Realization of symmetry in the ERG approach to QFT

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What to be discussed

Based on ERG, we would like to study gauge theory non-pertubatively.

In ERG approach to QFT, we start from a path integral with a UV momentum cutoff.

Questions:

1. How do we obtain a QFT, even perturbatively, from a path integral with a cutoff?
2. Gauge symmetry would be broken with the cutoff.
3. Non-perturbative study? Phase structure?

We will explain the present status. But it is still insufficient to reach our goal. If the homotopy algebra resolve even a part of the problem, that would be very nice!!

What to be discussed in answering the above questions

1. How do we obtain a QFT, even perturbatively, from a path integral with a cutoff?

   Taking a scalar theory for simplicity, we explain basic ideas such as Wilsonian action, flow equation, etc. Then we discuss the pertubative renormalizability to define a scalar FT.

2. Gauge symmetry would be broken with the cutoff.

   We will find the following:

   - The situation is like the realization of the chiral symmetry on the lattice. The lattice regularization is not compatible with a naive chiral symmetry.
   - The BV antifield formalism is quite useful.

3. Non-perturbative study? Phase structure?

   We dimensionless formulation of the flow equation to observe the phase structure.

Note

In the lierature, we encounter different words for ERG, i.e., non-perturbative renormalization group or functional renormalization group. All three of them mean basically the same thing. Each of them has its own hisorical or conceptual backgrond.

The content

1. Wilsonian action and Polchinski eq.

2. Perturbative renormalizability

3. 1PI action and its flow equation

4. Relations of 1PI and Wilsonican actions

5. Symmetry vs Regularization, BV formalism

   - Notion of composite operators

6. Dimensionless formulation for flow eq.

7. An example: application to QED

8. Summary and Discussion: we need to achieve ...

Wilsonian action and Polchinski eq.

We take a scalar field theory to explain basic ideas of ERG. Subjects related to symmetry realization is to be discussed in later parts of the talk.

Consider a Euclidean path integral with the regulator function K introduced to the kinetic term. $K(x)$ is a function of $x = p^2/\Lambda_0^2$ with a momentum cutoff Λ_0.

$$Z_{\Lambda_0} = \int [d\phi]\ e^{S_{\Lambda_0}[\phi]}$$

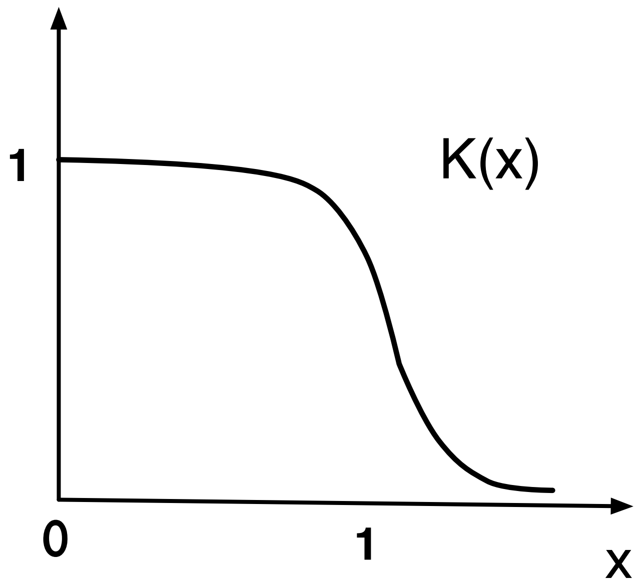
$$S_{\Lambda_0}[\phi] = -\frac{1}{2}\int_p \frac{p^2+m^2}{K(p/\Lambda_0)}\phi(p)\phi(-p) + S_{I,\Lambda_0}[\phi], \qquad \left(\int_p = \int \frac{d^D p}{(2\pi)^D} \right)$$

S_{I,Λ_0} defines the interaction at the scale Λ_0. For the moment, we do not need to assume a particular form for S_{I,Λ_0}. The propagator is $\frac{K(p/\Lambda_0)}{p^2+m^2}$ and the function K determines the propagating modes. We choose K to allow only modes with momenta below Λ_0 to propagate. For that purpose we assume that the regulator function $K(x)$ have the following properties:

- $K(0) = 1$ and $K \sim 1$ for $0 \leq x = p^2/\Lambda_0^2 \leq 1$

- $K(x)$ is a Taylor expandable and monotonically decreasing function of x.

- Rapidly decreasing as $x = p^2/\Lambda_0^2 \to \infty$



The propagator $\frac{K(p/\Lambda_0)}{p^2+m^2}$ allows the modes up to Λ_0^2 to propagate.

$$Z_{\Lambda_0} = \int [d\phi] \; e^{S_{\Lambda_0}[\phi]}$$

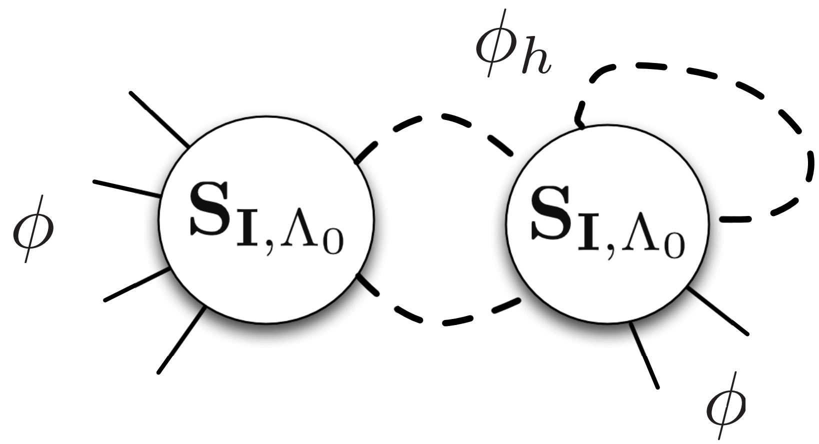
$$S_{\Lambda_0}[\phi] = -\frac{1}{2} \int_p \frac{p^2 + m^2}{K(p/\Lambda_0)} \phi(p)\phi(-p) + S_{I,\Lambda_0}[\phi], \qquad \left(\int_p = \int \frac{d^D p}{(2\pi)^D} \right)$$

In evaluating the path integral, we generate Feyman diagrams with the interaction vertices from S_{I,Λ_0} and propagators whose momenta restricted as $p^2 \lesssim \Lambda_0^2$.

Now we introduce another cutoff Λ that is less than Λ_0 and separate the modes $\phi_l(p)$ with $p^2 \lesssim \Lambda_0^2$ and ϕ_h with $\Lambda \lesssim p^2 \lesssim \Lambda_0^2$, i.e., the low momentum modes and the high momentum modes. It is easy to see the propagator split into two parts:

$$\frac{K(p/\Lambda_0)}{p^2 + m^2} = \frac{K(p/\Lambda)}{p^2 + m^2} + \frac{K(p/\Lambda_0) - K(p/\Lambda)}{p^2 + m^2}$$

We integrate over ϕ_h in the path integral to find the action $S_\Lambda[\phi_l]$: in terms of Feyman diagram, we collect all the graphs connected by internal lines of high momentum modes. We show an example in the next figure.



- Interacton vertices are provided by S_{I,Λ_0}.

- The external lines are for lower momentum modes ϕ_l. The above figure contributes to the interaction part of the action $S_\Lambda[\phi_l]$.

Rewiting ϕ_l as ϕ, we find

$$S_\Lambda[\phi] = -\frac{1}{2}\int_p \frac{p^2+m^2}{K(p/\Lambda)}\phi(-p)\phi(p) + S_{I,\Lambda}[\phi]$$

$$\exp S_{I,\Lambda}[\phi] \equiv \int [d\phi_h] \exp\left\{-\frac{1}{2}\int_p \frac{p^2+m^2}{K(p/\Lambda_0)-K(p/\Lambda)}\phi_h(-p)\phi_h(p) + S_{I,\Lambda_0}[\phi+\phi_h]\right\}$$

The scale dependent action $S_\Lambda[\phi]$ is the Wilsonian action.

Summary: definition of Wilsonian action via path integral

Start from a theory defined at the scale Λ_0,

$$Z_{\Lambda_0} = \int [d\phi]\ e^{S_{\Lambda_0}[\phi]}, \qquad S_{\Lambda_0}[\phi] = -\frac{1}{2}\int_p \frac{p^2+m^2}{K(p/\Lambda_0)}\phi(p)\phi(-p) + S_{I,\Lambda_0}[\phi]\,.$$

Separate ϕ into the high momentum modes ϕ_h and the low momentum mode ϕ_l and integrate over ϕ_h. We find a theory with the lower cutoff Λ.

$$Z_{\Lambda_0} = \int [d\phi_l][d\phi_h]\ e^{S_{\Lambda_0}[\phi]} = \int [d\phi_l]\ e^{S_\Lambda[\phi_l]}$$

$$S_\Lambda[\phi] = -\frac{1}{2}\int_p \frac{p^2+m^2}{K(p/\Lambda)}\phi(-p)\phi(p) + S_{I,\Lambda}[\phi]$$

$$\exp S_{I,\Lambda}[\phi] \equiv \int [d\phi_h] \exp\Big\{-\frac{1}{2}\int_p \frac{p^2+m^2}{K(p/\Lambda_0)-K(p/\Lambda)}\phi_h(-p)\phi_h(p) + S_{I,\Lambda_0}[\phi+\phi_h]\Big\}$$

This procedure defines the renormalization transformation of ERG.

Polchinski eq.: functional differentical form for the flow equation

We now introduce a functional differential equation that describes an infinitesimal change of the cutoff, $\Lambda \to \Lambda - \delta\Lambda$.

We obtain $S_{\mathrm{I},\Lambda-\delta\Lambda}$ from $S_{\mathrm{I},\Lambda}$ by connecting vertices in $S_{\mathrm{I},\Lambda}$ with high momentum modes propagator:

$$\frac{K(p/\Lambda) - K(p/(\Lambda-\delta\Lambda))}{p^2+m^2} \sim \delta\Lambda \frac{\partial K}{\partial \Lambda}\frac{1}{p^2+m^2} = \frac{\delta\Lambda}{\Lambda}\frac{\Delta(p/\Lambda)}{p^2+m^2}\,,$$

$$\Delta(p/\Lambda) \equiv \Lambda \frac{\partial K(p/\Lambda)}{\partial \Lambda}\,.$$

In order to find $S_{I,\Lambda-\delta\Lambda} - S_{I,\Lambda}$ in the 1st order of $\delta\Lambda$, we consider diagrams with a single high momentum propagator. There are only two types of Feyman diagrams shown below. There the high momentum propagator is indicated by the dotted lines.

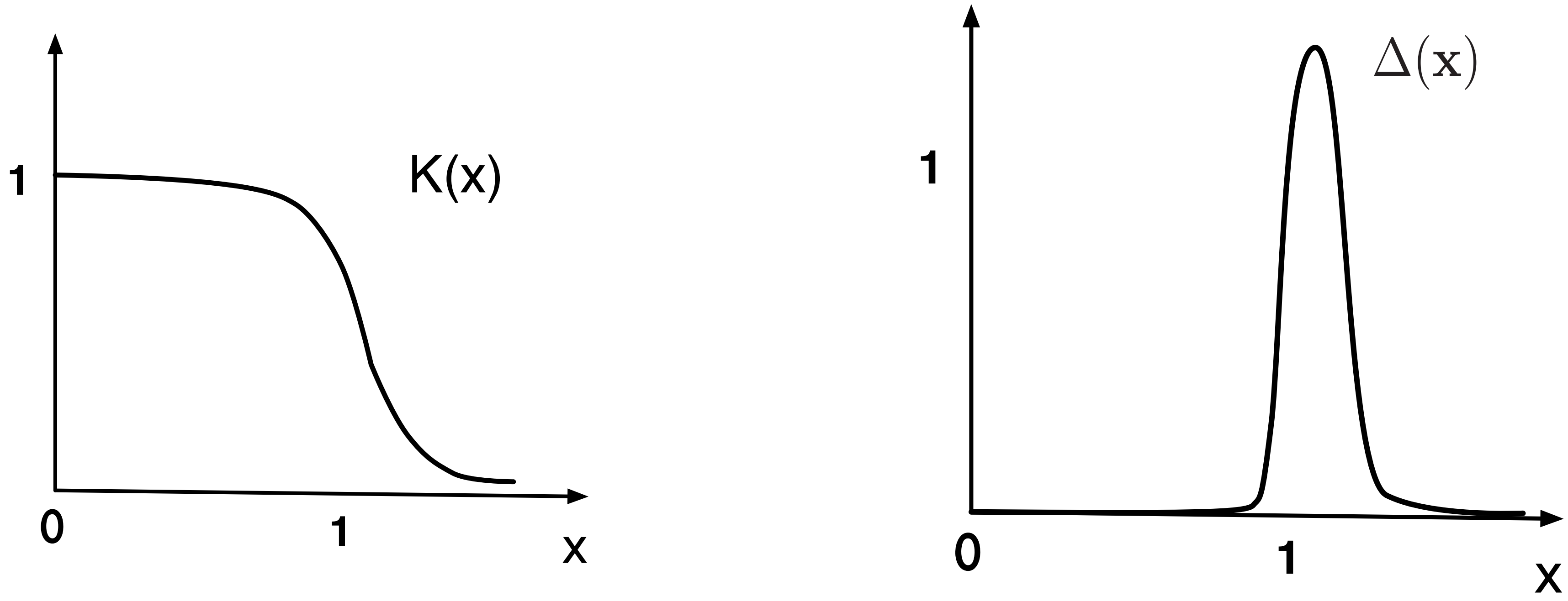
Figure 1: Two types of diagrams that contribute to Polchinski equation

We reach the functional flow equation called Polchinski equation (1984)

$$-\Lambda\frac{\partial}{\partial\Lambda}S_{I,\Lambda}[\phi] = \frac{1}{2}\int_p \frac{\Delta(p/\Lambda)}{p^2+m^2}\Big(\frac{\partial S_{I,\Lambda}}{\partial\phi(p)}\frac{\partial S_{I,\Lambda}}{\partial\phi(-p)} + \frac{\partial^2 S_{I,\Lambda}}{\partial\phi(p)\partial\phi(-p)}\Big)$$

Of course, the Polchinski equation is equivallent to the formulation with the path integral when the initial condition is provided as $S_{I,\Lambda}|_{\Lambda=\Lambda_0} = S_{I,\Lambda_0}$.

The function $\Delta(p/\Lambda)$ takes the form shown in the figure. Δ is a derivative of K and it takes its value around $p^2 \sim \Lambda^2$.



