EL_∞-algebras: Definition and Applications

Christian Saemann
Maxwell Institute and
School of Mathematical and Computer Sciences
Heriot–Watt University, Edinburgh

Workshop
"Homotopy Algebra of Quantum Field Theory and Its Application",
Yukawa Institute, 29.3.2021

Based on:

- arXiv:2104.????? with Leron Borsten and Hyungrok Kim


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Motivation: Five Questions

A first question

What is the algebraic structure underlying Courant algebroids?

Answers in the literature:

- Roytenberg (2002):
  An (exact) Courant algebroid is the symplectic dg-manifold
  $$\mathcal{V}_2 = T^*[2]T[1]M\ ,\quad \omega = \mathrm{d}x^\mu \wedge \mathrm{d}p_\mu + \mathrm{d}\xi^\mu \wedge \mathrm{d}\zeta^\mu\ ,$$
  $$Q = \{S, -\}\ ,\quad S = \xi^\mu p_\mu$$
  for M some manifold. Dorfman and Courant brackets:
  $$[X,Y]_D = \{QX, Y\}\ ,\quad [X,Y]_C = \tfrac{1}{2}\big(\{QX,Y\} - \{QY,X\}\big)$$
- cf. Rogers (2011):
  - $[-,-]_C$ part of L_∞-algebra
  - $[-,-]_D$ part of dg-Leibniz algebra.
- Is there more to it?

Relevance of first question

This may be seem as a niche question, but:

- Courant algebroids underlie Hitchin's Generalized Geometry
- Application in supergravity: Generalized tangent bundle
- All generalized tangent bundles are symplectic L_∞-algebroids
- Dorfman bracket structure relevant in tensor hierarchies
- Currently relevant: Double and Exceptional Field Theory.

In order to further understand supergravity:
understand symplectic L_∞-algebroids!

What is the alg. structure underlying multisymplectic manifolds?

Answers in the literature:

- Multisymplectic/p-plectic manifold (M,ω):
  $$\omega \in \Omega^{p+1}(M)~,\quad \mathrm{d}\omega = 0~,\quad \iota_X\omega = 0 \Leftrightarrow X = 0$$
  comes with Hamiltonian $p-1$-forms α with X_α:
  $$\iota_{X_\alpha}\omega = \mathrm{d}\alpha$$
  This leads to brackets:
  $$[\alpha,\beta]_h = \mathcal{L}_{X_\alpha}\beta~,\quad [\alpha,\beta]_s = \iota_{X_\alpha}\iota_{X_\beta}\omega$$
- cf. Baez, Hoffnung, Rogers (2008), Rogers (2011):
  - $[\alpha,\beta]_s$ is part of L_∞-algebra
  - $[\alpha,\beta]_h$ for $p=2$ is part of hemistrict Lie 2-algebra.

Relation to symplectic L_∞-algebroids:

- The L_∞-algebra of (M,ω) sits inside the L_∞-algebra of ω-twisted $T^*[2]T[1]M$.

Relevance of second question

Again, it sounds like a niche question, but:

- Higher version of function algebras over symplectic manifolds.
- Objects to be quantized in higher geometric quantization.
- Crucial for emergent geometry from noncommutative spaces:
  - Traditional NC geometry: mostly Kähler manifolds
  - More general: Nambu-Poisson manifold
  - Physical arguments for these.
  - More general: multisymplectic manifolds
  - Abstract Nonsense suggests these
- Also: gauge algebras for M2-brane models (fuzzy funnels)

In order to further understand emergent spacetimes
as well as M2-brane models:
understand multisymplectic geometry!

A third question

How to construct "good" curvatures for non-abelian gauge potentials in presence of B-field?

