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Homotopy algebra & symmetry generators in QFT

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Last week, we learned …

Lagrangian's homotopy algebraic structure : μ_{bv}

- For a given Lagrangian, we can solve the BV master equation $\Delta e^{S_{bv}[\varphi]} = 0$,
  which tells us Lagrangian's homotopy algebra $\mu_{bv} = \mu_1 + \mu_2 + \cdots$

$$S_{bv}[\varphi] = \frac{1}{2}\omega\big(\,\varphi, \mu_1(\varphi)\,\big) + \frac{1}{3!}\omega\big(\,\varphi, \mu_2(\varphi,\varphi)\,\big) + \cdots$$

- Homological perturbation lemma describes the Feynman graph expansion.
  Hence, the path-integral *P* preserves the nilpotent property $P\,\mu_{bv} = \mu_{effective}\,P$.

P : homotopy alg. of the original QFT → (loop) homotopy alg. of its effective QFT

Reminder of how to get μ_{bv}

Quick review & notation in this talk

- Consider a master action $S_{bv}[\varphi] = S_{cl}[\phi] + \cdots$, which solves $\Delta\, e^{S_{bv}[\varphi]} = 0$.

- Write φ^a for all of fields and antifields collectively.

  e.g. For QED, $\varphi^a = A_\mu \,, c \,, \psi \,, \bar{\psi}$ (if any, antighosts & auxiliary fields) and their antifields.

- Rewrite our action into the contracted form :

$$S_{bv}[\varphi] = \sum_n \frac{1}{(n+1)!} \int dx\, \mu_{a_0 a_1 \dots a_n}\, \varphi^{a_n} \dots \varphi^{a_1} \varphi^{a_0}$$

$$= \sum_n \frac{1}{(n+1)!} \int dx\, \varphi^{a_0}\, \omega_{a_0 b} \left(\mu^b{}_{a_1 \dots a_n}\, \varphi^{a_n} \dots \varphi^{a_1} \right).$$

BV symplectic form : ω_{ab}

Reminder of how to get μ_{bv}

How to get Lagrangian's L_∞

- We can always start with the contracted form of the BV action :

$$S_{bv}[\varphi] = \sum_n \frac{1}{(n+1)!}\int dx\, \varphi^{a_0}\, \omega_{a_0 b}\, \left(\mu^b{}_{a_1\ldots a_n}\, \varphi^{a_n}\ldots\varphi^{a_1}\right) \quad .$$

- We **assume** that $\mu_{a_0 a_1\ldots a_n}$ is **graded symmetric** $\mu_{\ldots ab\ldots} = (-)^{ab}\mu_{\ldots ba\ldots}$,

  which ensures the "cyclic property" $\mu_{a_0 a_1\ldots a_n} = (-)^{a_0(a_1+\cdots+a_n)}\mu_{a_1\ldots a_n a_0}$.

- Then, the condition $\Delta\, e^{S_{bv}} = 0$ gives **the (quantum) L∞ relations** :

$$\hbar\,\omega^{ab}\mu^c{}_{ab\underline{a_n\ldots a_1}} + \frac{1}{2}\sum_m \frac{1}{m!(n-m)!}\,\mu^c{}_{\underline{a_n\ldots a_{m+1}}b}\ \mu^b{}_{\underline{a_m\ldots a_1}} = 0$$

underline denotes the right sum

Reminder of how to get μ_{bv}

You can weaken L∞ 's assumption :

- We can always start with the contracted form of the BV action :

$$S_{bv}[\varphi] = \sum_{n} \frac{1}{(n+1)!} \int dx \, \varphi^{a_0} \, \omega_{a_0 b} \left(\mu^{b}{}_{a_1 \dots a_n} \, \varphi^{a_n} \dots \varphi^{a_1} \right) \quad .$$

- We **assume** that $\mu_{a_0 a_1 \dots a_n}$ is **graded symmetric** $\mu_{\dots a b \dots} = (-)^{ab} \mu_{\dots b a \dots}$,

  which ensures the "cyclic property" $\mu_{a_0 a_1 \dots a_n} = (-)^{a_0 (a_1 + \dots + a_n)} \mu_{a_1 \dots a_n a_0}$.

- Then, the condition $\Delta \, e^{S_{bv}} = 0$ gives **the (quantum) L∞ relations**.

→ *When we relax this assumption, we get (quantum) A∞ .*

Reminder of how to get μ_{bv}

How to get Lagrangian's A_∞

- We can start with the contracted form of the BV action :

$$S_{bv}[\varphi] = \sum_n \frac{1}{n+1} \int dx \; \varphi^{a_0} \, \omega_{a_0 b} \left(\mu^b{}_{a_1 \ldots a_n} \, \varphi^{a_n} \ldots \varphi^{a_1} \right) \quad .$$

- We just **assume** the "cyclic property" $\mu_{a_0 a_1 \ldots a_n} = (-)^{a_0(a_1 + \cdots + a_n)} \mu_{a_1 \ldots a_n a_0}$ only.

- Then, the condition $\Delta \, e^{S_{bv}} = 0$ gives **the (quantum) A∞ relations**.

