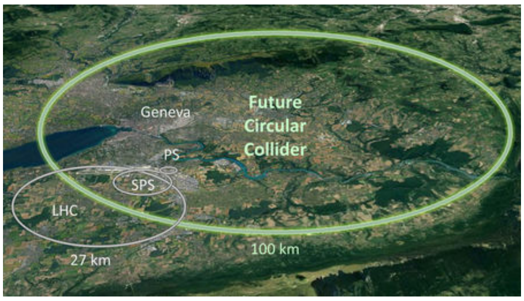
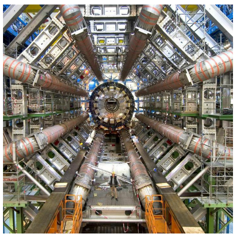
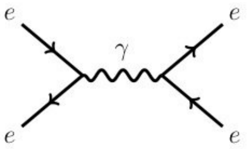
Scattering Amplitudes & Feynman Diagrams from Homotopy Algebras

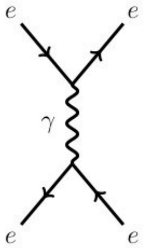
Literature: arXivs:

1809.09899 – BV & L_∞-algebras
2002.11168 – Scattering Amplitudes
2009.12616 – Detailed Feynman Diags
2102.11390 – Summary & Application



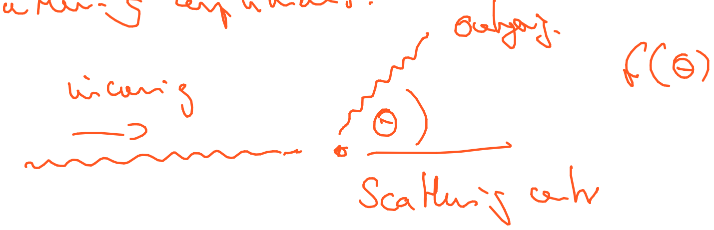




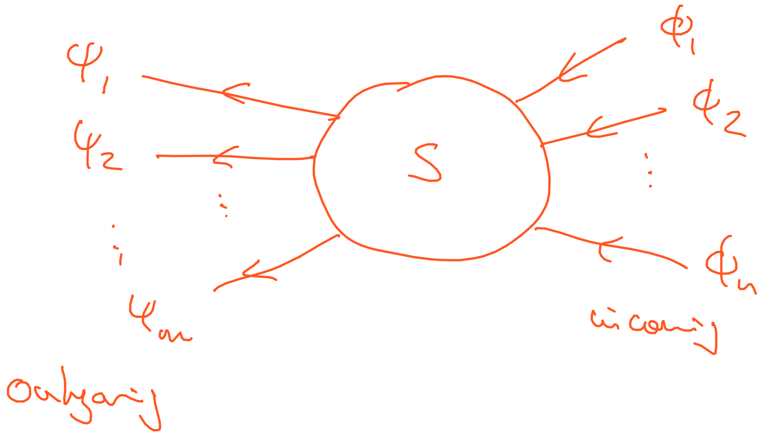


Scattering Amplitudes & Homotopy Algebras
Overview

Scattering amplitudes:



QFT: More generally



$\langle \psi_1, \dots, \psi_m | S | \phi_1, \phi_2, \dots, \phi_n \rangle$

$\nearrow$ S-matrix

$S = \mathbb{1} + iT$

$\nearrow$ Computed by Feynman diagrams.

QFT (complicated)

- Field theory: action functional
- path integral
- Wick rotation
- Propagators + Feynman rules
- Wick contractions
- Feynman diagrams
- LSZ-reduction

Homotopy algebras

- Field theory $\rightarrow$ L_∞-algebra
- Any L_∞-algebra has a minimal model
- Higher products in minimal model are scattering amplitudes.

Actions → Homotopy Algebras

Abstractly:

$$S_{\text{classical}} \xrightarrow{BV} S_{BV},\ \{-,-\},\ Q_{BV} = \{S_{BV}, -\} \xrightarrow{\text{dual}} \text{cyclic } L_\infty\text{-algebra.}$$

Concretely: $S = \int d^4x \left(\frac{1}{2} \varphi (\Box - m^2) \varphi - \frac{\kappa}{3!} \varphi^3 - \frac{\lambda}{4!} \lambda^4 \right)$

identify with a hom. Maurer-Cartan theory.

$$S_{hMC} = \langle a, \tfrac{1}{2} \mu_1(a) + \tfrac{1}{3!} \mu_2(a,a) + \tfrac{1}{4!} \mu_3(a,a,a) + \dots \rangle$$

$a \in L_1$ of some L_∞-algebra L.

$a \Leftrightarrow \varphi$

$L_1 = C^\infty(\mathbb{R}^{1,3})$

$\mu_1 : L_1 \to L_2 \qquad \varphi \mapsto (\Box - m^2)\varphi \in L_2$

$L_2 = C^\infty(\mathbb{R}^{1,3})$

$\mu_2 : L_1 \wedge L_1 \to L_2 \qquad (\varphi_1, \varphi_2) \longmapsto -\kappa \varphi_1 \varphi_2 \in L_2$

$\mu_3 : L_1 \wedge L_1 \wedge L_1 \to L_2 \qquad (\varphi_1, \varphi_2, \varphi_3) \longmapsto -\lambda \varphi_1 \varphi_2 \varphi_3 \in L_3$

$\langle -,- \rangle : L_1 \otimes L_2 \to \mathbb{C} \qquad (\varphi_1, \varphi_2^+) \longmapsto \int d^4x\, \varphi_1 \varphi_2^+$