Answers in the literature:

- Bergshoeff et al. (1982), Chapline et al. (1983): Use Chern-Simons terms:

$$F = \mathrm{d}A + \tfrac{1}{2}[A,A]~, \quad H = \mathrm{d}B + (A, \mathrm{d}A) + \tfrac{1}{3}(A,[A,A])$$

- This is at odds with the "conventional" non-abelian gerbes:

$$F = \mathrm{d}A + \tfrac{1}{2}[A,A]~, \quad H = \mathrm{d}B - \tfrac{1}{3}(A,[A,A])$$

- Sati, Schreiber (2009): Solution: twist definition of curvatures

$$F = \mathrm{d}A + \tfrac{1}{2}[A,A]~, \quad H = \mathrm{d}B + (A,F) - \tfrac{1}{3}(A,[A,A])$$

- Where does $(-,-)$ come from?

Relevance of third question

Observation:

- Traditional non-abelian gerbes: fake curvature $\mathcal{F} = 0$
- All naive definitions problematic unless $\mathcal{F} = 0$:
  - Parallel transport
  - Topological invariants
  - Higher non-flat theories
- Then: can gauge away non-abelian parts.

Need this geometry:

- Tensor hierarchies of gauge supergravity
- Heterotic supergravity
- 6d superconformal field theories

Fourth question (mathematics)

What is a small cofibrant replacement for the operad $\mathcal{L}ie$ when working over finite characteristic?

Just briefly:

- Operads underlie and determined all kinds of algebras
- Cofibrant replacement: replace by weakly equivalent object with nice homotopy properties
- Over characteristic 0, can replace $\mathcal{L}ie$ by $\mathcal{L}ie_\infty$, the cobar construction applied to the Koszul dual cooperad $\mathcal{L}ie^{¡}$.
- Algebras over $\mathcal{L}ie_\infty$ are just the usual L_∞-algebras

Answers in the literature:

- Higher cobar-bar adjunction, but result too big.
- Dehling (2017): Step-by-step construction
- Explicit for up to degree 3 ("Weak Lie 3-algebras")

Fifth question (mathematics)

Asked by Loday (1993):

What do Leibniz algebras integrate to ("coquecigrue problem")?

Recall:

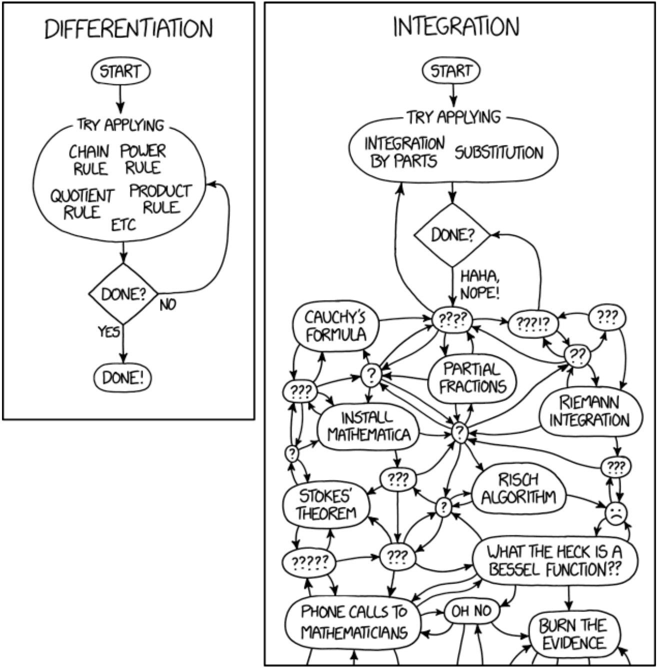
- Leibniz algebra $\mathfrak{a}$:

$$\{-,-\} \; : \; \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{a} \,,$$
$$\{a,\{b,c\}\} = \{\{a,b\},c\} + \{b,\{a,c\}\}$$

Answer in the Literature:

- Kinyon (2004): Lie racks

NB:
Coquecigrue: an imaginary creature regarded as an embodiment of absolute absurdity.