~~Reminder of how to get μ_{bv}~~

How to get Lagrangian's A_∞

- We can start with the contracted form of the BV action :

$$S_{bv}[\varphi] = \sum_n \frac{1}{n+1} \int dx \; \varphi^{a_0} \, \omega_{a_0 b} \left(\mu^b{}_{a_1 \dots a_n} \; \varphi^{a_n} \dots \varphi^{a_1} \right) \quad .$$

- We just **assume** the "cyclic property" $\mu_{a_0 a_1 \dots a_n} = (-)^{a_0(a_1 + \cdots + a_n)} \mu_{a_1 \dots a_n a_0}$ only.

- Then, the condition $\Delta \, e^{S_{bv}} = 0$ gives **the (quantum) A∞ relations**.

→ *Lagrangian's (quantum) A∞ algebra does not need an additional "matrix-like structure" or "space-time non-commutativity".*

But, when $\mu_{\dots a \, b \dots} = (-)^{ab} \mu_{\dots b \, a \dots}$ comes from physics, A∞ may be physically redundant.

My notation

The relation between $\mu^b{}_{a_1 \ldots a_n}$ *and* $\mu_{bv} = \mu_1 + \mu_2 + \cdots$

- We can get the L∞ relation $\sum_m \frac{1}{m!(n-m)!} \mu^c{}_{\underline{a_n \ldots a_{m+1}} b}\, \mu^b{}_{\underline{a_m \ldots a_1}} = 0$ from $(S_{bv}, S_{bv}) = 0$,

  These give a "component" expression.

- As we can switch from $\partial_\mu j^\mu \approx 0$ to $d j^{D-1} \approx 0$ ($j^{D-1} = j^\mu \star dx^\mu$: (D-1)-form) ,

  we can switch from $\mu^b{}_{a_1 \ldots a_n}$ to $\mu_n : H^{\otimes n} \to H$ (coder $\mu_n : T(H) \to T(H)$) .

  (Now, instead of dx^μ , we need to consider $d\varphi^a$ as bases of *H*.)

- Then, we can obtain Lagrangian's homotopy algebra $(\mu_{bv})^2 = 0$

  where $\mu_{bv} \equiv \mu_1 + \mu_2 + \mu_3 + \cdots$ is a coderivation acting on $T(H)$ or $S(H)$.

These are what we learned last week & my notation.

What I would like to tell you today is as follows . . .

Today, I would like to tell you …

Symmetry's homotopy algebraic structure : μ_{sym}

1. Homotopy algebras μ_{sym} also appear in realization of given symmetries.

2. We can incorporate symmetry's μ_{sym} into Lagrangian's μ_{bv} and get

$$\big(\underbrace{\mu_{sym} + \mu_{bv} + \cdots}_{\equiv \, \mu_{total}} \big)^2 = 0 \, .$$

3. The Feynman graph expansion $P \equiv \int \mathscr{D}[\phi] e^{S_{free}[\phi]} / Z$ **preserves** this

$\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$ in the sense that $P \, \mu_{total} = \mu'_{total} \, P$ with $(\mu'_{total})^2 = 0$.

Today, I would like to tell you …

What we can read from μ_{sym}

4. Homotopy algebraic structure μ_{sym} or $(\mu_{total})^2 = (\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$

- tells us how to realize symmetries in every “effective” theory.

- naturally includes 1-form symmetries, etc.

- may explain why symmetry or anomaly remains under the path-integral,
  even if it may break the manifest invariance.

Plan

(i) Homotopy algebra μ_{sym} in the realization of symmetries

& How to incorporate μ_{sym} into $(\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$

(ii) Behavior of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$ under the path-integral

& Applications to several models

1. Homotopy algebra μ_{sym} in the realization of symmetries

We consider . . .

- First, we explain μ_{sym} intuitively within the canonical formalism.

  momentum π & the Poisson bracket $\{A, B\} \equiv A\left[\frac{\overleftarrow{\delta}}{\delta\phi^a}\frac{\delta}{\delta\pi_a} - \frac{\overleftarrow{\delta}}{\delta\pi_a}\frac{\delta}{\delta\phi^a}\right]B$

- Next, we switch to the BV formalism and explain it more precisely.

  Antifields ϕ^* & the BV bracket $(A, B) \equiv A\left[\frac{\overleftarrow{\delta}}{\delta\phi^a}\frac{\delta}{\delta\phi^*_a} - \frac{\overleftarrow{\delta}}{\delta\phi^*_a}\frac{\delta}{\delta\phi^a}\right]B$

This tells us how to incorporate it into Lagrangian's homotopy alg.

1. Homotopy algebra μ_{sym} in the realization of symmetries

Intuitive explanation : the canonical formalism

- We consider a Lagrangian $S[\phi]$ without gauge degree:

  the canonical form $S[\phi] \longrightarrow S[\phi,\pi] = \int dx \left(\pi \cdot \dot{\phi} - H \right)$.

- Suppose that $S[\phi]$ is invariant under $\delta\phi = \epsilon^a \cdot \delta_a \phi$ (ϵ^a : constants) .

- These global symmetries may or may not be linearly realized :

  The Poisson bracket gives $\epsilon^a \cdot \delta_a \phi = \epsilon^a \left\{ S_a[\phi,\pi] \, , \phi \right\}$.

- This $S_a[\phi,\pi] \sim \int dx \, \pi \cdot \delta_a \phi + \cdots$ is a realization of symmetry generator.

