Colour–Kinematics Duality and Double Copy using Homotopy Algebras
ホモトピー代数による
カラー=キネマティクス・デュアリティ　ダブル・コピー
色と運動因子の双対性と二重複写

Hyungrok Kim
キム・ヒョンロク
金炯録

Homotopy Algebra of QFT and its Application
場の理論のA_∞/L_∞代数とその応用
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Joint work (2007.13803; 2102.11390) with
L Borsten, B Jurčo, H Kim, T Macrelli, C Saemann, M Wolf
L・ボーステン、B・ユルチョ、T・マクレッリ、C・ゼーマン、M・ヴォルフ
との共同研究（2007.13803、2102.11390）

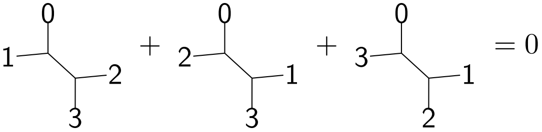
Summary | 概要

- On-shell kinematic Jacobi : Off-shell kinematic Jacobi ::
  O_∞-algebra : O-algebra
  オンシェル運動因子のヤコビ恒等式：オフシェル〃 ::
  O_∞ 代数：O 代数
- On-shell kinematic Jacobi identity for all fields (longitudinal gluons, ghosts, antighosts)
  すべての場（非横波グルーオン・（反）幽霊場）についてのオンシェル運動因子のヤコビ恒等式
- Factorise the theory into colour, kinematics, and a "base theory"
  理論を色代数・運動因子代数・底理論に分解

$$\text{YM ampl.} = \frac{c_i^{a_1\dots a_n} n_i^{\mu_1\dots\mu_n}}{d_i} \qquad \text{GR ampl.} = \frac{n_i^{\mu_1\dots\mu_n} n_i^{\nu_1\dots\nu_n}}{d_i}$$

$$S_{\text{YM}} = c_{ab} C_{ij} A^{ai} \,\square\, A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$
$$S_{\text{GR}} = C_{ij} C_{i'j'} A^{ii'} \,\square\, A^{jj'} + F_{ijk} F_{i'j'k'} A^{ii'} A^{jj'} A^{kk'}$$

Antelapsarian world | 好天☀の世界

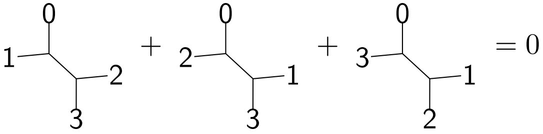


$$\nu_2(\nu_2(x_1, x_2), x_3) + \nu_2(\nu_2(x_2, x_3), x_1) + \nu_2(\nu_2(x_3, x_1), x_2) = 0$$

4-pt off-shell tree ⟹ n-pt off-shell tree/loop

4 点オフシェルトリー ⟹ n 点オフシェルトリーとループ

Antelapsarian world | 好天☀の世界



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4-pt off-shell tree
4 点オフシェルトリー
$\Longrightarrow$
n-pt off-shell tree/loop
n 点オフシェルトリーとループ

Antelapsarian world | 好天☀の世界

$$0 \to \underset{\deg=1}{\text{field}} \xrightarrow{\mu_1} \underset{\deg=2}{\text{antifield}} \to 0$$

The BCJ vertex ν_2 has degree -1:

$$\nu_2 \colon \underset{\deg=1}{\text{field}} \otimes \underset{\deg=1}{\text{field}} \to \underset{\deg=1}{\text{field}}$$

But the μ_2 encoding the cubic action term has degree 0:

$$\mu_2 \colon \underset{\deg=1}{\text{field}} \otimes \underset{\deg=1}{\text{field}} \to \underset{\deg=2}{\text{antifield}}$$

$$S = \langle \mathsf{field}_a, \mu_1(\mathsf{field}^a) \rangle + f_{abc} \langle \mathsf{field}^a, \mu_2(\mathsf{field}^b, \mathsf{field}^c) \rangle$$