From the shape of Δ, it is easy to understand that the momentum integration on the r.h.s. is taken only around $p^2 \sim \Lambda^2$ and it is finite. There is no infinity one might expect in a FT calculation.

$$-\Lambda\frac{\partial}{\partial\Lambda}S_{I,\Lambda}[\phi] = \frac{1}{2}\int_p \frac{\Delta(p/\Lambda)}{p^2+m^2}\Big(\frac{\partial S_{I,\Lambda}}{\partial\phi(p)}\frac{\partial S_{I,\Lambda}}{\partial\phi(-p)} + \frac{\partial^2 S_{I,\Lambda}}{\partial\phi(p)\partial\phi(-p)}\Big)$$

For the sake of completeness, we give the flow equation for S_Λ including the kinetic term. Clearly, the flow for $S_{I,\Lambda}$ is better to remember.

$$-\Lambda\frac{\partial}{\partial\Lambda}S_\Lambda[\phi] = \int_p \Big[\frac{\Delta(p/\Lambda)}{K(p/\Lambda)}\phi(p)\frac{\partial S_\Lambda[\phi]}{\partial\phi(p)} + \frac{1}{2}\ \frac{\Delta(p/\Lambda)}{p^2+m^2}\Big(\frac{\partial S_\Lambda}{\partial\phi(p)}\frac{\partial S_\Lambda}{\partial\phi(-p)} + \frac{\partial^2 S_\Lambda}{\partial\phi(p)\partial\phi(-p)}\Big)\Big]$$

Flows in coupling space

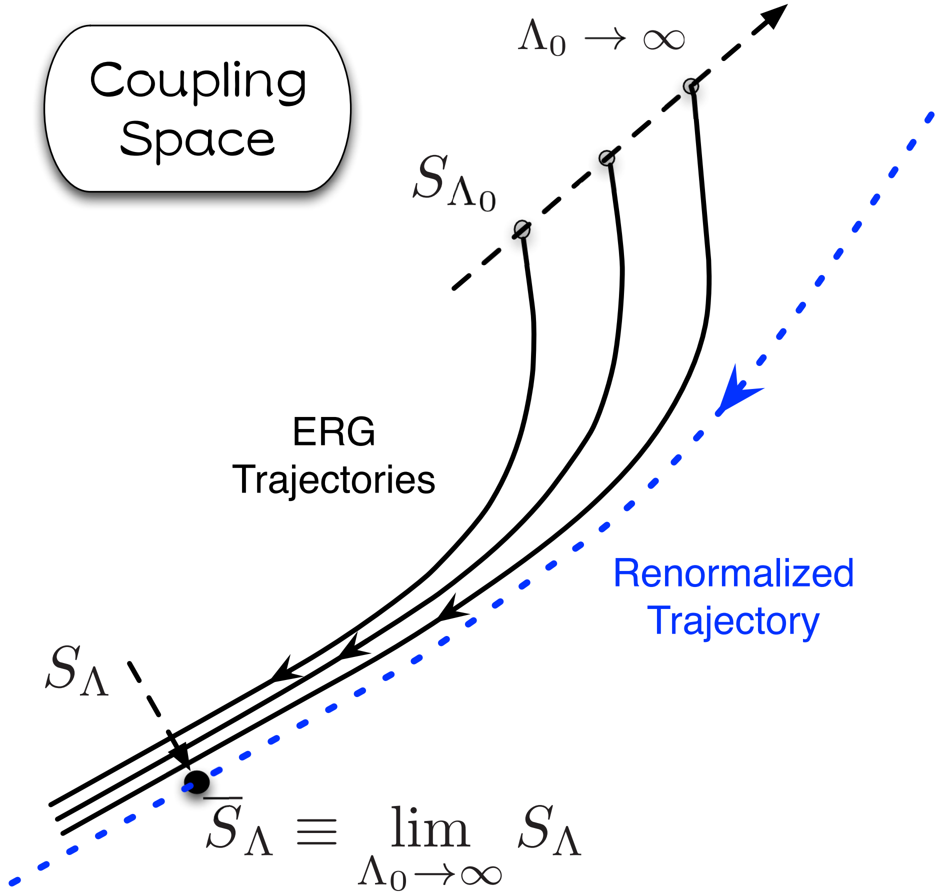
- The interaction action is a sum of operators $\mathcal{O}_i[\phi]$ multiplied by couplings g_i,

$$S_{I,\Lambda}[\phi] = \sum_i g_i \cdot \mathcal{O}_i[\phi]$$

- g_i are calculated via path integration over the modes $\Lambda^2 \lesssim p^2 \lesssim \Lambda_0^2$ and therefore they are functions of Λ, $g_i(\Lambda)$.

- $\{g_i(\Lambda)\}$ defines a flow in the coupling space starting from an initial point $\{g_i(\Lambda_0)\}$.

- We will come back to this figure later and explain the details.



Renormalizability

Up to now, we discussed the scale dependence of Wilsonican or 1PI action with an initial action defined at Λ_0. When we find a way to take a continuum limit or remove this initial cutoff dependence, we may identify this theory with a field theory. This corresponds to the renormalizability.

Here we discuss the point perturbatively by taking a scalar theory.

$$S_{\Lambda_0} = -\frac{1}{2}\int_p \frac{p^2+m^2}{K(p/\Lambda_0)}\phi(-p)\phi(p) + S_{I,\Lambda_0}$$

$$S_{I,\Lambda_0} = -\int d^4x\Big[\ \Delta m^2 \cdot \frac{\phi^2}{2} + \Delta z \cdot \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + (\lambda+\Delta\lambda)\cdot\frac{\phi^4}{4!}\ \Big].$$

The terms with $\Delta m^2,\ \Delta z,\ \Delta\lambda$ are so-called conterterms that are to be determined for an appropriate continuum limit.

Comments are in order.

- We chose S_{I,Λ_0} as an initial action at Λ_0. It contain $\lambda\phi^4$ interaction other than the counter terms. These terms are sufficient for a perturbative calculation.

- In principle, we could have started from an initial action with more interaction terms. It would have higher derivative terms with appropriate inverse powers of Λ_0.

- S_{I,Λ_0} is the action at Λ_0 and λ is the coupling defined at the same scale. There is no concept of 'bare coupling' in this formulation. The following perturbative calculation is not so different from that in field theory textbooks technically. However let us emphasize the points:

  – it is a perturbation with respect to λ defined at the scale Λ_0;
  – all the calculation is finite and there is no U.V. divergence:

We integrate out the high momentum modes in the formula,

$$\exp S_{I,\Lambda}[\phi] \equiv \int [D\phi_h] \exp\Big[-\frac{1}{2}\int_p \frac{p^2+m^2}{K_0-K}\phi_h(-p)\phi_h(p) + S_{I,\Lambda_0}[\phi+\phi_h]\Big] .$$

Expand $S_{I,\Lambda}$ in powers of field to define the coefficient $\mathcal{V}_{2n}$

$$S_{I,\Lambda}[\phi] = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int_{p_1+\cdots+p_{2n}=0} \mathcal{V}_{2n}(\Lambda; p_1, \cdots, p_{2n})\phi(p_1)\cdots\phi(p_{2n}) .$$

$\mathcal{V}_{2n}$ is obtained in perturbative expansion of λ. As a simple example, we calculate $\mathcal{V}_2$ in one loop level, $\mathcal{V}_2^{(1)}$. The following diagrams contribute to it.

$$\frac{K(q/\Lambda_0) - K(q/\Lambda)}{q^2 + m^2}$$

and $\left(\Delta m^2\right)^{(1)}$

Figure 2: $\mathcal{V}_2$ in one loop

$$\mathcal{V}_2^{(1)}(\Lambda; p, -p) = -\frac{\lambda}{2}\int_q \frac{K(q/\Lambda_0) - K(q/\Lambda)}{q^2 + m^2} - (\Delta m^2)^{(1)} \qquad (\bar{q} \equiv q/\Lambda)$$

$$= -\frac{\lambda}{2}\Big[(\Lambda_0^2 - \Lambda^2)\int_{\bar{q}} \frac{K(\bar{q})}{\bar{q}^2} - m^2 \frac{2}{(4\pi)^2} \ln \frac{\Lambda_0}{\Lambda} + \frac{m^4}{\Lambda^2}\int_{\bar{q}} \frac{K(\bar{q}\Lambda/\Lambda_0) - K(\bar{q})}{\bar{q}^4(\bar{q}^2 + m^2/\Lambda^2)}\Big] - (\Delta m^2)^{(1)}$$

Please pay attension to the Λ_0 dependence.

Details of the calculation

$$\int_q \frac{K(q/\Lambda_0) - K(q/\Lambda)}{q^2+m^2} = \int_q \Big(K(q/\Lambda_0) - K(q/\Lambda)\Big)\Big(\frac{1}{q^2} - \frac{m^2}{q^4} + \frac{m^4}{q^4(q^2+m^2)}\Big)$$

- The 1st integral $1/q^2$ terms on the r.h.s may be rewritten as follows by making the replacement as $q = \Lambda_0 \bar{q}$ or $q = \Lambda \bar{q}$

$$\text{1st term} = (\Lambda_0^2 - \Lambda^2) \int_{\bar{q}} \frac{K(\bar{q})}{\bar{q}^2}\,.$$

The above integral is a finite number.

$$\int_{\bar{q}} \frac{K(\bar{q})}{\bar{q}^2} = \int \frac{d^4\bar{q}}{(2\pi)^4}\frac{K(\bar{q})}{\bar{q}^2} = \frac{1}{16\pi^2}\int_0^\infty d\bar{q}^2 K(\bar{q})\,.$$

- <u>The 2nd integral</u>
  It is easy to roughly estimate the 2nd integral since $K_0 - K$ restrict p^2 in the region $\Lambda^2 < p^2 < \Lambda_0^2$.

$$F(\Lambda_0, \Lambda) = \int_q \frac{K(q/\Lambda_0) - K(q/\Lambda)}{q^4} \sim \int_{\Lambda^2}^{\Lambda_0^2} q^2 dq^2 \frac{1}{q^4} = \ln \frac{\Lambda_0^2}{\Lambda^2}\,.$$

  The exact calculation is in our review article.

- <u>The 3rd integral</u> is finite as $\Lambda_0 \to \infty$

$$\text{3rd term} = \frac{m^4}{\Lambda^2} \int_{\bar{q}} \frac{K(\bar{q}\Lambda/\Lambda_0) - K(\bar{q})}{\bar{q}^4(\bar{q}^2 + m^2/\Lambda^2)} \quad \to \quad \frac{m^4}{\Lambda^2} \int_{\bar{q}} \frac{1 - K(\bar{q})}{\bar{q}^4(\bar{q}^2 + m^2/\Lambda^2)} \quad : \text{finite}$$

Let us write our result again.

$$\mathcal{V}_2^{(1)}(\Lambda; p, -p) = -\frac{\lambda}{2}\int_q \frac{K(q/\Lambda_0) - K(q/\Lambda)}{q^2+m^2} - (\Delta m^2)^{(1)} \qquad\qquad (\bar{q} \equiv q/\Lambda)$$

$$= -\frac{\lambda}{2}\Big[(\Lambda_0^2 - \Lambda^2)\int_{\bar{q}} \frac{K(\bar{q})}{\bar{q}^2} - m^2\frac{2}{(4\pi)^2}\ln\frac{\Lambda_0}{\Lambda} + \frac{m^4}{\Lambda^2}\int_{\bar{q}} \frac{K(\bar{q}\Lambda/\Lambda_0) - K(\bar{q})}{\bar{q}^4(\bar{q}^2 + m^2/\Lambda^2)}\Big] - (\Delta m^2)^{(1)}$$

We choose the counter term $(\Delta m^2)^{(1)}$ to remove the divergent terms in $\mathcal{V}_2^{(1)}$ as $\Lambda_0 \to \infty$.

$$(\Delta m^2)^{(1)} = \frac{\lambda}{2}\Big(-\Lambda_0^2\int_{\bar{q}} \frac{K(\bar{q})}{\bar{q}^2} + m^2\frac{2}{(4\pi)^2}\ln\frac{\Lambda_0}{\mu}\Big).$$

An arbitrary scale μ is introduced here. By adding the two, we find the finite result in the limit:

$$\begin{aligned}\bar{\mathcal{V}}_2^{(1)} &\equiv \lim_{\Lambda_0\to\infty}\Big(\mathcal{V}_2^{(1)} + (\Delta m^2)^{(1)}\Big) \\ &= -\frac{\lambda}{2}\Big[-\Lambda^2\int_{\bar{q}} \frac{K(\bar{q})}{\bar{q}^2} + m^2\frac{2}{(4\pi)^2}\ln\frac{\Lambda}{\mu} + \frac{m^4}{\Lambda^2}\int_{\bar{q}} \frac{1-K(\bar{q})}{\bar{q}^4(\bar{q}^2+m^2/\Lambda^2)}\Big]\end{aligned}$$

In a similar manner, we can choose Δz and $\Delta\lambda$ so that all the couplings $\bar{\mathcal{V}}_{2n}^{(1)}$ are Λ_0 dependent. That defines the action $\bar{S}_\Lambda$:

$$\overline{S}_\Lambda \equiv \lim_{\Lambda_0 \to \infty} S_\Lambda \,.$$

This theory is called perturbatively renormalizabile.

- In the right figure, we find a couple of ERG trajectories starting from different initial actions S_{Λ_0} at Λ_0.

- The figure shows how we tune S_{Λ_0}, or choose the counter terms, and find a well-defined action $\bar{S}_\Lambda$ as $\Lambda_0 \to \infty$.

- $\bar{S}_\Lambda$ as a function of Λ gives a curve in the coupling space. The curve is called as the renormalized trajectory.

- The renormalized action $\bar{S}_\Lambda$ does not know the scale Λ_0.

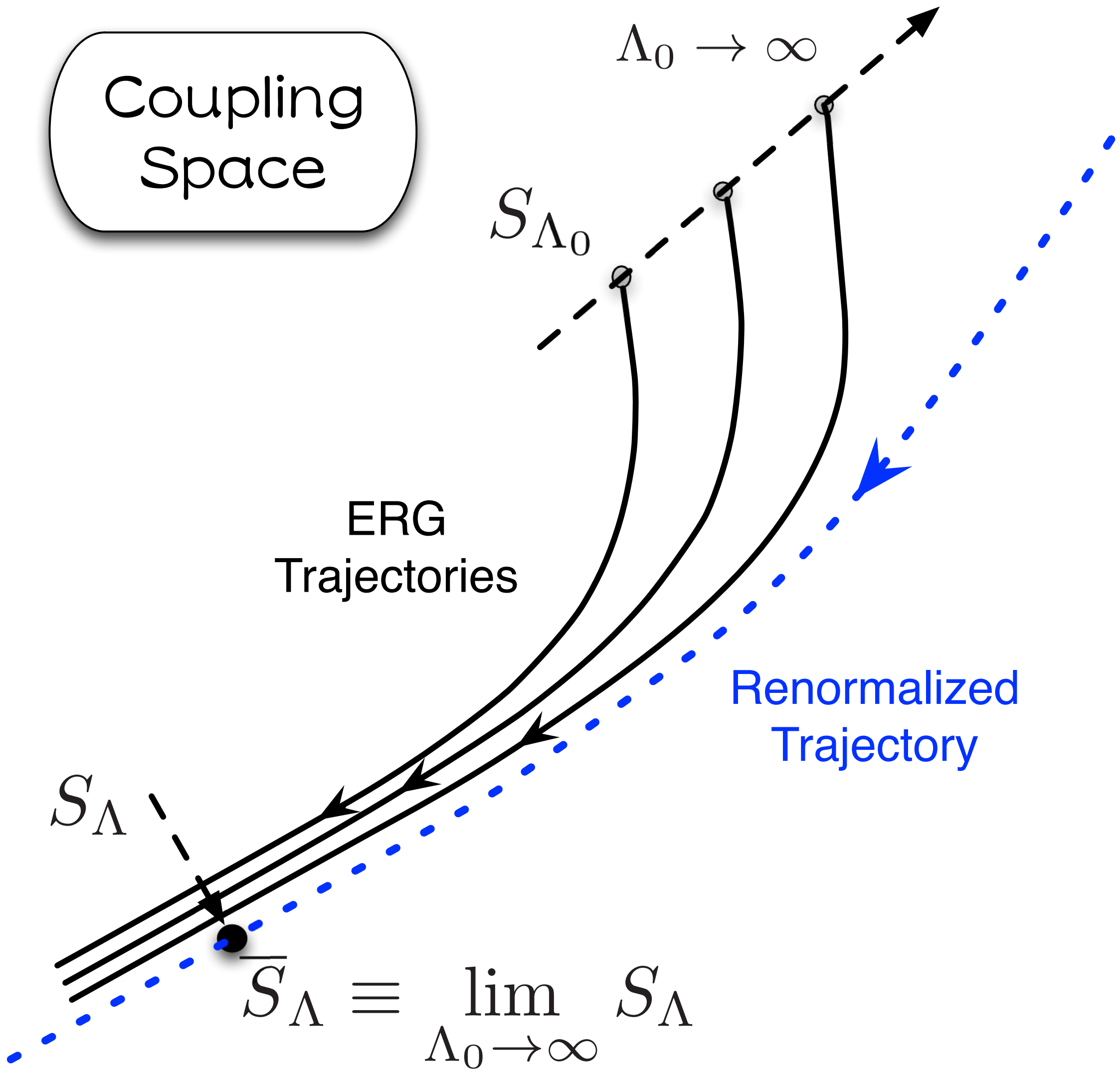


Figure 3: The renormalized trajectory

The effective average or 1PI action $\Gamma_{\Lambda,\Lambda_0}$ and its flow equation

We consider another action with the properties:

1. in the limit of $\Lambda \to 0$, it reduces to the ordinary effective action;

2. it is the sum of 1PI Feynman diagrams produced with vertices in S_{I,Λ_0} and the high momentum propagtor.