1. Homotopy algebra μ_{sym} in the realization of symmetries

Intuitive explanation : the canonical formalism

- Notice that the action $S = S[\phi, \pi]$ generates trivial transformations

$$\{ S , F[\phi, \pi] \} = \left(\frac{d\phi}{dt} - \frac{\delta H}{\delta \pi} \right) \cdot \frac{\delta F}{\delta \phi} + \left(\frac{d\pi}{dt} + \frac{\delta H}{\delta \phi} \right) \cdot \frac{\delta F}{\delta \phi} \approx 0$$

- Suppose that a Lie algebra $[\hat{T}_a, \hat{T}_b] = f_{ab}{}^c \hat{T}_c$ is realized on-shell :

$$\{ S_a[\phi, \pi] , S_b[\phi, \pi] \} \approx f_{ab}{}^c S_c[\phi, \pi] \qquad \text{(equality up to e.o.m.)}$$

- By using functionals $S_{ab}[\phi, \pi]$, we can get the off-shell equality :

$$\{ S_a[\phi, \pi] , S_b[\phi, \pi] \} = f_{ab}{}^c \, S_c[\phi, \pi] + \{ S , S_{ab}[\phi, \pi] \}$$

1. Homotopy algebra μ_{sym} in the realization of symmetries

Intuitive explanation : the canonical formalism

- Take $\{ S_c , \quad \}$ of $\{ S_a , S_b \} = f_{ab}{}^c S_c + \{ S , S_{ab} \}$ and consider the cyclic sum :

$$\Big\{ S_c , \{ S_a , S_b \} \Big\} + (cyclic) = \Big\{ S_c , f_{ab}{}^d S_d + \{ S , S_{ab} \} \Big\} + (cyclic)$$

- After some calculations, we get

$$\underbrace{S_k[\phi,\pi]\, f_{\underline{la}}{}^k f_{\underline{bc}}{}^l}_{Jacobi\ id.} = \Big\{ S \,,\, f_{\underline{ab}}{}^k S_{\underline{kc}}[\varphi] - \{ S_{\underline{a}}[\phi,\pi] \,, S_{\underline{bc}}[\phi,\pi] \} \Big\}$$

- Both sides of this equality vanish separately.

- Notice that the r.h.s. takes the $\{ S , \quad \}$-exact form.

1. Homotopy algebra μ_{sym} in the realization of symmetries

Intuitive explanation : the canonical formalism

- By using functionals $S_{abc}[\phi,\pi]$, this (r.h.s.)=0 implies

$$\{\, S_{\underline{a}}[\phi,\pi]\,, S_{\underline{bc}}[\phi,\pi]\,\} = f_{\underline{ab}}{}^{k}\, S_{k\underline{c}}[\phi,\pi] + \frac{1}{3} f_{\underline{abc}}{}^{k}\, S_{k}[\phi,\pi] + \frac{1}{3}\{\, S\,, S_{\underline{abc}}[\phi,\pi]\,\}$$

- We get a higher structure constant $f_{abc}{}^{d}$.

  We can repeat the same calculations by introducing a set of $\{S, S_a, S_{ab}\,, S_{abc}\,, \dots\}$:

$$\underbrace{\Big\{\, S_{\underline{a_0}}[\phi,\pi], \{\, S_{\underline{a_1\dots}}[\phi,\pi]\,, S_{\underline{\dots a_k}}[\phi,\pi]\,\}\Big\}}_{\text{order } k \text{ Jacobi id. off-shell}} = \underbrace{\{\, S\,, \;\dots \;\}}_{\approx 0}$$

  Again, (l.h.s.) and (r.h.s.) vanish separately.

1. Homotopy algebra μ_{sym} in the realization of symmetries

Intuitive explanation : the canonical formalism

- We will get a set of structure constants $\{ f_{ab}{}^{c}, f_{abc}{}^{d}, f_{abcd}{}^{e}, \dots \}$, a set of generators $\{ S, S_a, S_{ab}, S_{abc}, \dots \}$, and a set of algebraic relations :

(r.h.s.) $$\sum_k \{ S_{\underline{a_1 \dots a_k}}[\phi, \pi] , S_{\underline{a_{k+1} \dots a_n}}[\phi, \pi] \} = \sum_l f_{\underline{a_1 \dots a_l}}{}^{b} S_{\underline{b a_{l+1} \dots a_n}}[\phi, \pi]$$

(l.h.s.) $$\sum_m \frac{1}{m!(n-m)!} f_{\underline{a_1 \dots a_m}}{}^{b} f_{b\, \underline{a_{m+1} \dots a_n}}{}^{c} = 0 \qquad \cdots \quad L_\infty\text{-relations}$$

- If there is no higher conservation low $\partial_\mu j^{\mu\nu} \approx 0$, higher $f_{abc}{}^{d}, \dots$ cannot occur.

1. Homotopy algebra μ_{sym} in the realization of symmetries

Switch to the BV formalism

- We first consider a Lagrangian $S[\phi]$ without gauge degree.

  Then, the BV master action is this $S[\phi]$ itself.

- Suppose that $S[\phi]$ is invariant under $\delta\phi = \epsilon^A \cdot \delta_A \phi$ (global sym) .

- For these constants ϵ^A , we introduce constant ghosts ξ^A .