Need an operator h of degree -1 that relates the action with BCJ vertex

$$\nu_2(x, y) = h(\mu_2(x, y)) - \underbrace{(\mu_2(hx, y) + \mu_2(x, hy))}_{=0 \text{ due to degree}}$$

Antelapsarian world | 好天☀の世界

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Antelapsarian world | 好天☀の世界

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Antelapsarian world | 好天☀の世界

$$\begin{aligned}\nu_2(x,y) &= h(\mu_2(x,y)) - (\mu_2(hx,y)+\mu_2(x,hy))\\ &= \text{failure of } h \text{ to be a derivation for } \mu_2\end{aligned}$$

h が μ_2 の微分となることの失敗量

If h is a derivation for $\nu_2 \to \nu_2$ satisfies Jacobi identity
h が ν_2 の微分 $\to$ ν_2 のヤコビ恒等式
☞ *Batalin–Vilkovisky (BV) algebra* | バタリン=ヴィルコヴィスキ代数

Unrelated to quantisation! | 量子化とは無関係!!

[Reiterer ’19]

Antelapsarian example | 好天☀の双随伴表現スカラー場

$$S = \int \phi^{aA} \Box \phi_{aA} + f_{abc} F_{ABC} \phi^{aA} \phi^{bB} \phi^{cC}$$

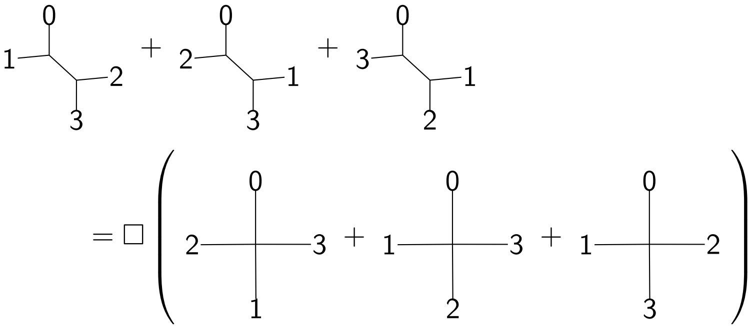
$$0 \to \Omega^0(\mathbb{R}^d) \otimes \mathfrak{g}_{\mathsf{L}} \otimes \mathfrak{g}_{\mathsf{R}} \xrightarrow{\Box} \Omega^0(\mathbb{R}^d) \otimes \mathfrak{g}_{\mathsf{L}} \otimes \mathfrak{g}_{\mathsf{R}} \to 0 \quad \text{(str. L}_\infty\text{=dgLa)}$$

Strip left flavour = “colour”; remains right flavour = “kinematics”

$$0 \to \Omega^0(\mathbb{R}^d) \otimes \mathfrak{g}_{\mathsf{R}} \underset{\mathrm{id}}{\overset{\Box}{\rightleftarrows}} \Omega^0(\mathbb{R}^d) \otimes \mathfrak{g}_{\mathsf{R}} \to 0 \qquad \text{(BV-alg.)}$$

$$\begin{aligned}
\mu_1 &= \Box && \text{(field} \to \text{antifield)} \\
h &= \mathrm{id} && \text{(antifield} \to \text{field)} \\
\mu_2(\phi, \phi')^A &= F^A{}_{BC} \phi^B \phi'^C && \text{(field} \otimes \text{field} \to \text{antifield)} \\
\nu_2(\phi, \phi')^A &= F^A{}_{BC} \phi^B \phi'^C && \text{(field} \otimes \text{field} \to \text{field)}
\end{aligned}$$

Postlapsarian world | 荒天 🌧 の世界



$$\nu_2(\nu_2(x_1, x_2), x_3) + \nu_2(\nu_2(x_2, x_3), x_1) + \nu_2(\nu_2(x_3, x_1), x_2)$$
$$= \Box(\nu_3(x_1, x_2, x_3) + \cdots)$$

Over coeff. ring (Hopf) $\mathbb{R}[\partial_\mu]$ | 係数環（ホップ）$\mathbb{R}[\partial_\mu]$

