Homological perturbation theory in Homological mirror symmetry

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Homotopy Algebra of Quantum Field Theory and Its Application

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Plan:

- Mirror symmetry and Homological mirror symmetry
- Kontsevich-Soibelman's proposal based on SYZ torus fibration
- Fukaya category $Fuk(\mathbb{R}^2)$ on $\mathbb{R}^2$

  (with (the standard) symplectic form)
- Applying HPT to obtain $Fuk(\mathbb{R}^2)$
- Other examples and related topics

Mirror symmetry

$$\{\text{symplectic mfds. } M\} \overset{\text{Mirror Symmetry}}{\Longleftrightarrow} \{\text{complex mfds. } \check{M}\}$$

• Mirror symmetry (duality) is formulated as equivalences of various structures on M and $\check{M}$.

• Physically, mirror symmetry is a duality between

A-twisted topological string (A-model) on M

and B-twisted topological string (B-model) on $\check{M}$.

ex. (tree) closed string: Gromov-Witten invariant from A-model, the space of deformation of complex structure from B-model

$\star$ **Homological mirror symmetry (HMS)** corresponds to the case of tree open strings.

So, naively, we may expect it to be formulated as an equivalence of the A_∞-structures

What is nontrivial is that we have various boundary conditions for topological open strings $\Rightarrow$ A-branes and B-branes

Thus, we should consider the correspondence

$$A_\infty\text{-category of A-branes on } M \overset{\text{HMS}}{\Longleftrightarrow} A_\infty\text{-category of B-branes on } \check{M}$$

Mathematical formulation of HMS (Kontsevich'94) is an equivalence

$$Tr(Fuk(M)) \simeq D^b(coh(\check{M}))$$

of triangulated categories, where

- $Fuk(M)$ is the Fukaya A_∞ category of Lagrangians (A-branes) in M,
- $D^b(coh(\check{M}))$ is the derived category of coherent sheaves (B-branes) on $\check{M}$,
- a way of constructing a triangulated category $Tr(\mathcal{C})$ from an A_∞-category $\mathcal{C}$ is also given there.

Usually, we can replace $D^b(coh(\check{M}))$ by

(a full subcategory of) the DG category $DG(hol(\check{M}))$
of holomorphic vector bundles

in the sense that it usually generates $D^b(coh(\check{M}))$ by Tr.

This $DG(hol(\check{M}))$ is thought of as a generalization of holomorphic Chern-Simons theory.

$=$ topological open SFT of the B-model

(cf. Lazaroiu'01: SFT and brane superpotentials.)

Claim: Fukaya category should be obtained

as a minimal model of $DG(hol(\check{M}))$!!

A way of understanding HMS in this direction is proposed by

Kontsevich-Soibelman'01: HMS and torus fibration.

More explicit formulation is in

H.K'14: On some deformations of Fukaya categories.

$$Fuk(M) \supset Fuk'(M) \xrightarrow{HPT} DG(hol(\check{M}))$$

Some of relevant works:

Stasheff'63; Homotopy associativity of H-spaces I, II.

Fukaya'93; Morse homotopy, A^{∞}-category, and Floer homologies.

Kontsevich'94; Homological algebra of mirror symmetry.

Fukaya'96; Morse homotopy and Chern-Simons perturbation theory.

Fukaya-Oh'97; Zero-loop open strings in the cotangent bundle and Morse homotopy.

Kontsevich-Soibelman'01: HMS and torus fibration.

Fukaya'05; Multivalued Morse theory, asymptopic analysis and MS.

A toy example: Fukaya category $Fuk(\mathbb{R}^2)$

For an **object** $a \in \mathrm{O}b(Fuk(\mathbb{R}^2))$,

we consider a line in $\mathbb{R}^2$ expressed as

$$L_a : y = t_a x + s_a \ , \qquad t_a, s_a \in \mathbb{R}.$$

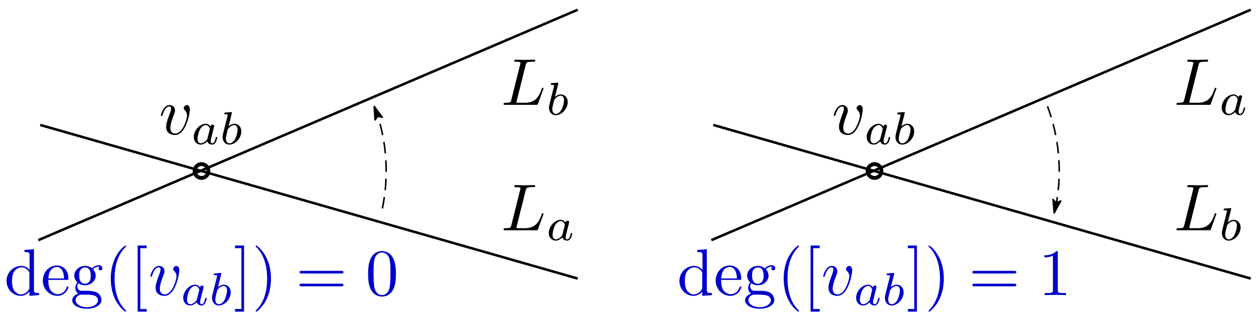
(called an affine Lagrangian section)

For $a, b \in \mathrm{O}b(Fuk(\mathbb{R}))$ s.t. $t_a \neq t_b$, the space V_{ab} of **morphisms** is a $\mathbb{Z}$-graded vector space generated by the intersection point $v_{ab} := L_a \cap L_b$.