Can be done for any polynomial actions

In general

$$\begin{tikzcd} \dots \arrow[r] & L_0 \arrow[r, bend left, "\text{gauge trafo}"] \arrow[r] & L_1 \arrow[r, bend left, "\text{eom}"] \arrow[r] & L_2 \arrow[r, bend left, "\mu_1 + \frac{1}{2}\mu_2 + \dots = 0"] \arrow[r] & L_3 \arrow[r] & \dots \end{tikzcd}$$

(higher ghosts) ghosts fields antifields Noether (higher Noether)

$\underbrace{\text{(higher ghosts) ghosts fields}}_{\text{dual to BRST}}$

dual to BRST

gauge structure / kinematics

$D^2 = 0$

$\langle -,- \rangle$ is compatible with μ_i

$$\langle \underbrace{\phi_0, \mu_n(\phi_1, \dots, \phi_n)}\rangle = \pm \langle \phi_n, \mu_n(\phi_0, \phi_1, \dots, \phi_{n-1}) \rangle$$

$\underbrace{\qquad\qquad\qquad\qquad}_{BV}$

BV

full classical field theory + dynamics.

Morphisms & Quasi-Isomorphisms

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ A & \arrow[u] & B \end{tikzcd}$$

3 pictures

Higher products

$m_1 : \mathcal{A} \to \mathcal{A}$

$m_2 : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$

$\vdots$

structure consts.

restrict + shift

differential graded algebras

$\odot^\bullet \mathcal{A}[1]^*$

$Q = D^*$

$Q^2 = 0$

duality $\longleftrightarrow$

Codifferential coalgebra

$\odot^\bullet \mathcal{A}[1]$

$D = \hat{m}_1 + \hat{m}_2 + \dots$

$D^2 = 0$

Morphisms of homotopy algs:

$=$ (Morphisms of dga $)^*$

Richer than graded vector space morphisms on $\mathcal{A}$

example: $\mathcal{A} = \mathcal{A}_{-1} \longrightarrow \mathcal{A}_0$

$\mathcal{A}[1]^*$ has basis t^α and r^a $\quad |t^\alpha| = 1$, $|r^a| = 2$

endomorphisms on $\odot^\bullet \mathcal{A}[1]^*$

$$\Phi(t^\alpha) = \phi^\alpha_\beta \tilde{t}^\beta$$

$$\Phi(r^a) = \phi^a_b r^b + \phi^a_{\alpha\beta} t^\alpha \odot t^\beta$$

dually: basis τ_α, σ_a on $\mathcal{A}$ $\quad |\tau_\alpha| = 0 \quad |\sigma_a| = -1$

$$\phi_1(\tau_\beta) = \phi^\alpha_\beta \tau_\alpha \ , \quad \phi_1(\sigma_b) = \phi^a_b \sigma_a \qquad |\phi_1| = 0$$

$$\phi_2 : (\tau_\alpha, \tau_\beta) \longmapsto \phi^a_{\alpha\beta} \sigma_a \qquad |\phi_2| = -1$$

generalize:

$\Phi : \mathcal{A} \to \mathcal{A} \qquad \phi_n : \otimes^n \mathcal{A} \to \mathcal{A} \qquad |\phi_n| = 1-n$

Isomorphism / equivalent

$$\begin{tikzcd} (\odot^\bullet \mathcal{A}[1]^*, Q) \arrow[r, bend left, "\Phi"] & (\odot^\bullet \tilde{\mathcal{A}}[1]^*, \tilde{Q}) \arrow[l, bend left, "\Psi"] \end{tikzcd}$$

such that

$\Psi \circ \Phi \cong \mathrm{id}$

$\Phi \circ \Psi \cong \mathrm{id}$

Quasi-Isomorphs & Field Theories

translate to homotopy algebra $\mathcal{A}$:

- m_1 in $\mathcal{A}$ is a differential
- ϕ_1 is a chain map. $\phi_n : \mathcal{A}^n \to \mathcal{A}$ $|\phi_n| = 1-n$

Quasi-isomorphism $\phi : \mathcal{A} \to \mathcal{B}$
is a morphism such that

$$\phi_1 : H^\bullet_{m_1}(\mathcal{A}) \xrightarrow{\cong} H^\bullet_{m_1}(\mathcal{B})$$

- Allows for integrating in / out of fields.
- BV trivial pairs $Q : \bar{c} \longrightarrow b$

$$C^\infty(\mathbb{R}^{1,3}) \xrightarrow{id} C^\infty(\mathbb{R}^{1,3})$$

cohomologically trivial

- Observables $\cong$ $\dfrac{\text{Functions on Field space}}{\text{gauge trafos} \qquad \text{eom}}$

$\uparrow$

this is what the cohomology does.

A_∞- and L_∞- algebras

3 pictures

Higher products

$m_1 : A \to A$

$m_2 : A \times A \to A$

$\vdots$

structure consts.

restrict & shift

differential graded algebras

$\bigcirc^\bullet A[1]^*$

$Q = D^*$

$Q^2 = 0$

duality $\longleftrightarrow$

Codifferential coalgebra

$\bigcirc^\bullet A[1]$

$D = \hat{m}_1 + \hat{m}_2 + \dots$

$D^2 = 0$

Specialize: $\bigcirc = \otimes$ A_∞-algebras

$\bigcirc = \odot$ L_∞-algebras.

Recall: Matrix algebras $A \cdot B$ antisymmetrize to Matrix Lie algebras $[A,B] = AB - BA$

A_∞-algebras m_i antisymmetrize to L_∞-algebras $\mu_i = \sum_{\sigma \in S_i} m_i \circ \sigma$

L_∞-algebras of interesting FT are antisyms of A_∞-algebras.

work here for pedagogical reasons.

Towards Minimal Models

m_1 : kinematical terms $\leftarrow$ Gaussian part.