  Then, the action $S[\phi]$ is still invariant under $\delta\phi = \xi^A \cdot \delta_A \phi$.

- We write φ for all ϕ , ϕ^* correctively.

1. Homotopy algebra μ_{sym} in the realization of symmetries

Switch to the BV formalism

- We can get generators $S_A[\varphi] = S_A[\phi, \phi^*]$ satisfying

$$\delta_A \phi = \big(\; S_A[\varphi] \; , \; \phi \; \big) \qquad \text{with the BV bracket} \; \big(A \, , B \big) \equiv A \left[\frac{\overleftarrow{\delta}}{\delta \phi^a} \frac{\delta}{\delta \phi_a^*} - \frac{\overleftarrow{\delta}}{\delta \phi_a^*} \frac{\delta}{\delta \phi^a} \right] B \; .$$

- These symmetry generators take $S_A[\varphi] \sim \int dx \, \phi_a^* \cdot \delta_A \phi^a + \cdots$

- These $S_A[\varphi]$ generate symmetries of the BV master action.

  We can always find these because $(S , \;)$ is acyclic: we have $\big(\; S_A[\varphi] \; , \; S \; \big) = 0$ now.

1. Homotopy algebra μ_{sym} in the realization of symmetries

Switch to the BV formalism

- In BV, we can always find functionals $S_{AB}[\varphi]$ giving

  the off-shell equality : $\big(S_A[\varphi] \, , S_B[\varphi] \big) = f_{AB}{}^{C} \, S_C[\varphi] + \big(S \; , S_{AB}[\varphi] \big)$.

- We can repeat the same calculations as before.

  (Every step is precise in BV, which is not intuitive one unlike before.)

  We get a set of algebras $\sum_k \big(S_{\underline{A_1 \ldots A_k}}[\varphi] \; , \; S_{\underline{A_{k+1} \ldots A_n}}[\varphi] \big) = \sum_l f_{\underline{A_1 \ldots A_l}}{}^{B} \, S_{B \, \underline{A_{l+1} \ldots A_n}}[\varphi]$

  & the L∞-relations $\sum_m \frac{1}{m!(n-m)!} f_{\underline{A_1 \ldots A_m}}{}^{B} \; f_{B \, \underline{A_{m+1} \ldots A_n}}{}^{c} = 0$.

1. Homotopy algebra μ_{sym} in the realization of symmetries

The BV master equation is now modified

- The relation $\sum_k \big(\underline{S_{A_1 \dots A_k}}[\varphi] \, , \, \underline{S_{A_{k+1} \dots A_n}}[\varphi] \big) = \sum_l \underline{f_{A_1 \dots A_l}}{}^{B} \, S_{B \, \underline{A_{l+1} \dots A_n}}[\varphi]$ provides that

  the action $S_{bv}[\varphi]$ and source terms $S_{source}[\varphi, \xi] \equiv \sum_k \frac{1}{k!} S_{A_1 \dots A_k}[\varphi] \, \xi^{A_k} \dots \xi^{A_1}$ satisfy

$$\frac{1}{2} \big(S_{bv}[\varphi] + S_{source}[\varphi, \xi] \, , \, S_{bv}[\varphi] + S_{source}[\varphi, \xi] \big) = - \sum_l \frac{1}{k!} \frac{\partial S_{source}[\varphi, \xi]}{\partial \xi^B} f^{B}{}_{A_1 \dots A_k} \, \xi^{A_k} \dots \xi^{A_1}$$

Comments on QFT with gauge degrees

- If your QFT has any gauge degree, first of all, you must solve the BV master equation and get a solution $S_{bv}[\varphi] = S[\phi] + \phi^* \delta\phi + \cdots$.

- You can apply the same calculations to $S_{bv}[\varphi]$, instead of $S[\phi]$.

  Then, you can see symmetries of gauge invariant QFTs.

- If you want to consider symmetries of a gauge-fixed theory $S_{BRS}[\phi]$,

  it is the same as QFTs without gauge degrees.

Comments on the relation to conservation lows

- We introduced constant ghosts ξ^A for $\delta\phi = \epsilon^A \cdot \delta_A \phi$.

  These ξ^A come from usual conservation lows $\partial_A j^A \approx 0$.

  In many cases, these ξ^A have ghost number "1".

- If there exist higher conservation lows $\partial_{\mu_1} j^{\mu_1 \mu_2 \cdots \mu_n} \approx 0$,

  constant ghosts which have ghost number "n" may appear.

So, when QFT has a 1-form symmetry, constant ghosts ξ^A which have ghost number "1" or "2" naturally appear in the above procedure.

Plan

(i) Homotopy algebra μ_{sym} in the realization of symmetries

& How to incorporate μ_{sym} into $(\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$

(ii) Behavior of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$ under the path-integral

& Applications to several models

2. How to incorporate μ_{sym} into $(\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$

Our Lagrangian's homotopy algebra

- For simplicity, we consider a QFT without gauge degree.

  (Or assume that we could perform the Legendre transformation / gauge-fixing and could obtain1PI action / path-integrable gauge-fixed action : S_{1PI} / S_{BRS} .

  Then, vertices of S_{1PI} / S_{BRS} may or may not have explicit $\hbar$ dependence.)