DIFFERENTIATION
START
TRY APPLYING
CHAIN RULE
POWER RULE
QUOTIENT RULE
PRODUCT RULE
ETC
DONE?
NO
YES
DONE!
INTEGRATION
START
TRY APPLYING
INTEGRATION BY PARTS
SUBSTITUTION
DONE?
HAHA, NOPE!
CAUCHY'S FORMULA
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PARTIAL FRACTIONS
RIEMANN INTEGRATION
INSTALL MATHEMATICA
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STOKES' THEOREM
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WHAT THE HECK IS A BESSEL FUNCTION??
PHONE CALLS TO MATHEMATICIANS
OH NO
BURN THE EVIDENCE

Relevance of fourth and fifth questions

If we understood algebras properly, we should be able to answer such questions, at least in principle, by Abstract Nonsense.

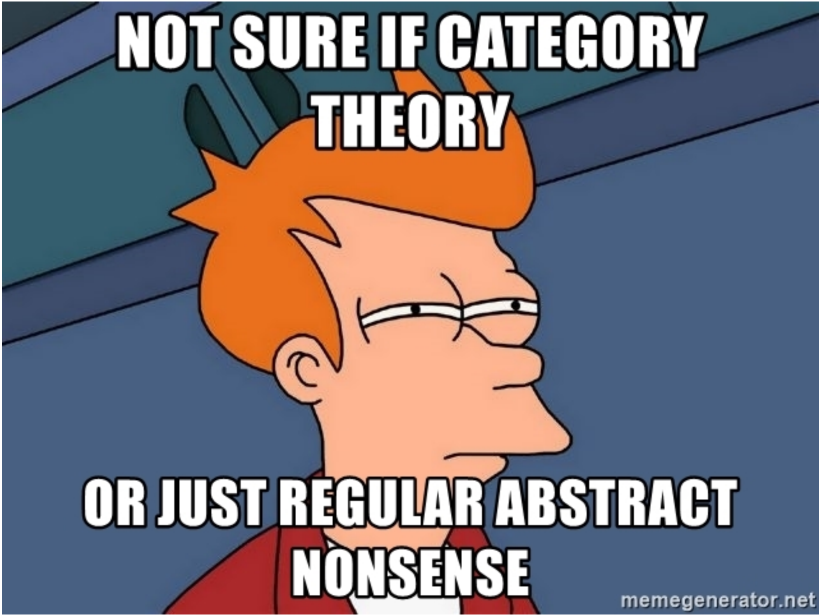
Summary

All these questions:

1) Algebraic structure underlying symplectic L_∞-algebroids?
2) Algebraic structure underlying multisymplectic manifolds?
3) Algebraic structure underlying higher curvature forms?
4) Cofibrant replacement of $\mathcal{L}ie$?
5) How do you integrate Leibniz algebras?

have a simple, unifying answer:

EL_∞-algebras


NOT SURE IF CATEGORY THEORY
OR JUST REGULAR ABSTRACT NONSENSE
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Starting point of construction

Hint: Start with 2-plectic manifolds and hemistrict Lie algebras.

- Math. Objects: Stuff + Structure + Structure Relations
- E.g.: Lie algebra: Vector space V with Lie bracket $[\cdot,\cdot]$:

  $[v,w]=-[w,v]$ and $[u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0$
- Internal categorification:
  - "stuff" $\rightarrow$ (small) category, objects and morphisms of "stuff"
  - "structure" $\rightarrow$ functors
  - structure relations hold "up to isomorphisms"
  - functors satisfy coherence axioms
- Weak Lie 2-algebra is a category $\mathcal{L}$: Roytenberg, 2007
  - objects and morphisms form vector spaces
  - endowed with functor $[\cdot,\cdot]:\mathcal{L}\times\mathcal{L}\rightarrow\mathcal{L}$
  - natural trafos: $\mathsf{Alt}:[v,w]\Rightarrow-[w,v]$
    $\mathsf{Jac}:[u,[v,w]]+[v,[w,u]]\Rightarrow-[w,[u,v]]$

