Some remarks on HPL

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HPL

(V, Q), (W, e)

dg v.s.

<u>Standard Situation</u> (SS)

$$k \circlearrowright (V, Q) \underset{i}{\overset{p}{\rightleftarrows}} (W, e)$$

$$Q^2 = 0 \qquad e^2 = 0 \qquad |Q| = |e| = 1$$

$$pQ = ep \qquad |p| = 0$$

$$ie = Qi \qquad |i| = 0$$

$$ip - 1_V = Qk + kQ \qquad |k| = -1$$

<u>Deformation retract</u> (DR) $\quad$ SS + $pi = 1_W$

<u>Special def. retract</u> (SDR) $\quad$ DR + $\begin{cases} pk = 0 \\ ki = 0 \\ k^2 = 0 \end{cases}$

PERTURBATION LEMMA

Consider SS

$$k \circlearrowright (V, Q) \underset{i}{\overset{p}{\rightleftarrows}} (W, e) \qquad (*)$$

+ "small" perturbation $\delta : V \longrightarrow V$ s.t.

$$(Q+\delta)^2 = 0,$$

Brown
Shih
Gugenheim
Hüebschmann
Lambe & Stasheff

then

$$k' \circlearrowright (V, Q+\delta) \underset{i'}{\overset{p'}{\rightleftarrows}} (W, e')$$

$$\left.\begin{aligned} e' &= e + p\,\delta(1-k\delta)^{-1} i \\ p' &= p(1-\delta k)^{-1} \\ i' &= (1-k\delta)^{-1} i \\ k' &= k(1-\delta k)^{-1} \end{aligned}\right\} \quad (**)$$

1. $(**)$ is a SS

2. if f/i is a quasi-isomorphism (surjective / injective on $H(V)$) then f'/i' is.

3. if $(*)$ is a DR then $(**)$ is a DR iff

$$f\,(Ak^2A + Ak + kA)\,i = 0$$

where $A = (1-\delta k)^{-1}\delta$

4. if $(*)$ is a SDR then $(**)$ is a SDR.

<u>lemma</u> (Hodge dec.) (V, Q, ω) a dg symplectic v.s. $|\omega| = -1$

then $V \cong H \oplus \mathrm{Im}\, Q \oplus C$

and $\exists$ maps k, i, p

$$\begin{tikzcd} H \oplus \mathrm{Im}\, Q \oplus C \arrow[loop above, "k"] \arrow[loop below, "Q"] \arrow[r, shift left, "p"] & (H, 0) \arrow[l, shift left, "i"] \end{tikzcd}$$ is a SDR

$\omega = \omega' + \omega''$

$\omega' = \omega|_H$

$\omega'' = \omega|_{\mathrm{Im}\, C \oplus Q}$

H – symplectic

$C, \mathrm{Im}\, Q$ – Lagrangian $\subset \mathrm{Im}\, Q \oplus C$

and $k = Q^{-1}|_{\mathrm{Im}\, Q}$

p, i – projection/inclusion of $H \subset V$

<u>Harmonious</u> Hodge decomposition

Explicitly Assume $V = \bigoplus_{k\in\mathbb{Z}} V_k$ V_k - finite dimensional

$\alpha^i, \beta^i, \gamma^i$ - homogeneous coordinates on M, $\operatorname{Im} Q$, C

$$\Delta = \Delta' + \Delta'' \qquad \Delta' = \frac{(-1)^{|\alpha^i|}}{2} \omega'^{ij} \frac{\partial_L^2}{\partial\alpha^i \partial\alpha^j} \qquad (\omega')^{ik} \omega'_{kj} = \delta^i_j$$

$$\Delta'' = (-1)^{|\gamma^i|} \omega''^{ij} \frac{\partial_L^2}{\partial\gamma^i \partial\beta^j}$$

the same for ω''

similarly $\{\cdot,\cdot\} = \{\cdot,\cdot\}' + \{\cdot,\cdot\}''$

$$\omega = \begin{pmatrix} \omega' & 0 & 0 \\ 0 & 0 & \omega'' \\ 0 & -\omega'' & 0 \end{pmatrix}$$

similarly for ω^{-1}

on $\mathcal{F}(V)$ (formal series in $\hbar$)

a "proper" subspace

Gwilliam

Lemma (SDR on $\mathcal{F}(V)$) There is a SDR

$$K \circlearrowright \left(\mathcal{F}(V), \{S_{free}, \cdot\}\right) \underset{i}{\overset{p}{\rightleftarrows}} \left(\mathcal{F}(H), 0\right)$$

$$\{S_{free}, \cdot\} = -\gamma^i Q^k_i \frac{\partial}{\partial \beta^k} \qquad Qc_i = Q^k_i b_k$$

$$K(x) = \frac{1}{\#\beta+\gamma} \beta^k Q^{-1\,i}_{\ k} \frac{\partial}{\partial \gamma^i}$$

$$i = \sum_{n \geq 0} (p^*)^{\otimes n} \qquad p = \sum_{n\geq 0} (i^*)^{\otimes n}$$