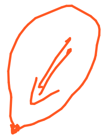
$m_2, m_3, \dots$: interaction terms

$\mathcal{A}^0 = H^\bullet_{m_1}(\mathcal{A})$

chain maps

$$\begin{tikzcd} (\mathcal{A}, m_1) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (\mathcal{A}^0, 0) \arrow[l, shift left, "e"] \end{tikzcd}$$

requires choice



$$p \circ e = \mathrm{id}_{\mathcal{A}^0}$$

$$\mathrm{id} - e \circ p = m_1 \circ h + h \circ m_1$$

$|h| = -1$

contracting homotopy.

Can choose h such that

$$p \circ h = h \circ e = h \circ h = 0$$

Coalgebra picture $\otimes^\bullet \mathcal{A}[1]^*$ Δ

codifferential D ...

$D^2 = 0$

m_i : higher products, lift to codifferentials.

$$\hat{m}_1(a_1 \otimes a_2) = m_1(a_1) \otimes a_2 + a_1 \otimes m_1(a_2)$$

$$\hat{m}_2(a_1 \otimes a_2 \otimes a_3) = m_2(a_1, a_2) \otimes a_3 + a_1 \otimes m_2(a_2, a_3)$$

$$\vdots$$

$$D = \underbrace{\hat{m}_1}_{D_0} + \underbrace{\hat{m}_2 + \hat{m}_3 + \dots}_{D_{\mathrm{int}}}.$$

$|D| = 1$ $D^2 = 0$

Tensor trick

$$\begin{tikzcd} (\mathcal{A}, m_1) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (\mathcal{A}^0, 0) \arrow[l, shift left, "e"] \end{tikzcd}$$

to

$$\begin{tikzcd} (\otimes^\bullet \mathcal{A}[1]^*, D_0) \arrow[loop above, "H_0"] \arrow[r, shift left, "P_0"] & (\otimes^\bullet \mathcal{A}^0[1]^*, 0) \arrow[l, shift left, "E_0"] \end{tikzcd}$$

with $P_n = p \otimes p \otimes \dots \otimes p$

$$E_n = e \otimes e \otimes \dots \otimes e$$

$$H_n = \sum_k \underbrace{\mathrm{id} \otimes \dots \otimes \mathrm{id}}_{k} \otimes h \otimes e \circ p \otimes \dots \otimes e \circ p$$

Then: again

$$\mathrm{id} - E_0 \circ P_0 = D_0 \circ H_0 + H_0 \circ D_0$$

$$P_0 \circ E_0 = \mathrm{id} \qquad P_0 \circ H_0 = H_0 \circ E_0 = H_0 \circ H_0 = 0$$

Homological Perturbation Lemma

$$\begin{tikzcd} (\otimes^\bullet \mathcal{A}[1], D_0) \arrow[loop left, "H_0"] \arrow[r, shift left, "P_0"] & (\otimes^\bullet \mathcal{A}^0[1], 0) \arrow[l, shift left, "E_0"] \end{tikzcd}$$

perturb to

$$\begin{tikzcd} (\otimes^\bullet \mathcal{A}[1], \mathcal{D} = D_0 + D_{int}) \arrow[loop left, "H"] \arrow[r, shift left, "P"] & (\otimes^\bullet \mathcal{A}^0[0], \mathcal{D}^0) \arrow[l, shift left, "E"] \end{tikzcd}$$

HPL:

$$P = P_0 \, (1 + D_{int} \circ H)^{-1}$$ ← regard as geometric series

$$E = (1 + H_0 \circ D_{int})^{-1} \circ E_0$$

$$H = H_0 \circ (1 + D_{int} \circ H_0)^{-1}$$

$$\mathcal{D}^0 = P_0 \, D_{int} \circ E_0$$

exist & satisfy same relations as $P_0, E_0, H_0,$

Consequences:

- $\mathcal{D}^0$ is a codifferential

  → Homotopy alg. structure on cohomology $\mathcal{A}^0$.

P, E are $\overset{\text{codiff}}{\vee}$ coalgebra morphisms.

$\mathcal{A}$ is quasi-isomorphic to $\mathcal{A}^0$

- $\mathcal{A}^0$ is called a minimal model for $\mathcal{A}$.

m_1 : kinematical part $\qquad m_1 = (\Box - m^2)\varphi$

h : "inverse"

$$\mathrm{id} - e \circ p = m_1 \circ h + h \circ m_1$$

on kernel of p

h : propagator $\qquad m_1(\phi) = 0$

$$\begin{tikzcd}
\text{fields} & \text{antifields} \\
\mathcal{A}_1 \arrow[r, "m_1"] & \mathcal{A}_2
\end{tikzcd} \qquad \begin{tikzcd}
\mathcal{A}_1^0 \arrow[r, "0"] & \mathcal{A}_2^0
\end{tikzcd}$$

$$\begin{tikzcd}
\text{free fields} \arrow[r, "0"] & \text{free fields} \arrow[r, "\text{quasi isomorphisms}"] & \text{free fields} \arrow[r, "0"] & \text{free fields.} \\
\oplus & \oplus & & \\
\text{interacting fields} \arrow[r, bend left, "m_1"] & \text{interacting fields} \arrow[l, bend left, "h = m_1^{-1}"] & &
\end{tikzcd}$$

Cohomologically trivial

minimal model.

higher products have to be scattering amplitudes.