※ $\mathrm{BV}_\infty^\Box$-algebra [Reiterer '19]

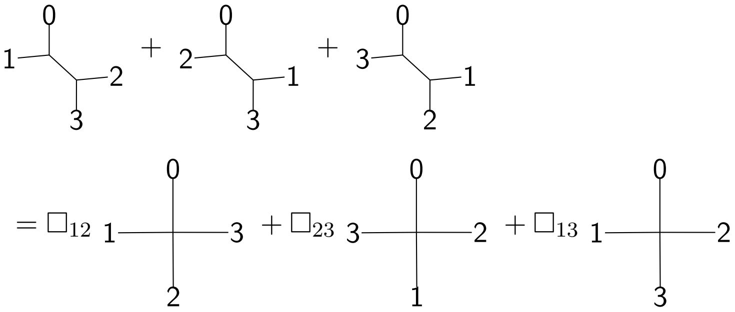
Rectifying the world | 荒天 ≅ 好天 🌦 〜蝦で鯛を釣る〜

If the kinematic Jacobi identity holds

- On-shell
- Fails to be off-shell, but in an algebraic way
- for all fields (longitudinal gluons, ghosts, antighosts)

then abstract nonsense says that a quasi-isomorphism to a strict BV-algebra exists, recovering off-shell kinematic Jacobi identity.
Physically: introduction of an infinite tower of auxiliaries.

Yang–Mills at 4 points | 楊（ヤン）=ミルズ理論の 4 点振幅



$$+ \underbrace{\mathcal{O}(\partial \cdot A)}_{\substack{\text{gauge-dependent} \\ \text{ゲージ依存}}}$$

BRST and all that | ＢＲＳＴ量子化

1. BRST symmetry | ＢＲＳＴ対称性

$$Q: \qquad A \mapsto \mathrm{D}c \qquad \underbrace{c}_{\substack{\text{ghost}\\ \text{幽霊場}}} \mapsto cc \qquad \overbrace{\bar{c}}^{\substack{\text{反幽霊場}\\ \text{antighost}}} \mapsto \underbrace{b}_{\substack{\text{Nakanishi–Lautrup}\\ \text{中西=ラウトルップ場}}} \mapsto 0$$

2. Gauge condition | ゲージ条件

$$G[A] = \partial \cdot A + \cdots$$

3. Gauge-fixed action | ゲージ固定作用

$$\begin{aligned} \mathcal{L} &= F^2 + Q\big((G[A] + \xi b)\bar{c}\big) \\ &= \underbrace{F^2}_{\substack{\text{physical}\\ \text{元の作用}}} + \underbrace{(G[A] + \xi b)b}_{\substack{\text{gauge-fixing}\\ \text{ゲージ固定項}}} + \underbrace{(\partial \cdot \mathrm{D}c)\bar{c} + \cdots}_{\substack{\text{ghost}\\ \text{幽霊項}}} \end{aligned}$$

Yang–Mills at 4 points | 楊（ヤン）=ミルズ理論の 4 点振幅

$$\xi b^2 + b\overbrace{\left(\partial \cdot A - 2\xi\frac{A\cdot(A\cdot\partial)A}{\Box}\right)}^{\substack{\text{ゲージ固定汎函数}\\ \text{gauge-fixing functional}}} \qquad\qquad b$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\xi b^2 + b\partial \cdot A + \underbrace{\frac{(\partial\cdot A)A\cdot(A\cdot\partial)A}{\Box} - \xi\left(\frac{A\cdot(A\cdot\partial)A}{\Box}\right)^2}_{\substack{\text{new terms in action}\\ \text{作用の新しい項}}} \qquad b + \underbrace{\frac{A\cdot(A\cdot\partial)A}{\Box}}_{\substack{\text{shift of NL field}\\ \text{中西=ラウトルップ場の再定義}}}$$