- In this case, we can find $(S_{bv} , S_{bv}) = 0$ and $\Delta S_{bv} = 0$.

- The classical BV master equation $(S_{bv} , S_{bv}) = 0$ gives

the (cyclic) L∞ relations $$\sum_{m} \frac{1}{m!(n-m)!} \, \mu^{c}{}_{\underline{a_n \ldots a_{m+1}} b} \, \mu^{b}{}_{\underline{a_m \ldots a_1}} = 0 \quad .$$

The relation between ${\mu^b}_{a_1 \dots a_n}$ *and* $\mu_{bv} = \mu_1 + \mu_2 + \cdots$

- The relation $\sum_m \frac{1}{m!(n-m)!} {\mu^c}_{\underline{a_n \dots a_{m+1}} b} {\mu^b}_{\underline{a_m \dots a_1}} = 0$ is a "component" expression.

- As $\partial_\mu j^\mu \approx 0$ and $d j^{D-1} \approx 0$,

  we can switch from ${\mu^b}_{a_1 \dots a_n}$ to $\mu_n : H^{\otimes n} \to H$ (coder $\mu_n : T(H) \to T(H)$) .

  (Now, instead of dx^μ , we need to consider $d\varphi^a$ as bases of *H*.)

- So, we can get $(\mu_{bv})^2 = (\mu_1 + \mu_2 + \cdots)^2 = 0$ from $(S_{bv}, S_{bv}) = 0$.

2. How to incorporate μ_{sym} into $(\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$

We know that $(S_{bv}, S_{bv}) = 0 \;\Leftrightarrow\; (\mu_{bv})^2 = 0$

- $(S_{bv}, S_{bv}) = 0$ gives $\sum_m \frac{1}{m!(n-m)!} \mu^c{}_{\underline{a_n \dots a_{m+1}} b}\, \mu^b{}_{\underline{a_m \dots a_1}} = 0$, which is $(\mu_{bv})^2 = 0$.

- Likewise, we consider $S_{sym}[\xi] = \sum_n \frac{1}{(n+1)!} \xi^*_B f^B{}_{A_1 \dots A_n} \xi^{A_n} \dots \xi^{A_1}$ and $(\;,\;)_\xi = \frac{\partial}{\partial \xi^A} \frac{\partial}{\partial \xi^*_A} - \frac{\partial}{\partial \xi^*_A} \frac{\partial}{\partial \xi^A}$.

  Then, we find that

  $(S_{sym}[\xi], S_{sym}[\xi])_\xi = 0$ gives $\sum_m \frac{1}{m!(n-m)!} f_{\underline{A_1 \dots A_m}}{}^B f_{B \underline{A_{m+1} \dots A_n}}{}^c = 0$, which is $(\mu_{sym})^2 = 0$.

- These pieces will give $(\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$.

We already obtained "…" of $(\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$

- We learned that the action $S_{bv}[\varphi]$ and source terms

$$S_{source}[\varphi, \xi] \equiv \sum_k \frac{1}{k!} S_{A_1 \ldots A_k}[\varphi]\, \xi^{A_k} \ldots \xi^{A_1}$$

satisfy $$\frac{1}{2}\Big(S_{bv}[\varphi] + S_{source}[\varphi, \xi] \ , \ S_{bv}[\varphi] + S_{source}[\varphi, \xi] \Big) = - \sum_l \frac{1}{k!} \frac{\partial S_{source}[\varphi, \xi]}{\partial \xi^B} f^B{}_{A_1 \ldots A_k}\, \xi^{A_k} \ldots \xi^{A_1} \quad .$$

- The nil-potency of the classical BV is obstructed by global symmetries.

- We can resolve it by adding $S_{sym}[\xi] = \sum_n \frac{1}{(n+1)!}\, \xi^*_B\, f^B{}_{A_1 \ldots A_n}\, \xi^{A_n} \ldots \xi^{A_1}$ and $(\ ,\ )_\xi = \frac{\partial}{\partial \xi^A} \frac{\partial}{\partial \xi^*_A} - \frac{\partial}{\partial \xi^*_A} \frac{\partial}{\partial \xi^A}$.

2. How to incorporate μ_{sym} into $(\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$

We already obtained "…" of $(\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$

- We consider the sum

$$S_{total}[\varphi,\xi] \equiv S_{bv}[\varphi] + \underbrace{\sum_k \frac{1}{k!} S_{A_1\ldots A_k}[\varphi]\,\xi^{A_k}\ldots\xi^{A_1}}_{S_{source}[\varphi,\xi]} + \underbrace{\sum_n \frac{1}{(n+1)!}\,\xi^*_B\, f^B{}_{A_1\ldots A_n}\,\xi^{A_n}\ldots\xi^{A_1}}_{S_{sym}[\xi]} \quad .$$

- We also consider the sum of the anti-brackets

$$(\ \ ,\ \ )_{\varphi,\xi} \equiv \left[\frac{\overleftarrow{\delta}}{\delta\phi^a}\frac{\delta}{\delta\phi^*_a} - \frac{\overleftarrow{\delta}}{\delta\phi^*_a}\frac{\delta}{\delta\phi^a}\right] + \left[\frac{\overleftarrow{\partial}}{\partial\xi^A}\frac{\partial}{\partial\xi^*_A} - \frac{\overleftarrow{\partial}}{\partial\xi^*_A}\frac{\partial}{\partial\xi^A}\right] \quad .$$

- Then, we obtain $\big(\, S_{total}[\varphi,\xi]\,,\, S_{total}[\varphi,\xi]\,\big)_{\varphi,\xi} = 0$, which is $(\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$.