The $\mathbb{Z}$-grading is attached as follows. $\forall a \neq b \in \mathfrak{F}_N$,

- $V_{ab}^0 = \mathbb{R} \cdot [v_{ab}]$, $\quad V_{ab}^1 = 0$, $\quad (t_a < t_b)$,

- $V_{ab}^0 = 0$, $\quad V_{ab}^1 = \mathbb{R} \cdot [v_{ab}]$, $\quad (t_a > t_b)$.



The A_∞-**structure** $\{m_n\}_{n\geq 1}$ is defined as follows.

For a fixed $n \geq 2$ and $a_1, ..., a_{n+1} \in \mathrm{O}b(Fuk(\mathbb{R}^2))$ s.t.

$$a_j \neq a_k \text{ for } j \neq k = 1, ..., n+1,$$

$m_n : V_{a_1a_2} \otimes \cdots \otimes V_{a_na_{n+1}} \to V_{a_1a_{n+1}}$ is set to be

$$m_n([v_{a_1a_2}], ..., [v_{a_na_{n+1}}]) = c_{a_1\cdots a_{n+1}}[v_{a_1a_{n+1}}],$$
$$c_{a_1\cdots a_k} = \pm e^{-Area(v)}$$

where, if $\vec{v} := (v_{a_1a_2}, ..., v_{a_na_{n+1}}, v_{a_{n+1}a_1})$ forms a clockwise convex $(n+1)$-gon,

($Area(\vec{v})$ is the area of the $(n+1)$-gon) and $c_{a_1 \cdots a_{n+1}} = 0$ otherwise.

$m_1 : V_{ab} \to V_{ab}$ is set to be $m_1 = 0\ \forall a \neq b$.

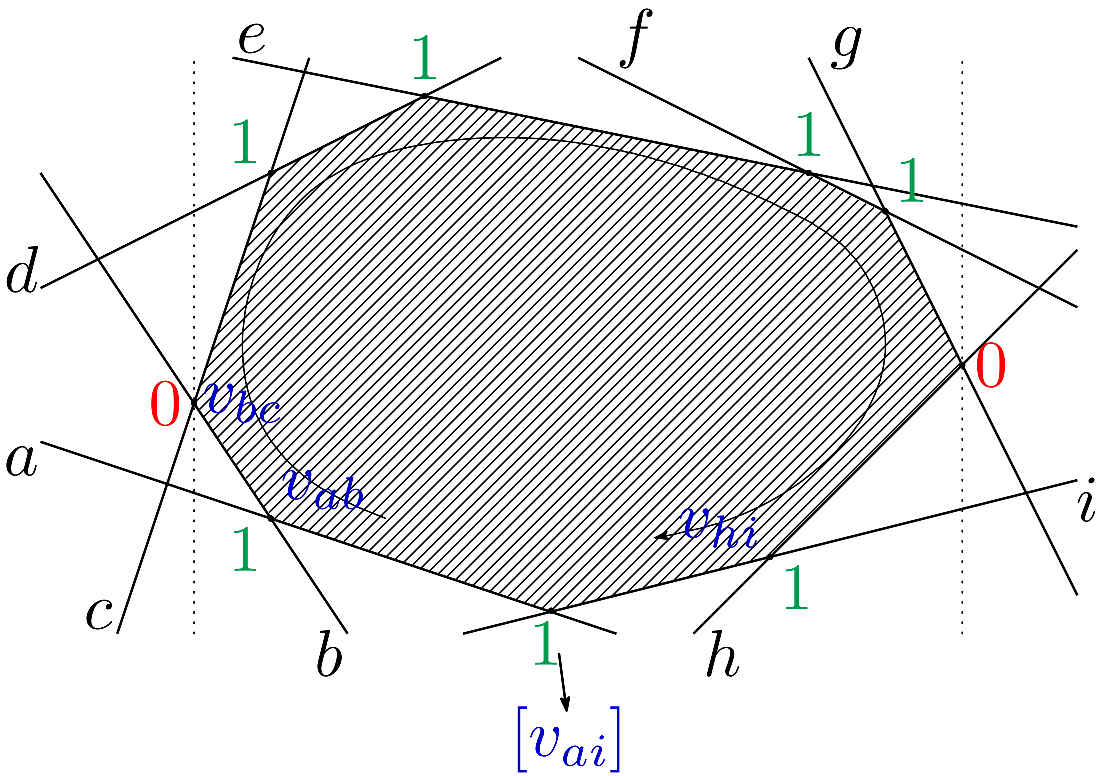
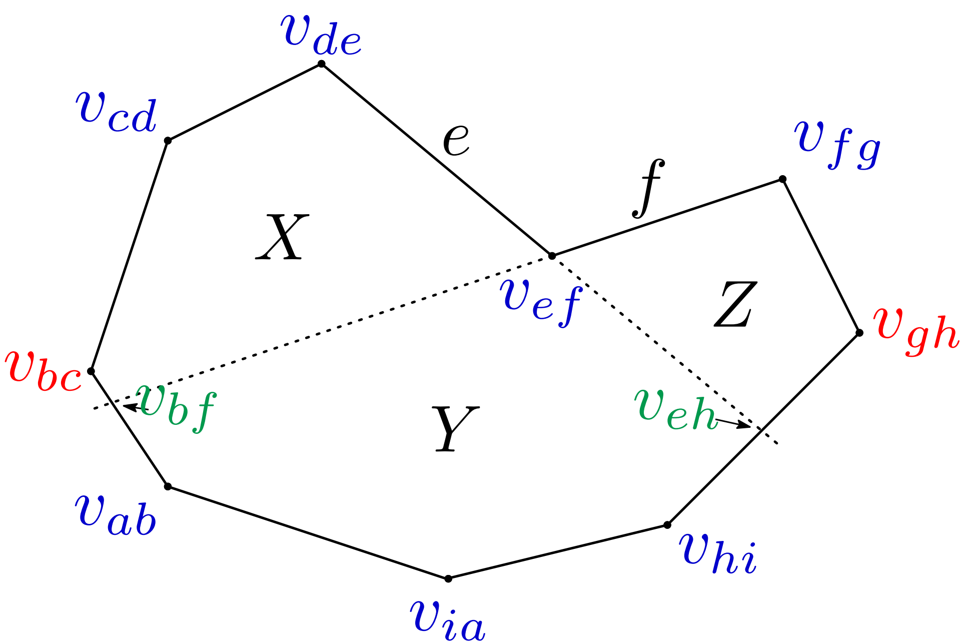