As we will observe later, the flow equation has a simple one-loop structure. Because of its simplicity and robustness, the flow equation has been used for various applications.

In order to realize the 2nd point above, we need to consider the kinetic term that produces high momentum propagator. That is achieved by replacing the kinetic term in S_{Λ_0} as

$$\frac{p^2+m^2}{K(p/\Lambda_0)} \to \frac{p^2+m^2}{K(p/\Lambda_0)-K(p/\Lambda)}$$

By this replacement, we obtain the action $S_{\Lambda_0,\Lambda}$ that is related to S_{Λ_0} as

$$S_{\Lambda,\Lambda_0}[\phi] = S_{\Lambda_0}[\phi] - \frac{1}{2}\int_p R_\Lambda(p)\phi(-p)\phi(p)$$

$$R_\Lambda(p) \equiv (p^2+m^2)\Big(\frac{1}{K(p/\Lambda_0)-K(p/\Lambda)} - \frac{1}{K(p/\Lambda_0)}\Big)$$

As $\Lambda \to 0$, $K(p/\Lambda) \to 0$ $\;(\text{for } p^2 \neq 0)$ and we find

$$\lim_{\Lambda\to 0} S_{\Lambda,\Lambda_0} = S_{\Lambda_0}\,.$$

Using $S_{\Lambda,\Lambda_0}[\phi]$, we introduce a generating functional W_{Λ,Λ_0} and its Legendre transform. The latter is often called as the effective average action. In this talk, we call it as the 1PI action.

$$W_{\Lambda,\Lambda_0}[J] \equiv \ln\Big(\int [d\phi] e^{S_{\Lambda,\Lambda_0}[\phi]+J\cdot\phi}\Big)$$

$$\Gamma_{\Lambda,\Lambda_0}[\Phi] \equiv W_{\Lambda,\Lambda_0}[J] - \int_p J(-p)\Phi(p), \qquad \Phi(p) = \frac{\delta W_{\Lambda,\Lambda_0}[J]}{\delta J(-p)}$$

As we stated already, $\lim_{\Lambda \to 0} S_{\Lambda,\Lambda_0} = S_{\Lambda_0}$. Therefore in the same limit $\Lambda \to 0$, we would obtain the effective action Γ_{Λ_0} to be calculated with S_{Λ_0}

$$\lim_{\Lambda \to 0} \Gamma_{\Lambda,\Lambda_0} = \Gamma_{\Lambda_0} \,.$$

With our understanding that its Λ_0 dependence would be taken care of in an appropriate continuum limit, the r.h.s. is the ordinary effective action. Therefore $\Gamma_{\Lambda,\Lambda_0}$ has the property 1.

A comment is in order. We denote the field for the Wilsonian action is ϕ (the small one) while for the 1PI action Φ (the large one).

Flow eq. for the 1PI action[1]

The 1PI action $\Gamma_{\Lambda,\Lambda_0}$ changes as we lower the cutoff Λ. We would find a flow equation that has the same information as the Polchinski eq. for the associated Wilsonian action.The interaction part of the 1PI action, $\Gamma_{I,\Lambda,\Lambda_0}$ is defined as

$$\Gamma_{I,\Lambda,\Lambda_0}[\Phi] \equiv \Gamma_{\Lambda,\Lambda_0}[\Phi] + \frac{1}{2}\int_p \frac{p^2+m^2}{K(p/\Lambda_0) - K(p/\Lambda)}\Phi(-p)\Phi(p)$$

As we will show shortly that $\Gamma_{I,\Lambda,\Lambda_0}$ satisfies the following flow eq.

$$-\Lambda\frac{\partial}{\partial\Lambda}\Gamma_{I,\Lambda,\Lambda_0}[\Phi] = \frac{1}{2}\int_p \Big(-\Lambda\frac{\partial}{\partial\Lambda}R_\Lambda(p)\Big)(\Gamma^{(2)}_{\Lambda,\Lambda_0})^{-1}(p,-p)$$

$(\Gamma^{(2)}_{\Lambda,\Lambda_0})^{-1}(p,-p)$ is often called as the full propagator. It is the inverse of the 2nd functional derivative of the 1PI action.

The flow equation has a simple one-loop structure. Because of its simplicity and robustness, the flow equation has been used for various applications.

[1] Bonini et. al. ' 93, Ellwanger '94, Morris ' 94, Wetterich ' 93

Remarks on the flow eq. of 1PI action

$$
\begin{aligned}
\Gamma_{\Lambda,\Lambda_0}[\Phi] &\equiv \Gamma_{I,\Lambda,\Lambda_0}[\Phi] - \frac{1}{2}\int_p \frac{p^2+m^2}{K(p/\Lambda_0)-K(p/\Lambda)}\Phi(-p)\Phi(p) \\
&= \Gamma_{I,\Lambda,\Lambda_0}[\Phi] - \frac{1}{2}\int_p \frac{p^2+m^2}{K(p/\Lambda_0)}\Phi(-p)\Phi(p) - \frac{1}{2}\int_p R_\Lambda(p)\Phi(-p)\Phi(p)\,, \\
&\qquad R_\Lambda(p) \equiv (p^2+m^2)\Big(\frac{1}{K(p/\Lambda_0)-K(p/\Lambda)} - \frac{1}{K(p/\Lambda_0)}\Big)\,.
\end{aligned}
$$

In the last expression of $\Gamma_{\Lambda,\Lambda_0}$, we know

- R_Λ goes to zero as $\Lambda \to 0$. The remaining two terms give rise to the ordinary effective action as $\Lambda \to 0$ and $\Lambda_0 \to \infty$.

- In this sense, $\Gamma_{\Lambda,\Lambda_0}[\Phi]$ consists of two parts, i.e., a pure regulator term with R_Λ and those directly related to the effective action.

Let us write the flow equation again.

$$-\Lambda\frac{\partial}{\partial\Lambda}\Gamma_{I,\Lambda,\Lambda_0}[\Phi] = \frac{1}{2}\int_p \Big(-\Lambda\frac{\partial}{\partial\Lambda}R_\Lambda(p)\Big)\,(\Gamma^{(2)}_{\Lambda,\Lambda_0})^{-1}(p,-p)\,,$$

we have only the interaction part on the l.h.s.

$(\Gamma^{(2)}_{\Lambda,\Lambda_0})^{-1}$ is the inverse of

$$\Gamma^{(2)}_{\Lambda,\Lambda_0}[\Phi] = \Big(\Gamma^{(2)}_{I,\Lambda,\Lambda_0}[\Phi] - \frac{p^2+m^2}{K(p/\Lambda_0)}\Big) - R_\Lambda(p)$$

We notice the following points:

1. $(\Gamma^{(2)}_{\Lambda,\Lambda_0})^{-1}$ is also a field dependent functional;
2. the regulator function $K(p/\Lambda)$ for the cutoff Λ appears only through R_Λ;
3. in $(\Gamma^{(2)}_{\Lambda,\Lambda_0})^{-1}$, we have the regulator function R_Λ.

In expanding $(\Gamma^{(2)}_{\Lambda,\Lambda_0})^{-1}$ in terms of fields, we find a set of differential equations for couplings.

$$-\Lambda\partial_\Lambda \Gamma_{I,\Lambda,\Lambda_0}[\Phi] = \frac{1}{2}$$

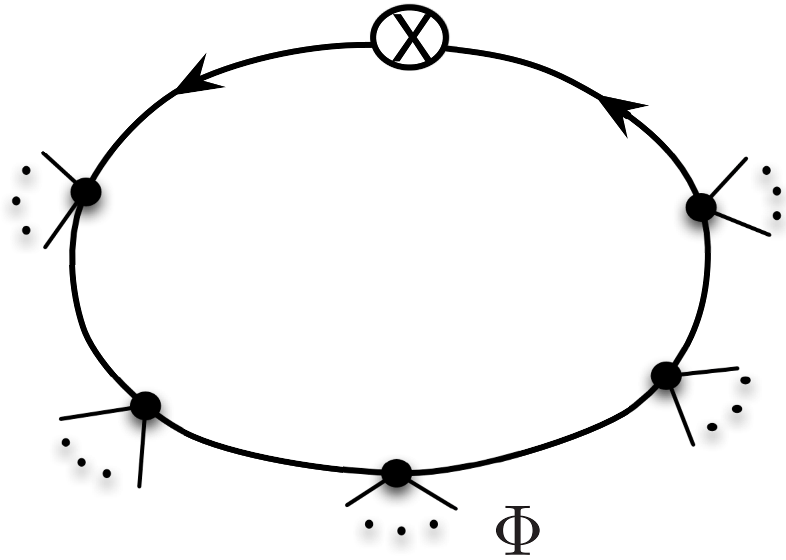


Figure 4: The flow of 1PI action

- $\bigotimes$ represents $-\Lambda\partial_\Lambda R_\Lambda$.
- The full propagator $(\Gamma^{(2)}_{\Lambda,\Lambda_0})^{-1}$ is shown as the arrowed line:
  - Blobs on the line are vertices; small dots are for fields Φ.

Derivation of the 1PI flow equation

Ignoring the subscript Λ_0 for simplicity, we write the 1PI action as

$$\Gamma_\Lambda = -\frac{1}{2}\int_p \Delta_H^{-1}(p)\Phi(-p)\Phi(p) + \Gamma_{I,\Lambda}$$

where

$$\Delta_H(p) = \frac{K(p/\Lambda_0) - K(p/\Lambda)}{p^2 + m^2}$$

is the high momentum propagator.

Expand $\Gamma_{I,\Lambda}$ to define Γ_{2n}

$$\Gamma_{I,\Lambda} = \sum_{n=1}^{\infty} \frac{1}{(2n)!}\int_{p_1,\cdots,p_{2n}} \Phi(p_1)\cdots\Phi(p_{2n})\ \Gamma_{2n}(\Lambda; p_1,\cdots,p_{2n})\ \ (2\pi)^D\delta(p_1+\cdots p_{2n})$$

$\Gamma_{2n}(\Lambda; p_1,\cdots,p_{2n})$ is the collection of 1PI diagrams with $2n$ external legs. We represent it by the following diagram

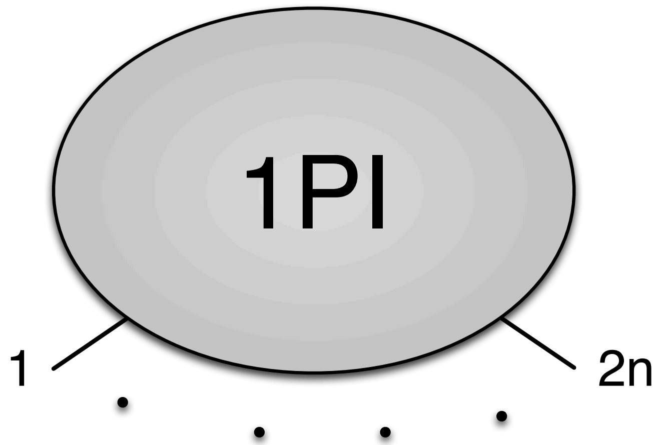
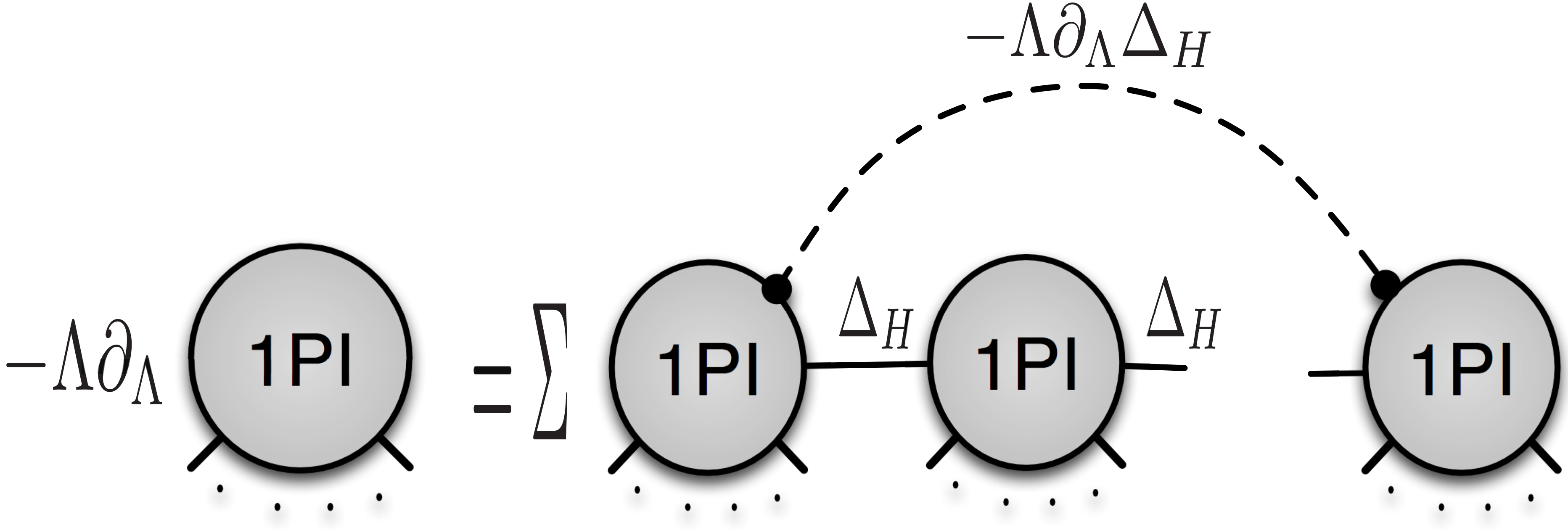


