Introduction to the BV Formalism and Homotopy Algebras

Motivation: Yang-Mills theory

The Yang-Mills action in R_ξ-gauge is

$$S = \int d^4x \left(-\frac{1}{4}(F^a_{\mu\nu})^2 - \bar{c}_a \partial^\mu (\nabla_\mu c)^a + \frac{\xi}{2}(b^a)^2 + b_a \partial^\mu A^a_\mu \right)$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{bc}{}^a A^b_\mu A^c_\nu$$

$$(\nabla_\mu c)^a = \partial_\mu c^a + g f_{bc}{}^a A^b_\mu c^c$$

c^a ghost	1
A^a_μ gauge potential	0
b^a Nakanishi-Lautrup	0
$\bar{c}^a$ anti-host	−1

This action is invariant under a fermionic symmetry:

$$Qc^a = -\frac{1}{2} g f_{bc}{}^a c^b c^c$$
$$QA_\mu^a = (\nabla_\mu c)^a$$
$$Qb^a = 0$$
$$Q\bar{c}^a = b^a$$

called the BRST symmetry, and we have $Q^2 = 0$.

The first ingredient is $\mathbb{Z}$-graded vector spaces

$$V = \bigoplus_{k\in\mathbb{Z}} V_k$$

$$\Rightarrow V \otimes W = \bigoplus_{k\in\mathbb{Z}} (V\otimes W)_k$$
$$(V\otimes W)_k := \bigoplus_{i+j=k} V_i \otimes W_j$$

$$\Rightarrow V^* = \bigoplus_{k\in\mathbb{Z}} (V^*)_k$$
$$(V^*)_k := (V_{-k})^*$$

$$\Rightarrow \bar{V}[\ell] = \bigoplus_{k \in \mathbb{Z}} (\bar{V}[\ell])_k$$

$$(\bar{V}[\ell])_k := \bar{V}_{k+\ell}$$

Given $\bar{V}$, $Q : \mathcal{C}^\infty(\bar{V}) \to \mathcal{C}^\infty(\bar{V})$ where $\mathcal{C}^\infty(\bar{V}) = \bigodot^\bullet \bar{V}^*$, which is of degree 1 and $Q^2 = 0$

$(\bar{V}, Q)$ is called a Q vector space.

Take an ordinary vector space $\mathfrak{g}$ w/ basis e_a and consider $\mathfrak{g}[1]$

$$(\mathfrak{g}[1])_k = \mathfrak{g}_{k+1} = \begin{cases} \mathfrak{g} & k = -1 \\ 0 & \text{else} \end{cases}$$

Let us introduce coordinates on $\mathfrak{g}[1]$, $\xi^a : \mathfrak{g}[1] \to \mathbb{R}$

Hence $|\xi^a| = 1$, $Q : \mathcal{C}^\infty(\mathfrak{g}[1]) \to$

$C^\infty(\mathfrak{g}[1])$

$$Q = \frac{1}{2} \xi^b \xi^c f_{cb}{}^a \frac{\partial}{\partial \xi^a}$$

$$\Rightarrow Q^2 = \frac{1}{2}[Q,Q] \overset{!}{=} 0$$

$\Leftrightarrow$ $f_{ab}{}^c$ are structure const. of a Lie algebra

Hence, the differential graded algebra $(\mathcal{C}^\infty(\mathfrak{g}[1]), Q)$ is the Chevalley-Eilenberg description of a Lie algebra $(\mathfrak{g}, [-,-])$

Define

$$a := \xi^a \otimes e_a \in (\mathfrak{g}[1])^* \otimes \mathfrak{g}$$

$$\Rightarrow Qa := Q\xi^a \otimes e_a$$
$$= \frac{1}{2} \xi^b \xi^c f_{cb}{}^a \otimes e_a$$

$$= -\frac{1}{2}\xi^b\xi^c \otimes f_{bc}{}^a e_a$$

$$= -\frac{1}{2}\xi^b\xi^c \otimes [e_b, e_c]$$

$$= -\frac{1}{2}[a, a]'$$

BV formalism

Consider $\mathcal{L}_{BRST}[1]$ $\mathbb{Z}$ graded vector space of fields of some theory. Say Φ^I are the fields of ghost number $|\Phi^I|_{gh}$. Then the anti field Φ_I^{+} has ghost number $|\Phi_I^+| = -1 - |\phi^I|_{gh}$,

$$Q_{BV}\Phi_I^+ = (-)^{|\phi^I|_{gh}} \frac{\delta S_{BRST}}{\delta \phi^I} + \dots$$