For later reference

$$K_0 = \#_{\beta+\gamma} K$$

$$i^*(\alpha) = \alpha \qquad i^*(\beta) = 0 = i^*(\gamma)$$

$$p^*(\alpha) = \alpha$$

tensor trick Eilenberg & Maclane

$\boxed{\text{1st perturbation}}$ $\quad \delta_1 \equiv \hbar\Delta$

gives <u>effective action</u> $\quad W \in \mathcal{F}(H)$

$$e^{\frac{W}{\hbar}} = P_1\left(e^{S_{int}/\hbar}\right) = P(1-\hbar\Delta K)^{-1} e^{S_{int}/\hbar}$$

<u>theorem</u>

- $\hbar\Delta' W + \frac{1}{2}\{W, W\}' = 0$ on $\mathcal{F}(H)$
- $e^{\frac{W}{\hbar}} = P e^{\frac{1}{2}\hbar\partial_P} e^{\frac{S_{int}}{\hbar}}$ with propagator

$$\partial_P = (-1)^{|y^i|} \omega''^{ij} Q^{-1\,l}{}_j \frac{\partial^2_L}{\partial y^i \partial y^l}$$

standard Feynman diagram expansion QFT

equivalently $\quad e^{\frac{W(\alpha)}{\hbar}} = \int_{\beta=0} d\beta\, d\gamma\, e^{S/\hbar}$ $\quad$ – BV path integral gauge fixing $\beta = 0$

• Similarly $f \in \mathcal{F}(V)$

• $p_1\left(e^{S_{int}/\hbar} f\right) = \int\limits_{\beta=0} e^{S} f \in \mathcal{F}(\mathcal{X})$

<u>unnormalized</u> path integral

2. Perturbation

$$\delta_2 = \hbar \Delta + \{S_{int}, \cdot\}$$

<u>Theorem</u>

- $P_2(f) = e^{-W/\hbar} P_1(e^{S/\hbar} f)$

  i.e. here HPL gives directly <u>normalized path integral</u>

- perturbed differential on $\mathcal{F}(H)$

$$E_2 = \Delta' + \{W, \cdot\}'$$

- another (equivalent) formula for W

$$W = \sum_{k=0}^{\infty} \frac{1}{\#\alpha} P(\delta_2 K)^k \#_\alpha S_{int}$$

$$\delta_2 = \cancel{\hbar\Delta} + \{S_{int}, \cdot\} \quad \text{tree level}$$

Costello

Homotopies

$\mathcal{F}(V) \otimes \Omega([0,1])$ ← forms on the unit interval

$|A| = 0$
$|B| = -1$

$A(t) + B(t)dt$ is a homotopy between $A(0)$ and $A(1)$ if $A(t)$ solves qME $\forall t \in [0,1]$

$$\hbar \Delta A(t) + \frac{1}{2}\{A(t), A(t)\} = 0$$

and

$$\dot{A}(t) + \{S_{free}, B(t)\} + \{A(t), B(t)\} + \hbar \Delta B(t) = 0$$

Theorem $S_0, S_1 \in \mathcal{F}(V)$. Then the following are equivalent

1. There exists a homotopy between S_0 and S_1
2. There exists a $F \in \mathcal{F}(V)$, $|F| = -1$ s.t.

$$e^{S_0/\hbar} - e^{S_1/\hbar} = \{S_{free}, F\} + \hbar \Delta F$$

3. There is a symplectic differ $\phi = 1 + \dots$ s.t.

$$e^{\frac{S_{free}+S_0}{\hbar}} d^{\frac{1}{2}}V = \phi^*\left(e^{\frac{S_{free}+S_1}{\hbar}} d^{\frac{1}{2}}V\right)$$

Remarks

- $2 \Rightarrow 1$ The simplest homotopy works $e^{A(t)/\hbar} \equiv (1-t)e^{S_0/\hbar} + t e^{S_1/\hbar}$

$$B(t) = -F e^{-A(t)/\hbar}$$

- ϕ is generated by $\{B(t), \cdot\}$

Homotopy between W and S from HPL

- Recall perturbation by $\delta_1 = \hbar\Delta$

the new SDR; in particular applying

$$\iota_1 P_1 - P_1 \iota_1 = Q_1 K_1 + K_1 Q_1$$

to $e^{\frac{S_{int}}{\hbar}}$

$$\Rightarrow e^{W(\hbar)/\hbar} - e^{S_{int}/\hbar} = Q_1 K_1 e^{S_{int}/\hbar} + 0$$

$(Q_1 = \hbar\Delta + \{S_{int}, \cdot\})$

$F = K_1 e^{S_{int}/\hbar}$ in the previous theorem

- In this case

$$e^{\frac{S_{tree} + I(W)}{\hbar}} d^{\frac{1}{2}}V = \phi^*\left(e^{\frac{S_{tree} + S_{int}}{\hbar}} d^{\frac{1}{2}}V\right)$$

ϕ is generated by

$$X(t) = -\hbar\{e^{-A(t)/\hbar} K_1(e^{S_{int}}), \cdot\}$$

i.e. HPL $\Rightarrow$ decomposition theorem for the corresponding quantum L_∞-alg. into the contractible part S_{tree} and the minimal part W

