Homotopy Algebras and String Theory

Homotopy algebras are a perturbative manifestation of the BV structure of some (dynamical) system.

More or less any Lagrangian field theory has a (possibly trivial) BV-extension.

In this sense, every perturbative field theory (with- or without gauge in-variance) gives rise to a homotopy algebra.

Not every BV-system necessarily derives from a Lagrangian system: In the geometric BV, one instead starts with a nilpotent vector field on some graded space $\mathcal{F}$.

If $\mathcal{F}$ admits a symplectic form one can (re) construct an action perturbatively. This situation arises in string theory.

A unifying characterisation of BV-algebras is provided by operads.

Rep: BRST → BV :

"boundary cond."

auxiliary

gauge fixing

$$\Psi = B(\Pi + F(A)) = \text{function on } \mathcal{F}$$

B-ghost

$$S_{BRST}[\Phi] = S_0[A] + s(\Psi)$$

$$\{\Phi\} = \{\overset{(0)}{A_\mu}, \overset{(-1)}{B}, \overset{(1)}{C}, \overset{(0)}{\Pi}\} \quad \text{fields}$$

$$\begin{cases} s(A) = D_A C \\ s(B) = \Pi \\ s(C) = [C, C] \end{cases}$$

affine conn. on $E \to M$ G-bdle

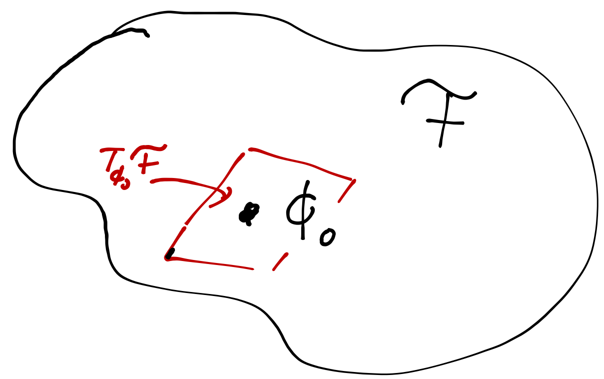
Lie bracket on $\mathfrak{g}$

$$S_{BV} = S_0[\Phi] + \int s(\Phi)\,\Phi^*$$

$$\{\Phi^*\} = \{\underset{(-1)}{A_\mu^*}, \underset{(0)}{B^*}, \underset{(-2)}{C^*}, \underset{(-1)}{\Pi^*}\} \quad \text{anti-fields} \quad \deg(\Phi^*) = -\deg(\Phi) - 1$$

$$\boxed{(S, S) = \frac{\partial_R S}{\partial \Phi}\frac{\partial_L S}{\partial \Phi^*} = 0 \quad \text{Master equation (classical)}}$$

Mathematical setting:



$S : \mathcal{F} \longrightarrow \mathbb{R}$ (BV-action)

$dS|_{\Phi_0} = 0$

BV-data $(\mathcal{F}, S, \Delta)$ $\quad \Delta$: 2nd order differential (BV-operator)

$$e^{-\frac{1}{\hbar}S} \Delta e^{\frac{1}{\hbar}S} =: \frac{2}{\hbar^2} (S, S) + \frac{1}{\hbar} \Delta S = 0$$ (Quantum BV-master eqn)

$v \equiv (S, \cdot)$ defines a homological vector field $v^2 = 0$ $(\hbar = 0)$

<u>Rem:</u> a symplectic form is not required at but if ω is a non-degenerate symp. form on $T\mathcal{F}$

$$dS = i_v \omega$$

$(dS)|_{\Phi_0} = 0$ (critical point) $\iff v(\Phi_0) = 0$

We can then expand v around Φ_0 :

$$v = \underbrace{v^{(0)}_{\Phi_0}}_{\neq 0} + \underbrace{v^{(1)}_{\Phi_0}}_{T_{\Phi_0}\mathcal{F} \to T_{\Phi_0}\mathcal{F}} + \underbrace{v^{(2)}_{\Phi_0}}_{T_{\Phi_0}\mathcal{F} \to T_{\Phi_0}\mathcal{F} \otimes T_{\Phi_0}\mathcal{F}} + \cdots$$

Then $v^2 = 0$ is the condition that $\{ v^{(n)}_{\Phi_0} \}_{n \geq 0}$

define a L_∞ - algebra.