Extended Fock space | 拡張フォック空間

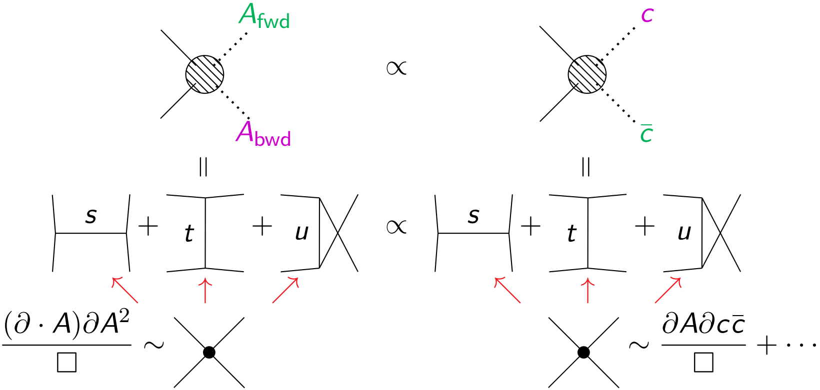
Extend the Fock space with on-mass-shell modes of unphysical gluons $A_{\text{fwd}}, A_{\text{bwd}}$; ghosts c; antighosts $\bar{c}$
フォック空間に非横波グルーオン $A_{\text{fwd}}, A_{\text{bwd}}$・（反）幽霊場 $c, \bar{c}$ のオンシェルモードを追加

$$|\text{state}\rangle = \underbrace{|\text{harmonic}\rangle}_{\substack{\text{physical}\\ \text{実在状態}}} + \underbrace{|\text{exact}\rangle + |\text{coexact}\rangle}_{\substack{\text{unphysical}\\ \text{非実在状態}}}$$

$$\begin{tikzcd}
\mathrm{gh} = 1 & 0 & c \arrow[d, bend right, "Q^\dagger"'] & \\
\mathrm{gh} = 0 & A_{\text{transv}} \arrow[u, "Q"'] \arrow[d, "Q^\dagger"] & A_{\text{fwd}} \arrow[u, bend right, "Q"'] & A_{\text{bwd}} = b \arrow[d, bend right, "Q^\dagger"'] \\
\mathrm{gh} = -1 & 0 & & \bar{c} \arrow[u, bend right, "Q"']
\end{tikzcd}$$

[九後、小嶋（京大）'79]

Ward identities | ウォード=高橋恒等式



YM at ≥ 5 points | 楊=ミルズ理論の 5 点以上の振幅

Gauge-fixing generates unwanted quintic, sextic terms →
higher-order Tolotti–Weinzierl-type terms
ゲージ固定中 5、6 次項が発生 → 高次トロッティ=ヴァインツィアール項

$$S = A\,\Box\,A + g(AA\partial A + c\partial\bar{c}A) + g^2\left(\frac{\partial^2 A^4}{\Box} + \frac{c\bar{c}\partial^2 A^2}{\Box}\right) + g^3\left(\frac{\partial^3 A^5}{\Box^2} + \cdots\right) + \cdots$$

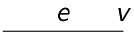
$$(\mu_1, h; \mu_2, \mu_3, \mu_4, \dots; \nu_2, \nu_3, \nu_4, \dots)$$

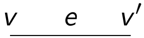
Rectification via auxiliary fields | 補助場による正規化（レクティフィケーション）

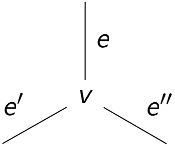
half-edge (e, v)
半辺

edge e
辺

vertex v
頂点







field
場

kinetic term
運動項

interaction term
相互作用項

$\phi_{e,v}$

$\phi_{e,v} \Box \phi_{e,v'}$

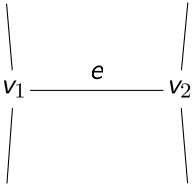
$$f_{abc} \underset{\uparrow}{F_v} \phi^a_{e,v} \phi^b_{e',v} \phi^c_{e'',v}$$