Minimal Model Example

$$D = \hat{m}_1 + \hat{m}_2 = D_0 + D_{int}$$

$$\hat{m}_2(a_1 \otimes a_2 \otimes a_3) = m_2(a_1, a_2) \otimes a_3 + a_1 \otimes m_2(a_2, a_3)$$

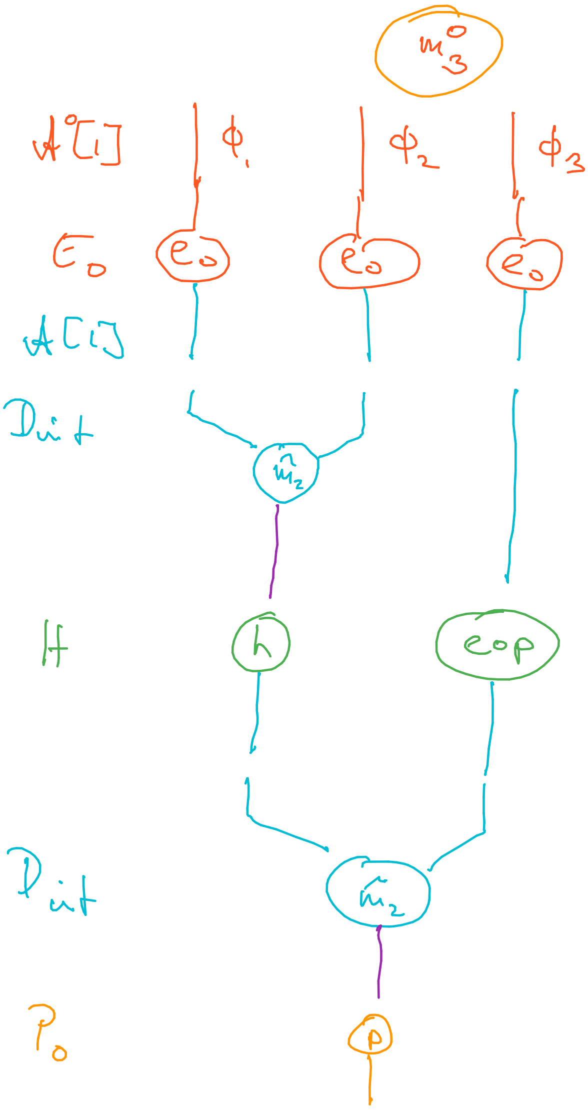
$$\hat{m}_2 : Y|\cdots| + |Y|\cdots| + \cdots + |\cdots|Y$$

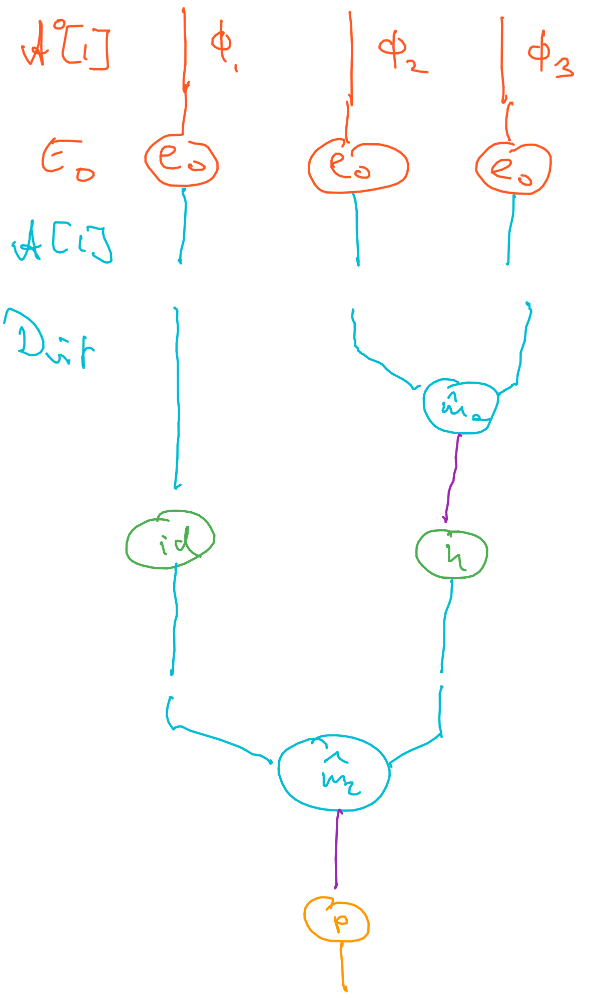
To identify minimal model, need to compute D^0

$D^0 = P \circ D_{int} \circ E_0$ $\qquad P = P_0 \circ (1 + D_{int} \circ H)^{-1}$

$$= P_0 \circ (1 - D_{int} \circ H + D_{int} \circ H \circ D_{int} \circ H - \cdots)$$

Let's compute $D^0 : \underline{\otimes^3 A[1] \to A[1]}$: $P_0 \circ D_{int} \circ H \circ D_{int} \circ E_0$.