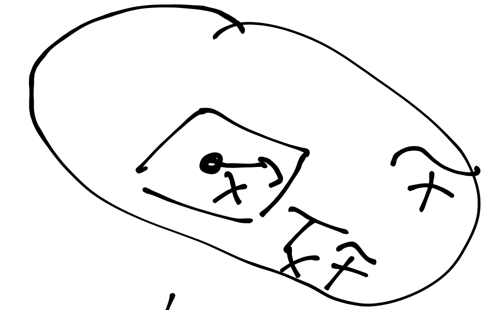
If $Q \equiv v^{(1)}_{\Phi_0}$ and H = coh (Q), then $\{ v^{(n)}_{\Phi_0} \}_{n \geq 0}$

induces on H the structure of a minimal L_∞- algebra: $\{ f^{(n)} \}_{n \geq 1}$

homological perturbation lemma

~ S-matrix = obstruction to e.o.m. on H

Construction: (Fukaya ?)



Let $\quad T_{\Phi_0}\mathcal{F} \ni \Psi = \overset{H}{\Psi_p} + \Psi_u$

$\Psi_u \stackrel{\perp}{=} Q^{-1}Q\Psi \in H^{\perp}$

and $\quad l_2 \equiv \tilde{v}^{(2)}_{\Phi_0}\,, \quad \tilde{v}^{(n>2)}_{\Phi_0} = 0$

$v = \underset{=0}{v^{(0)}} + \underset{Q}{\overset{\shortparallel}{v^{(1)}}} + \underset{l}{\overset{\shortparallel}{v^{(2)}}} +$

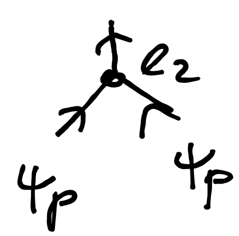
e.w. $\quad 0 = Q\Psi_u + l_2(\Psi,\Psi) = Q\Psi_u + l_2(\Psi_p+\Psi_u, \Psi_p+\Psi_u)$

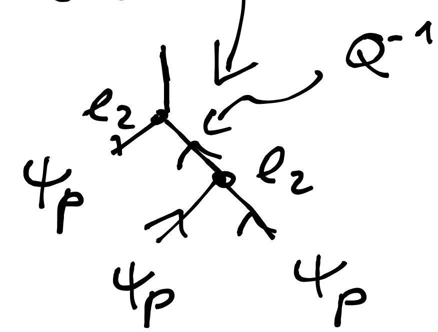
Q^{-1}: $\quad \Psi_u = -Q^{-1}l_2(\Psi_p + Q^{-1}l_2(\dots), \Psi_p + Q^{-1}l_2(\dots))$

Obstructions: $\quad \underbrace{Pl_2(\Psi_p,\Psi_p)}_{s^{(2)}}\,, \quad \underbrace{Pl_2(\Psi_p\, Q^{-1}l_2(\Psi_p,\Psi_p) + \dots)}_{s^{(3)}}\,, \quad \dots$

$P \xrightarrow{pr} H$

$\#$





Rem: hom. perturb. lemma has wider range of applicability: quantum, elimination of d.o.f's

Summary:

vertices

L_∞ $\quad$ cdc $\quad$ univ L_∞

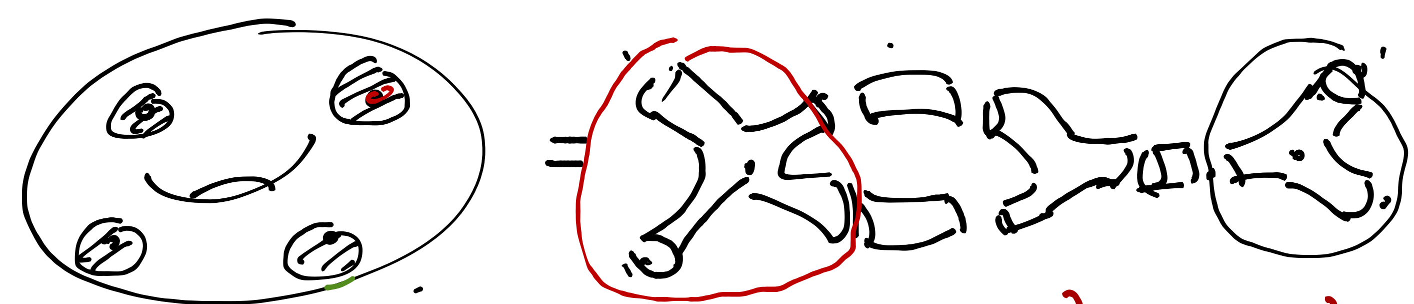
$$(\mathcal{F}, \Delta, S) \rightsquigarrow v \longrightarrow \{v^{(n)}\} \rightsquigarrow \{s^{(n)}\}$$

how v.f $\longleftarrow$? $\longleftarrow$ S.F.T $\Longleftarrow$ $\underbrace{}_{\text{S.T.}}$