Formally, this doubling corresponds to

$$\mathcal{L}_{BV}[1] := T^*[-1]\left(\mathcal{L}_{BRST}[1]\right)$$

This comes with a canonical symplectic structure

$$\omega := \delta\phi^+_I \wedge \delta\phi^I$$

with $|\omega| = -1$. Hamiltonian vector fields V_F for F a function $V_F \lrcorner\, \omega = \delta F$. Then the Poisson bracket

$$\{F, G\} := V_F \lrcorner\, V_G \lrcorner\, \omega$$

Then

$$\{F, G\} = (-)^{|\phi^I|_{gh}(|F|_{gh}+1)} \frac{\delta F}{\delta \phi^I} \frac{\delta G}{\delta \phi^+_I}$$

anti-bracket

$$- (-)^{(|\Phi^I|_{gh}+1)(|F|_{gh}+1)} \frac{\delta F}{\delta \Phi^+_I} \frac{\delta G}{\delta \Phi^I}$$

We have

$$\{F, G\} = (-)^{(|F|_{gh}+1)(|G|_{gh}+1)} \{G, F\}$$

A BV action $S_{BV} \in C^\infty(\mathcal{E}_{BV}[1])$ subject to

$$\{S_{BV}, S_{BV}\} = 0$$

known as the classical master equation.

$$S_{BV}\big|_{\Phi^+_I=0} = S_0$$

original action

$$(Q_{BV}\Phi^I)\big|_{\Phi^+_I=0} = Q_{BRST}\,\Phi^I$$

and

$$Q_{BV} = \{S_{BV}, -\}$$

Note $Q_{BV}^2 = 0 \iff \{S_{BV}, S_{BV}\} = 0$

Let's move on to L_∞-algebras.

As before

$$\begin{aligned} a &= a^{\underline{I}} \otimes e_{\underline{I}} \\ &= \phi^I \otimes e_I + \phi^+_I \otimes e^I \\ &\in (\mathcal{L}_{BV}[1])^* \otimes \mathcal{L}_{BV} \end{aligned}$$

of degree 1.

$$\begin{aligned} Q_{BV} a &= \{S_{BV}, a\} \\ &= - f(a) \\ &= - \sum_{i \geq 1} \frac{1}{i!} \mu_i'(a, \dots, a) \end{aligned}$$

The μ_i' encode i-ary graded antisymmetric linear maps

$$\mu_i : \mathcal{L}_{BV} \times \dots \times \mathcal{L}_{BV} \to \mathcal{L}_{BV}$$

$$\mu_1'(a) := (-)^{|a^I|_{gh}} a^I \otimes \mu_1(e_I)$$

$$\mu_i'(a, \dots, a) := (-)^{\left[i \sum_{j=1}^{i} |a^{I_j}|_{gh} + \sum_{j=2}^{i} |a^{I_j}|_{gh} \sum_{k=1}^{j-1} |e_{I_k}|_{\mathcal{L}_{BV}} \right]} a^{I_1} \dots a^{I_i} \otimes \mu_i(e_{I_1}, \dots, e_{I_i})$$

Then $Q^2_{BV} = 0$, we get the homotopy Jacobi identities

$$\sum_{i_1 + i_2 = i} \sum_{\sigma \in \overline{Sh}(i_1; i)} (-1)^{i_2} \chi(\sigma; l_1, \dots, l_i) \mu_{i+1}(\mu_i(l_{\sigma(1)} \dots, l_{\sigma(i)}), l_{\sigma(i+1)}, \dots, l_{\sigma(i)}) = 0$$

$\forall\, l_1, \dots, l_i \in \mathcal{L}_{BV}$. Here

$$l_1 \wedge \dots \wedge l_i = \varepsilon(\sigma; l_1, \dots, l_i)$$
$$l_{\sigma(1)} \wedge \dots \wedge l_{\sigma(i)}$$

The sum is over all $(i_1; i)$ unshuffles σ which consist of permutations σ of $\{1, \dots, i\}$ so that the first i_1 and the last $i - i_1$ images of σ are ordered.