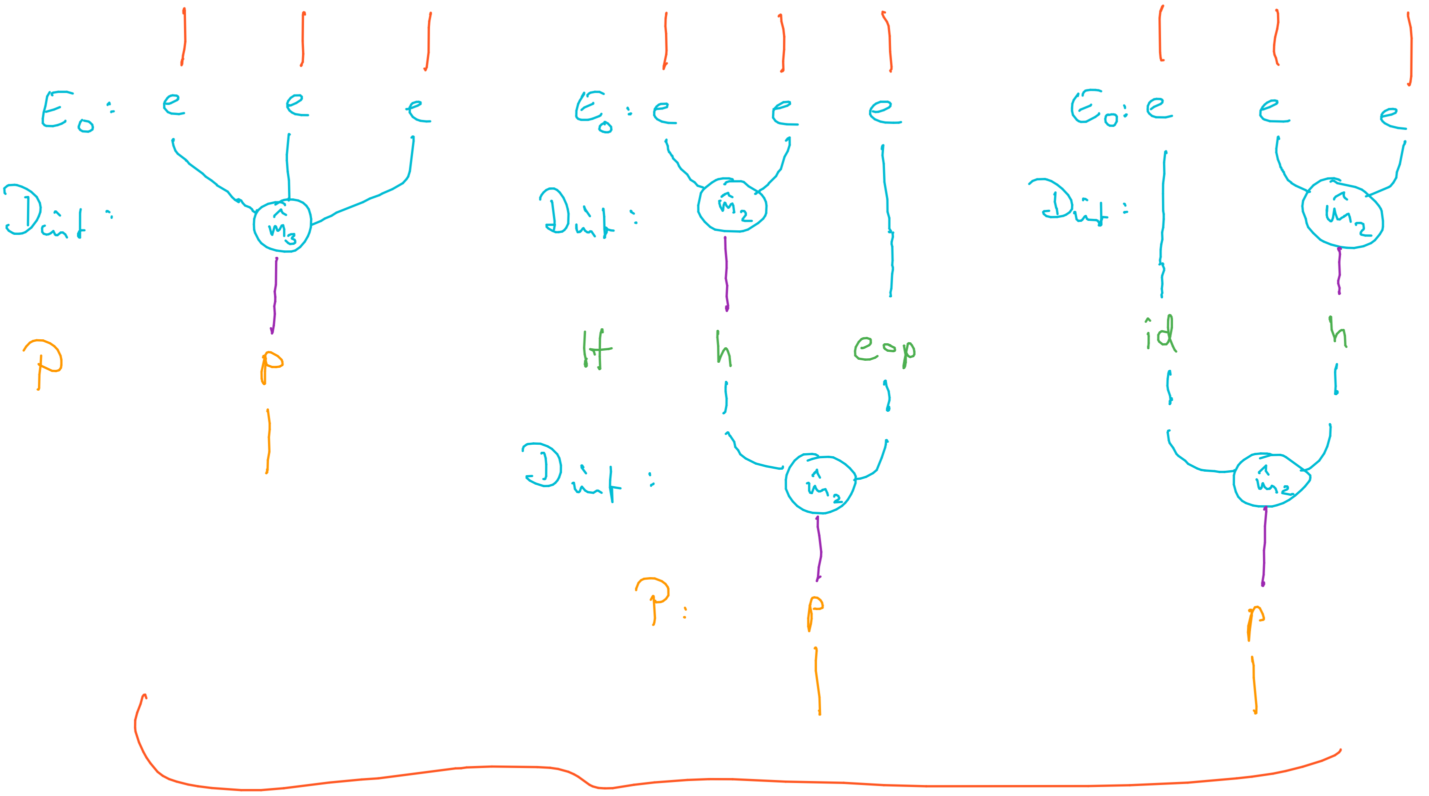




In this way: get all kinds of binary trees.
Looks like Feynman diagrams.

Another Example:

$D = \hat{m}_1 + \hat{m}_2 + \hat{m}_3$ $\qquad D^\infty = \dots + P_0 \circ D_{int} \circ E_0 + P_0 \circ D_{int} \circ H \circ D_{int} \circ E_0 + \dots$

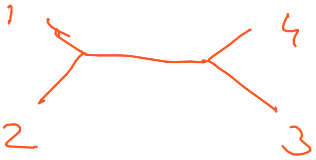


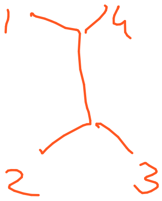
m_3^∞ in A^∞.

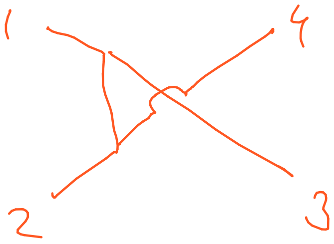
Note: # of terms ≠ # of Feynman diagrams.

s, t, u - channels

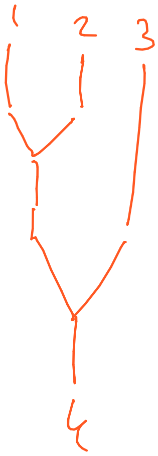
want



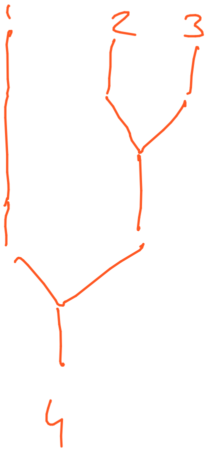




HPL:

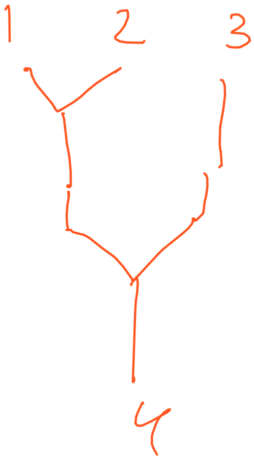


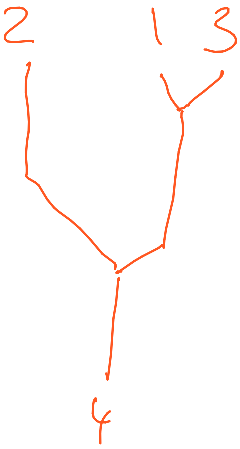
+

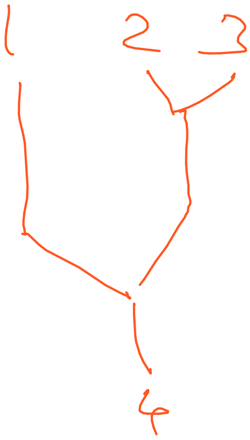


But need to consider L_∞ - algebra products

$$\underline{\mu_3^0(\phi_1,\phi_2,\phi_3)} = \sum_{\sigma\in S_n} m_3(\phi_{\sigma(1)},\phi_{\sigma(2)},\phi_{\sigma(3)})$$