Figure of a cloxkwise convex polygon

The A_∞-**relation** follows from a polygon having one nonconvex vertex.

There exist two ways to divide it into two convex polygons.



In this figure, the area $X + Y + Z$ is divided into

(i) $X + (Y + Z)$ or (ii) $(X + Y) + Z$.

Corresponding to (i) and (ii) one has

$$(i):\ +m_5(v_{ab},m_4(v_{bc},v_{cd},v_{de},v_{ef}),v_{fg},v_{gh},v_{hi})$$
$$= e^{-X-(Y+Z)}v_{ai}\ ,$$

$$(ii):\ -m_6(v_{ab},v_{bc},v_{cd},v_{de},m_3(v_{ef},v_{fg},v_{gh}),v_{hi})$$
$$= -e^{-(X+Y)-Z}v_{ai}\ .$$

Thus, we obtain

$$0 = +m_5(v_{ab},m_4(v_{bc},v_{cd},v_{de},v_{ef}),v_{fg},v_{gh},v_{hi})$$
$$-m_6(v_{ab},v_{bc},v_{cd},v_{de},m_3(v_{ef},v_{fg},v_{gh}),v_{hi})\ ,$$

which is just one of the A_∞-relations.

The DG-category model $\mathcal{C}_{DR}(\mathbb{R})$ of $Fuk(\mathbb{R}^2)$

- The **objects** are the same: $\mathrm{O}b(\mathcal{C}_{DR}(\mathbb{R})) = \{a, b, \dots\}$

- $\forall a, b \in \mathrm{O}b(\mathcal{C}_{DR}(\mathbb{R}))$, the space of **morphsims** is set to be $\mathcal{C}_{DR}(\mathbb{R})(a,b) = \oplus_{r=0,1}\Omega^r_{ab}(\mathbb{R})$, $\Omega^0_{ab} := \mathcal{S}(\mathbb{R})$, $\Omega^1_{ab} := \mathcal{S}(\mathbb{R}) \cdot dx$, where $\mathcal{S}(\mathbb{R})$ is the space of rapidly decreasing smooth functions ;

- a differential $d_{ab} : \Omega^0_{ab} \to \Omega^1_{ab}$ is given by

$$d_{ab} := d - df_{ab}\wedge = e^{f_{ab}} d e^{-f_{ab}},$$

  where $f_{ab} := f_a - f_b$, $f_a := (1/2)(t_a x + s_a)^2$;

- a product $\Omega^{r_{ab}}_{ab} \otimes \Omega^{r_{bc}}_{bc} \to \Omega^{r_{ab}+r_{bc}}_{ac}$ by the usual wedge product $\wedge$.

This $\mathcal{C}_{DR}(\mathbb{R})$ is thought of as a topological open SFT model

(though we do not have cyclicity in this case).

Just by rewriting d as $\bar{\partial}$,

$\mathcal{C}_{DR}(\mathbb{R})$ turns out to be a subcategory of $DG(hol(\check{M} = \mathbb{C}))$ consisting of line bundles.

This is thought of as the topological open SFT of B-model

(though $\mathbb{R}^2 \leftrightarrow \mathbb{C}$ is not a mirror pair in the usual sense).

Homological perturbation theory (HPT) (1986$\sim$)

(Kadeishvili, Gugenheim, Lambe, Stasheff, Huebschmann,...)

For an A_∞-algebra $(A, \mathfrak{m})$,

strongly deformation retract (SDR) data is

$$(V, d) \underset{\pi}{\overset{\iota}{\rightleftarrows}} (A, m_1) \ , \qquad h : A^r \to A^{r-1}$$

s.t. $m_1 h + h m_1 = id_A - \iota \circ \pi, \qquad \pi \circ \iota = id_V$.