What are inputs of these μ_{sym} & μ_{total} ?

- We got the L∞-relations : $\sum_m \frac{1}{m!(n-m)!} f_{\underline{A_1 \ldots A_m}}{}^{B} f_{B\,\underline{A_{m+1} \ldots A_n}}{}^{c} = 0$, which gives μ_{sym} .

  Q. What is the vector space H_ξ on which μ_{sym} acts ?

  A. The vector space of constants ghosts $\xi = \xi^A \cdot e_A$ (e_A are “bases”)

  or its (symmetrized) tensor algebra $S(H_\xi)$.

- Inputs of $\mu_{total} = \mu_{sym} + \mu_{bv} + \cdots$ are $S(H) \otimes S(H_\xi)$.

μ_{total} gives an open-closed homotopy algebra

when we use A∞ description for Lagrangian’s μ_{bv} .

Plan

(i) Homotopy algebra μ_{sym} in the realization of symmetries

& How to incorporate μ_{sym} into $(\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$

(ii) Behavior of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$ under the path-integral

& Applications to several models

2. Behavior of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$

$(\mu_{total})^2 = 0$ *in "effective" theories*

- We first split $S_{bv}[\varphi]$ into the kinetic part $S_{free}[\varphi]$ and interacting part $S_{int}[\varphi]$:

$$S_{bv}[\varphi] = S_{free}[\varphi] + S_{int}[\varphi] \quad , \quad \text{which provides} \quad \mu_{bv} = \mu_1 + \overbrace{\mu_2 + \dots}^{\mu_{int}} \ .$$

- We split fields $\phi = \phi' + \phi''$ and define a generic "effective" action by integrating out ϕ'' ,

$$P : \ S[\phi' + \phi''] \longmapsto A[\phi'] \equiv \ln \int D[\phi'']\, e^{S[\phi'+\phi'']} \ .$$

- Homological perturbation lemma guarantees that an effective one $\hbar\,\Delta' + (A[\phi'], \ \ )'$ is nilpotent, which gives $(\mu_{effective})^2 = 0$.

→ We can obtain $(\mu'_{sym})^2 = 0$ and $(\mu'_{total})^2 = 0$ recursively, as $(\mu'_{bv})^2 = 0$.

2. Behavior of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$

$(\mu_{total})^2 = 0$ is preserved under the path-integral

- We know

  $(S_{free}[\varphi] , \ \ )$ is nilpotent , which is $(\mu_1)^2 = 0$.

  $(S_{bv}[\varphi] , \ \ ) = (S_{free}[\varphi] + S_{int}[\varphi] , \ \ )$ is nilpotent , which is $(\mu_1 + \mu_{int})^2 = 0$.

- Now, we got

  $(S_{total}[\varphi, \xi] , \ \ )_{\varphi,\xi} = (S_{bv}[\varphi] , \ \ ) + (S_{source}[\varphi, \xi] + S_{sym}[\xi] , \ \ )_{\varphi,\xi}$ is nilpotent,

  which is $(\mu_{total})^2 = (\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$.

2. Behavior of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$

$(\mu_{total})^2 = 0$ *is preserved under the path-integral*

- We also know

$$\hbar\,\Delta + (\, S_{free}[\varphi]\,, \quad) \text{ is nilpotent , which is } (\,\hbar\Delta + \mu_1\,)^2 = 0\,.$$

$$\hbar\Delta + (\, S_{bv}[\varphi]\,, \quad) = \hbar\Delta + (\, S_{free}[\varphi] + S_{int}[\varphi]\,, \quad) \text{ is nilpotent , which is } (\,\hbar\Delta + \mu_{bv}\,)^2 = 0\,.$$

- As long as symmetries $\delta\phi$ are not anomalous, $\int D[\phi]\, e^{S[\phi]} = \int D[\phi + \delta\phi]\, e^{S[\phi+\delta\phi]}$, we may get

$$\hbar\Delta + (\, S_{total}[\varphi, \xi]\,, \quad)_{\varphi,\xi} = \hbar\Delta + (\, S_{bv}[\varphi]\,, \quad) + (\, S_{source}[\varphi, \xi] + S_{sym}[\xi]\,, \quad)_{\varphi,\xi} \text{ is nilpotent,}$$

$$\text{which is } (\,\hbar\Delta + \mu_{total}\,)^2 = (\,\mu_{sym} + \hbar\Delta + \mu_{bv} + \cdots\,)^2 = 0\,.$$

→ We consider the Homological Perturbation connecting these.

2. Behavior of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$

Free theories give the Gaussians, which fixes the ambiguity

- Since we can solve free QFTs, we start from a deformation retract of free theories :

$$h'' \circlearrowright \Big(\text{state space}\,,\ \underbrace{(\,S_{\text{free}}\,,\quad)}_{\mu_1'+\mu_1''} \Big) \underset{i''}{\overset{p''}{\rightleftarrows}} \Big(\underbrace{\text{on shell of } \phi''}_{\text{cohomology of } \hat{\mu}_1''},\ \underbrace{(A_{free},\ )}_{\mu_1'} \Big)$$

where a BV propagator h'' gives a Hodge decomposition : $\mu_1'' h'' + h'' \mu_1'' = 1 - i'' p''$.

