A presymplectic BV formalism for the abelian $\mathscr{N} = (2,0)$ multiplet

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This talk is based on joint work with Brian R. Williams, building on some ideas I thought about together with Johannes Walcher and Richard Eager.

I'm also grateful to Martin Cederwall and Kevin Costello (among others) for conversations.

A rough outline:

- Some unconventional reminders about classical BV theories, and about supersymmetry
- A machine for producing classical (BRST or BV) supermultiplets up to homotopy
- An unusual output in six dimensions and its presymplectic interpretation
- Consistency checks, nonperturbative generalization, electric/magnetic duality

We applied our formalism to compute all twists of the abelian tensor multiplet, though I won't discuss that in detail today.

Underlying big idea:

A classical field theory consists of a sheaf: the solutions to the equations of motion, considered up to gauge equivalence.

A *resolution* replaces a (complicated) sheaf by a chain complex $\mathscr{E}^\bullet$ of simpler ones: locally free sheaves, for example.

Free field theories can be resolved by chain complexes of vector bundles. The BRST formalism (Chevalley–Eilenberg complex) resolves the quotient by gauge symmetries; the BV formalism resolves the critical locus of the action functional.

A field theory admits an action of the automorphisms of the manifold where it is defined. In particular, a field theory on affine space admits an action of the affine (Poincaré) group.

In a *supersymmetric* field theory, a larger super Lie algebra containing the affine algebra acts. In fact, it admits a $\mathbb{Z}$ grading.

In six dimensions, one such (complexified) algebra looks as follows:

$$\mathfrak{p} = \mathfrak{so}(V) \oplus \mathfrak{sp}(2) \oplus (S_+ \otimes R_2)[-1] \oplus V[-2]. \tag{1}$$

Here $V \cong \mathbb{C}^6$ are the (complex) translations, S_+ is the chiral spinor representation of $\mathfrak{so}(6)$, and $R_2 \cong (\mathbb{C}^4, \omega)$ the defining representation of $\mathfrak{sp}(2)$. The interesting bracket uses $\wedge^2 S_+ \cong V$, tensored with ω.

On the BRST or BV theory, this action is witnessed by an L_∞-module structure. (This module structure is rarely strict.) An L_∞ module structure can be defined on the BRST theory only by using auxiliary fields; these are not necessary in the BV theory, and are thought not to exist in general.

The *pure spinor superfield formalism* uses the space $\mathrm{MC}(\mathfrak{p})$ of Maurer–Cartan elements in $\mathfrak{p}$ to provide "bigger and better" resolutions of supermultiplets:

- Everything is resolved locally freely, not just over affine space, but over the *superspace* $\mathbb{A}(\mathfrak{p}_+)$.
- Correspondingly, the L_∞ $\mathfrak{p}$-module structure is *strict* (and trivial to write down).
- In certain examples (gauge multiplets), the resolution is in fact *multiplicative* (by a super-commutative differential graded algebra over functions on $\mathbb{A}(\mathfrak{p}_+)$).
- Every equivariant sheaf over $\mathrm{MC}(\mathfrak{p})$ produces a supermultiplet. (It may be a BRST or BV multiplet, depending on context.)
- There is a spectral sequence whose E_1 page consists of the component fields of the multiplet.

What happens in our example?

We use the structure sheaf $\mathscr{O}_{\mathrm{MC}(\mathfrak{p})}$. The corresponding super-commutative differential graded algebra is

$$A^\bullet = \left(C^\infty(\mathbb{A}(\mathfrak{p}_+)) \otimes_{\mathbb{C}} \mathscr{O}_{\mathrm{MC}(\mathfrak{p})},\ D = \lambda^a D_a\right). \qquad (2)$$

Here $\lambda^a \in \mathfrak{p}_1^\vee$ are the linear coordinate functions that generate $\mathscr{O}_{\mathrm{MC}(\mathfrak{p})}$, and D_a are the odd vector fields defining the right action of $\mathfrak{p}_1$ on $\mathbb{A}(\mathfrak{p}_+)$.

Supersymmetry acts strictly via the left action on $\mathbb{A}(\mathfrak{p}_+)$. But note that $\mathfrak{p}_0$ acts diagonally.