Given SDR, there exists an A_∞-algebra $(V, \mathfrak{m}')$ with $m'_1 = d (= \iota \circ m_1 \circ \pi)$ and ι, π lift to A_∞-quasi-isomorphisms.

Note that:

- There exists an explicit construction of $(V, \mathfrak{m}')$ and $\mathfrak{f} : (V, \mathfrak{m}') \to (A, \mathfrak{m})$ (the lift of ι) using "Feynman graphs",

  where the homotopy operator h play the role of the propagator.

- If $d = 0$, then $(V, \mathfrak{m}')$ is a minimal A_∞-algebra, i.e.,

  HPT reduces to Kadeishvili's minimal model theorem.

- HPT holds true just in a similar way for an A_∞-category.

$\star$ Let us apply HPT to $(A, \mathfrak{m}) = \mathcal{C}_{DR}(\mathbb{R})$.

Let us construct homotopy operators h_{ab} on $\mathcal{C}_{DR}(\mathbb{R})(a, b) = \Omega_{ab}$.

$\circ$ For $a \neq b \in \mathrm{O}b(\mathcal{C}_{DR}(\mathbb{R}))$, fix $\epsilon \in (0, 1]$ and define $d^{\dagger}_{\epsilon;ab} : \Omega^{r}_{ab} \to \Omega^{r-1}_{ab}$ by

$$d^{\dagger}_{\epsilon;ab} = \epsilon\, d^{\dagger} - \iota_{\mathrm{grad}(f_{ab})}.$$

We see that $H_{\epsilon} := d_{ab} d^{\dagger}_{\epsilon;ab} + d^{\dagger}_{\epsilon;ab} d_{ab}$ has only non-negative real eigenvalues.

In particular,

[for $\epsilon = 1$], H_1 is the Hamiltonian of a harmonic oscillator,

(cf. Witten's Morse theory)

[for $\epsilon =$ ‘0‘], $H_0 = e^{f_{ab}} \mathcal{L}_{\mathrm{grad}(f_{ab})} e^{-f_{ab}}$.

(**cf.** $d_{ab} := d - df_{ab} \wedge \ = e^{f_{ab}} \cdot d \cdot e^{-f_{ab}}$.)

Let $\psi_t : \Omega^r_{ab} \to \Omega^r_{ab}$, $t \in [0, \infty)$, be a linear map satisfying $\psi_0 = Id$ and

$$\frac{d\psi_t}{dt} = -H_\epsilon \psi_t.$$

Integrating the above equation over $[0, \infty)$, we obtain

$$d_{ab} h_{\epsilon;ab} + h_{\epsilon;ab} d_{ab} = Id_{\Omega_{ab}} - P_{\epsilon;ab},$$

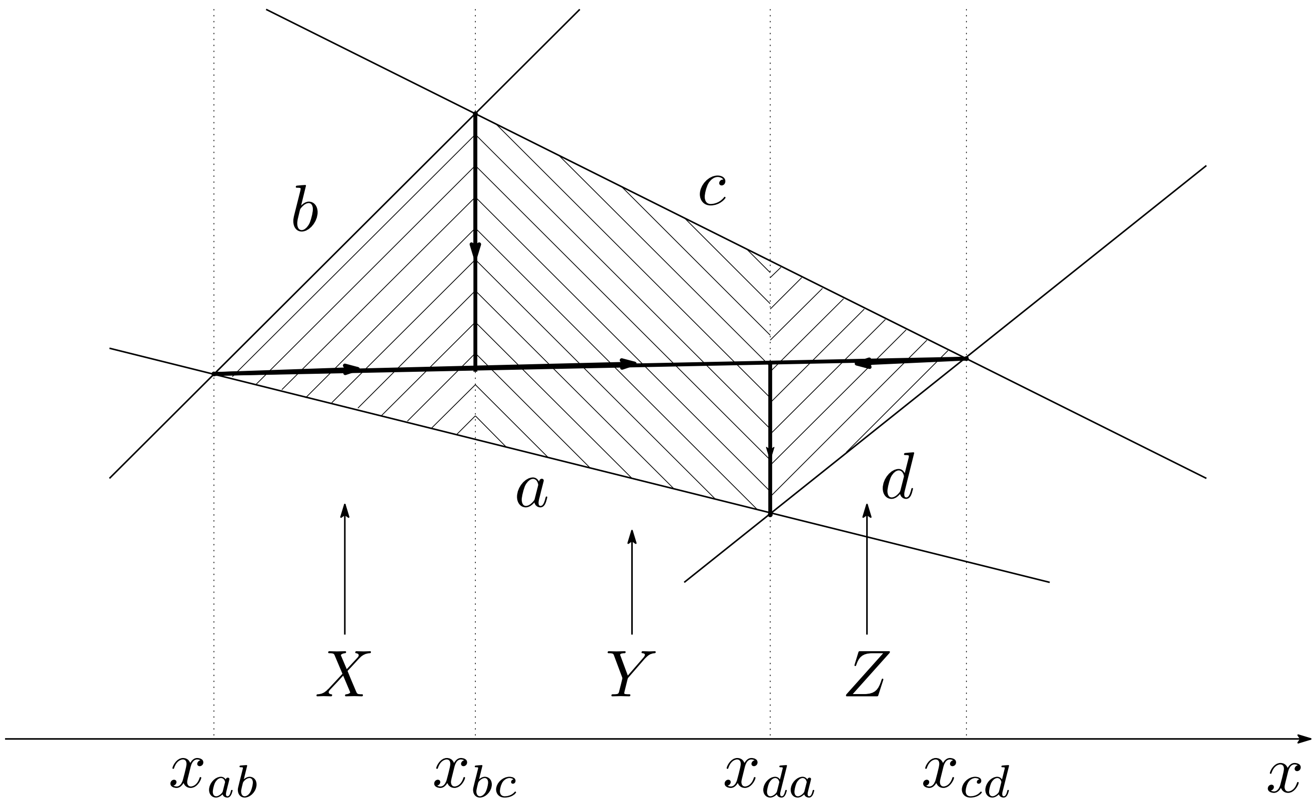
$$h_{\epsilon;ab} := \int_0^\infty dt \; d^\dagger_{\epsilon;ab} \psi_t, \quad P_{\epsilon;ab} := \lim_{t\to\infty} \psi_t.$$