Remarks • ϕ – is a Poisson diffeomorphism

$$(V, \omega, S_{free} + I(W)) \longrightarrow (V, \omega, S_{free} + S_{int})$$

an example of a quantum Lso-iso

$$(\hbar\Delta + \{S_{free} + I(W), \cdot\})\phi^*$$
$$= \phi^*(\hbar\Delta + \{S_{free} + S_{int}, \cdot\})$$

• δ_n action $\quad S_{free} + I(W) = \phi^*(S_{free} + S_{int} - \frac{1}{2}\hbar \log Ber(\phi))$

Quasiomorphism (Minimal model) $\quad p \circ \phi^{-1} : (V, \omega, S) \to (H, \omega', W)$

Another example of homotopy ERGF (Toy model finite dim)

Costello. Endlicher

ε- cut off $\Delta_\varepsilon = \Delta' + \Delta''_\varepsilon$ $(\Delta_0 = \Delta)$

$\dot{\Delta}_\varepsilon = -\Delta''_\varepsilon$ corresponding $\dot{\partial}_{P_\varepsilon} = -\partial_{P_\varepsilon}$

ERGF

S_ε - one parameter family of actions

$$\left(\frac{d}{d\varepsilon} + \partial_{P_\varepsilon}\right) e^{S_\varepsilon} = 0 \qquad (*)$$

"renormalized" scale ε BV-equation $(**)$

$$\Delta_\varepsilon e^{S_\varepsilon} = 0$$

Simple claim:

$S_{\varepsilon'}$ solves scale ε' QME

$\Downarrow$

S_ε solves scale ε QME

S_ε is almost a homotopy in the above sense

$$\frac{dS_\varepsilon}{d\varepsilon} + QB_\varepsilon + \Delta_{\varepsilon} B_\varepsilon + \{S_\varepsilon, B_\varepsilon\}_{\varepsilon} + D B_\varepsilon = 0$$

$D = \#_{\alpha+\beta}$ $\qquad B_\varepsilon = - K_0 \, S_\varepsilon$

$$e^{\tilde{S}_\varepsilon} := e^{-\frac{1}{2}\varepsilon D} e^{S_\varepsilon}$$

$$\Rightarrow \qquad \Delta e^{\tilde{S}_\varepsilon} = 0$$

$$\frac{d}{d\varepsilon} \tilde{S}_\varepsilon + Q \tilde{B}_\varepsilon + \Delta \tilde{B}_\varepsilon + \{\tilde{S}_\varepsilon, \tilde{B}_\varepsilon\} = 0$$

$$\tilde{B}_\varepsilon = -K_0 \tilde{S}_\varepsilon$$

describes a homotopy

Morphisms². $\underline{QOSC}$ Quantum odd symplectic "cat."

<u>Definition</u> QOSC

- objects – odd symplectic supermanifolds (ω – odd symplectic form)

(M, ω)

- morphisms $(M_1, \omega_1) \longrightarrow (M_2, \omega_2)$

generalized (distributional) half densities

$$\mathrm{Dens}^{\frac{1}{2}}(\overline{M_1} \times M_2)$$

on $(\overline{M_1} \times M_2, -\omega_1 + \omega_2)$

supported on Lagrangian $L \subset \overline{M_1} \times M_2$

- Composition

$$\mathrm{Dens}^{\frac12}(\bar M_1\times M_2)\times \mathrm{Dens}^{\frac12}(\bar M_2\times M_3) \xrightarrow[M_2]{\int} \mathrm{Dens}^{\frac12}(\bar M_1\times M_3)$$

— on half-densities on an odd symplectic manifold there is a BV differential Δ

<u>Prop</u> Composition of morphism is compatible with Δ, i.e. composition of Δ-closed $\frac12$densities is Δ-closed

<u>Definition</u> $F \in \mathrm{Dens}^{\frac{1}{2}}(M)$ s.t. $\Delta F = 0$

is called a solution to the

QME

Notation for pairs (M, F)

(M, F) and (N, G) for such pairs

Morphism $(M, F) \longrightarrow (N, G)$

is a Δ-closed $\rho \in \mathrm{Dens}^{\frac{1}{2}}(\overline{M}, N)$ s.t.

$$\int_M F\rho = G$$

Example 1 (M, F) and $\phi : M \to M$ symplectomorphism

then the graph of ϕ

$$\{(m, \phi(m)), m \in M\} \subset \overline{M} \times M$$

Lagrangian

δ_{graph} is a morphism

$$(M, F) \longrightarrow (M, \phi_* F)$$

Example 2 — Minimal model

$$V = H \oplus \operatorname{Im}\mathbb{Q} + \mathbb{C}$$

$$\mathrm{Dens}^{\frac{1}{2}}(\bar{V} \times H) \ni \delta_{\mathbb{C} \times H_{diag}}$$

$$\int_V e^S \, \delta_{\mathbb{C} \times H_{diag}} \, d^{\frac{1}{2}} V = e^W d^{\frac{1}{2}} H$$