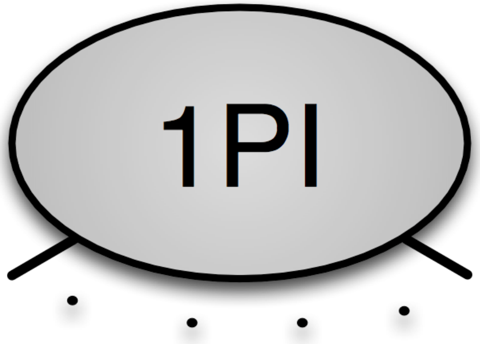
Figure 5: A blob representing 1PI vertex

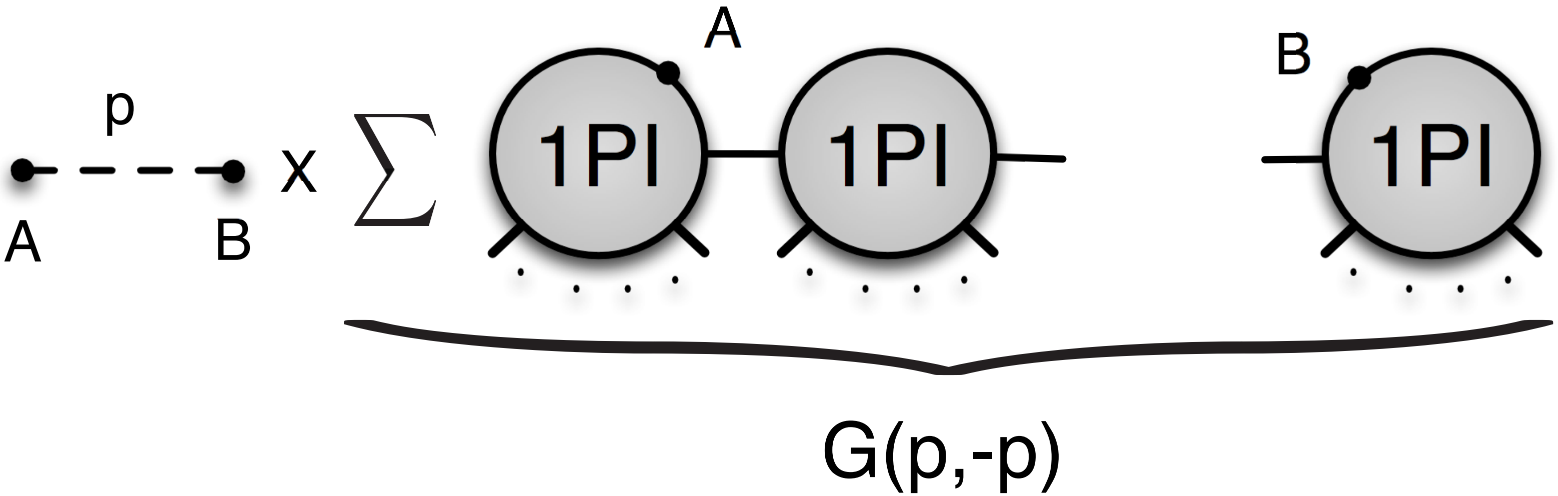


- The derivative acting on a high momentum propagator to give a dotted line that represents

$$-\Lambda\partial_\Lambda\Delta_H(p) = \frac{\Delta(p/\Lambda)}{p^2+m^2}$$

- The derivative $-\Lambda\partial_\Lambda$ acts on all possible Δ_H in the 1PI diagram. The sum is over the possible internal Δ_H.

$-\Lambda\partial_\Lambda$ 

$= \frac{1}{2}\int_p$ 

We introduce the notation $G(p,-p)$ for the above sum and show that it is proportional to the inverse of $\Gamma^{(2)}$.

$$G(p,-q) = \Gamma^{(2)}_{I,\Lambda}(p,-q) + \int_r \Gamma^{(2)}_{I,\Lambda}(p,-r)\Delta_H(r)\Gamma^{(2)}_{I,\Lambda}(r,-q) + \cdots$$

that may be expressed simply

$$G = \Gamma^{(2)}_{I,\Lambda} + \Gamma^{(2)}_{I,\Lambda} \Delta_H \Gamma^{(2)}_{I,\Lambda} + \cdots .$$

Now let us calculate $(\Gamma^{(2)})^{-1}$. Starting from $\Gamma^{(2)}_{\Lambda} = -\Delta_H^{-1} + \Gamma^{(2)}_{I,\Lambda}$, we find

$$\begin{aligned}
\left(\Gamma^{(2)}_{\Lambda}\right)^{-1} &= \left(-\Delta_H^{-1} + \Gamma^{(2)}_{I,\Lambda}\right)^{-1} = \left[\left(1 - \Gamma^{(2)}_{I,\Lambda}\Delta_H\right)\left(-\Delta_H^{-1}\right)\right]^{-1} \\
&= -\left(\Delta_H + \Delta_H \Gamma^{(2)}_{I,\Lambda} \Delta_H + \cdots\right) \\
&= -\left(\Delta_H + \Delta_H G \Delta_H\right) \sim -\Delta_H G \Delta_H
\end{aligned}$$

We ignored the Δ_H term that produces a field independent term to the 1PI flow equation.

$$-\Lambda \partial_\Lambda \Gamma_{I,\Lambda} = \frac{1}{2} \int_p \frac{\Delta}{p^2+m^2} G(p,-p)$$

$$= \frac{1}{2} \int_p (p^2+m^2) \frac{\Delta}{(K(p/\Lambda_0) - K(p/\Lambda)^2} \cdot \Delta_H G(p,-p) \Delta_H$$

$$\sim -\frac{1}{2} \int_p (p^2+m^2) \frac{\Delta}{(K(p/\Lambda_0) - K(p/\Lambda))^2} \left(\Gamma^{(2)}_{\Lambda}\right)^{-1}$$

On the last line, a field independent term is discarded. Finally, we observe

$$-\Lambda\partial_\Lambda R_\Lambda(p) = -(p^2+m^2)\frac{\Delta}{(K(p/\Lambda_0)-K(p/\Lambda))^2}$$

and reach the flow equation for 1PI action,

$$-\Lambda\partial_\Lambda \Gamma_{I,\Lambda} = \frac{1}{2}\int_p \big(-\Lambda\partial_\Lambda R_\Lambda(p)\big)\big(\Gamma^{(2)}_\Lambda(p,-p)\big)^{-1}$$

Relations of 1PI and Wilsonian actions

Rewriting relations appeared in the Legendre transformation, we find

$$\Gamma_{I,\Lambda,\Lambda_0}[\Phi] = S_{I,\Lambda}[\phi] + \frac{1}{2}\int_p \Delta_H^{-1}(p)(\Phi - \phi)(-p)(\Phi - \phi)(p) \tag{1}$$

$$\Phi(p) = \phi(p) + \Delta_H(p)\frac{\delta S_{I,\Lambda}}{\delta\phi(-p)}, \qquad \phi(p) = \Phi(p) - \Delta_H(p)\frac{\delta\Gamma_{I,B,\Lambda}}{\delta\Phi(-p)}\,.$$

The first equation (1) relates vertices or couplings of two actions.

Symmetry vs Regularization

Now we would like to consider a field theory with gauge symmetry.

The regularization with a **momentum cutoff is not compatible with the gauge symmetry in a standard form**. We have encountered a similar situation in the Lattice theory, namely, the chiral symmetry on lattice.

A lesson from chiral symmetry on lattice

In a chiral symmetric theory, we have

$$S_F = \bar{\psi} D \psi, \quad D\gamma_5 + \gamma_5 D = 0\,,$$
$$\delta\psi = \gamma_5 \psi\,, \quad \delta\bar{\psi} = \bar{\psi}\gamma_5\,, \tag{2}$$

The difficulty to realize a chiral theory on lattice, a particular regularization for a field theory, is summarized as the Nielsen-Ninomiya's No-go theorem (1981).

Ginsparg and Wilson stated that we should modify the above relation by the $O(a)$

breaking term,

$$D\gamma_5 + \gamma_5 D = aDR\gamma_5 D, \quad \text{or} \quad \gamma_5 D^{-1} + D^{-1}\gamma_5 = aR\gamma_5$$

where R is an appropriately defined local operator. The $O(a)$ breaking depends on the U.V. action to define a theory and is expected to dissapear in the continuum limit.

Now a solution to the GW relation is known as the Lüscher symmetry (1988)

$$\delta\psi = \gamma_5\Big(1 - a\frac{R}{2}D\Big)\psi, \quad \delta\bar{\psi} = \bar{\psi}\Big(1 - a\frac{R}{2}D\Big)\gamma_5$$
$$\delta S = \bar{\psi}\Big(\gamma_5 D + D\gamma_5 - aDR\gamma_5 D\Big)\psi = 0$$

The lesson here is clear. The chiral symmetry survives in a modified form.

We will find the same story for gauge symmetry in ERG. **Gauge symmetry survives in a modified form even with the presence of the momentum cutoff.**

The anti-field formalism a la Batalin-Vilkovisky

For a classical gauge fixed action $S_{\rm cl}[\phi]$ for a generic gauge theory, define an extended action as

$$\bar{S}_{\rm cl}[\phi, \phi^*] \equiv S_{\rm cl}[\phi] + \phi^*_A \delta\phi^A$$

- Here antifields ϕ^*_A are introduced as sources for the BRST transformations $\delta\phi^A$.

the canonical structure via the antibracket for any field variables X and Y, we define

$$(X, Y) \equiv \frac{\partial^r X}{\partial\phi^A}\frac{\partial^l Y}{\partial\phi^*_A} - \frac{\partial^r X}{\partial\phi^*_A}\frac{\partial^l Y}{\partial\phi^A}$$

$$(\bar{S}_{\rm cl}, \bar{S}_{\rm cl}) = 2(\delta S_{\rm cl} + \phi^*_A \delta^2\phi^A)$$

Classical master equation (CME): $(\bar{S}_{\rm cl}, \bar{S}_{\rm cl}) = 0 \Leftrightarrow$ action invariance *and* the nilpotency.

Generalize the consideration for $\bar{S}[\phi, \phi^*]$ that defines a quantum system via the functional integration over ϕ.

$$\int D\phi \; e^{\bar{S}[\phi,\phi^*]}$$

Under the BRST transformation of fields $\delta\phi^A \equiv (\phi^A, \bar{S}) = \frac{\partial^l \bar{S}}{\partial \phi^*_A}$, the changes of the action and the functional measure are summed up to **the quantum master operator**:

$$\bar{\Sigma}[\phi, \phi^*] \equiv \frac{\partial^r \bar{S}}{\partial \phi^A}\frac{\partial^l \bar{S}}{\partial \phi^*_A} + \frac{\partial^r}{\partial \phi^A}\delta\phi^A = \frac{1}{2}(\bar{S},\ \bar{S}) + \Delta\bar{S}\,, \qquad \Delta \equiv (-)^{\epsilon_A+1}\frac{\partial^r}{\partial \phi^A}\frac{\partial^r}{\partial \phi^*_A}$$

where $\epsilon_A \equiv \epsilon(\phi^A)$ is the Grassmann parity.

The system is BRST invariant quantum mechanically if the two contributions cancel:

$$\bar{\Sigma}[\phi, \phi^*] = 0\ . \quad \text{(QME)}$$

The quantum BRST transformation as

$$\delta_Q X \equiv (X, \bar{S}) + \Delta X$$

We have two important algebraic identities without assuming QME:

$$\delta_Q \bar{\Sigma}[\phi, \phi^*] = 0\,,$$
$$\delta_Q^2 X = (X, \bar{\Sigma}[\phi, \phi^*])\,.$$

The quantum BRST transformation is nilpotent if and only if QME holds.

Also useful to remember that QME $=$ WT identity $+$ nilpotency.

$\bar{\Sigma}[\phi, \phi^* = 0] = 0$ is the WT identity.

Partition functions with different scales are related as

$$\bar{Z}_B[J, \phi^*] = N_J \bar{Z}_\Lambda[J, \Phi^*]$$

$$\bar{Z}_{\Lambda_0}[J, \phi^*] = \int D\phi \exp\left(\bar{S}_{\Lambda_0}[\phi, \phi^*] + K_0^{-1} J \cdot \phi\right)$$

$$\bar{S}_{\Lambda_0}[\phi, \phi^*] \equiv -\frac{1}{2}\phi \cdot K_0^{-1} D \cdot \phi + S_{I,\Lambda_0}[\phi] + \phi^* \cdot \delta\phi$$

$$\bar{Z}_\Lambda[J, \Phi^*] = \int D\Phi \exp\left(\bar{S}_\Lambda[\Phi, \Phi^*] + K^{-1} J \cdot \Phi\right)$$

$$\bar{S}_\Lambda[\Phi, \Phi^*] \equiv -\frac{1}{2}\Phi \cdot K^{-1} D \cdot \Phi + \bar{S}_{I,\Lambda}[\Phi, \Phi^*]$$

$$N_J \equiv \exp\left(-\frac{(-)^{\epsilon_A}}{2} J_A K_0^{-1} K^{-1} \Delta_H^{AB} J_B\right)$$

$$K\Phi_A^* = K_0 \phi_A^*$$

N_J is produced via path integral over the high momentum modes: two J are connected by Δ_H. N_J turns out to be very important. (Φ^A, Φ_A^*) is a canonical pair at Λ.

Symmetry under the scale change

The QM operator has an important property: it satisfies the linearized Polchinski equation

$$-\Lambda \frac{\partial}{\partial \Lambda} \bar{\Sigma}_\Lambda = \bar{\mathcal{D}} \bar{\Sigma}_\Lambda$$

where

$$\bar{\mathcal{D}} \equiv \int_p \Big[\big(K^{-1}\Delta\big) \Big(\Phi^A \frac{\partial^l}{\partial \Phi^A} - \Phi^*_A \frac{\partial^l}{\partial \Phi^*_A} \Big) + (-)^{\epsilon_A} \big(D^{-1}\Delta\big)^{AB} \Big(\frac{\partial^l \bar{S}_\Lambda}{\partial \Phi^B} \frac{\partial^r}{\partial \Phi^A} + \frac{1}{2} \frac{\partial^l \partial^r}{\partial \Phi^B \partial \Phi^A} \Big) \Big] .$$

Therefore once we find $\bar{\Sigma}_\Lambda = 0$ at some scale, it vanishes under the scale change. The RG flow keeps the gauge symmetry.

Scale change of BRST transformation

Taking ϕ^* derivative of $\bar{Z}_B[J,\phi^*] = N_J \bar{Z}_\Lambda[J,\Phi^*]$ and using $K\Phi^*_A = K_0 \phi^*_A$, we find

$$\langle K_0^{-1}\delta\phi^A\rangle_{\bar{S}_{\Lambda_0},K_0^{-1}J} = N_J \langle K^{-1}\delta\Phi^A\rangle_{\bar{S}_\Lambda,K^{-1}J}\,.$$

Recall

$$\frac{1}{K}\frac{\partial^l \bar{S}_\Lambda}{\partial\Phi^*_A} = K^{-1}\delta\Phi^A\,.$$

The above relation tells us the scale dependence of BRST transformation.