The pair $(\mathcal{L}_{BV}, \mu_i)$ is called an L_∞-algebra. Since we have ω, this L_∞-algebra comes w/ an inner product

$$\langle l_1, l_2 \rangle := (-1)^{|l_1|_{\mathcal{L}_{BV}}} \omega(\sigma(l_1), \sigma(l_2))$$

where $\sigma : \mathcal{L}_{BV} \to \mathcal{L}_{BV}[1]$. This inner product has ghost number 0 and L_∞-degree -3 and it's cyclic

$$\langle \ell_1, \mu_i(\ell_2, \dots, \ell_{i+1})\rangle$$

$$= \pm \langle \ell_{i+1}, \mu_i(\ell_1, \dots, \ell_i)\rangle$$

Now, using $\langle\, ,\rangle$

$$\underset{0}{\underset{\uparrow}{S}} = \sum_i \frac{1}{(i+1)!} \langle \underset{\in \mathcal{L}_{BV,1}}{\underset{\uparrow}{\overset{1}{\overset{\downarrow}{a}}}}, \overset{2-i}{\overset{\downarrow}{\mu_i}} (\overbrace{a, \dots, a}^{i}) \rangle_{\leftarrow -3}$$

$$\delta S = 0 \iff f(a) = \sum_i \frac{1}{i!} \mu_i(a, \dots, a)$$

$$= 0$$

$$= \mu_1(a) + \frac{1}{2}\mu_2(a,a) + \dots$$

Take $c_0 \in \mathcal{L}_{BV,0}$

$$\delta a = \mu_1(c_0) + \mu_2(a, c_0) + \frac{1}{2!}\mu_3(a, a, c_0) + \cdots$$

The action is invariant under these transformations, which we call gauge transformations

Take $c_{-1} \in \mathcal{L}_{BV,-1}$

$$\delta c_0 = \mu_1(c_{-1}) + \mu_2(a, c_{-1}) + \cdots$$

These are called higher gauge transformations.

Take

$$S = \int H_{\mu\nu\lambda} H^{\mu\nu\lambda}$$

$\rightarrow H_{\mu\nu\lambda} = \partial_{[\mu} B_{\nu\lambda]}$

$\rightarrow B_{\mu\nu} \mapsto B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$

$$\rightarrow \Lambda_\mu \mapsto \Lambda_\mu - \partial_\mu \lambda$$

<u>Rmk</u>: $[\delta_{c_0}, \delta_{c_0'}] a$

$$= \delta_{c_0''} a + \sum_{i\geq 0} \frac{1}{i!} (-)^i \mu_{3+i}(f, a, \dots, a, c_0, c_0')$$

where

$$c_0'' = \sum_{i\geq 0} \frac{1}{i!} \mu_{i+2}(a, \dots, a, c_0, c_0')$$

In the BV formalism, we can write

$$S = \sum_{i\geq 1} \frac{1}{(i+1)!} \langle a, \mu_i'(a, \dots, a) \rangle'$$

where

$$a = a^I \otimes e_I = \phi^I \otimes e_I + \phi_I^+ \otimes e^I$$

<u>Example</u>: Yang-Mills theory

$$c, A, b, \bar{c} \rightsquigarrow c^+, A^+, b^+, \bar{c}^+$$

$$c \overset{\mu_1}{\longmapsto} \partial c$$

$$\begin{pmatrix} A \\ b \\ \bar{c} \end{pmatrix} \overset{\mu_1}{\longmapsto} \begin{pmatrix} \partial(\partial\cdot A) - \Box A \\ \bar{c} \\ b \end{pmatrix}$$