We thus obtain a family of SDRs for $\mathcal{C}_{DR}(\mathbb{R})$ (parametrized by ϵ).

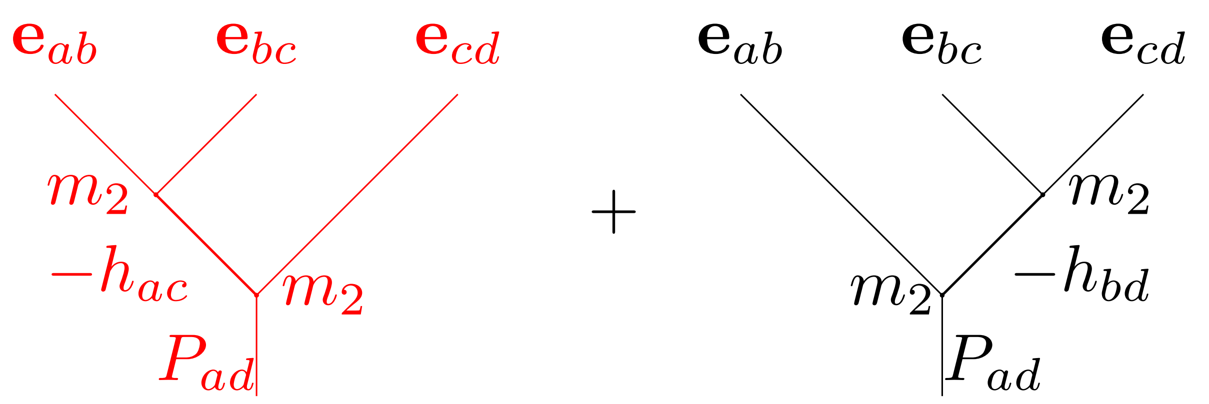
In the limit $\epsilon \to 0$, HPT derives the A_∞-products of $Fuk(\mathbb{R}^2)$.

- For example, for $m'_3(\mathbf{e}_{ab}, \mathbf{e}_{bc}, \mathbf{e}_{cd})$,

$$\left(P_{0;ab}\mathcal{C}_{DR}(\mathbb{R})(a,b) \ni \mathbf{e}_{ab} \quad \longleftrightarrow \quad [v_{ab}]\right)$$



HPT implies $m'_3(\mathbf{e}_{ab}, \mathbf{e}_{bc}, \mathbf{e}_{cd}) =$



$$= \quad -e^{-(X+Y+Z)} \cdot \mathbf{e}_{ad} \quad + \quad 0\ .$$

The concrete construction is

in H.K'09; An A_∞-structure for lines in $\mathbb{R}^2$.

The explanation is in H.K'11; HPT and HMS.

To summarize,

Fukaya category is obtained

as **a particular limit** $\leftarrow$ "singular !!"

of a family of minimal models

of the topological open SFT of B-model

via HPT.

This strategy actually works well for T^2, i.e.,

M is a symplectic two-torus and $\check{M}$ is the mirror dual elliptic curve (T^2 with a complex structure)

since $\mathbb{R}^2$ is the covering space of T^2.

H.K'21; Fukaya categories of two-tori revisited.

Remark: For this T^2 case, a relation of an A_∞-product and an exact triangle is explained in the above paper and

Kobayashi'17; On exact triangles consisting of stable vector bundles on tori.

(Kobayashi also discusses its generalization to higher dim. tori.)

General setting of Kontsevich-Soibeolman's proposal

is based on the Strominger-Yau-Zaslow (SYZ) torus fibration:

Construct M and $\check{M}$ as T^n-fibration of the same base space B

so that M and $\check{M}$ are related by the T-duality of the fiber T^n.

By this T-duality, a Lagrangian (multi-)section in M is transformed to a holomorphic vector bundle on $\check{M}$.

In Leung-Yau-Zaslow'00; From special Lagrangians to Hermitian-Yang-Mills via Fourier-Mukai trnasform.

and Leung'05; Mirror symmetry without corrections.

- In general, M and $\check{M}$ have singular fibers.

A way of treating these singular fibers is discussed in

Fukaya'05; Multivalued Morse theory, asymptopic analysis and mirror symmetry

but it seems hard to construct a HMS functor explicitly in general.