Now let us find out the scale dependence

Suppose BRST transformations at Λ_0, Λ are given by the following functionals,

$$K_0^{-1}\delta\phi^A = \mathcal{R}^A[\phi, \phi^*], \qquad K^{-1}\delta\Phi^A = R^A[\Phi, \Phi^*]$$

Rewriting the relation $\langle K^{-1}\delta\Phi^A\rangle_{\bar{S}_\Lambda, K^{-1}J} = N_J^{-1}\langle K_0^{-1}\delta\phi^A\rangle_{\bar{S}_{\Lambda_0}, K_0^{-1}J}$, we find

$$\langle R^A[\Phi, \Phi^*]\rangle_{\bar{S}_\Lambda, K^{-1}J} = N_J^{-1}\langle \mathcal{R}^A[\phi, \phi^*]\rangle_{\bar{S}_{\Lambda_0}, K_0^{-1}J}$$

$$\text{l.h.s.} = R^A[K\partial_J^l, \Phi^*]\bar{Z}_\Lambda[J, \Phi^*]$$

$$\text{r.h.s.} = N_J^{-1}\mathcal{R}^A[K_0\partial_J^l, \phi^*]\bar{Z}_{\Lambda_0}[J, \phi^*] = N_J^{-1}\mathcal{R}^A[K_0\partial_J^l, \phi^*]N_J \cdot \bar{Z}_\Lambda[J, \Phi^*]$$

We observe here that J derivative acts on N_J and that produces the change.

Let us assume the transformation at Λ_0 as

$$K_0^{-1}\delta\phi^A = \mathcal{R}^{(1)A}{}_B(\Lambda_0)\phi^B + \frac{1}{2}\mathcal{R}^{(2)A}{}_{BC}(\Lambda_0)\phi^B\phi^C$$

Here the coefficients $\mathcal{R}^{(1)A}{}_{B}(\Lambda_0),\ \mathcal{R}^{(2)A}{}_{BC}(\Lambda_0)$ are given at Λ_0.

We are interested in how the r.h.s. changes under the flow.

- As explained shortly, the linear term changes as

$$[\Phi^A]^* \equiv \Phi^A + (K_0 - K)(D^{-1})^{AB} \frac{\partial^l \bar{S}_{I,\Lambda}}{\partial \Phi^B}$$

- The bilinear term $\phi^B \phi^C$ changes as

$$[\Phi^A \Phi^B]^* \equiv [\Phi^A]^* [\Phi^B]^* + (K_0 - K)(D^{-1})^{AC} (K_0 - K)(D^{-1})^{BD} \frac{\partial^l \partial^l \bar{S}_{I,\Lambda}}{\partial \Phi^C \partial \Phi^D}$$

We find the BRST transformation at Λ as

$$\delta \Phi^A = K \Big(\mathcal{R}^{(1)A}{}_{B}(\Lambda_0)[\Phi^B]^* + \frac{1}{2} \mathcal{R}^{(2)A}{}_{BC}(\Lambda_0)[\Phi^B \Phi^C]^* \Big)$$

Derivation of $[\Phi^A]^*$ and notion of composite operators

Observe the relation of partition functions at Λ and Λ_0

$$\int D\phi\; e^{\bar{S}_{\Lambda_0}+K_0^{-1}J\cdot\phi} = N_J \int D\Phi\; e^{\bar{S}_\Lambda + K^{-1}J\cdot\Phi}\,,$$

$$N_J \equiv \exp\Big(-\tfrac{(-)^{\epsilon_A}}{2} J_A K_0^{-1} K^{-1} (\Delta_H)^{AB} J_B\Big)\,.$$

Now acting $K_0 \partial^l/\partial J_A$, we find

$$\begin{aligned}
\langle \phi^A \rangle_{\Lambda_0, K_0^{-1}J} &= \int D\phi\; \phi^A\; e^{\bar{S}_{\Lambda_0}+K_0^{-1}J\cdot\phi} \\
&= N_J \int D\Phi \Big[\frac{K_0}{K}\Phi + \underbrace{\Big(N_J^{-1} K_0 \frac{\partial^l}{\partial J_A} N_J\Big)}_{-(-)^{\epsilon_A} K^{-1}(\Delta_H)^{AB} J_B}\Big] e^{\bar{S}_\Lambda + K^{-1}J\cdot\Phi} \\
&= N_J \langle \Big(\Phi^A + (\Delta_H)^{AB} \frac{\partial^l \bar{S}_{I,\Lambda}}{\partial \Phi^B}\Big)\rangle_{\Lambda, K^{-1}J}
\end{aligned}$$

From the above derivation, we know the scale dependence of the quantity in the brace,

$$\left[\Phi^A\right]^* \equiv \Phi^A + (\Delta_H)^{AB} \frac{\partial^l \bar{S}_{I,\Lambda}}{\partial \Phi^B}$$

In referring to the Polchinski equation, we may characterize the scale dependence of $\left[\Phi^A\right]^*$ by saying that it follows the linearized Polchinski equation. This is the notion of the composite operator due to Becchi.

There are couple of convenient composite opratoers. Those are explained in our review. Obviouly, $\bar{\Sigma}$ is the most important composite operator that tells us the presence of BRST symmetry.

The 1PI action $\bar{\Gamma}_{\Lambda_0,\Lambda}$, QME and the modified ST identity

Introduce antifield dependence to 1PI action

$$\exp\Big(\bar{W}_{\Lambda_0,\Lambda}[J,\phi^*]\Big) \equiv \int \mathcal{D}\phi \exp\Big(-\frac{1}{2}\phi\cdot(K_0-K)D\cdot\phi + \bar{S}_{I,\Lambda_0}[\phi,\phi^*] + K_0^{-1}J\cdot\phi\Big)$$

Define the effective average action as

$$\bar{\Gamma}_{\Lambda_0,\Lambda}[\varphi_\Lambda,\phi^*] \equiv \bar{W}_{\Lambda_0,\Lambda}[J,\phi^*] - K_0^{-1}J\cdot\varphi_\Lambda, \qquad \varphi_\Lambda(p) \equiv K_0(p)\frac{\partial^l \bar{W}_{\Lambda_0,\Lambda}[J,\phi^*]}{\partial J(-p)}$$

QME and the modified ST identity

The path integral average of the QM operator $\bar{\Sigma}_{\Lambda_0}[\phi, \phi^*]$

$$\bar{\Sigma}^{1PI}_{\Lambda_0,\Lambda}[\varphi_\Lambda, \phi^*] \equiv \int \mathcal{D}\phi \; \bar{\Sigma}_{\Lambda_0}[\phi, \phi^*] \exp\Big(\bar{S}_{\Lambda_0,\Lambda}[\phi, \phi^*] + K_0^{-1} J \cdot \phi\Big) \;/\; \exp[\bar{W}_{\Lambda_0,\Lambda}[J, \phi^*]]$$

$$= \frac{\partial^r \bar{\Gamma}_{\Lambda_0,\Lambda}}{\partial \varphi^A_\Lambda} \frac{\partial^l \bar{\Gamma}_{\Lambda_0,\Lambda}}{\partial \phi^*_A} + \underbrace{[R_\Lambda]_{BA}\Big(-(\bar{\Gamma}^{(2)})^{-1}_{\Lambda_0,\Lambda} \frac{\partial^l}{\partial \varphi^C_\Lambda} \frac{\partial^l \bar{\Gamma}_{\Lambda_0,\Lambda}}{\partial \phi^*_A} + \varphi^B_\Lambda \frac{\partial^l \bar{\Gamma}_{\Lambda_0,\Lambda}}{\partial \phi^*_A}\Big)}$$

$$[R_\Lambda(p)]_{BA} \equiv D_{BA}(p)\Big(\frac{1}{K_0 - K} - \frac{1}{K_0}\Big) \qquad \to 0 \;\; \text{as} \;\; \Lambda \to 0$$

- $\bar{\Sigma}^{1PI}_{\Lambda_0,\Lambda} = 0$ is the modified Slavnov-Taylor identity. (Ellwanger 1994)

- It gives the Zinn-Justin equation in $\Lambda \to 0$.

Antifield dependence of the Wilson action

Assume the following action at Λ_0

$$\begin{aligned}\bar{S}_{\Lambda_0}[\phi,\ \phi^*] &= -\frac{1}{2}\phi^A K_0^{-1} D_{AB}\phi^B + \bar{S}_{I,\Lambda_0}[\phi,\ \phi^*]\ ,\\ \bar{S}_{I,\Lambda_0}[\phi,\ \phi^*] &= S_{I,B}[\phi] + K_0\phi^*_A \mathcal{R}^A[\phi]\ ,\\ \mathcal{R}^A[\phi] &= \mathcal{R}^{(1)A}{}_B\phi^B + \frac{1}{2}\mathcal{R}^{(2)A}{}_{BC}\phi^B\phi^C\ .\end{aligned}$$

we obtain our final expression for the Wilsonian action:

$$\begin{aligned}\bar{S}_\Lambda[\Phi,\ \Phi^*] = &-\frac{1}{2}\Phi^A K^{-1} D_{AB}\Phi^B + K\Phi^*_A\mathcal{R}^A[\Phi]\\ &+\ln\Bigg[\exp\Big\{\frac{K}{2}\Phi^*_A\mathcal{R}^{(2)A}{}_{BC}\frac{\partial^l}{\partial\mathcal{J}_B}\frac{\partial^l}{\partial\mathcal{J}_C}\Big\}\\ &\qquad\times\exp\Big\{\frac{1}{2}(-)^{\epsilon(\mathcal{J})}\mathcal{J}\cdot(K_0-K)D^{-1}\cdot\mathcal{J} + S_{I,\Lambda}[\Phi']\Big\}\Bigg].\end{aligned}$$

$$
\begin{aligned}
\bar{S}_\Lambda[\Phi,\ \Phi^*] &= -\frac{1}{2}\Phi^A K^{-1} D_{AB}\Phi^B + K\Phi^*_A \mathcal{R}^A[\Phi] \\
&+ \ln\Bigg[\exp\Big\{\frac{K}{2}\Phi^*_A \mathcal{R}^{(2)A}{}_{BC}\frac{\partial^l}{\partial \mathcal{J}_B}\frac{\partial^l}{\partial \mathcal{J}_C}\Big\} \\
&\qquad \times \exp\Big\{\frac{1}{2}(-)^{\epsilon(\mathcal{J})}\mathcal{J}\cdot(K_0 - K)D^{-1}\cdot\mathcal{J} + S_{I,\Lambda}[\Phi']\Big\}\Bigg].
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_A &\equiv K\Phi^*_B\left(\mathcal{R}^{(1)B}{}_A + \mathcal{R}^{(2)B}{}_{CA}\Phi^C\right), \\
\Phi'^A &\equiv \Phi^A + \mathcal{J}_B(K_0 - K)(D^{-1})^{BA}.
\end{aligned}
$$

Note we need to know the initial Wilsonian action $\bar{S}$ or the action satisfying the WT identity and the BRST transformation at Λ_0.

For QED, the Wilsonian action is simplified since $R^{(2)A}_{BC} = 0$.

$$\bar{S}_\Lambda[\Phi,\ \Phi^*] = -\frac{1}{2}\Phi^A K^{-1} D_{AB} \Phi^B + K\Phi^*_A \mathcal{R}^A[\Phi]$$
$$+\frac{1}{2}(-)^{\epsilon(\mathcal{J})}\mathcal{J}\cdot(K_0 - K)D^{-1}\cdot\mathcal{J} + S_{I,\Lambda}[\Phi']\,.$$

$$\mathcal{J}_A \equiv K\Phi^*_B \mathcal{R}^{(1)B}{}_A\,,$$
$$\Phi'^A \equiv \Phi^A + \mathcal{J}_B(K_0 - K)(D^{-1})^{BA}$$
$$= \Phi^A + K\Phi^*_C \mathcal{R}^{(1)C}{}_B(K_0 - K)(D^{-1})^{BA}$$

- The field variable in $S_{I,\Lambda}$ is shifted by a term linear in Φ^*.

A remark: Variations of Wilsonian flow equations

- We have explained the Polchinski equation with a single regularization function $K(p/\Lambda)$. There are other ways to introduce regulator functions for the Wilsonian RG flow equation: Bervillier (2004,2013,2014); Ball et al (1995); Rosten (2011); Osborn and Twigg (2012), for example.

- Though they look very different each other in their appearances, one can understand them in an organized manner as different implementation of regularization functions (Igarashi, KI, Sonoda 2016).

We do not go into further details.

Dimensionless formulation

In the formulation we have described, the Wilsonian and 1PI actions keep changing along flows. We do not find any fixed points from flow equations. In order to find the phase structure, we have to move to dimensionless formulation by using the momentum cutoff Λ. Only the dimensional flow equation can be put on a computer.

Let us consider the flow equation:

$$\Lambda\partial_\Lambda \Gamma_{I,\Lambda} = \frac{1}{2}\int_p \mathrm{Str}\left[\Lambda\partial_\Lambda R_\Lambda(p)\left(\Gamma_\Lambda^{(2)}\right)^{-1}(p,-p)\right].$$

- Easy to count mass dimensions for the quantities in the flow equation.
- We know that there could appear logarithmic terms like $\ln(\Lambda/\mu)$ as we have seen in a perturbative calculation. The wave function renormalization is of this type.

- $\bar{x}_\mu = \Lambda x_\mu$ and $\bar{p}_\mu = p_\mu/\Lambda$, $\delta^d(\bar{p}) = \Lambda^d \delta^d(p)$.

- Introduce the parameter t as

$$\partial_t = \Lambda \partial_\Lambda \,, \quad t = \ln(\Lambda/\mu) \,.$$

- The dimensionless fields

$$\bar{\Phi}^A(\bar{x}) = \sqrt{Z_A}\ \Lambda^{-d_A}\ \Phi(x) \,,$$
$$\bar{\Phi}^A(\bar{p}) = \sqrt{Z_A}\ \Lambda^{d-d_A}\ \Phi(p) \,.$$

- $Z_A(t) = Z_A(\ln(\Lambda/\mu))$ and we define the corresponding anomalous dimension η_A as

$$\eta_A = -\partial_t \ln Z_A = -\Lambda \partial_\Lambda \ln Z_A \,.$$

Now we need to make all the coefficients in expanding the 1PI by fields.

As a preparation, we write the 1PI action in two ways: they differ only in the kinetic terms.

$$\Gamma_\Lambda = \Gamma_{I,\Lambda} - \frac{1}{2}\Phi^A(\Delta_H^{-1})_{AB}\Phi^B \equiv \hat{\Gamma}_\Lambda - \frac{1}{2}\Phi^A(R_\Lambda)_{AB}\Phi^B$$

$\hat{\Gamma}_\Lambda$ is given as

$$\hat{\Gamma}_\Lambda = -\frac{1}{2}\Phi^A\frac{D_{AB}}{K(p/\Lambda_0)}\Phi^B + \Gamma_{I,\Lambda} \quad \xrightarrow{\Lambda_0\to\infty} \quad -\frac{1}{2}\Phi^A D_{AB}\Phi^B + \Gamma_{I,\Lambda}$$

In understanding the UV side of a theory may be properly taken care of as we have seen in a perturbative calculation, we send Λ_0 to infnity to simplify expressions.