$$(A^+) \overset{\mu_1}{\longmapsto} \partial\cdot A^+$$

$$(c,c) \overset{\mu_2}{\longmapsto} [c,c]$$

$$(A,c) \overset{\mu_2}{\longmapsto} [A,c]$$

$$(A^+,c) \overset{\mu_2}{\longmapsto} [A^+,c]$$

$$(A,A) \overset{\mu_2}{\longmapsto} \partial\cdot[A,A] + [A;\partial A]$$

$$(c,c^+) \overset{\mu_2}{\longmapsto} [c,c^+]$$

$$(A,A^+) \overset{\mu_2}{\longmapsto} [A,A^+]$$

$$(A,A,A) \overset{\mu_3}{\longmapsto} [A;[A,A]]$$

$$\langle a, \mu_1(a)\rangle + \langle a, \mu_2(a,a)\rangle + \langle a, \mu_3(a,a,a)\rangle$$

$$\int \left(dA + \tfrac{1}{2}[A,A]\right) \wedge * \left(dA + \tfrac{1}{2}[A,A]\right)$$

$$\int dA \wedge * dA \rightsquigarrow \int A \wedge \underbrace{d * d}_{\mu_1} A$$

$$\int dA \wedge * [A,A] \rightsquigarrow \int A \wedge d * [A,A]$$

$$\int [A,A] \wedge * [A,A] \rightsquigarrow \int A \wedge [A, *[A,A]]$$

Gauge fixing

Gauge fixing is implemented by a canonical transformation

$$(\phi^I, \phi^+_I) \mapsto (\tilde{\phi}^I, \tilde{\phi}^+_I)$$

$$= \left(\phi^I, \phi^+_I + \frac{\delta \Psi}{\delta \phi^I}\right)$$

with Ψ of ghost number -1,

called the gauge fixing fermion.

We need to introduce fields (and their antifields) of negative ghost number, e.g. the anti-ghost $\bar{c}$ and the Nakanishi-Lautrup field b. In order not to change the Q_{BV} cohomology, they need to transform trivially

$$Q_{BV}\bar{c} = b \quad , \quad Q_{BV}\, b = 0$$

$$Q_{BV}\bar{c}^{+} = 0 \quad , \quad Q_{BV}\, b^{+} = -\bar{c}^{+}$$

The gauge fixed action

$$S^{gf}_{BV}[\hat{\Phi}^I, \hat{\Phi}^+_I] := S_{BV}\left[\Phi^I, \Phi^+_I + \frac{\delta \Psi}{\delta \Phi^I}\right]$$

$$\Psi = \int \bar{c}\,(b - \xi \partial A)$$

$\boxed{\text{How do we do gauge fixing at the path integral level?}}$

Note we define the BV Laplacian

$$\Delta F := (-)^{|\phi^I|_{gh} + |F|_{gh}} \frac{\delta^2 F}{\delta \phi_I^+ \delta \phi^I}$$

Consider

δ-distribution

$$Z_\Psi = \int \mu(\phi, \phi^+)\, \delta\left(\phi_I^+ - \frac{\delta \Psi}{\delta \phi^I}\right) e^{\frac{i}{\hbar} S^{\hbar}_{qBV}[\phi, \phi^+]}$$

measure compatible w/ ω

quantum BV action

$$S^{\hbar}_{qBV}\big|_{\hbar=0} = S_{BV}$$

In order for Z_Ψ to be independent of Ψ, i.e. $Z_{\Psi+\delta\Psi} = Z_\Psi$. This boils down to requiring

$$\boxed{\Delta e^{\frac{i}{\hbar} S^{\hbar}_{qBV}} = 0}$$

$$\Longleftrightarrow$$

$$\boxed{\{S^{\hbar}_{qBV}, S^{\hbar}_{qBV}\} - 2i\hbar \Delta S^{\hbar}_{qBV} = 0}$$

This called the quantum master equation.

Define

$$Q_{qBV} := \{S^{\hbar}_{qBV}, -\} - 2i\hbar\Delta$$

Note $Q^2_{qBV} = 0 \Longleftrightarrow$ quantum master eq.