The resulting multiplet (in components):

$$\begin{array}{cccccccc}
-2 & & -1 & & 0 & & 1 \\
\hline
\Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{\pi_+ d} & \Omega^3_+ \\
& & & & \Omega^0 \otimes_{\mathbb{C}} W & \xrightarrow{d \star d} & \Omega^6 \otimes W \\
& & & & S_- \otimes R_2 & \xrightarrow{\partial\!\!\!/} & S_+ \otimes R_2
\end{array}$$

W is the five-dimensional representation of $\mathfrak{sp}(2)$. (This cohomology was first studied by Cederwall, Nilsson, and Tsimpis; they were stymied by the lack of an obvious BV structure.)

This is clearly resolving the correct object, with its linear equations of motion. Furthermore, it has an explicit L_∞ $\mathfrak{p}$-module structure with only ρ_2-type correction terms (worked out in our paper).

To interpret it as some kind of BV theory, we need to make sense of the pairing.

Idea: Think of the self-dual two-form as obtained from a nondegenerate two-form via a *self-duality constraint*.

A simple motivating example:

The underlying sheaf of the classical, two-dimensional chiral boson just consists of holomorphic functions. This has an obvious smooth resolution by the Dolbeault complex:

$$\mathscr{O}_{\mathrm{hol}}(\mathbb{C}) \cong \left(\Omega^{0,0} \xrightarrow{\bar{\partial}} \Omega^{0,1} \right). \tag{3}$$

(In fact, this is the same as the self-dual complex.) It admits a map to the BV theory of the free boson:

$$\begin{array}{ccc} \Omega^{0,0} & \xrightarrow{\partial\bar{\partial}} & \Omega^{1,1} \\ {\scriptstyle \mathrm{id}}\uparrow & & \uparrow{\scriptstyle \partial} \\ \Omega^{0,0} & \xrightarrow{\bar{\partial}} & \Omega^{0,1} \end{array} \tag{4}$$

(Shifted) symplectic structures do not pull back, but (shifted) *presymplectic* structures do! We can thus define a pairing on the Dolbeault complex by the formula

$$\omega(\alpha, \alpha') = \int \alpha \wedge \partial \alpha'. \tag{5}$$

In fact, this is "close" to being nondegenerate; viewing ω as a skew map from $\mathscr{E}$ to $\mathscr{E}^![-1]$, its cone is the de Rham complex of $\mathbb{C}$—so that the kernel just consists of constant functions.

In the usual symplectic world, there's an equivalence

$$\{\text{functions/constants}\} \leftrightarrow \{\text{Hamiltonian vector fields}\}, \qquad (6)$$

coming from the correspondence defined by the *Hamiltonian pairs*

$$\mathrm{Ham}(M,\omega) = \{(X,f) : i_X\omega = df\} \subseteq \mathrm{Vect}(M) \oplus C^\infty(M). \qquad (7)$$

Giving the quadratic action functional S and the linearized BV differential $\{S,\cdot\}$ are thus equivalent.

In the presymplectic setting, this correspondence still makes sense, but no longer includes all functions; there is a notion of *Hamiltonian observable*. We get the BV differential from the pure spinor setup for free.

We can similarly define a presymplectic BV pairing on the self-dual complex by mapping it into the complex representing the nondegenerate theory of abelian two-forms:

$$\begin{array}{ccccccccccc}
\Omega^0 & \longrightarrow & \Omega^1 & \longrightarrow & \Omega^2 & \xrightarrow{d\star d} & \Omega^4 & \longrightarrow & \Omega^5 & \longrightarrow & \Omega^6 \\
{\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle d}\uparrow & & & & \\
\Omega^0 & \longrightarrow & \Omega^1 & \longrightarrow & \Omega^2 & \xrightarrow{\pi_+ d} & \Omega^3_+ & & & &
\end{array}$$

As above, the pairing is almost nondegenerate, up to a single copy of constant functions in the kernel.

Armed with this formalism, we can start computing things. For the experts: both twists of the abelian multiplet are now known. The holomorphic twist is

$$\Omega^{\leq 1,*}(X)[2] \oplus \left(\Omega^{0,*}(X) \otimes K^{1/2} \otimes \Pi R_1[1]\right), \quad \omega = \partial + \mathrm{id}; \tag{8}$$

it is defined on any complex threefold X. Similarly, there is a non-minimal twist, defined on the product of a Riemann surface and a smooth four-manifold. It consists of the chiral boson with values in the shift by two of the de Rham cohomology of the four-manifold:

$$\Omega^{0,*}(\Sigma) \otimes_{\mathbb{C}} \Omega^*_{\mathrm{dR}}(M^4)[2], \quad \omega = \partial_\Sigma \otimes \mathrm{id}_{M^4}\,. \tag{9}$$

Does this theory pass the usual consistency checks?