- Even if the path-integral of ϕ'' breaks the manifest invariance, we can read (non-linear) realization of symmetries in effective theories.

→ HPL tells us recursive relations

Tree part : realization of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$ *in effective theories*

$$h'' \circlearrowright \Big(\text{state space}, \underbrace{(S_{\text{free}}, \quad)}_{\mu'_1+\mu''_1} \Big) \underset{i''}{\overset{p''}{\rightleftarrows}} \Big(\underbrace{\text{on shell of } \phi''}_{\text{cohomology of } \hat{\mu}''_1}, \underbrace{(A_{free},)}_{\mu'_1} \Big)$$

perturbation : $(S_{\text{free}}, \quad) \longmapsto (S_{\text{bv}}, \quad) \equiv (S_{\text{free}}, \quad) + (S_{\text{int}}, \quad)$ gives the tree graph expansion

$$h_{tree} \circlearrowright \Big(\text{state space}, \underbrace{(S_{\text{bv}}, \quad)}_{\mu'_1+\mu''_1} \Big) \underset{i_{tree}}{\overset{P_{tree}}{\rightleftarrows}} \Big(\underbrace{\text{on shell of } \phi''}_{\text{cohomology of } \hat{\mu}''_1}, \underbrace{(A[\phi'],)}_{\mu'_{bv}} \Big)$$

perturbation : $(S_{\text{bv}}, \quad) \longmapsto (S_{total}, \quad)_{\phi,\xi} \equiv (S_{bv}, \quad) + (S_{source} + S_{sym}, \quad)_{\phi,\xi}$

$$\tilde{h}_{tree} \circlearrowright \Big(\text{state space}, \underbrace{(S_{total}, \quad)_{\phi,\xi}}_{\mu_{total}} \Big) \underset{\tilde{i}_{tree}}{\overset{\tilde{P}_{tree}}{\rightleftarrows}} \Big(\underbrace{\text{on shell of } \phi''}_{\text{cohomology of } \hat{\mu}''_1}, \underbrace{(A_{total}[\phi', \xi],)_{\phi',\xi}}_{\mu'_{total}} \Big)$$

As the BG-current relation in a generic QFT,
we can get $\mu'_{total} \equiv \mu'_{sym} + \mu'_{bv} + \cdots$ from recursive relations.

Tree + loop : realization of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$ in effective theories

$$h'' \circlearrowleft \Big(\text{state space}, \underbrace{(S_{\text{free}}, \quad)}_{\mu'_1 + \mu''_1} \Big) \underset{i''}{\overset{p''}{\rightleftarrows}} \Big(\underbrace{\text{on shell of } \phi''}_{\text{cohomology of } \hat{\mu}''_1}, \underbrace{(A_{free},)}_{\mu'_1} \Big)$$

perturbation : $(S_{\text{free}}, \quad) \longmapsto \hbar\,\Delta + (S_{\text{free}}, \quad)$ gives the Wick theorem

$$h_{\text{Wick}} \circlearrowleft \Big(\text{state space}, \underbrace{\hbar\Delta + (S_{\text{free}}, \quad)}_{\hbar\Delta' + \mu'_1 + \hbar\Delta'' + \mu''_1} \Big) \underset{I_{\text{Wick}}}{\overset{P_{\text{Wick}}}{\rightleftarrows}} \Big(\underbrace{\text{on shell of } \phi''}_{\text{cohomology of } \hat{\mu}''_1}, \underbrace{\hbar\Delta' + (A[\phi'],)}_{\hbar\Delta' + \mu'_1} \Big)$$

perturbation to obtain $\hbar\Delta + (S_{total}, \quad)_{\phi,\xi}$

$$\tilde{h}_{\text{Wick}} \circlearrowleft \Big(\text{state space}, \underbrace{\hbar\Delta + (S_{total}, \quad)_{\phi,\xi}}_{\hbar\Delta + \mu_{total}} \Big) \underset{\tilde{i}_{\text{Wick}}}{\overset{\tilde{P}_{\text{Wick}}}{\rightleftarrows}} \Big(\underbrace{\text{on shell of } \phi''}_{\text{cohomology of } \hat{\mu}''_1}, \underbrace{\hbar\Delta' + (A_{total}[\phi', \xi],)_{\phi',\xi}}_{\hbar\Delta' + \mu'_{q-total}} \Big)$$

We can get $\mu'_{q-total} \equiv \mu'_{sym} + \mu'_{bv} + \cdots$, which includes $\hbar$, from recursive relations.

Plan

(i) Homotopy algebra μ_{sym} in the realization of symmetries

& How to incorporate μ_{sym} into $(\mu_{sym} + \mu_{bv} + \cdots)^2 = 0$

(ii) Behavior of $\mu_{total} \equiv \mu_{sym} + \mu_{bv} + \cdots$ under the path-integral

& ~~Applications to several models~~

Two examples & comments

Applications

Examples of μ_{total} : Maxwell's theory

- We consider the Maxwell theory : $S_{bv}[\varphi] = \int dx \Big[\frac{-1}{4} F_{\mu\nu} F^{\mu\nu} + A^{*\mu} \partial_\mu C \Big]$.

