Connected sum & Beilinson-Drinfeld algebras

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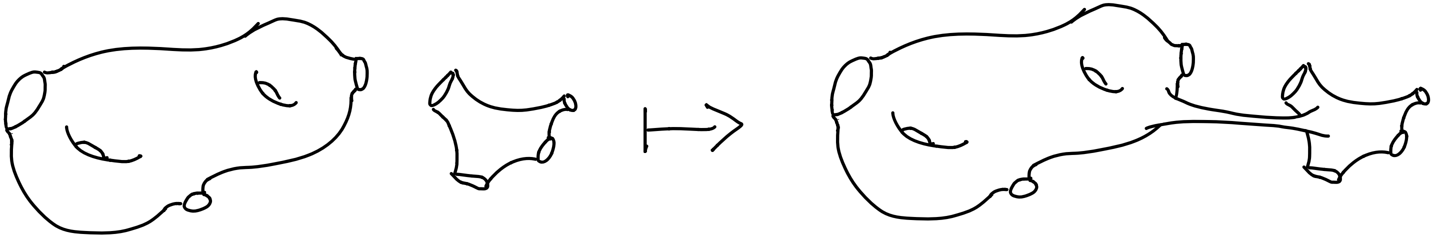
soon on arxiv

What this talk is about:

Algebraic consequence of the existence of the connected sum of surfaces



a Beilinson-Drinfeld algebra
[BD, Costello-Gwilliam]

Def $\underbrace{A, \cdot}_{\text{cga}, \mathbb{R}[[\hbar]] \text{ module}}, \underbrace{\Delta}_{\substack{\Delta^2 = 0 \\ \text{second order} \\ \text{odd}}}, \underbrace{\{\,,\,\}}_{\substack{\text{odd Poisson} \\ \text{(Gerstenhaber)}}}$ such that

$$\Delta(X \cdot Y) = \Delta X \cdot Y + (-1)^X X \cdot \Delta Y + (-1)^X \hbar \{X, Y\}$$

yes, that's the only difference

modular operads : encode possible Feynman diagrams

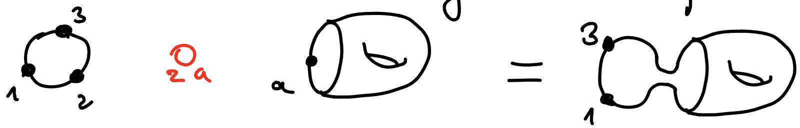
3 basic examples

1) **QC** : homeomorphism classes of orientable, connected surfaces with boundary (specify g and no. of boundary components)

operadic composition : sewing together boundaries $N = n$

2) **QO** : homeomorphism classes of orientable, connected surfaces with boundary and with marked points on the boundary

operadic composition: connecting marked points $N = n_{\text{marked pts}}$



3) End_V : $\left\{ \text{functions } V^{\otimes n} \longrightarrow \mathbb{C}[[\hbar]] \right\}_{n \geq 0}$, V has <u>odd</u> symplectic form ω

odd ⟶ operadic composition : contractions using $\omega^{-1} \in V \otimes V$ $N = n$

bigradin b (no. of things to glue, generalized ...)

SLOGAN: Each modular operad $\mathcal{P}$ determines a BV algebra

[Barannikov]

solutions of QME : algebras over $\mathcal{F}\mathcal{P}$

a bar construction of $\mathcal{P}$

[Zwiebach, Markl : QC case
Doubek-Jurčo-Münster : QO, QOC]

more precisely! for a modular operad $\mathcal{P}$ and (V,ω) as before

one defines

$$\mathrm{Fun}_{\mathcal{P}}(V) = \bigoplus_{N,G} \Big(\mathcal{P}(N,G) \otimes (V^*)^{\otimes N} \Big)^{S_N}$$

← invariants under permutations

which is canonically a (non-commutative) BV algebra, using the operad structure of $\mathcal{P}$ and ω.

our examples

$$\mathrm{Fun}_{QC}(V) \cong \mathrm{Sym}^\bullet(V^*)[[\hbar]]$$

: $\mathcal{P}(n,G)$ is the trivial rep of S_n

$$\Delta = \hbar \frac{\partial^2}{\partial\phi^i \partial\phi_i^+}$$

$$\mathrm{Fun}_{QO}(V) \cong \mathrm{Sym}^\bullet(\mathrm{Cyc}(V^*))[[\hbar]]$$

[Schedler, Barannikov, Hájek]

the product is missing !

one solution: disconnected surfaces : e^S more natural nc modular operads

[A. Schwarz, R. Kaufmann-Ward-Zúñiga]