differential operator
微分作用素

Auxiliary fields (quartic) | 4 次項の補助場

$$\frac{\partial^2 A^4}{\Box}$$

$$\Downarrow$$



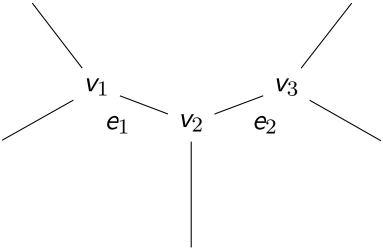
$$\Downarrow$$

$$\phi_{a\mu\nu\rho e,v_1} \Box \phi^{a\mu\nu\rho}_{e,v_2} + gf_{abc}(\partial_\mu \phi^{a\mu\nu\rho}_{e,v_1}) A^b_\nu A^c_\rho + gf_{abc}(\partial_\mu \phi^{a\mu\nu\rho}_{e,v_2}) A^b_\nu A^c_\rho$$

Auxiliary fields (quintic) | 5 次項の補助場

$$\frac{\partial^3 A^5}{\Box^2}$$

$$\Downarrow$$



$$\Downarrow$$

$$\begin{gathered}\phi_{e_1 v_1} \Box \phi_{e_1 v_2} + \phi_{e_2 v_2} \Box \phi_{e_2 v_3} \\ + f_{abc} \left(F_{v_1} \phi^a_{e_1 v_1} A^b A^c + F_{v_2} \phi^a_{e_1 v_2} \phi^b_{e_2 v_2} A^c + F_{v_3} \phi^a_{e_2 v_3} A^b A^c \right)\end{gathered}$$

Cubic action | 3 次作用

$$S = c_{ab} C_{ij} A^{ai} \Box A^{aj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$

where

- a, b, c: adjoint colour index 色の随伴表現の添え字
- i, j, k: DeWitt index over gluon, (anti)ghost, auxiliaries
  すべての場のデウィット添え字
- c_{ab}, f_{abc}: structure constants of colour Lie algebra
  色のリー代数の構造定数
- C_{ij}, F_{ijk}: “structure const. of kinematic algebra” (diff. op.)
  「運動因子のリー代数の構造定数」（微分作用素）

Cf. literature on “kinematic structure constants”: [Monteiro–O’Connell 1105.2565, 1311.1151; Bjerrum-Bohr–Damgaard–M–O’C 1203.0944; Chen–Johansson–Teng–Wang 1906.10683]

Cubic action | 3 次作用

For a given trivalent graph Γ_i, | 三次グラフ Γ_i について、

$$\sum \text{all Feynman diagrams for topology } \Gamma_i$$

$$= \epsilon^{(1)}_{a_1\mu_1} \cdots \epsilon^{(n)}_{a_n\mu_n} \frac{c_i^{a_1 \ldots a_n} n_i^{\mu_1 \ldots \mu_n}}{|\mathrm{Aut}(\Gamma_i)| d_i}$$

$$(\mathrm{mod} \text{ terms} \propto \textstyle\sum_i p_i \text{ or} \propto p_i^2 \text{ for external } p_i)$$

Double copy | 二重複写（ダブル・コピー）

$$\text{YM amplitude} = \epsilon^{(1)}_{a_1\mu_1}\cdots\epsilon^{(n)}_{a_n\mu_n}\int_{\text{loops}}\sum_{i\in\Gamma_{n,l}}\frac{c_i^{a_1\dots a_n}n_i^{\mu_1\dots\mu_n}}{|\mathrm{Aut}(i)|d_i}$$