- Let us consider translations $\delta A_\mu = \epsilon^\nu \partial_\nu A_\mu$ and shifts $\delta A_\mu = \epsilon_{\mu\nu} x^\nu$ with $\epsilon_{\mu\nu} + \epsilon_{\nu\mu} = 0$.
  (The commutator is the gauge transformation with $\epsilon^\mu \epsilon_{\mu\nu} x^\nu$.)

  Usual currents ⟶ constant ghosts ξ_μ , $\xi_{\mu\nu}$ which have ghost # 1 appear.

- The Maxwell theory has higher order currents $\epsilon\, \partial_\mu F^{\mu\nu} \approx 0$, which gives constant shifts.
  ⟶ a constant ghost η which has ghost # 2 appears.

- $S_{sym}[\xi] = \int dx \Big[\underbrace{-\eta^* \xi^\mu \xi^\nu \xi_{\mu\nu}}_{f^d{}_{abc}} \Big]$ and $S_{source}[\varphi, \xi] = \int dx \Big[A^{*\mu} \big(\partial_\nu A_\mu \, \xi^\nu + x^\nu \, \xi_{\mu\nu} \big) + C^* \big(\partial_\mu C \, \xi^\mu + \underbrace{x^\mu \, \xi_{\mu\nu} \xi^\mu}_{f^c{}_{ab}} + \eta \big) \Big]$

(Likewise, 2-form abelian gauge theory $\int dx\, F_{\mu\nu\rho} F^{\mu\nu\rho}$ gives more interesting result.)

Applications

Examples of μ_{total}-transfer : Lorentz sym of light-cone SFT

- We consider Witten's open SFT : $S_{bv}[\varphi] = \frac{1}{2}\omega\big(\varphi, Q_{BRST}\varphi\big) + \frac{1}{3}\omega\big(\varphi, \mu_2(\varphi,\varphi)\big)$.
- This is manifestly Lorentz covariant : $\delta\varphi = \epsilon_{\mu\nu}\int d\sigma\, X^{\mu}(\sigma)\frac{\delta}{\delta X^{\nu}(\sigma)}\varphi$, which gives $S_{total}[\varphi,\xi]$.
- BRST operator has a similarity transformation $Q_{BRST} = e^{-R}\Big(\overbrace{c_0\,L_0^{lightcone}}^{\mu_1'} - p^{+}\overbrace{\sum_{n\neq 0} c_{-c}\,a_n^{+}}^{\mu_1''}\Big)\,e^{R}$.
- This gives Kato-Ogawa's no-ghost theorem:

$$h^{long} \circlearrowright \Big(\text{covariant states}\,,\ \underbrace{Q_{BRST}}_{\mu_1'+\mu_1''}\Big) \ \underset{i_{long}}{\overset{p^{long}}{\rightleftarrows}}\ \Big(\underbrace{\text{lightcone states}\,,}_{\text{cohomology of }\hat{\mu}_1''}\ \underbrace{c_0\,L_0^{lightcone}}_{\mu_1'}\Big)$$

Applications

Examples of μ_{total}-transfer : Lorentz sym of light-cone SFT

- As a result of the perturbation,

$$\tilde{h}^{long} \circlearrowleft \Big(\text{covariant states}\,,\ \underbrace{Q_{BRST} + \mu_2}_{\mu_{bv}} + \mu_{source+sym} \Big) \underset{\tilde{i}_{long}}{\overset{\tilde{p}^{long}}{\rightleftarrows}} \Big(\underbrace{\text{lightcone states}}_{\text{cohomology of } \hat{\mu}_1''}\,,\ \underbrace{c_0\, L_0^{lightcone} + \mu_{int}^{lightcone}}_{\mu_{lightcone}} + \mu_{source+sym}^{lightcone} \Big)\,,$$

  we obtain a Witten-type light-cone SFT with nonlinear Lorentz invariance.

- Classical light-cone action :

$$S_{lightcone}[\varphi_{phys}] = \frac{1}{2}\omega\big(\varphi_{phys}, c_0\, L_0^{lightcone}\varphi_{phys}\big) + \sum_{n=2}^{\infty}\frac{1}{n+1}\omega\big(\varphi_{phys}, \mu_n^{lightcone}(\varphi_{phys}, \dots, \varphi_{phys})\big)$$

- Nonlinear Lorentz transformation :

$$\delta\,\varphi_{phys} = \mu_{Lorentz}[\varphi_{phys}, \xi] \equiv p^{long}\frac{1}{1-(\mu_{total} - c_0 L_0^{lc}\,)\,h^{long}}\,i_{long}\,\delta\varphi^{cov}$$

- Lorentz symmetry follows from $[\,\mu_{lc}\,, \mu_{Lorentz}\,] = 0$ and cyclic property of $\mu_{Lorentz}$.

Applications

Comments

- When we consider not Euclidean but topologically non-trivial space-time,

  μ_{total} takes different forms, unlike μ_{bv} .

- If you know nice toy models, please let me know.

  Any models are welcome : we can study them.

Thank you for your attention !

Please enjoy the YITP workshop

"Homotopy Alg. of QFT & Its Appl." .