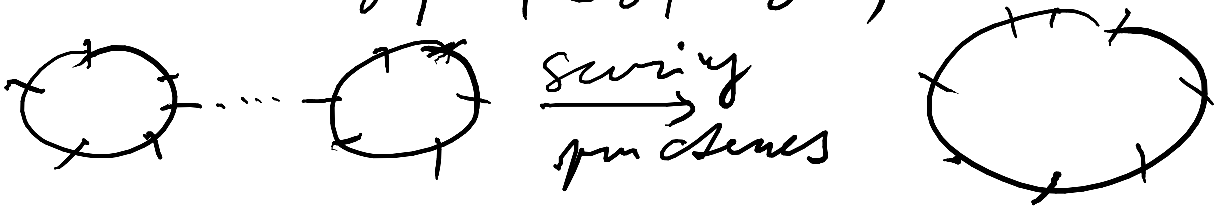
closed SFT $\qquad IBL_\infty - \mathcal{L}_c$

$(V_c, \Delta_c + \{S_c, \cdot\})$

open SFT $\qquad IBL - \mathcal{L}_o$

(cyclic cochains on V_o, $[\,,\,]_o$, δ_o)

bracket



cobracket

• open-closed SFT

$\mathcal{L}_c + \mathcal{L}_{oc}$ — all but closed vertices

<u>Theorem</u> $\mathcal{L}_{oc}$ is an IBL_∞ morphism from $\mathcal{L}_c$ to $\mathcal{L}_o$

Münster
Sachs

classical version

Kajiura
Stasheff

I didn't discuss:

HPL $\longrightarrow$ Explicit homotopy between W and S

$\longrightarrow$ Morphism between corresponding quantum homotopy algebras

$\longrightarrow$ Odd symplectic category

$\longrightarrow$ works for IBL_∞ & generalizations (algebras over cobar construction of properads)

RG in (generalized) BV-theories

Zwiebach's construction
of SFT – operadic interpretation

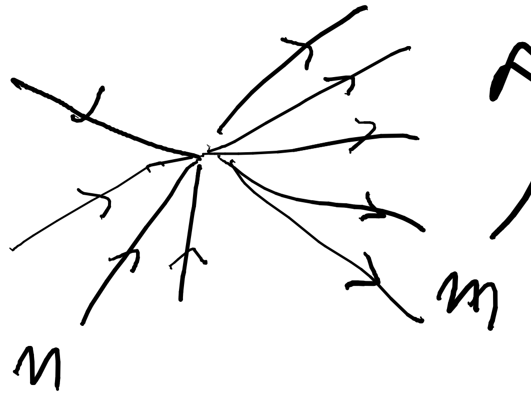
$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{FP} \arrow[r, "\text{decomposition}", "\substack{\text{of moduli space}\\ \text{(geom. vertices)}}"'] \arrow[dr, "\substack{\text{SFT}\\ \text{action}\\ \text{(alg. vertices)}}"'] & \substack{\text{chains}\\ \text{on a moduli space}} \arrow[d, "\text{CFT}"] \\
& \mathrm{End}\, V
\end{tikzcd}$$

philosophy adopted in:
M. Doubek, M. Markl, I. Sachs, BJ   LNP 973

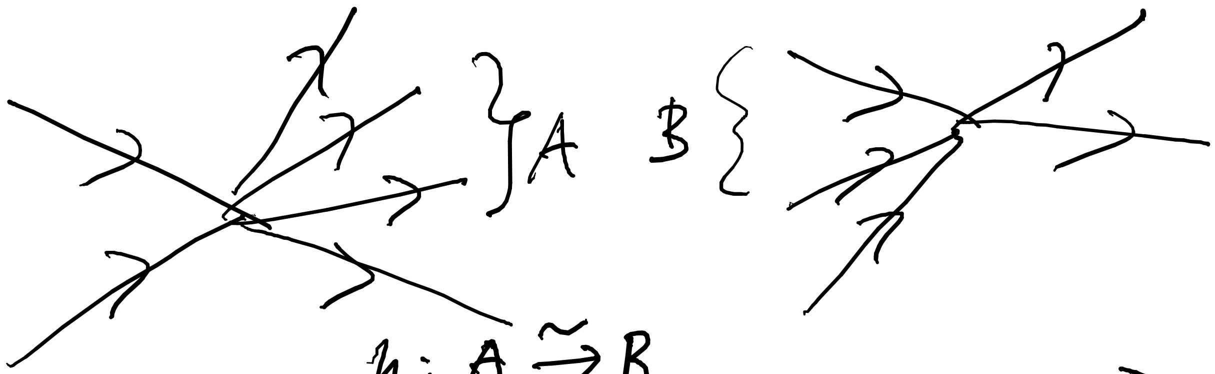
Properads

$\mathrm{DCor} := \mathrm{Cor} \times \mathrm{Cor}$

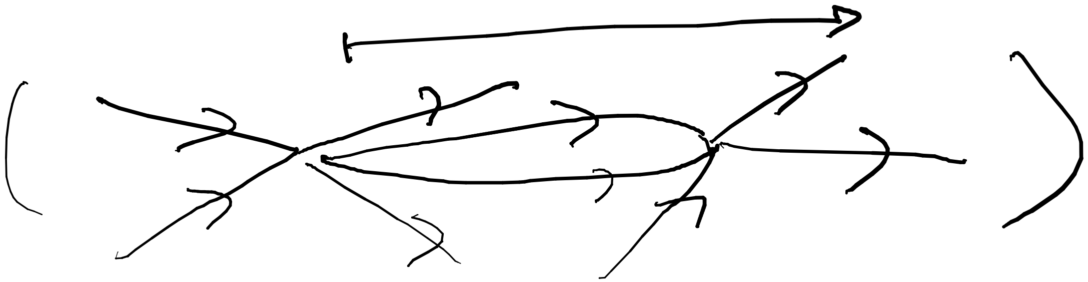
category of directed corollas (inputs, outputs)