SFT:

$\Sigma_{g,n}$

geometry:



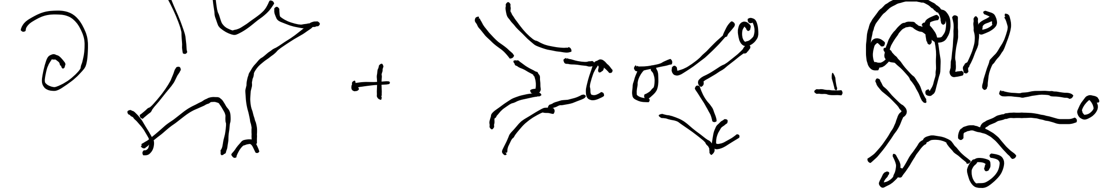
Consistent decomposition of $\mathcal{M}_{g,n} \Rightarrow$ genus, # punct.

bound. op $\quad$ chains of

$$\partial V_4 + \{V_3, V_3\} + \hbar \Delta V_6 = 0$$

(Master eqn)
(Zwiebach)

$+ \; \ldots$

$g = 0$ $\qquad \mathcal{V}_n \qquad \partial$ 

$$\Big\downarrow \; CFT_{\Sigma_{0,n}} \quad \text{chain map}$$

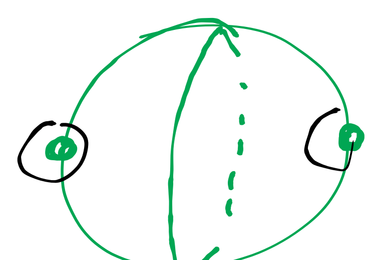
vertex op. (plus inf)

$$\int_{\mathcal{M}_{\mathcal{V}_n}} \underbrace{C(a_1, \dots, a_n)}_{\text{CFT-corr. fu}} : \underline{\underline{V}}^{\otimes n} \longrightarrow \mathbb{C}$$

$$\text{S.t.} \quad \int_{\partial \mathcal{M}_{\mathcal{V}_n}} C(a_1 \cdots) = \int_{\mathcal{M}_{\mathcal{V}_n}} C(Q a_1 \cdots)$$

$v^{(1)}_{\Phi_0}$

from CFT

symplectic form: 

$\mathcal{V}_{02} \xrightarrow{CFT} \omega(\cdot,\cdot)$ on V

Result: $S[\Phi] = C_2(\Phi, Q\Phi) + \sum_{n\geq 2} C_n(\Phi^{\otimes n})$

$$= \frac{1}{2}\omega(\Phi, Q\Phi) + \sum_{n\geq 2} \frac{1}{n+1}\omega(\Phi, l_n(\Phi^n))$$

CFT_Σ is a BV-morphism $\Rightarrow$ $(S, S) = O(\hbar)$

$$\Rightarrow l_2(l_2(a,b),c) \pm l_2(a, l_2(b,c)) =$$

$$\pm Q l_3(a,b,c) \pm l_3(Qa,b,c) \pm l_3(a,Qb,c) \pm l_3(a,b,Qc)$$

$\cdots$

(Stasheff, Kajiura)

Or, more economically, as coderivations,

$$L = (Q\otimes 1\!\!1 \otimes \cdots \otimes 1\!\!1) + (1\!\!1 \otimes Q \otimes \cdots) + (l_2 \otimes 1\!\!1 \otimes \cdots) + (l_3 \otimes \cdots\cdots)$$

$$\boxed{L^2 = 0}$$

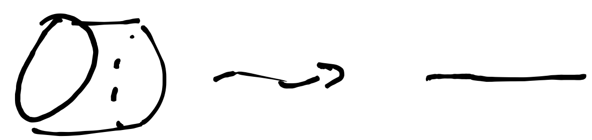
(plus cyclicity)

Definition: SFT is a morphism between BV algebras

- is it well defined?
- is it unique?

elimination of p

Toy model:



world line $\quad I \to \int \sqrt{g_{\mu\nu}\dot{q}^\mu \dot{q}^\nu}$

Lorentzian metric on $(\mathbb{R}^n, g)$

$$I = \int_a^b \left(p\dot{q} - \frac{1}{2e}\left(g^{-1}(p,p) - m^2 \right) \right) d\tau$$

$$q : [a,b] \longrightarrow \mathbb{R}^n$$

reparam.