Def: A connected sum for a modular operad $\mathcal{P}$ is a pair of maps

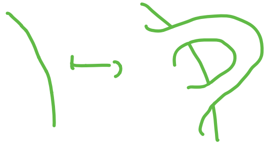
$$\#_2 : \mathcal{P}(N_1, G_1) \otimes \mathcal{P}(N_2, G_2) \longrightarrow \mathcal{P}(N_1+N_2, G_1+G_2+1)$$

$$\#_1 : \mathcal{P}(N, G) \longrightarrow \mathcal{P}(N, G+2) \quad \text{(adding a handle)}$$

satisfying some axioms obvious from pictures

Examples:

- the connected sum of surfaces QC, QO, $\#_1$:

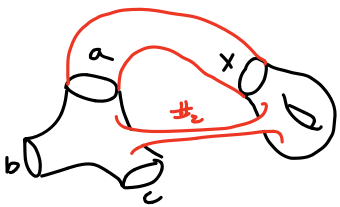


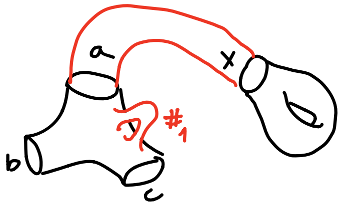
- or $(V^*)^{\otimes n} \otimes (V^*)^{\otimes m} \longrightarrow (V^*)^{\otimes n+m}$ for End_V

  $\#_1$: multiplication by $\hbar$

the most important axiom

$$\circ_{ax} \circ \#_2 = \#_1 \circ {}_a\circ_x$$





$\bigoplus_{n,G}\left(\mathbb{P}(n,G)\otimes(V^*)^{\otimes n}\right)^{S_n}$

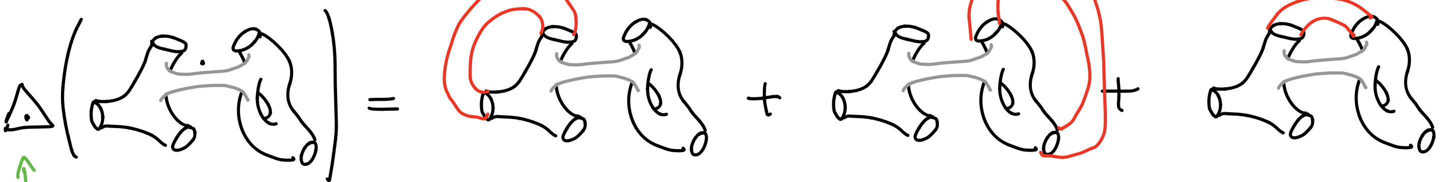
Theorem: If $\mathbb{P}$ has a connected sum, then $\mathrm{Fun}_{\mathbb{P}}(V)$ becomes a Beilinson-Drinfeld algebra. If $\#_1$ is injective, then the QME ($\Delta S + \frac{1}{2}\{S,S\} = 0$) for $S \in \mathrm{Fun}_{\mathbb{P}}(V)$ is equivalent to

$$\Delta e^{S/\hbar} = 0 \quad \text{in } \mathrm{Fun}_{\mathbb{P}}(V)((\hbar)) \big/ (\hbar X - \#_1 X)$$

In our examples: this recovers the obvious product on $\mathrm{Sym}(V^*)$ and $\mathrm{Sym}(\mathrm{Cyc}(V^*))$ with $\Delta = \hbar \frac{\partial^2}{\partial\phi\,\partial\phi^+}$

Why BD?

$$\Delta(X\cdot Y) = \Delta X\cdot Y + (-1)^X X\cdot\Delta Y + (-1)^X \hbar\{X,Y\}$$



sum over all possible connections

Final remarks

- funky numbering

$$G = 2g + \frac{n_{bnd}}{2} - 1 \qquad (QC)$$

$$G = 2g + n_{bnd} - 1 = 1 - \chi_{\Sigma} \quad (QO)$$

come from open-closed $G = \alpha g + \frac{\alpha}{2} n_{open} + \frac{\alpha - 1}{2} n_{closed} + \frac{\alpha - 2}{4} n_{marked\,pts} + 1 - \alpha$

do you recognize these? and $\alpha = 2$ s.t.  has $G=0$

(sorry, $\alpha = 1$ was a mistake during the talk)

- more examples?

$QC = \mathrm{Mod}\,(\mathrm{Com})$

$QO = \mathrm{Mod}\,(\mathrm{Ass})$

$Q? = \mathrm{Mod}\,(\mathrm{Lie})$ ⟵ what is it? does it admit a connected sum?

- WIP: $e^{W/\hbar} = P(1 - \Delta H)^{-1} e^{S_{int}/\hbar}$: the minimal model.

Thank you for your attention!