Now we get

$$Q_{qBV}\, a = -\sum_{\substack{L\geq 0\\ i\geq 1}} \frac{\hbar^L}{i!} \mu'_{i,L}(a, \dots, a)$$

The $\mu'_{i,L}$ are the higher products of what is known as a quantum L_∞-algebra, and $Q^2_{qBV} = 0$, yields a quantum generalisation of the homotopy Jacobi identities.

Morphisms of L_∞-Algebras

Goal: generalise a morphism of Lie algebras to L_∞-algebras

Given $(\mathcal{L}, \mu_i)$ and $(\tilde{\mathcal{L}}, \tilde{\mu}_i)$.

A (curved) morphism $\phi : \mathcal{L} \to \tilde{\mathcal{L}}$ it's

$$\phi_i : \underbrace{\mathcal{L} \times \dots \times \mathcal{L}}_{i\text{-times}} \to \tilde{\mathcal{L}}$$

of degree $1-i$ and totally graded anti-symmetric, for $i \geq 0$.

$$\phi_0 : \mathbb{R} \to \tilde{\mathcal{L}}_1$$

is constant map

$$0 = \sum_{i \geq 1} \frac{1}{i!} \tilde{\mu}_i(\phi_0, \dots, \phi_0)$$

i.e. ϕ_0 is a Maurer-Cartan element.

$$\phi_1(\mu_1(l_1)) = \tilde{\mu}_1(\phi_1(l_1)) + \sum \frac{1}{i!} \tilde{\mu}_{i+1}(\phi_0, \dots, \phi_0, \phi_1(l_1))$$

$$\vdots$$

If we define

$$\tilde{\mu}_i^{\phi_0}(\tilde{\ell}_1, \dots, \tilde{\ell}_i)$$

$$:= \sum_{j \geq 0} \frac{1}{j!} \tilde{\mu}_{i+j}(\phi_0, \dots, \phi_0, \tilde{\ell}_1, \dots, \tilde{\ell}_i)$$

for ϕ_0 a Maurer Cartan element in $\tilde{\mathcal{L}}$.

then $(\tilde{\mathcal{L}}, \tilde{\mu}_i^{\phi_0})$ is an L_∞-algebra.

A curved morphism between $(\mathcal{L}, \mu_i)$ and $(\tilde{\mathcal{L}}, \tilde{\mu}_i)$ is the same a morphism (i.e. $\phi_0 \equiv 0$) between $(\mathcal{L}, \mu_i)$ and $(\tilde{\mathcal{L}}, \tilde{\mu}_i^{\phi_0})$

How do Maurer-Cartan elements behave under morphisms?

Suppose $\phi : (\mathcal{L}, \mu_i) \to (\tilde{\mathcal{L}}, \tilde{\mu}_i)$
Pick $a \in \mathcal{L}_1$ and consider

$$\tilde{a} = \Phi_0 + \Phi_1(a) + \frac{1}{2}\Phi_2(a,a) + \cdots$$

Then $$= \sum_{i\geq 0} \frac{1}{i!}\Phi_i(a,\dots,a)$$

$$\tilde{f} = \sum_{i\geq 1} \frac{1}{i!}\tilde{\mu}_i(\tilde{a},\dots,\tilde{a})$$

$$= \sum_{i\geq 0} \frac{1}{i!}\phi_{i+1}(a,\dots,a,f)$$

If now $a \mapsto a + \delta_{c_0} a$ and

$\tilde{a} \mapsto \tilde{a} + \delta_{\tilde{c}_0}\tilde{a}$. If

$$\tilde{c}_0 = \Phi_1(c_0) + \Phi_2(a,c_0) + \dots \in \tilde{\mathcal{L}}_0$$

Then

$$\delta_{\tilde{c}_0}\tilde{a} = -\sum_{i\geq 0}\frac{1}{i!}\Phi_{i+2}(a,\dots,a,f,c_0)$$

$$+ \sum_{i\geq 0} \frac{1}{i!}\Phi_{i+1}(\delta_{c_0}a, a,\dots,a)$$

$$\Rightarrow \sum_{i\geq 0}\frac{1}{i!}\Phi_i(a+\delta_{c_0}a, \dots, a+\delta_{c_0}a)$$

$$= \tilde{a} + \delta_{\tilde{\xi}_0} \tilde{a} + \sum_{i \geq 1} \frac{1}{i!} \phi_{i+2}(a, \dots, a, f, \xi_0)$$

Hence gauge equivalence classes of Maurer Cartan elements are mapped to gauge equivalence classes of Maurer Cartan elements.