Reparametrisation invariance : $\delta q = \frac{p\,\varepsilon}{e}$; $\delta e = \dot{\varepsilon} \sim \dot{c}$

BRST quantisation of the world line $\rightsquigarrow$ $Q_{BRST} = c\,\underbrace{(g^{-1}(p,p) - m^2)}_{\equiv \mathcal{H}}$

$$\begin{cases} \delta e = -i\dot{c}\,, \quad \delta q = ip\,\frac{c}{e} \\ \delta b = \mathcal{H}\,; \quad \delta c = \delta p = 0 \end{cases}$$

$\sim$ states

To identify the vector space $\boxed{V}$, we look for a representation of the graded asscociative algebra

$$\mathcal{A} = \{p, q, b, c\} \qquad [p,q] = 1$$

$$[b,c] = 1$$

$$Q = cH \qquad p = -i\,\partial_q\,, \quad b = \frac{\partial}{\partial c} \qquad c^2 = 0$$

$$H = (p^2 - m^2)$$

a generic state $\Phi = \phi(q) + \phi^*(q)\,c\,, \quad \deg(\Phi) = 0$

$$H = coh(Q)\big|_0 = \{\phi \in C^\infty(\mathbb{R}^n) \mid (\Box - m^2)\phi = 0\}$$

$= Q_{BRST} \hookrightarrow \partial$

$\mathcal{F}$

$\rightsquigarrow$ massive scalar field ϕ plus anti field ϕ^*

Symplectic form : $\omega(\Phi,\Psi) = \int dc \int d^n q \, \bar{\Phi} \, \Psi$

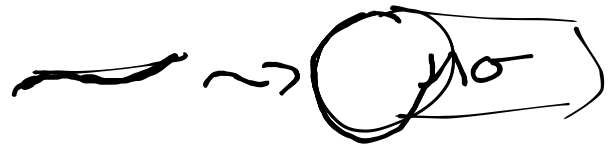
BV-action : $S[\Phi] = \omega(\Phi \, Q \, \Phi)$

$\therefore$ *no gauge invariance.*

<u>Rem</u>: At this point one may forget about the world line and simply consider representations of the graded associative algebra generated by {p,q,b,c} with a nilpotent element Q. $\mathcal{A}$

This leads to interesting generalisations describing Yang-Mills theory, Gravity, ...

(Dai, Huang, Siegel; Bonezzi, Meyer, I.S.)



For the string, $\{q, b, c\}$ are replaced by

$q \in C^\infty(S^1 \times [a,b], \mathbb{R}^n)$, $\begin{cases} c \in T\Sigma_{gn}, \text{ odd v.f. on } \Sigma_{gn} \\ b \in (T\Sigma)^{\otimes 2} \text{ symm. tensor} \end{cases}$

$c(\sigma)$

$\rightsquigarrow$ $\mathcal{A}$ is infinite dimensional... $[L, L]_+ = 0$

$[L+\delta L, L+\delta L] = 0$

coderivation

Uniqueness: $L \rightsquigarrow L + \delta L$

$[L, \delta L] = 0$

$\rightsquigarrow$ $d_c \equiv [L, \cdot\,]$ Chevalley-Eilenberg differential

with non-trivial deformations of L given by the cohomology of d_c

Thm. (w/N.Moeller) : coh(d_c)= $\emptyset$

$\rightsquigarrow$ (local) background independence. $\mathcal{A}$

$$\nu^2 = 0 \overset{\text{Pert.}}{\Longrightarrow} [L,L] = 0 \quad \checkmark$$

$$\Longleftrightarrow (S,S) = o(\hbar) = \hbar \Delta S$$

Rem: Again, one may forget about the world sheet and replace $\mathcal{A}$ by some other (infinite dimensional) graded Lie algebra and construct a coderivation L such that [L,L]=0.

perturbative substitute for $\nu^2=0$

Example: Superstring $\quad \mathcal{A} = \{(q,\theta),(b,\beta),(c,\gamma)\}$

superpartners

geometric vertices on supermoduli space are obscure

Here, [L,L]= 0 can be solved recursively with suitable initial conditions on ℓ_2

(w/ Erler, Konopka; ....)