- For a compact toric manifold $\check{M}$ ($\longleftarrow$ non-Calabi-Yau),

we can still discuss mirror symmetry of SYZ type.

In this case, we need to modify the notion of Fukaya category $Fuk(M)$ since $\partial(B) \neq \emptyset$.

Then, HMS is shown by applying HPT when

∘ $\check{M}$ is $\mathbb{C}P^n$ or their products

K-Futaki'20; HMS of $\mathbb{C}P^n$ and their products via Morse homotopy.

∘ $\check{M}$ is the Hirzebruch surface $\mathbb{F}_1$

K-Futaki'20 preprint; HMS of $\mathbb{F}_1$ via Morse homotopy.

Thank you !!

Appendix

- A_∞-algebras
- A_∞-categories
- On the DG-structure $DG(hol(\check{M}))$
- The set-up for general cases
- The idea is to interpolate Morse homotopy
- Appendix for the SDR data we construct for $\mathcal{C}_{DR}(\mathbb{R})$

Def. [A_∞-algebra (Stasheff'63)]

$(V, \mathfrak{m} := \{m_n\}_{n\geq 1})$ *is an* A_∞-**algebra** $\Leftrightarrow$

$V = \oplus_{r\in\mathbb{Z}} V^r$ *:* $\mathbb{Z}$*-graded vector space,*

$\mathfrak{m} := \{m_n : V^{\otimes n} \to V\}_{n\geq 1}$ *: a collection of degree* $(2-n)$ *multilinear maps s.t.*

$$0 = \sum_{k+l=n+1} \sum_{j=0}^{k-1} \pm\, m_k(v_1, \cdots, v_j, m_l(v_{j+1}, \cdots, v_{j+l}), v_{j+l+1}, \cdots, v_n)\ ,$$

for $n = 1, 2, \ldots,$

$$0 = \sum_{k+l=n+1} \sum_{j=0}^{k-1} \pm m_k(v_1, \cdots, v_j,$$

$$m_l(v_{j+1}, \cdots, v_{j+l}), v_{j+l+1}, \cdots, v_n) ,$$

for $n = 1, 2, ...,$

where $v_i \in V^{|v_i|}$, $i = 1, ..., n$, *and* $|m_n| = (2 - n)$ *implies*

$$|m_n(v_1, ..., v_n)| = (2 - n) + |v_1| + \cdots + |v_n|.$$

The A_∞-relations for $n = 1, 2, 3$:

for $m_1 = d$, $m_2 = \cdot$, $\quad x, y, z \in V$:

$$
\begin{aligned}
i) &\quad d^2 = 0 \ , \\
ii) &\quad d(x \cdot y) = d(x) \cdot y + (-1)^{|x|} x \cdot d(y) \ , \\
iii) &\quad (x \cdot y) \cdot z - x \cdot (y \cdot z) = d(m_3)(x, y, z).
\end{aligned}
$$

$i) \Leftrightarrow (V, d)$ forms a complex.

$ii) \Leftrightarrow$ Leibniz rule of d w.r.t. to product $\cdot$.

$iii)$ $\cdot$ is associative **up to homotopy**.

In particular, if $m_3 = 0$, the product $\cdot$ is strictly associative. An A_∞-algebra $(V, \mathfrak{m})$ with $m_3 = m_4 = \cdots = 0$ is called a differential graded (DG) algebra.

Def. [A_∞-morphism]

Given two A_∞-algebras $(V, \mathfrak{m})$ and $(V', \mathfrak{m}')$, an A_∞-morphism $\mathfrak{f} : (V, \mathfrak{m}) \to (V', \mathfrak{m}')$ is a collection of degree $(1-k)$ multilinear maps

$$\mathfrak{f} := \{f_k : V^{\otimes k} \to V'\}_{k \geq 1} \text{ s.t.}$$

$$\sum_{i \geq 1} \sum_{k_1 + \cdots + k_n = n} \pm m'_i(f_{k_1} \otimes \cdots \otimes f_{k_i})(v_1, ..., v_n)$$

$$= \sum_{\substack{i+1+j=k \\ i+l+j=n}} \pm f_k(\mathbf{1}^{\otimes i} \otimes m_l \otimes \mathbf{1}^{\otimes j})(v_1, ..., v_n)$$

for $n = 1, 2, ...$

Note: the above relation for $n = 1$ implies $f_1 : V \to V'$ forms a chain map

$$f_1 : (V, \mathfrak{m}) \to (V', \mathfrak{m}').$$

Def. *An* A_∞*-morphism* $\mathfrak{f} : (V, \mathfrak{m}) \to (V', \mathfrak{m}')$ *is called an* A_∞-**quasi-isomorphism** *iff* $f_1 : (V, m_1) \to (V', m_1')$ *induces an isom. on the cohomologies of the two complexes.*

Remark. *For a given* A_∞*-quasi-isomorphism* $\mathfrak{f} : (V, \mathfrak{m}) \to (V', \mathfrak{m}')$*, there always exists an inverse* A_∞*-quasi-isomorphism*

$$\mathfrak{f}' : (V', \mathfrak{m}') \to (V, \mathfrak{m}).$$

Thus, A_∞*-quasi-isomorphisms define (homotopy) equivalence between* A_∞*-algebras.*

We need a categorical version of these terminologies.