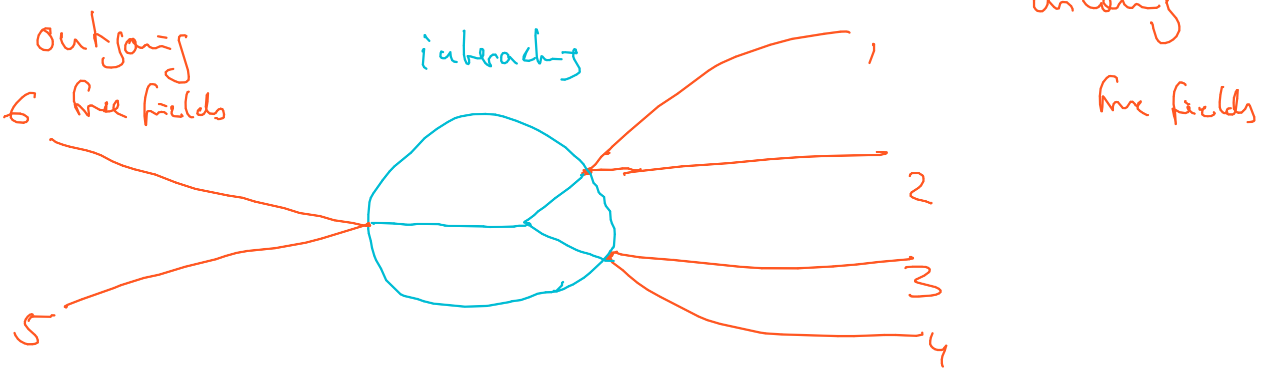
Scattering Amplitudes

$$\mathcal{A}(\phi_0, \phi_1, \dots, \phi_n) = \langle \phi_0, \mu_n^\circ(\phi_1, \dots, \phi_n)\rangle^\circ$$

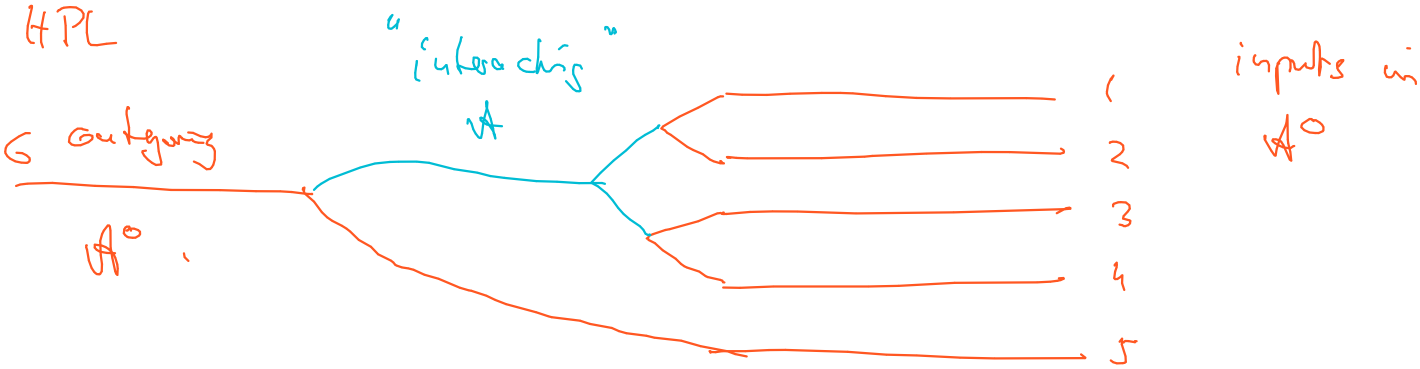
$$= \sum_{\sigma \in S_n} \langle \phi_0, \mu_n^\circ(\phi_{\sigma(1)}, \dots, \phi_{\sigma(n)})\rangle^\circ$$

Comparison

QFT: Feynman diagrams.



HPL



perturbative QFT
(tree level) $\longleftrightarrow$ HPL.

What about Loops?

$$\underbrace{Q_{BV} = \{S_{BV}, -\},\ \{S_{BV}, S_{BV}\} = 0}_{\text{classical}} \longrightarrow \underbrace{i\hbar\Delta + \{S_{BV}, -\},\ 2\hbar\Delta S_{BV} + \{S_{BV}, S_{BV}\} = 0}_{\text{quantum.}}$$

BV Laplacian $\Delta = \frac{\partial^2}{\partial\phi^I \partial\Phi^+_I}$ I: DeWitt indices.

Dually:

$$\Delta^*(\phi_1 \otimes \cdots \otimes \phi_n) = \sum_{i=0}^{n} \sum_{j=i}^{n} \phi_1 \otimes \cdots \otimes \phi_i \otimes \phi^I \otimes \phi_{i+1} \otimes \cdots \otimes \phi_j \otimes \phi^+_I \otimes \phi_{j+1} \otimes \cdots \otimes \phi_n$$

$$\begin{tikzcd} (\otimes^\circ A[1], D_0) \arrow[loop left, "H"] \arrow[r, shift left, "P_0"] & (\otimes^\circ A^\circ[1], d) \arrow[l, shift left, "E_0"] \end{tikzcd}$$