1. Define coefficients by expanding $\hat{\Gamma}_\Lambda$ in terms of fields

2. Replace all the quantities by their dimensionless forms

3. We may define the dimensionless coefficients

$$\hat{\Gamma}_\Lambda = \sum_{n=2}^{\infty} \int \frac{d^d p_1}{(2\pi)^d} \cdots \frac{d^d p_n}{(2\pi)^d} (2\pi)^d \delta^d(p_1 + \cdots p_n) \Gamma_{A_1, \cdots, A_n}(\Lambda;\ p_1, \cdots, p_n) \Pi_{i=1}^n \Phi^{A_i}(p_i)$$

$$= \sum_{n=2}^{\infty} \int_{\bar{p}_i} \Lambda^{nd} \Lambda^{-d} (2\pi)^d \delta^d(\bar{p}_1 + \cdots \bar{p}_n) \Gamma_{A_1, \cdots, A_n}(\Lambda;\ p_1, \cdots, p_n) \Pi_{i=1}^n \Lambda^{-d+d_{A_i}} \frac{\bar{\Phi}^{A_i}(\bar{p}_i)}{\sqrt{Z_{A_i}}} .$$

Define the dimensionless coefficients $\bar{\Gamma}_{A_1, \cdots, A_n}$

$$\bar{\Gamma}_{A_1, \cdots, A_n}(t; \bar{p}_1, \cdots, \bar{p}_n) \equiv \frac{\Lambda^{\sum_i d_{A_i} - d}}{\sqrt{Z_{A_1} \cdots Z_{A_n}}} \Gamma_{A_1, \cdots, A_n}(\Lambda;\ p_1, \cdots, p_n) .$$

Taking the t derivative of the dimensionless coefficients

$$\bar{\Gamma}_{A_1,\cdots,A_n}(t;\bar{p}_1,\cdots,\bar{p}_n) \equiv \frac{\Lambda^{\sum_i d_{A_i}-d}}{\sqrt{Z_{A_1}\cdots Z_{A_n}}}\,\Gamma_{A_1,\cdots,A_n}(\Lambda;\ p_1,\cdots,p_n)\,.$$

we find

$$\partial_t\bar{\Gamma}_{A_1,\cdots,A_n}(t;\bar{p}_1,\cdots\bar{p}_n) = \Big(\sum_i\big(d_{A_i}+\frac{\eta_{A_i}}{2}\big)-d\Big)\bar{\Gamma}_{A_1,\cdots,A_n}$$
$$+\frac{\Lambda^{\sum_i d_{A_i}-d}}{\sqrt{Z_{A_1}\cdots Z_{A_n}}}\,\big(\partial_t\Gamma_{A_1,\cdots,A_n}|_p+\sum_i\bar{p}_i^\mu\partial_{\bar{p}_i^\mu}\Gamma_{A_1,\cdots,A_n}\big)\,.$$

- The red part was included in the dimensionful flow equation.
- The t derivative on a Z factor produces an anomalous dimension $\eta_A = -\partial_t \ln Z_A$.

$$\eta_A \equiv -\partial_t \ln Z_A$$

- p derivative acts on $\Gamma_{A_1,\cdots,A_n}$ and it does not act on the delta function.

Dimensionless flow equation

Finally, we reach the dimensionless flow equation

$$\partial_t \hat{\Gamma}_\Lambda[\Phi] = (-)^{\epsilon_A} \frac{1}{2} \int_p \left(\partial_t R - \eta R\right)_{AB}(p) \left[\left(\Gamma_\Lambda^{(2)}\right)^{-1}\right]^{AB}(p,-p)$$

$$-d\, \hat{\Gamma}_\Lambda[\Phi] + (d_A + \eta_A/2) \int_p \Phi^A(p) \frac{\partial^l \hat{\Gamma}_\Lambda[\Phi]}{\partial \Phi^A(p)} + \int_p \Phi^A(p)\; p \cdot \frac{\partial}{\partial p} \left(\frac{\partial^l}{\partial \Phi^A(p)}\right)' \hat{\Gamma}_\Lambda[\Phi] \;,$$

where

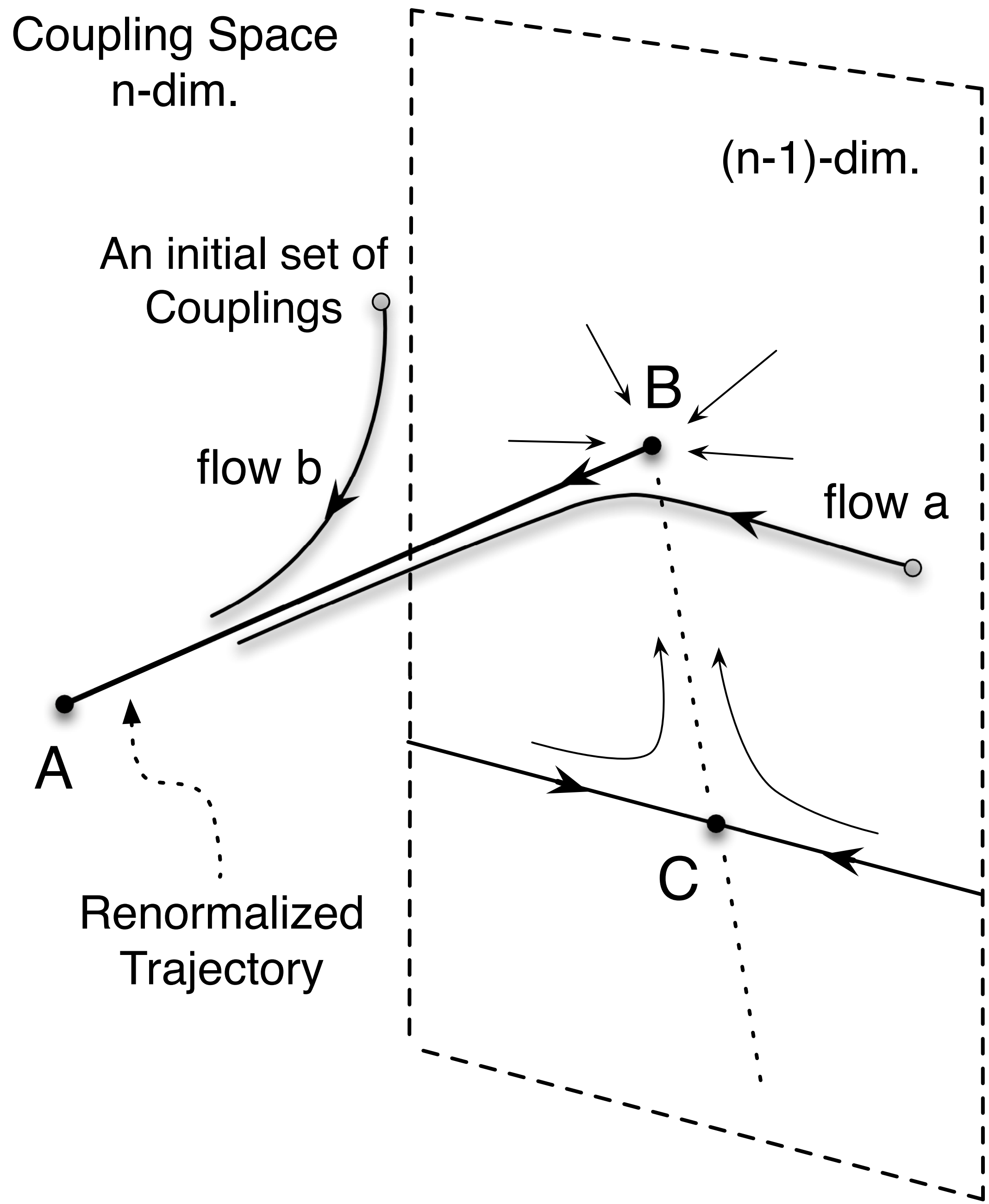
$$\left(\partial_t R - \eta R\right)_{AB}(p) = \partial_t R_{AB} - \eta_A R_{AB} = -r_A(p) D_{AB}(p)$$

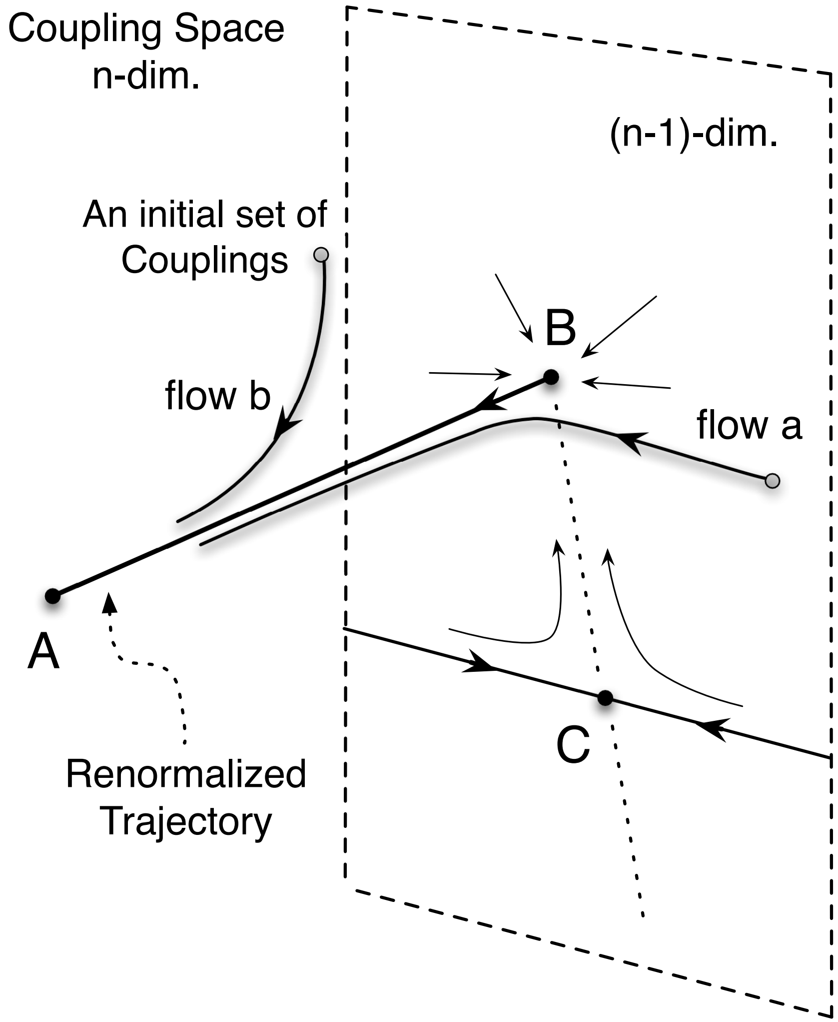
$$r_A(p) = -\partial_t \left(\frac{K}{1-K}\right) + \eta_A \frac{K}{1-K} = \frac{2x\; K'(x)}{\left(1-K(x)\right)^2} + \eta_A \frac{K(x)}{1-K(x)} \;.$$

- The bar to indicate dimensionless quantities are ignored here.
- Expanding the dimensionless flow equation in terms of fields, we find a set of differential equations for the dimensionless couplings defined above.

Hierarchical Structure of Phase Space

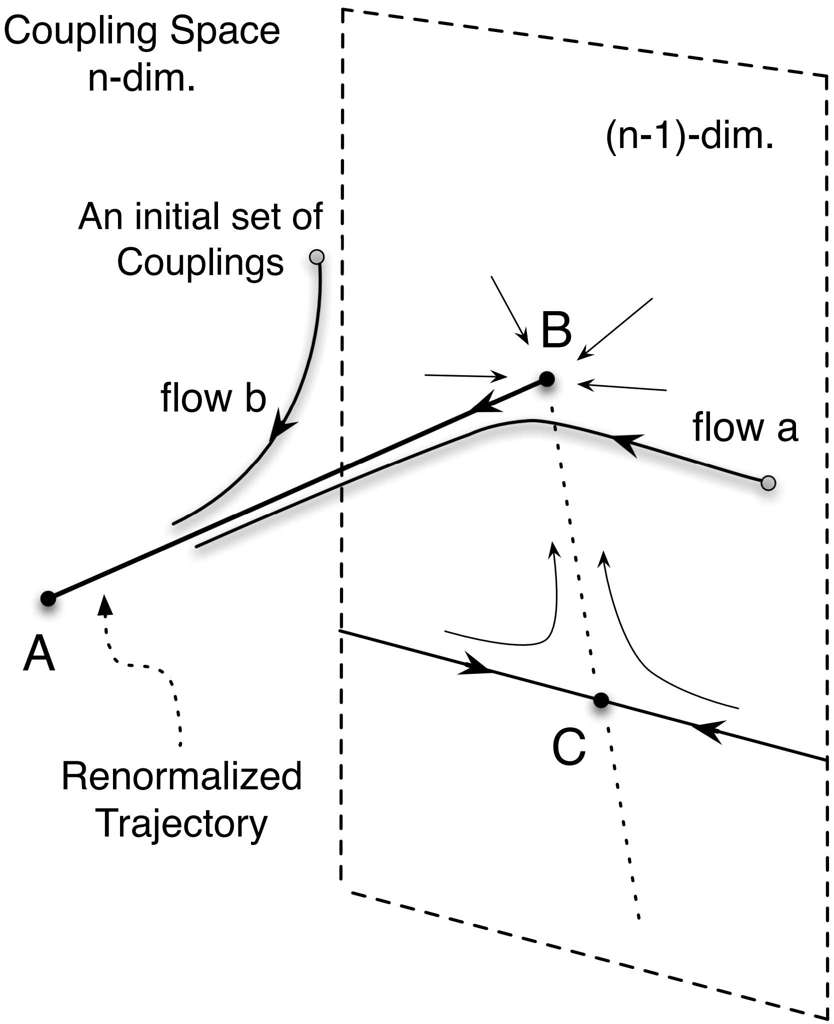
- Coupling space is, generally speaking, an infinite dimensional space. The figure is meant to be drawn for a n dimensional coupling space.
- The figure shows a typical sets of renormalization flows obtained with dimensionless formulation of the flow equation.
- Later we will see this structure for QED.





Explanations

- Starting from two sets of initial couplings, we have drawn the flows a and b.
- Adding more and more flows, we start to observe structures.
  - Three points, A, B and C, are fixed points of the flow equation.
  - The point A is a sink of all the flows starting from initial points located on the left of the $(n-1)$ dimensional boundary manifold, the critical surface.
  - On the $(n-1)$ subspace, we find a similar structure: the point B as a sink and the $(n-2)$ dim. critical surface (the line in the figure). The point C is again a sink.



- The point C has $(n-2)$ irrelevant (shrinking) directions and two relevant (extending) directions, one on the $(n-1)$ dim. plane and the other coming out of the plane.
- The point B has one relevant and $(n-1)$ irrelevant directions.
- The flow from B to A is the renormalized trajectory that defines a field theory: the points A and B are the I.R. and U.V. fixed points respectively for the field theory.

Application to QED

This part is based on an on-going work in collaboration with

Y. Echigo, Y. Igarashi, J. Pawlowski and Y. Takahashi

- We consider QED with a massless fermion with ERG.
- Dimensionaless flow equations are solved numerically to find the phase structure.
- To see the consistency with some earlier results, the anomalous mass dimension γ_m is evaluated at the UV fixed point.
- Conditions out of the modified WT identity are considered.

Properties of Wilsonian and 1PI actions

	Wilson action	1PI action
	$S_\Lambda[\phi]$	$\Gamma_\Lambda[\Phi]$
Diagrams	Connected	1PI
Gauge symmetry	mod. WT id.	'mod. Slavnov-Taylor id.'
RG flow eq.	Polchinski eq.	1PI flow eq.