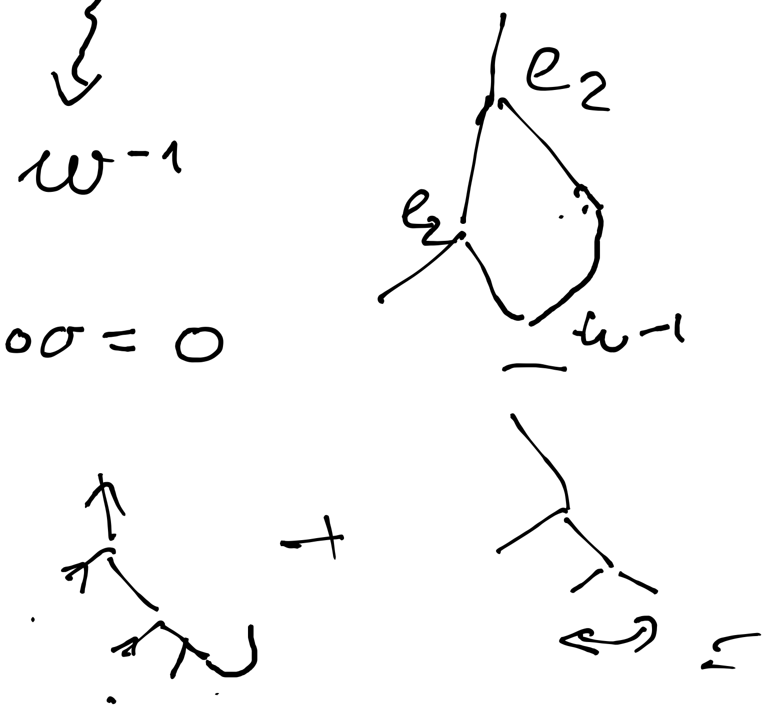
Rem: This problem can also be adressed directly within the BV-master equation. (Sen)

Quantum BV: $\partial V_4 + \{V_3, V_3\} + \boxed{\hbar \Delta V_6} = 0$

$\underline{\underline{Q}}$ $\quad \underline{\underline{l_4}}$ $\quad \underline{e_3}$ $\quad \omega^{-1}$

$$[L, L]\big|_n = 0 \iff \sum_{i_1+i_2=n} \sum_\sigma l_{i_1+1}\circ\Big(l_{i_2} \wedge \underbrace{1\wedge\cdots\wedge 1}_{i_1}\Big)\circ\sigma = 0$$

perm.



becomes under the inclusion of Δ:

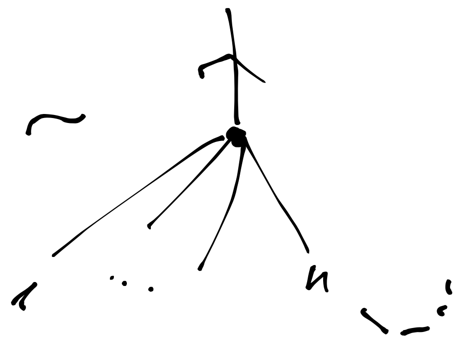
$$\sum_{\substack{g_1+g_2=g\\ i_1+i_2=n}} \sum_\sigma l^{g_1}_{i_1+1}\circ\Big(l^{g_2}_{i_2}\wedge\underbrace{1\wedge\cdots\wedge 1}_{i_1}\Big)\circ\sigma + l^{g-1}_{n+2}\Big(\omega^{-1}\wedge\underbrace{1\wedge\cdots\wedge 1}_{n-1}\Big) = 0$$

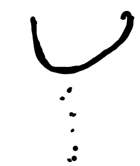
Loop algebra (Markl)

ω symplec

Rem: cyclicity is allways assumed here

alternatively, one may lift ω^{-1} to a (strict) coderivation of order 2: $D(\omega^{-1})$

$\ell_n^{\#} \sim$ 

$\underbrace{D(\omega^{-1})}_{\text{lift of } \omega \text{ to coder}^2} \sim$ 

then

$$h = L + D(\omega^{-1}) \in \underbrace{CODER}_{co}(\hbar, SV)$$

$$= \bigoplus_{n=1} \hbar^{n-1} \, Coder^{n}(SV)$$

$$V \otimes V \otimes V \cdots$$

with $[h, h] = 0$ $\quad$ IBL$_\infty$-algebra (Cieliebak, Fukaya, Latschev)

Summary:

Feynman transform of mod. operads

BV action

$(F_?, \Delta_?, S_?)$

?

perturb. exp'n

algebras

$\partial \nu + \{\nu, \nu\} + \hbar \Delta \nu = 0$

decomposition of mod. space

SFT

CFT

homotopy algebra

$[L, L] = 0$

l_n