Lie 2-algebras are equivalent to differential graded algebras $\mathfrak{L}$ with

$$\varepsilon_2 : \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L}\ , \quad |\varepsilon_2| = 0\ , \quad \mathsf{alt} : \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L}\ , \quad |\mathsf{alt}| = -1$$

Generalize, preserving differential compatibility: $h\mathcal{L}ie$-algebras

$h\mathcal{L}ie$-algebras

Graded vector space $\mathfrak{L}$ with

$$\varepsilon_1 \ : \ \mathfrak{L} \to \mathfrak{L}\ , \quad |\varepsilon_1| = 1\ , \quad \varepsilon_2^i \ : \ \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L}\ , \quad |\varepsilon_2^i| = -i$$

such that

$$\begin{aligned}
\varepsilon_1(\varepsilon_1(x_1)) &= 0\ ,\\
\varepsilon_1(\varepsilon_2^i(x_1,x_2)) &= \pm\varepsilon_2^i(\varepsilon_1(x_1),x_2) \pm \varepsilon_2^i(x_1,\varepsilon_1(x_2)) + \varepsilon_2^{i-1}(x_1,x_2) \mp \varepsilon_2^{i-1}(x_2,x_1)\\
\varepsilon_2^i(\varepsilon_2^i(x_1,x_2),x_3) &= \pm\varepsilon_2^i(x_1,\varepsilon_2^i(x_2,x_3)) \mp \varepsilon_2^i(x_2,\varepsilon_2^i(x_1,x_3)) \mp \varepsilon_2^{i+1}(x_2,\varepsilon_2^{i-1}(x_3,x_1))\\
\varepsilon_2^j(\varepsilon_2^i(x_1,x_2),x_3) &= \pm\varepsilon_2^{i+1}(x_2,\varepsilon_2^{j-1}(x_3,x_1))\\
\varepsilon_2^i(\varepsilon_2^j(x_1,x_2),x_3) &= \pm\varepsilon_2^j(x_1,\varepsilon_2^i(x_2,x_3)) \mp \varepsilon_2^i(x_2,\varepsilon_2^j(x_1,x_3)) \pm \varepsilon_2^{i+1}(x_3,\varepsilon_2^{j-1}(x_1,x_2))
\end{aligned}$$

Generalizes hemistrict Lie 2-algs and specializes dg-Leibniz algs.

Recall:

- Koszul dual of $\mathcal{L}ie$ is $\mathcal{C}om$
- Therefore:

$$\begin{array}{ccc} L_\infty\text{-algebras} & \leftrightarrow & \text{dg-com algebras} \\ \mu_i & \leftrightarrow & Q \end{array}$$

Here:

- Koszul dual of $h\mathcal{L}ie$ is $\mathcal{E}ilh$
- $\mathcal{E}ilh$ has products $\oslash_i$ of degree i
- Relations dual to the one shown before (generalised Zinbiel)

EL_∞-algebras

EL_∞-algebra are homotopy $h\mathcal{L}ie$-algebras.
That is, graded vector space $\mathfrak{L}$ with higher products

$$\begin{aligned}
\varepsilon_1 &: \mathfrak{L} \to \mathfrak{L} , \quad |\varepsilon_1| = 1 , \\
\varepsilon_2^i &: \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L} , \quad |\varepsilon_2^i| = -i \\
\varepsilon_3^{ij} &: \mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L} , \quad |\varepsilon_3^{ij}| = -i - j , \\
&\vdots \qquad \vdots
\end{aligned}$$

such that

$$\begin{aligned}
\varepsilon_1(\varepsilon_1(x_1)) &= 0 , \\
\varepsilon_1(\varepsilon_2^i(x_1, x_2)) &= \pm\varepsilon_2^i(\varepsilon_1(x_1), x_2) \pm \varepsilon_2^i(x_1, \varepsilon_1(x_2)) + \varepsilon_2^{i-1}(x_1, x_2) \mp \varepsilon_2^{i-1}(x_2, x_1) \\
&\vdots \qquad \vdots
\end{aligned}$$

amounting to $Q^2 = 0$ in the corresponding dual $\mathcal{E}ilh$-algebra.