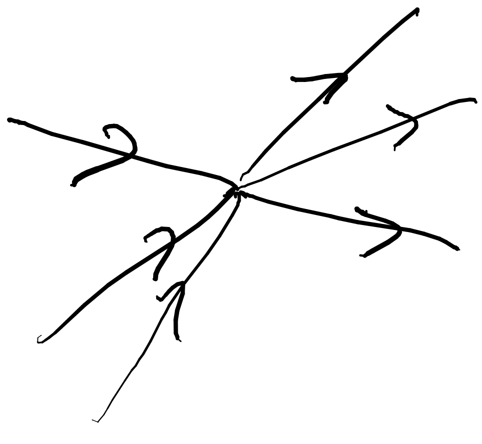
$\Sigma_n \ni \rho \Big($  $\Big) \ni \sigma \in \Sigma_m$

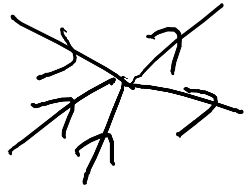
$A \overset{n}{\circ} B$



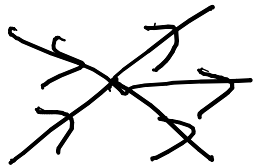
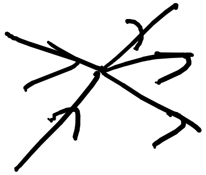
$\eta : A \xrightarrow{\sim} B$

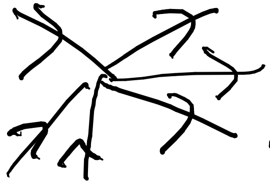
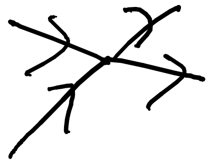
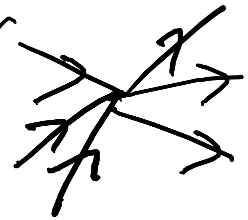




Properad $P($ n  m $)$ – collection of d.g.v.s.

\+ morphisms $\deg = 0$

$P($  $)$ $\xrightarrow{P(\rho,\sigma)}$ $P($ $\rho\,($  $)\,\sigma$ $)$ $\qquad \Sigma \times \Sigma$ action

$P($  A $)$ $\otimes$ $P($ B  $)$ $\xrightarrow{A \overset{\eta}{\circ} B}$ $P($  $)$

$A \overset{\eta}{\circ} B$

- $\Sigma \times \Sigma$ equivariant
- associative

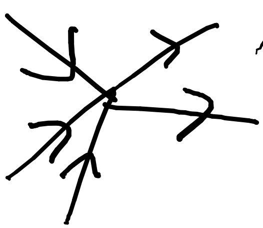
Assume additional grading by $G \in \mathbb{N}_0$

"Euler characteristic" $\chi := 2G + n + m - 2$

$\mathcal{P}(n, m, G) \sim \mathcal{P}(n, m, \chi) \neq \emptyset$ only if $\chi > 0$ (stability)

$G = 0$ $n + m \geq 3$, $G = 1$ $n + m \geq 1$

<u>Ex. 1</u> Closed Frobenius properad $\mathcal{F}$

$\mathcal{P}\Big($  $\chi \Big) = \Big\langle$  $\chi \Big\rangle$ $\quad \chi > 0$

$\Sigma \times \Sigma$ action – trivial

n
$\circ$
$A \quad B$ – obvious (do not depend on A, B and n)

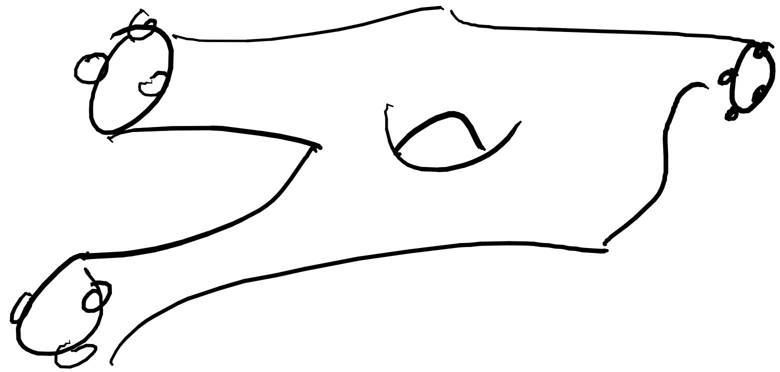
(homeomorphism classes) of oriented surfaces with n-input and m-output punctures and genus G

$\amalg$ — gluing of subsets A (outgoing), B (incoming)

$A \circ B$ — punctures of two surfaces and

<u>Ex. 2</u> Open Frobenius OF properad

Surfaces with input and output boundaries with punctures on the boundaries



$G = 2g + b - 1$

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B. J.

- glueing of products mixes input and output boundaries
- well defined splitting of mixed boundaries

Ex. 3 OCF combination of 1+2 (2-colored properad)

Ex. 4 End_V

$$P(n,m,X) = Hom(V^{\otimes n}, V^{\otimes m})$$

$\Sigma \times \Sigma$ action – permutations of tensor factors

$A \overset{n}{\circ} B$ – obvious using (partial) compositions

Feynman transform $\to$ cobar construction

Analogue of Barannikov's construction

$\to$ homological differential operation

Ex. BV-operator $\Delta + \{S, \cdot\}$

Thank you for your attention