Two actions are related via a Legendre transformation.

$$\Gamma_{I,\Lambda}[\Phi] = S_{I,\Lambda}[\phi] - \frac{1}{2}(\Phi - \phi){\Delta_H}^{-1}(\Phi - \phi)$$

$$\Phi - \phi = -\Delta_H \frac{\partial^l S_{I,\Lambda}}{\partial \phi} = -\Delta_H \frac{\partial^l \Gamma_{I,\Lambda}}{\partial \Phi}$$

where $\Delta_H \equiv (1-K)D^{-1}$ is the high momentum propagator that allows momentum modes above the cutoff k to propagate. (A remark:Λ_0 is sent to the infinity and we have $(1-K)$ in Δ_H)

Our strategy Use WT id. for Wilson action and find RG flows for 1PI action.

WT identity: $\Sigma_\Lambda[\phi] = 0$.

(cf. Ellwanger (1994), Ellwanger et. al.(1996), Igarashi et.al. (2009) and (2016))

$$\Sigma_\Lambda[\phi] = Z_e Z_3^{1/2} \int_p \left[\frac{\partial S_\Lambda}{\partial a_\mu(p)} (-ip_\mu) c(p) + \frac{\partial^r S_\Lambda}{\partial \bar{c}(p)} \xi^{-1} p_\mu a_\mu(p) \right]$$
$$-i\ e \int_{p,q} \left[\frac{\partial^r S_\Lambda}{\partial \psi_\alpha(q)} \frac{K(q)}{K(p)} \psi_\alpha(p) - \frac{K(p)}{K(q)} \bar{\psi}_{\hat{\alpha}}(-q) \frac{\partial^l S_\Lambda}{\partial \bar{\psi}_{\hat{\alpha}}(-p)} \right] c(q-p)$$
$$-i\ e \int_{p,q} U_{\beta\hat{\alpha}}(-q,p) \left[\frac{\partial^l S_\Lambda}{\partial \bar{\psi}_{\hat{\alpha}}(-p)} \frac{\partial^r S_\Lambda}{\partial \psi_\beta(q)} - \frac{\partial^l \partial^r S_\Lambda}{\partial \bar{\psi}_{\hat{\alpha}}(-p) \partial \psi_\beta(q)} \right] c(q-p)$$

$$U_{\beta\hat{\alpha}}(-q,p) \equiv \left[K(q) \frac{\big(1 - K(p)\big)}{\not{p}} - \frac{K(p)\big(1 - K(q)\big)}{\not{q}} \right]_{\beta\hat{\alpha}}$$

- ϕ^A are also renormalised fields with the same Z factors as Φ^A.
- Z_e is for a finite renormalisation of the gauge coupling.
- $Z_e Z_3^{1/2}$ appears in the WT identity.

- Why Z factors in the WT identity? In deriving the WT identity, we start from the known symmetry at some scale and observe how it changes under the scale change.

- The flow equation and the WT identity (or the master equation in BV formalism) are formally compatible if we include all the possible couplings.

  1. Perturbative renormalizability has been discussed (e.g. Bonini et al (1997), Igarashi, Sonoda, KI (2009)).
  2. The Wilson action with a finite cut-off may be constructed explicitly based on the flow and master equations (Igarashi, Morris, KI(2019)). **Tim Morris will explain this in the next talk.**

- **For a practical calculation, we must restrict the number of possible couplings, a truncation of the action is inevitable. We do not yet know a scheme for the truncation that keep consistency of the flow equation and symmetry requirement.**

Ansatz to 1PI action for renormalised field Φ in Local Potential Approximation

(Igarashi et. al. J. Phys. A (2016))

$$\Gamma_\Lambda[\Phi] = \frac{1}{2}\Phi^A D_{AB}\Phi^B + \Gamma_{I,\Lambda}[\Phi]$$

$$\Gamma_{I,\Lambda}[\Phi] = \frac{h^{(aa)}_{\mu\nu}}{2}A_\mu A_\nu + 0\cdot\bar{\Psi}\Psi + e\cdot\bar{\Psi}A\Psi + G_S \text{ and } G_V\cdot\bar{\Psi}\Psi\bar{\Psi}\Psi\,,$$

$$h^{(aa)}_{\mu\nu}(t,p) = P^T_{\mu\nu}\mathcal{T}(t,p^2) + P^L_{\mu\nu}\mathcal{L}(t,p^2)\ .$$

- $P^T_{\mu\nu} = \delta_{\mu\nu} - p_\mu p_\nu/p^2,\ P^L_{\mu\nu} = p_\mu p_\nu/p^2$, the projection operators.

- $\mathcal{T}(t,p^2),\ \mathcal{L}(t,p^2),\ \alpha(t)\equiv e^2(t),\ G_S(t)$ and $G_V(t)$ are scale dependent and to be determined with flow equations and WT identity.

Ansatz to 1PI action in concrete

$$\Gamma_\Lambda[\Phi] = \frac{1}{2}\Phi^A D_{AB}\Phi^B + \Gamma_{I,\Lambda}[\Phi]$$

$$= \frac{1}{2}\int_p A_\mu(-p)\left[P^T_{\mu\nu}\left\{p^2 + \mathcal{T}(t,p^2)\right\} + P^L_{\mu\nu}\left\{\xi^{-1}p^2 + \mathcal{L}(t,p^2)\right\}\right]A_\nu(p)$$

$$+ \int_p \left[\bar{C}(-p)ip^2C(p) + \bar{\Psi}(-p)\not{p}\Psi(p)\right]$$

$$-e(t)\int_{p,q}\bar{\Psi}(-p)\not{A}_\mu(p-q)\Psi(q) + \frac{1}{2}\int_{p_1,\cdots,p_4}(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)$$

$$\times\Big[G_S(t)\Big\{\left(\bar{\Psi}(p_1)\Psi(p_2)\right)\left(\bar{\Psi}(p_3)\Psi(p_4)\right) - \left(\bar{\Psi}(p_1)\gamma_5\Psi(p_2)\right)\left(\bar{\Psi}(p_3)\gamma_5\Psi(p_4)\right)\Big\}$$

$$+G_V(t)\Big\{\left(\bar{\Psi}(p_1)\gamma_\mu\Psi(p_2)\right)\left(\bar{\Psi}(p_3)\gamma_\mu\Psi(p_4)\right) + \left(\bar{\Psi}(p_1)\gamma_5\gamma_\mu\Psi(p_2)\right)\left(\bar{\Psi}(p_3)\gamma_5\gamma_\mu\Psi(p_4)\right)\Big\}\Big]$$

Flow equation for photon two-point part of 1PI action, $\mathcal{T}$ and $\mathcal{L}$

$\mathcal{T}$ and $\mathcal{L}$ are functions of $x = p^2$ and $t = \ln \Lambda/\mu$.

By using the notation

$$D_x \equiv x\frac{\partial}{\partial x} - 1, \quad D_t \equiv \frac{\partial}{\partial t} - \eta_A(\alpha(t), \xi(t)) \,,$$

the flow equations are

$$(D_x - \tfrac{1}{2}D_t)\mathcal{T}(t,x) + \tfrac{\eta_A}{2}x = \alpha \; C_T^{(0)}(x) + \alpha\eta_\psi \; C_T^{(1)}(x) \,,$$
$$(D_x - \tfrac{1}{2}D_t)\mathcal{L}(t,x) - \tfrac{x}{2}D_t\xi(t)^{-1} = \alpha \; C_L^{(0)}(x) + \alpha\eta_\psi \; C_L^{(1)}(x) \,,$$

- $\alpha(t)$ and $\eta_{A,\psi}(\alpha(t), \xi(t))$ are t-dependent coefficients.
- $C_{T,L}^{(i)}$ are coefficients functions of $x = p^2$.
- Extracting x-linear terms from flow eqs., we find:

– Anomalous dimentions η_A are determined algebraically.
– Flow eq. of ξ, the gauge parameter, $D_t\xi(t)^{-1} = 0$, or

$$\partial_t \xi = -\eta_A \xi$$

The Landau gauge $\xi = 0$ is consistent with the flow of ξ.

- The rest are differential equations for $\mathcal{T}(t, x = p^2)$ and $\mathcal{L}(t, x)$.

Choosing the regulator function as $K(x) = \exp(-x)$,
we find differential equations for $\mathcal{T}$, $\mathcal{L}$ in the lowest order in α,

$$(x\partial_x - 1)\mathcal{T}(t,x) = -\frac{\alpha(t)}{8\pi^2 x^2}\left\{4 + \frac{2x^3}{3} - \left(4 + 2x - x^2\right)\exp(-x/2)\right\},$$
$$(x\partial_x - 1)\mathcal{L}(t,x) = \frac{\alpha(t)}{8\pi^2 x^2}\left\{12 - 8x - \left(12 - 2x - x^2\right)\exp(-x/2)\right\},$$

where $\alpha \equiv e^2$.

These differential equations can be solved analytically to give

$$\mathcal{T}(t,x) = \frac{\alpha(t)}{6\pi^2 x^2}\left[1 - \left(1 + \frac{x}{2} - x^2\right)\exp(-x/2) + \frac{x^3}{2}\int_0^x \frac{e^{-u/2}-1}{u}du\right] ,$$

$$\mathcal{L}(t,x) = -\frac{\alpha(t)}{2\pi^2 x^2}\left[1 - x - \left(1 - \frac{x}{2}\right)\exp(-x/2)\right] .$$

Two constants of integrations: one is related to the finite amount of wave function renormalisation for the photon field and the other is fixed by comparing the functional forms of $\mathcal{L}(t,x)$ here and that obtained from the WT identity.

The gauge mass term

Both $\mathcal{T}$ and $\mathcal{L}$ produce a constant term for small $x = p^2$, $3\alpha/16\pi^2$,
that will be a gauge mass $(3\alpha/16\pi^2)\Lambda^2$ once the dimensionality is recovered.
The gauge mass term vanishes as the IR cutoff k goes to zero.

Flow equations for $\alpha = e^2,\ G_S,\ G_V\ :\ t \equiv \ln\Lambda/\mu$

$$\frac{\partial \alpha}{\partial t} = (\eta_A + 2\eta_\psi)\alpha - \frac{6\alpha}{(4\pi)^2}(1 - \frac{2}{9}\eta_\psi)(G_S - 4G_V) + 2\alpha^2 \xi I^{(4)}$$

$$\frac{\partial G_S}{\partial t} = 2(1+\eta_\psi)G_S - \frac{3}{(4\pi)^2}\left(1 - \frac{2}{9}\eta_\psi\right)(3G_S - 8G_V)G_S$$
$$+\alpha G_S\left(\eta_A s^{(1)} + \eta_\psi s^{(2)} + s^{(3)}\right) + \alpha^2\left(\eta_A s^{(4)} + \eta_\psi s^{(5)} + s^{(6)}\right)$$

$$\frac{\partial G_V}{\partial t} = 2(1+\eta_\psi)G_V + \frac{3}{2(4\pi)^2}\left(1 - \frac{2}{9}\eta_\psi\right)G_S^2$$
$$+\alpha G_V\left(\eta_A v^{(1)} + \eta_\psi v^{(2)} + v^{(3)}\right) + \alpha^2\left(\eta_A v^{(4)} + \eta_\psi v^{(5)} + v^{(6)}\right)$$

- $I^{(4)},\ s^{(i)},\ v^{(i)}$ are coefficients functions of α and ξ that depend on the regulator function K.

- The anomalous dimensions, $\eta_i \equiv -\partial_t \ln Z_i$ are also functions of α and ξ.

Hierarchical Phase Structure: a funnel halved by the $\alpha = 0$ plane.