Yes—and in a way that teaches us new things.

The main test is dimensional reduction to 5d $\mathscr{N} = 2$ Yang–Mills theory. The scalars and fermions are obvious, so it reduces to considering dimensional reduction of the self-dual complex, and getting the pairing right.

Upon reduction to five dimensions, our complex $\Omega_+^{\leq 3}(\mathbb{R}^6)$ becomes isomorphic to

$$\begin{array}{ccccccc}
-2 & & -1 & & 0 & & 1 \\
\hline
\Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 \\
 & & \Omega^0 & \xrightarrow{d} & \Omega^1 & \overset{\star d}{\nearrow} &
\end{array}$$

If $\beta \in \Omega^{\leq 3}[2]$ and $\alpha \in \Omega^{\leq 1}[1]$, the pairing is

$$\omega(\alpha, \beta) = \int_{\mathbb{R}^5} \alpha \wedge d\beta.$$

This maps to the standard perturbative BV complex for five-dimensional abelian Yang–Mills theory:

$$
\begin{array}{ccccccccc}
-2 & & -1 & & 0 & & 1 & & 2 \\
\hline
\Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 & & \\
 & & & & & \nearrow_{\star d} & & & \\
 & & \Omega^0 & \xrightarrow{d} & \Omega^1 & & \downarrow d & & \\
 & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & & & \\
 & & \Omega^0 & \longrightarrow & \Omega^1 & \xrightarrow{d\star d} & \Omega^4 & \longrightarrow & \Omega^5
\end{array}
$$

It is clear that the map is compatible with the (pre)symplectic structures. Again, the cone of this map is a single copy of the constant sheaf.

What's behind the annoying constant sheaves everywhere?

You might be tempted to think that these are mismatches, or should be corrected somehow. In fact, they are there for a good reason; in a sense, the failure of our arguments to work perturbatively, on the nose, is a sign that Dirac quantization is important.

It helps to spell out, in words, what sort of object our dimensionally reduced complex is describing: a pair of a one-form α and a two-form β, up to the usual gauge invariances, and *subject to the single condition, that*

$$F_\alpha = \star F_\beta.$$

This suggests a clear nonperturbative generalization, inspired by Deligne cohomology: we should consider the complex

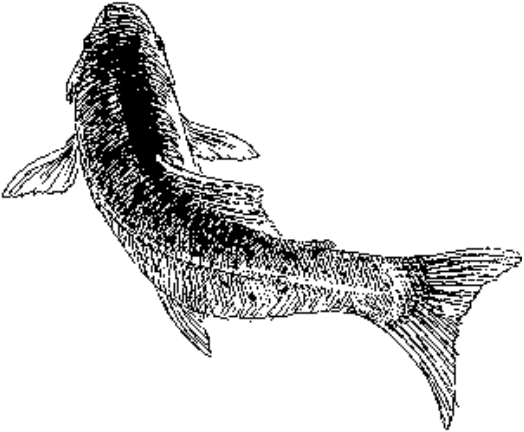
$$\underline{\mathbb{Z}} \longrightarrow \Omega^0 \longrightarrow \Omega^1 \longrightarrow \Omega^2 \xrightarrow{\pi_+ d} \Omega^3_+.$$

Its dimensional reduction then consists of a *pair* of smooth Deligne cohomology groups, representing the electric and magnetic connections, and subject to the single constraint that the curvatures are related by Hodge duality. In a precise sense, the electric and magnetic degrees of freedom are one another's antifields.

Note that interpreting this sheaf as a BV theory is only possible in the presymplectic world!

This sheds light both on the absence of a continuous coupling constant in the 6d theory, and on the inverse dependence of the 5d coupling constant on the compactification radius.

There are many more things to be done. The obvious $10,000 question is about using this formalism to study interactions. $A^\bullet$, of course, wants to be tensored with some kind of Lie 2-algebra…



Thanks for your attention!

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