Remember that $\mu_1^2 = 0$. We can therefore consider the cohomology

$$H^\bullet_{\mu_1}(\mathcal{L}) = \bigoplus_{k \in \mathbb{Z}} H^k_{\mu_1}(\mathcal{L})$$

$$H^k_{\mu_1}(\mathcal{L}) := \frac{\ker \mu_1|_{\mathcal{L}_k}}{\operatorname{im} \mu_1|_{\mathcal{L}_{k-1}}}$$

A curved morphism $\phi : (\mathcal{L}, \mu_i) \to (\tilde{\mathcal{L}}, \tilde{\mu}_i)$ is called a quasi-isom. whenever ϕ_1 induces an isomorphism on cohomologies. i.e.

$$H^\bullet_{\mu_1}(L) \cong H^\bullet_{\hat{\mu}_1, \phi_0}(\tilde{L})$$

Then there is a bijection between the moduli spaces of solutions to the Maurer Cartan equations.

Two theories are physically equivalent whenever there is a quasi isom. between their underlying L_∞ algebras.

Thm:

- Every L_∞ algebra is quasi-isomorphic to an L_∞-algebra with $\mu_1 = 0$, this is called the minimal model
- Every L_∞ algebra is quasi

isomorphic to one where $\mu_{i\geq 3} = 0$, and this is called strictification.

Tree-level scattering

There is an L_∞-structure on $H_{\mu_1}(\mathcal{L}_{BV}) =: \mathcal{L}_{BV}^\circ$ with products μ_i°

$$h \circlearrowright (\mathcal{L}_{BV}, \mu_1) \underset{e}{\overset{p}{\rightleftarrows}} (\mathcal{L}^\circ_{BV}, 0)$$

where h is called the contracting homotopy of degree -1 s.t.

$$1 = \mu_1 h + h\mu_1 + ep$$

$$pe = 1$$
$$ph = he = h^2 = 0$$
$$p\mu_1 = 0 = \mu_1 e$$

In the Chevalley–Eilenberg picture

$$H_0 \circlearrowright \Big(C^\infty(\mathcal{E}_{BV}[1]), \underbrace{Q_{BV,0}}_{\text{free part}}\Big) \underset{P_0}{\overset{E_0}{\rightleftarrows}} \big(C^\infty(\mathcal{E}^\circ_{BB}[1]), 0\big)$$

If we regard the non-linear part as

$$\delta := Q_{BV} - Q_{BB,0}$$

then the homological perturbation lemma guarantees that

$$H \circlearrowright \big(C^\infty(\mathcal{E}_{BV}[1]), Q_{BV}\big) \underset{P}{\overset{E}{\rightleftarrows}} \big(\mathcal{C}^\infty(\mathcal{E}^\circ_{BV}[1]), Q^\circ_{BV}\big)$$

$$E = E_0(1+\delta H_0)^{-1} \qquad P = P_0 - H\delta P_0$$

$$H = H_0(1+\delta H_0)^{-1} \qquad Q^\circ_{BV} = E\delta P_0$$

The relation for Q^0_{BV} is a recursion relation (in the perturbation), which yields the formulae μ_i^0 on $\mathcal{L}^0_{BV}$.

Then, for $a_1^0, \ldots, a_n^0 \in H_{\mu_1}(\mathcal{L}_{BV})$ the tree-level scattering amplitudes are given by

$$\mathcal{A}_n = \langle a_1^0, \mu_{n-1}^0(a_2^0, \ldots, a_n^0) \rangle$$