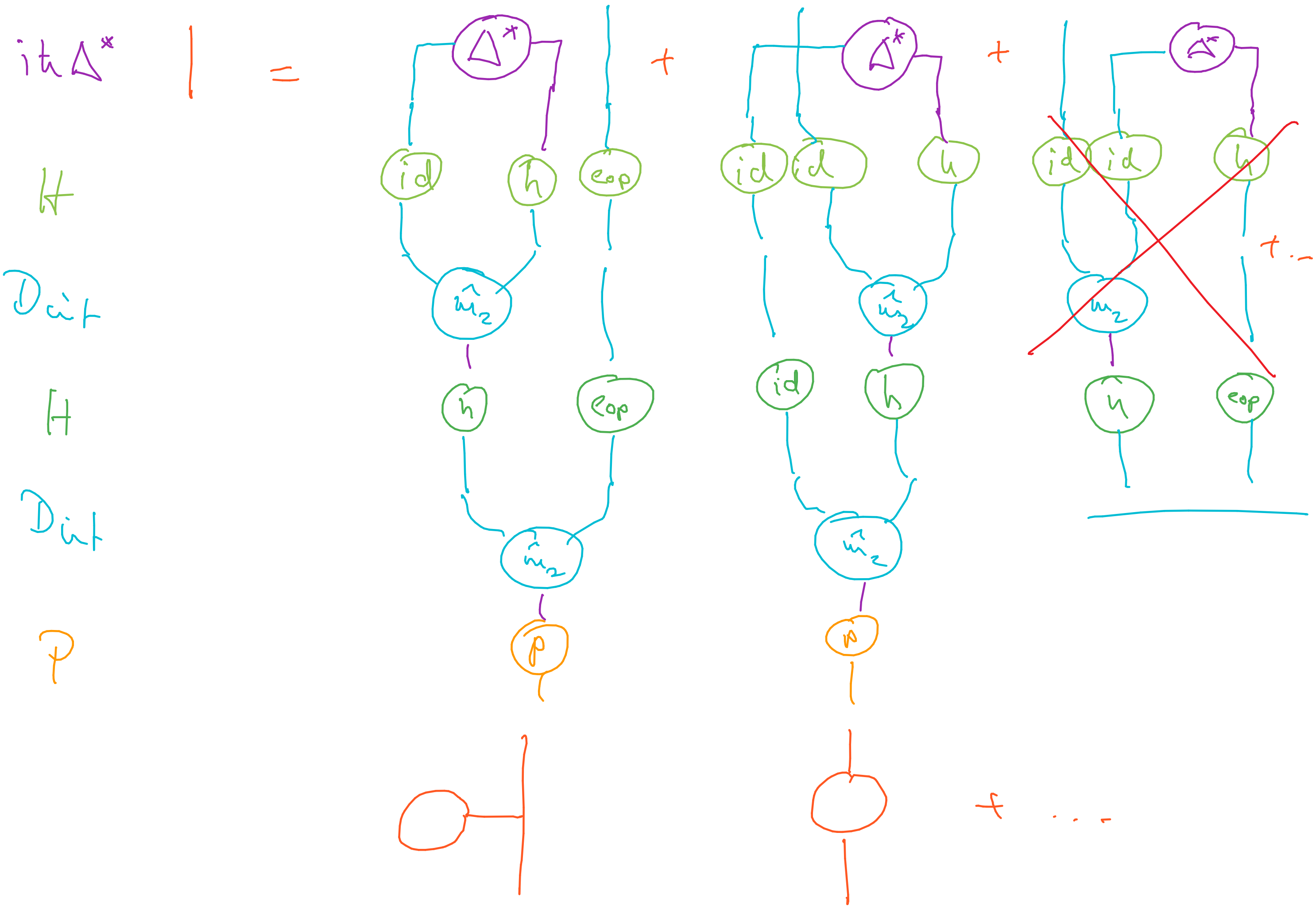
is perturbed by $D_0 \to D_0 + D_{int} + i\hbar\Delta^*$

$$\begin{tikzcd} (\otimes^\circ A[1], D_0 + D_{int} + i\hbar\Delta^*) \arrow[loop left, "H"] \arrow[r, shift left, "P"] & (\otimes^\circ A^\circ[1], D^\circ) \arrow[l, shift left, "E"] \end{tikzcd}$$

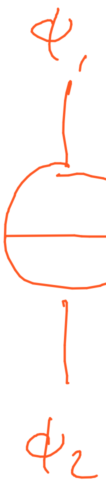
$D^\circ = P \circ (D_{int} + i\hbar\Delta^*) \circ E_0$ $P = P_0 (1 + (D_{int} + i\hbar\Delta^*) \circ H)^{-1}$

This creates indeed loop diagrams, with the loop order controlled by $\hbar$.

e.g. $D_{int} = \hat{m}_2$ $D^\circ : A^\circ \to A^\circ$ $H = id \ldots id\ h\ cop \ldots cop$