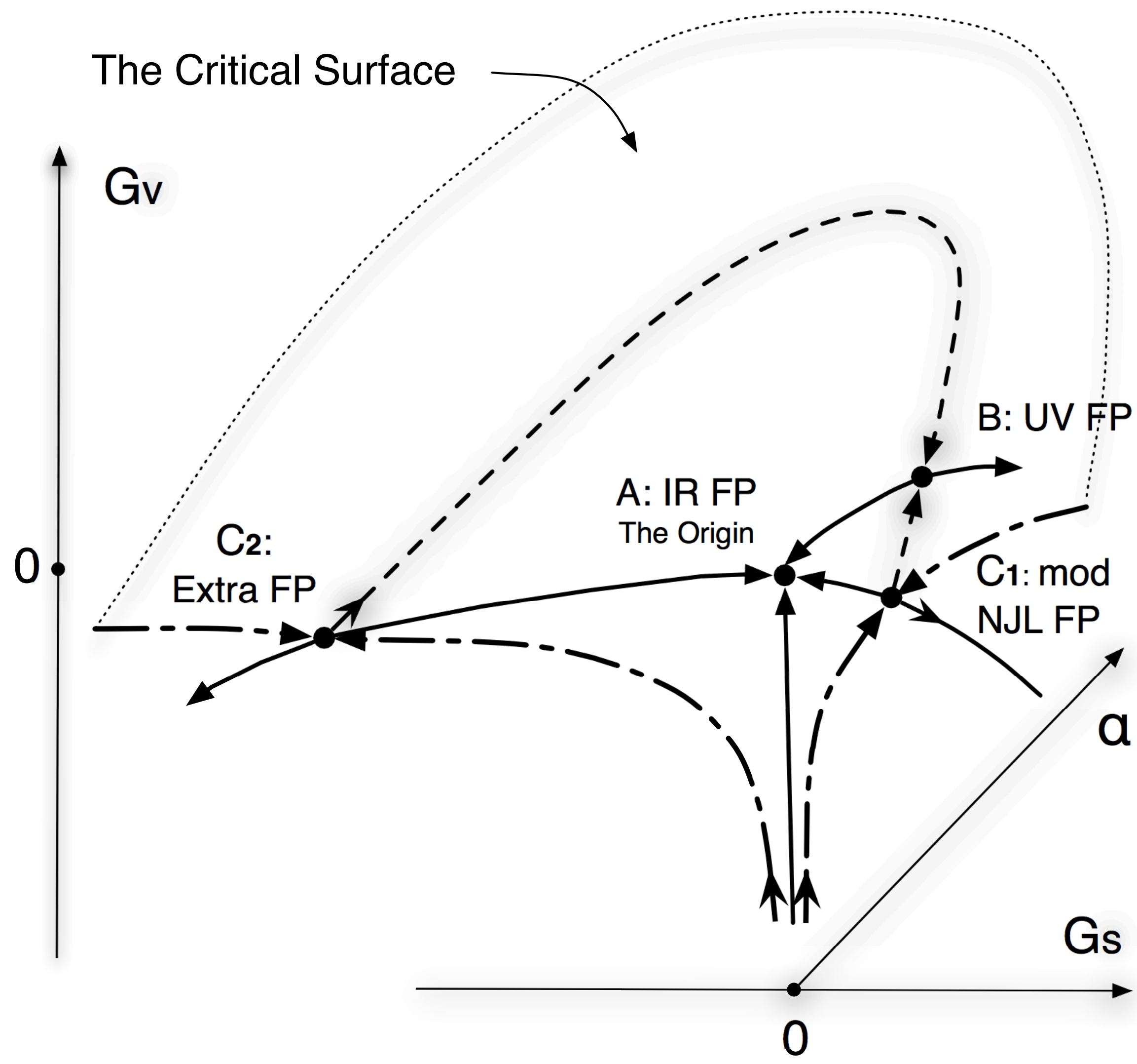
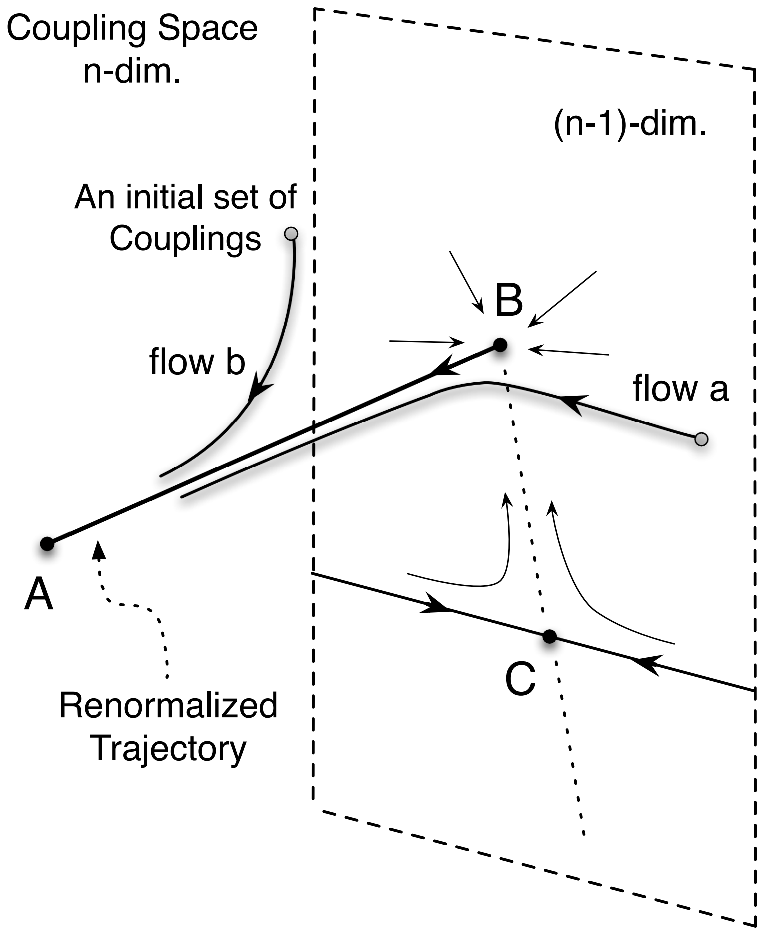
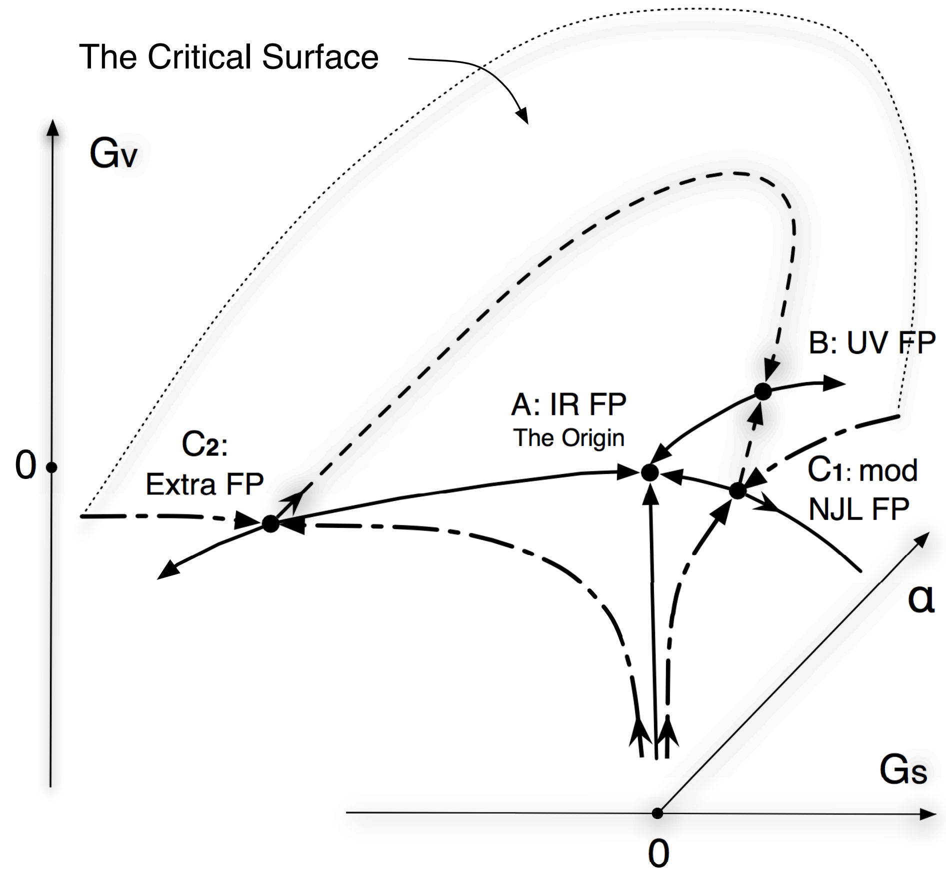


Figure 6: Four fixed points and critical surface and lines



Coupling Space
n-dim.
(n-1)-dim.
An initial set of
Couplings
B
flow b
flow a
A
C
Renormalized
Trajectory



The Critical Surface
Gv
0
B: UV FP
A: IR FP
The Origin
C2:
Extra FP
C1: mod
NJL FP
α
Gs
0

Four fixed points

Fixed Point	# of Rel. ops.	(α, G_S, G_V)
A: I.R.	0	The origin
B: U.V.	1	$(13.5,\ 6.99,\ 0.573)$
C_1: mod. NJL	2	$(0,\ 26.3,\ -3.29)$
C_2: extra	2	$(0, -105,\ -52.6)$

- The numerical calculations are done in Landau gauge.
- Flows on the $\alpha = 0$ plane stay on the plane. That is the modified NJL model.
- Aoki et. al.[2] studied the same system with different regularisation functions and identified the critical coupling without flowing the gauge coupling.
  - Overall phase structure is similar.
  - Some higher order terms are also included here.

[2] PTP **97** (1997) 479

Flows in general gauge

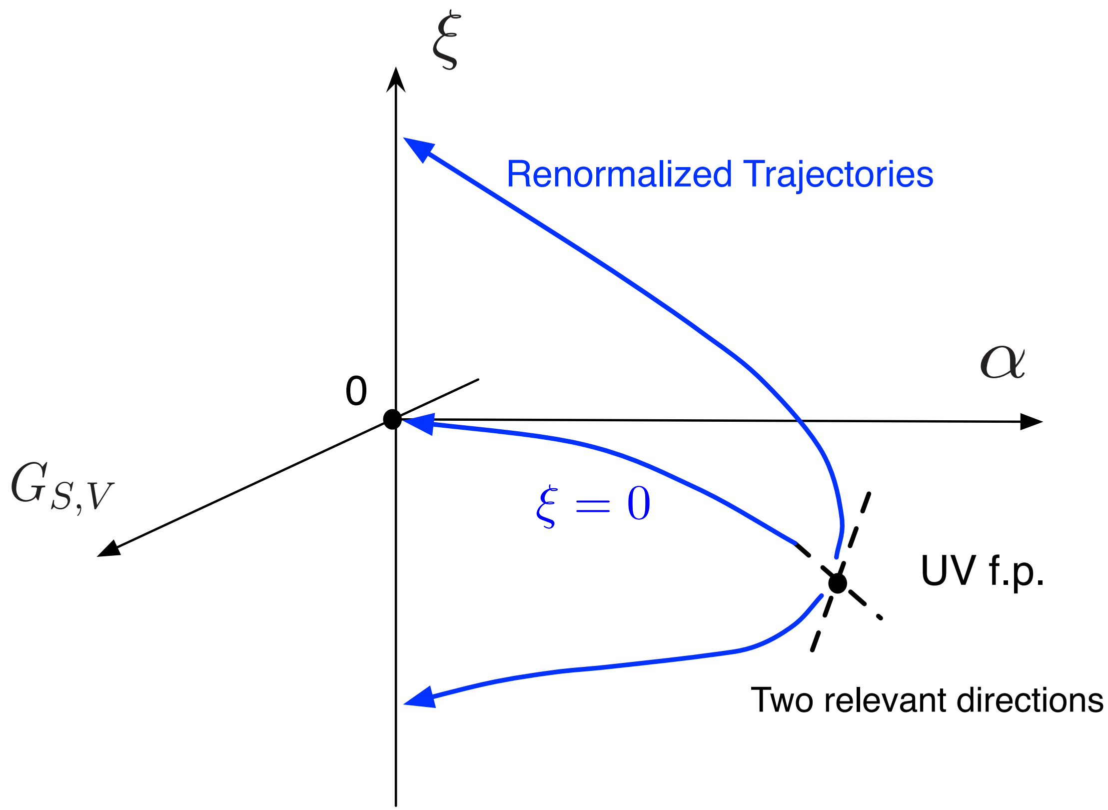


Figure 7: Flows with ξ

- Unique UV fixed point with $\xi = 0$.
- At the UV fixed point,
  there are two relevant directions:
  one stays on $\xi = 0$ plane;
  another has non-zero ξ component.
- The ξ-axis is the IR fixed line.

Anomalous mass dimension γ_m at the UV fixed point

Add $m_f\bar{\psi}\psi$ to $\Gamma_{I,k}$ and find γ_m defined by the flow, $\partial_t m_f = -(1+\gamma_m)m_f + O(m_f^3)$.

$$\gamma_m = -\eta_\psi + \frac{3G_S}{4\pi^2}(1-\frac{2}{9}\eta_\psi)$$

$$-\frac{\alpha}{(4\pi)^2}\int_0^\infty dx K(1-K)^2\Big[x\mathcal{K}_A(3T^2+\xi L^2) + 2\mathcal{K}_\psi(3T+\xi L)\Big]$$

$$\text{where } \mathcal{K}_{A(\psi)} \equiv -2x + \eta_{A(\psi)}(1-K),$$

$$T(x) \equiv \Big[x + (1-K(x))\mathcal{T}(x)\Big], \quad L(x) \equiv \Big[x + \xi(1-K(x))\mathcal{L}(x)\Big]$$

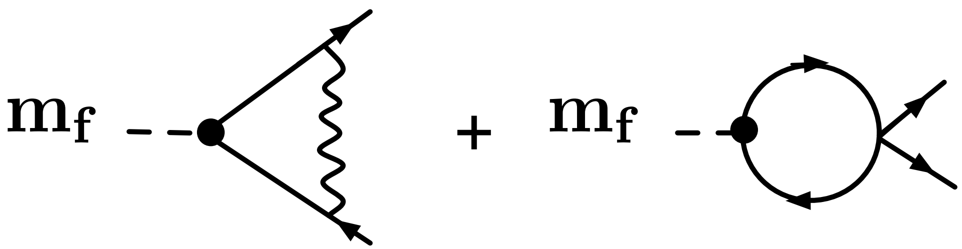


Figure 8: Blobs indicate $m_f\bar{\psi}\psi$ operators. To obtain γ_m, the regulator R_k should be inserted to internal propagators.

Our value, $\gamma_m = 1.078$, is to be compared to 1.177 (Aoki et. al.). Differences are : 1) correct identi--fication of the UV fixed point; 2) choice of regulator function; 3) higher order corrections are included.

Conditions from the WT identity in Landau gauge

1st WT relation: $A_\mu C$ terms in the WT id.

For the ansatz $\Gamma_{I,k} = h^{(aa)}_{\mu\nu} A_\mu A_\nu/2 + e \cdot \bar{\Psi} A \Psi + G_S$ and $G_V \cdot \bar{\Psi}\Psi\bar{\Psi}\Psi$,
$h^{(aa)}_{\mu\nu}(p) = P^T_{\mu\nu}\mathcal{T}(p) + P^L_{\mu\nu}\mathcal{L}(p)$,
we obtain the condition on $\mathcal{L}$ as 1st WT relation,

$$p_\mu h^{(aa)}_{\mu\nu}(p) = p_\nu \mathcal{L}(t,p) = \frac{e^2(t)}{Z_e Z_3^{1/2}} \int_q \mathrm{Tr}\left[U(p-q,q)\gamma_\nu\right] .$$

This gives rise the same funciton obtained from the RG flow eq. except the proportionality constant.

$$\mathcal{L}(t,x) = \frac{1}{Z_e Z_3^{1/2}} \cdot \frac{\alpha(t)}{2\pi^2 x^2}\left\{1 - x - \left(1 - \frac{x}{2}\right)\exp(-x/2)\right\} .$$

- The same function for $\mathcal{L}(t,x)$ are obtained via flow equation as well as WT identity, except the proportionality constant. We may regard this as an evidence of perturbative consistency.

- Yet, flow equations for $\mathcal{T}$ and $\mathcal{L}$ have higher order contributions in $\alpha = e^2$.

2nd WT relation: $\bar{\Psi}\Psi C$ terms in the WT id.

$$Z_e Z_3^{1/2} = 1 - \frac{3}{32\pi^2}\Big(G_S - 4G_V + \alpha I(\alpha)\Big)$$

$$\text{where} \quad I(\alpha) \equiv \int_0^\infty dx K(x)\big(1 - K(x)\big) x T(x,\alpha)$$

- The 2nd term is the one-loop contribution in the $\bar{\Psi}\Psi C$ terms of the WT id.

$$(1 - Z_e Z_3^{1/2})(\not{p} - \not{q}) - \int_l \Big[\big(2G_S - 8G_V\big) + 2e^2(1 - K(l^2))T(l^2)\Big] U(-q-l, p+l)$$

$$-e^2 \int_l \frac{(1-K(l))}{l^2}\{T(l^2) - \xi L(l^2)\}\not{l}\, U(-q-l, p+l)\not{l} = 0$$

Conditions from the WT identity in Landau gauge

- 1st relation: $Z_e Z_3^{1/2} = 1$.

- 2nd relation: $Z_e Z_3^{1/2} = 1 - \frac{3}{32\pi^2}\Big(G_S - 4G_V + \alpha I(\alpha)\Big)$

  where $I(\alpha) \equiv \int_0^\infty dx K(x)\big(1 - K(x)\big) x T(x,\alpha)$

Remarks on our study of QED

- From 80' to 90's there was a big activity studying strong coupling gauge theories via SD approach as well as functional RG. The present system, QED with massless fermion, was a typical example.

  To my knowledge, however, in earlier studies of the system flow equations were solved only partially and, strictly speaking, the UV fixed point had not been identified.

- As a physical quantity, the mass anomalous dimension γ_m is calculated and it found out to be close to the known results.

- The modified WT identity is also investigated.

  Flow eq. and WT id. produced the same funcion $\mathcal{L}$ in the lowest order in perturbation.

Summary and Discussion: we need to achieve ...

- We have explained

  1. Wilsonian and 1PI actions, their flow equations and the continuum limit
  2. BV formalism is powerful enough to treat the modified gauge symmetry due to the cutoff.
  3. Dimensionless formalism to find a phase structure

- We observed the formal compatibility of flow equations and QME. The compatibility can be confirmed in a perturbative expansion. This will be explained in Tim Morris' talk.

- However, for a practical application beyond a perturbative study, we need to introduce a truncated action as an ansatz. This truncation causes the real problem.

A systematic approach to improve the situation is much needed.

Other RG approaches, consistent with gauge symmetry

- Improvement in numerical approach: at the ERG 2020 conference, Coralie Schneider gave a talk on modified Slavnov-Taylor identity in YM theory and explained a newly developed Mathematica package to study the problem numerically. Their paper is now in the arXiv.

- Gauge invariant regularization is proposed (Morris 2000) and studied perturbatively.

- Proposal of a new exact renormalization group based on the idea of gradient flow (Sonoda and Suzuki 2021)

Thank you for your attention