Note: if $\varepsilon_k^I = 0$ for $I \neq (0, 0, \ldots, 0)$, then this is L_∞-algebra.

Properties of EL_∞-algebras

They generalize:

- (dg) Lie algebras
- L_∞-algebras
- Roytenberg's hemistrict and semistrict Lie 2-algebras
- Dehlings weak Lie 3-algebras
- $\Rightarrow$ They are weak Lie ∞-algebras

They specialize:

- Leibniz algebras
- homotopy Leibniz algebras

Properties:

- Modified homotopy transfer (modified tensor trick)
- Minimal model and strictification theorems
- EL_∞-algebras antisymmetrize to L_∞-algebras
- An L_∞-algebras in each quasi-isomorphism class

1) Algebraic structure underlying symplectic L_∞-algebroids?

- Note: symplectic L_∞-algebroids come with dg-Lie algebra
- Theorem: Any dg-Lie algebra induces an $h\mathcal{L}ie$-algebra.
- This $h\mathcal{L}ie$-structure produces Dorfman bracket.
- Antisymmetrization produces Courant bracket.

2) Algebraic structure underlying multisymplectic manifolds?

- Note: multisymplectic manifold come with dg-Lie algebra extends the Lie algebra of Hamiltonian vector fields

$$\mathsf{L}(M,\varpi) = \left(\underbrace{\Omega^0(M)}_{\mathsf{L}(M,\varpi)_{-n}} \xrightarrow{\mathrm{d}} \underbrace{\Omega^1(M)}_{\mathsf{L}(M,\varpi)_{1-n}} \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \underbrace{\Omega^{n-1}_{\mathrm{Ham}}(M)}_{\mathsf{L}(M,\varpi)_{-1}} \xrightarrow{\delta} \underbrace{\mathfrak{X}(M)}_{\mathsf{L}(M,\varpi)_0} \right)$$

- Induced $h\mathcal{L}ie$-bracket yields hemistrict brackets includes dg-Leibniz structure observed before.
- Antisymmetrization recovers L_∞-algebra structure

Answers to the questions II

3) Algebraic structure underlying higher curvature forms?
   - $\exists$ homotopy Maurer–Cartan theory, but not much richer.
   - Need to extend to Weil algebras to define curvatures
   - In all examples: adjusted Weil algebras exist.
   - EL_∞-algebras allow for "nice, good" curvatures
   - Conjecture: in every quasi-isomorphism class, there is one representant with an adjusted Weil algebra.
   - $\Rightarrow$ Good curvatures for any higher gauge algebra.

NB:

- Literature: all kinds of variations of Leibniz algebras.
- Some get close to special cases of $h\mathcal{L}ie$.
- Gauge algebras: infinitesimal symmetries, need to integrate!
- EL_∞-algebras do this by Abstract Nonsense.

Answers to the questions III

4) Cofibrant replacement of $\mathcal{L}ie$?
   - Yes: $h\mathcal{L}ie_\infty$.
5) How do you integrate Leibniz algebras?
   - Theorem: Any Leibniz algebra is an $h\mathcal{L}ie$-algebra.
   - Any $h\mathcal{L}ie$-algebra is a Lie 2-algebra and thus integrable by Abstract Nonsense.
   - Conjecture: Any homotopy Leibniz algebra is an EL_∞-algebra, and thus integrable by Abstract Nonsense.

Summary

- Constructed EL_∞-algebras, special kind of homotopy algebra
- Generalize L_∞-algebras
- They are higher Lie algebras, describe infinitesimal symmetries
- Appear in many contexts in maths/physics. In particular:
  - Higher gauge theory
  - Tensor hierarchies of supergravity

Thank You!