Def. [A_∞-category (Fukaya'93)]

An A_∞-**category** $\mathcal{C}$ $\Leftrightarrow$

$\mathrm{Ob}(\mathcal{C}) = \{a, b, \cdots\}$ *: a set of objects*

$V_{ab} := \mathrm{Hom}_{\mathcal{C}}(a, b)$ *: $\mathbb{Z}$-graded vector space for* $\forall a, b \in \mathrm{Ob}(\mathcal{C})$

a collection of multilinear maps

$$\mathfrak{m} := \{m_n : V_{a_1 a_2} \otimes \cdots \otimes V_{a_n a_{n+1}} \to V_{a_1 a_{n+1}}\}_{n \geq 1}$$

degree $(2-n)$ *defining an* A_∞*-structure.*

In particular, $\mathcal{C}$ *with* $m_3 = m_4 = \cdots = 0$ *is called a* **DG-category**.

Def. *Given two A_∞-categories $\mathcal{C}$ and $\mathcal{C}'$, $\mathfrak{f} := \{f, f_1, f_2, ...\} : \mathcal{C} \to \mathcal{C}'$ is an A_∞-***functor** $\Leftrightarrow$

$f : \mathrm{O}b(\mathcal{C}) \to \mathrm{O}b(\mathcal{C}')$ *a map of objects;*

a collection of multilinear maps

$$f_k : \mathrm{Hom}_{\mathcal{C}}(a_1, a_2) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(a_k, a_{k+1}) \\ \to \mathrm{Hom}_{\mathcal{C}'}(f(a_1), f(a_{k+1})), \quad k = 1, 2, ...$$

degree $(1-k)$ satisfying the defining equation of an A_∞-morphism.

*We call $\mathfrak{f}$ an A_∞-***quasi-isomorphism** *iff $f : \mathrm{O}b(\mathcal{C}) \to \mathrm{O}b(\mathcal{C}')$ is bijection and $f_1 : \mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{C}'}(f(a), f(b))$ induces an isom. on the cohomologies for $\forall a, b \in \mathrm{O}b(\mathcal{C})$.*

The DG structure for $DG(hol(\check{M}))$

is thought of a generalization of

DGA $(\Omega(\check{M}), d, \wedge)$ of differential forms on $\check{M}$

$\vdots$

DGA $(\Omega^{0,*}(\check{M}), \bar{\partial}, \wedge)$ of anti-holomorphic differential forms on $\check{M}$

$\vdots$

DG category $DG(hol(\check{M}))$, where each holomorphic vector bundle has a holomorphic structure $D := \bar{\partial} + A^{0,1}$, $D^2 = 0$. The differential (on the space of morphisms) is defined in a natural way by using the corresponding holomorphic structures.

Hope we can take full subcategories

$$\mathcal{C} \subset Fuk(M), \qquad \mathcal{C}' \subset DG(hol(\check{M}))$$

such that $Tr(\mathcal{C}) \simeq Tr(Fuk(M)),\ Tr(\mathcal{C}') \simeq Tr(DG(hol(\check{M})))$,

and $\mathcal{C} \simeq \mathcal{C}'$ as A_∞-categories. $\Rightarrow$ This implies $Tr(\mathcal{C}) \simeq Tr(\mathcal{C}')$.

In particular,

we hope to obtain the A_∞-quasi-isomorphism $\mathfrak{f} : \mathcal{C} \to \mathcal{C}'$ via HPT.

Outline of the plan to obtain $\mathfrak{f} : \mathcal{C} \to \mathcal{C}'$:

B: n-dim. mfd (equipped with tropical affine, Hessian structures!)

$\cdots \to T^*B$: symplectic manifold

$\cdots \to M := T^*B/\mathbb{Z}^n$: symplectic torus fibration

A_∞**-category** $M(B)$ **of Morse homotopy on** B:

$Ob(M(B)) = C^\infty(B)$,

For $f, g \in Ob(M(B))$, $Hom(f, g)$ is the Morse complex of $f - g$.

• Fukaya, Oh'93,'97: $M(B)$ is equivalent to the full subcategory of $Fuk(T^*B)$ consisting of Lagrangian sections $graph(df)$.

- $M(B)$ is A_∞-quasi-isomorphic to **a DG category** $DG(B)$

  via HPT

  where $Ob(DG(B)) = Ob(M(B))$,

  $$Hom_{DG(B)}(f,g) = \Omega(B), \qquad D = d + df\wedge$$

- Extend these stories to torus fibrations

- The DG structure in $DG(B)$ corresponds to that in $DG(hol(\check{M}))$

  where $\check{M} := TB/\mathbb{Z}^n$ is the dual torus fibration of M.