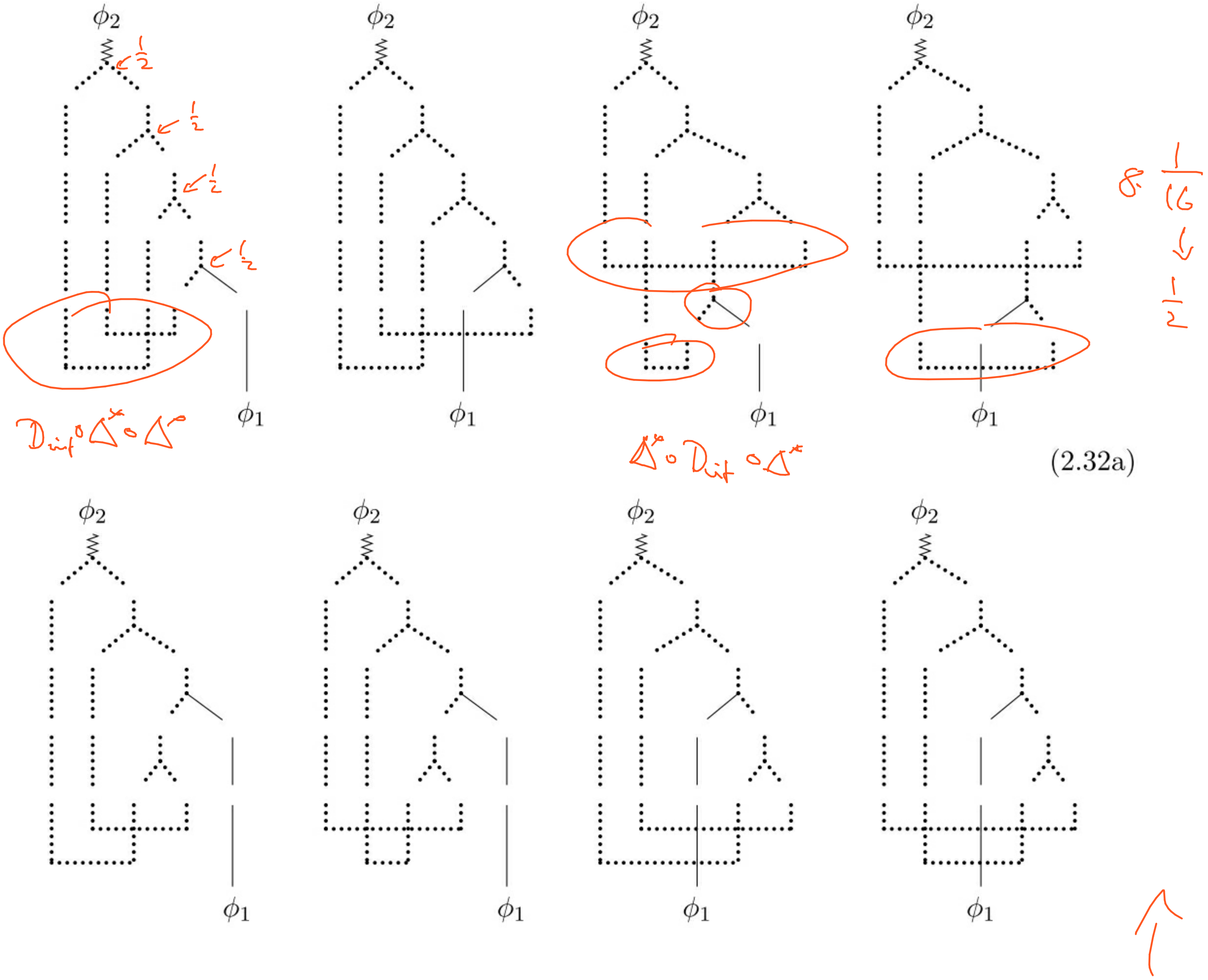


In this way, we get all loop diagrams and with correct symmetry factor ← non-trivial.



Example

Symmetry factor: $\frac{1}{2}$



$8 \cdot \frac{1}{16}$

$= \frac{1}{2}$

(2.32a)

each with $\frac{1}{2}$.

Remarks

- D^0 from HPL in loop case no longer a codifferential, i.e. no longer an A_∞-/ L_∞-algebra, but still $(D^0)^2=0$ defines quantum or loop A_∞-/L_∞-algebra.
- The recursion relation from HPL produces indeed the correct symmetry factors. 2009.1261
- Recursion relations are very useful:

  tree level:

  $$D^0 = P\circ D_{int}\circ E_0 \qquad P = P_0\,(1+D_{int}\circ H)^{-1}$$

  $$\underbrace{P = P_0 - P\circ D_{int}\circ H_0}_{\text{recursion}}$$

For Yang-Mills theory, at tree level:

Berends-Giele recursion relation for currents

↳ Parke-Taylor formula.

→ exists for <u>any</u> perturbative field theory at tree level.

Loops:

$$D^0 = P_0\,(D_{int} + i\hbar\Delta^*)\circ E_0$$

$$P = P_0 - P\circ(D_{int}+i\hbar\Delta^*)\circ H_0$$

For each order $\hbar$ (loop order) and g (coupling constant), need expressions for P lower in loop or coupling order.

→ Recursion relations.

exist for all perturbative QFTs at tree & loop level.

Outlook: Homotopy Algebras & QFT

Original motivation: higher gauge theory.

Classically: equivalence <—> quasi-isomorphisms.

e.g. 1st & 2nd order YM-theory.

strictifications exist (strictification theorem) <—> reformulate any classical FT in terms of exclusively cubic interactions.

attention: strictification may break quantum equivalence.

Colour decomposition of Yang-Mills theory   factorisation of homotopy algebras,

$$\underbrace{\mathcal{L}_{YM}}_{L_\infty\text{ algebra of Yang-Mills.}} = \underbrace{\mathfrak{g}}_{\text{gauge algebra}} \otimes \underbrace{\mathcal{C}}_{C_\infty\text{-algebra}}$$

tree level recursion.

Quantum level:

- Perturbative QFT = HPL for BV L_∞-algebras.
- Easy to implement in computer algebra programmes.
- Ideal tool for establishing combinatorial relations for amplitudes, e.g. 1-loop structure in Yang-Mills. Kleiss-Kuijf relations.
- Double copy: —> next week's talks.

$$\left.\begin{aligned}\mathcal{L}_{YM} &\cong \mathfrak{g} \otimes \text{kin} \otimes_\tau \text{scal}\\ \mathcal{L}_{grav} &\cong \text{kin} \otimes_\tau \text{kin} \otimes_\tau \text{scal}\end{aligned}\right\}$$

- Special theory with special amplitudes "enhanced" L_∞-algebras, e.g.

Reiterer: $BV_\infty^\square$-algebra.

Far-Outlook

- Homotopy algebras   QFT $\rightarrow$ Mathematicians (e.g. Costello, ...)
- ST dualities : (quantum) quasi-isomorphism.
- Renormalization group + homotopy algebras?
- Integrable models : recursions should be solvable or particularly nice.
- Quantum quasi-isomorphisms $\Leftrightarrow$ universality classes in Stat. Mech.