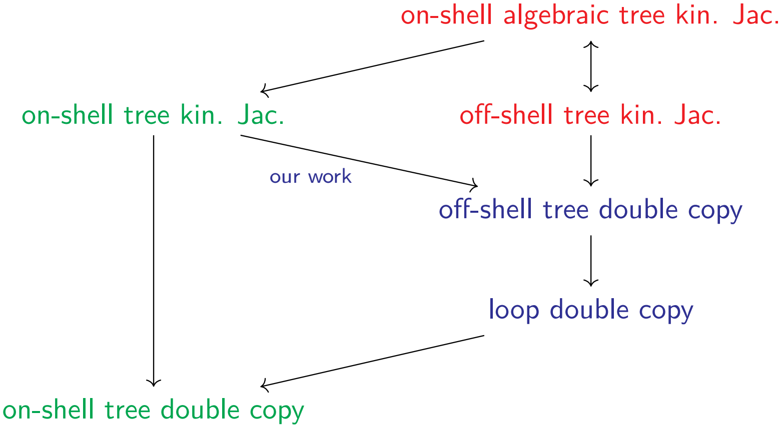
$$\text{Biadj. scal. amplitude} = \epsilon^{(1)}_{a_1b_1}\cdots\epsilon^{(n)}_{a_nb_n}\int_{\text{loops}}\sum_{i\in\Gamma_{n,l}}\frac{c_i^{a_1\dots a_n}c_i^{b_1\dots b_n}}{|\mathrm{Aut}(i)|d_i}$$

$$\text{Gravity amplitude} = \epsilon^{(1)}_{\mu_1\nu_1}\cdots\epsilon^{(n)}_{\mu_n\nu_n}\int_{\text{loops}}\sum_{i\in\Gamma_{n,l}}\frac{n_i^{\mu_1\dots\mu_n}n_i^{\nu_1\dots\nu_n}}{|\mathrm{Aut}(i)|d_i}$$

$$\epsilon_{\mu\nu} = \text{graviton} + \text{Kalb–Ramond 2-form} + \text{dilaton}$$

[Bern–Carrasco–Johansson ’08, ’10; Bern–Dennen–Huang–Kiermaier ’10]

Double copy | 二重複写（ダブル・コピー）



on-shell algebraic tree kin. Jac.

on-shell tree kin. Jac.

off-shell tree kin. Jac.

our work

off-shell tree double copy

loop double copy

on-shell tree double copy

Not Proven	Newly Proven	Proven
未証明	新証明	証明

ダブル・コピー

Double-copied action | 二重複写作用

$$\mathcal{L}(\mathfrak{YM}) = c_{ab} C_{ij} A^{ai} \square A^{bj} + f_{abc} F_{ijk} A^{ai} A^{bj} A^{ck}$$

$$\mathcal{L}(\mathfrak{Biadj}) = c_{ab} c_{a'b'} A^{aa'} \square A^{bb'} + f_{abc} f_{a'b'c'} A^{aa'} A^{bb'} A^{cc'}$$

$$\mathcal{L}(\mathfrak{GR}) = C_{ij} C_{i'j'} A^{ii'} \square A^{jj'} + F_{ijk} F_{i'j'k'} A^{ii'} A^{jj'} A^{kk'}$$

By construction (tree-level double copy), double-copied action gives correct tree-level amplitudes of $\mathcal{N} = 0$ supergravity and biadjoint scalar.

Does it give correct loop amplitudes?

$\Leftrightarrow$ Is it correctly quantised (i.e. has BRST)?

Cf. [Bern–Grant '99, Hohm '11]

Factorisation | 作用の分解

$$\begin{aligned}
\underset{\text{マクスウェル理論}}{\mathfrak{Maxwell}} &= \underset{\text{運動因子代数}}{\mathfrak{kinematic}} \otimes_{\mathbb{R}[\partial_\mu]} \underset{\text{スカラー場の理論}}{\mathfrak{Scalar}} \\
\underset{\text{ヤン=ミルズ理論}}{\mathfrak{YM}} &= \underset{\text{色代数}}{\mathfrak{colour}} \otimes \mathfrak{kinematic} \otimes_{\mathbb{R}[\partial_\mu]} \mathfrak{Scalar} \\
\underset{\mathcal{N}=0\ \text{超重力}}{\mathfrak{GR}} &= \mathfrak{kinematic} \otimes_{\mathbb{R}[\partial_\mu]} \mathfrak{kinematic} \otimes_{\mathbb{R}[\partial_\mu]} \mathfrak{Scalar}
\end{aligned}$$

Thanks for listening!

おおきに。