  (cf. forms on $B \leftrightarrow$ anti-hol. forms on TB)

This strategy works well for $M = \mathbb{R}^2$ and T^2 (H.K)

Appendix for the SDR data

$P_{\epsilon;ab}$ defines a projection;

$$P_{\epsilon;ab}\Omega^0_{ab} = \mathrm{Ker}(d_{ab} : \Omega^0_{ab} \to \Omega^1_{ab}),$$
$$P_{\epsilon;ab}\Omega^1_{ab} = \mathrm{Ker}(d^\dagger_{\epsilon;ab} : \Omega^1_{ab} \to \Omega^0_{ab}).$$

The cohomologies $P_{\epsilon;ab}\Omega_{ab} := \oplus_{r=0,1} P_{\epsilon;ab}\Omega^r_{ab}$ are spanned by the gaussians:

$$P_{\epsilon;ab}\Omega_{ab} = \{\ const \cdot e^{f_{ab}}\ \}, \qquad t_a < t_b$$
$$P_{\epsilon;ab}\Omega_{ab} = \{\ const \cdot e^{-\frac{1}{\epsilon}(f_{ab})} dx\ \}, \quad t_a > t_b.$$

We choose bases $\mathbf{e}_{\epsilon;ab}$ of $P_{\epsilon;ab}\Omega_{ab}$ by normalizing

$$\mathbf{e}_{\epsilon;ab}(x_{ab}) = 1 \ , \qquad t_a < t_b$$
$$\int_{-\infty}^{\infty} \mathbf{e}_{\epsilon;ab} = 1 \ , \qquad t_a > t_b.$$

In the limit $\epsilon \to 0$, the degree one base $\mathbf{e}_{\epsilon;ab}$ $(t_a > t_b)$ becomes the **delta function** localized at the point x_{ab} $(= x(v_{ab}))$.

In the limit $\epsilon \to 0$, $h_{ab} := \lim_{\epsilon \to 0} h_{\epsilon;ab}$ and $P_{ab} := \lim_{\epsilon \to 0} P_{\epsilon;ab}$ turn out to be

$$h_{ab} = \int_0^\infty dt e^{f_{ab}} \varphi_t^* (e^{-f_{ab}} \ \iota_{\mathrm{grad}(f_{ab})}),$$

$$P_{ab} = \lim_{t \to \infty} e^{f_{ab}} \varphi_t^* e^{-f_{ab}} \ ,$$

where $\varphi_t : \mathbb{R} \to \mathbb{R}$ is the flow defined by

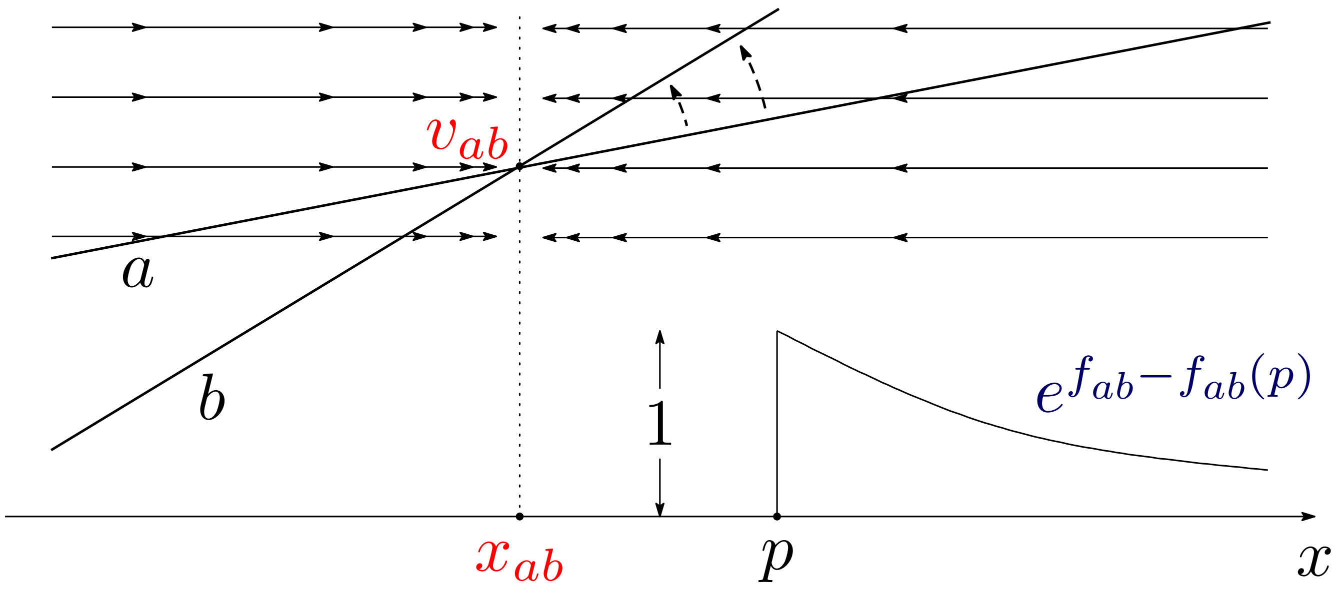
$$\frac{d\varphi_t}{dt} = \mathrm{grad}(f_{ab}), \qquad \varphi_0 = Id.$$

For example, for the following case:

$$
\begin{aligned}
&h_{ab}(\delta(x-p)dx) \\
&= \int_0^\infty dt e^{f_{ab}} \varphi_t^* e^{-f_{ab}} \delta(x-p) \frac{df_{ab}}{dx}(x) \\
&= e^{f_{ab}} (\varphi_t^* e^{-f_{ab}})|_{\varphi_t(x)=p}(x),
\end{aligned}
$$

$h_{ab}(\delta(x-p)dx)$ for $t_a < t_b$ and $x_{ab} < p$ turns out to be

Flow of $\mathrm{grad}(f_{ab})$



(**step function** twisted by $e^{f_{ab}}$).