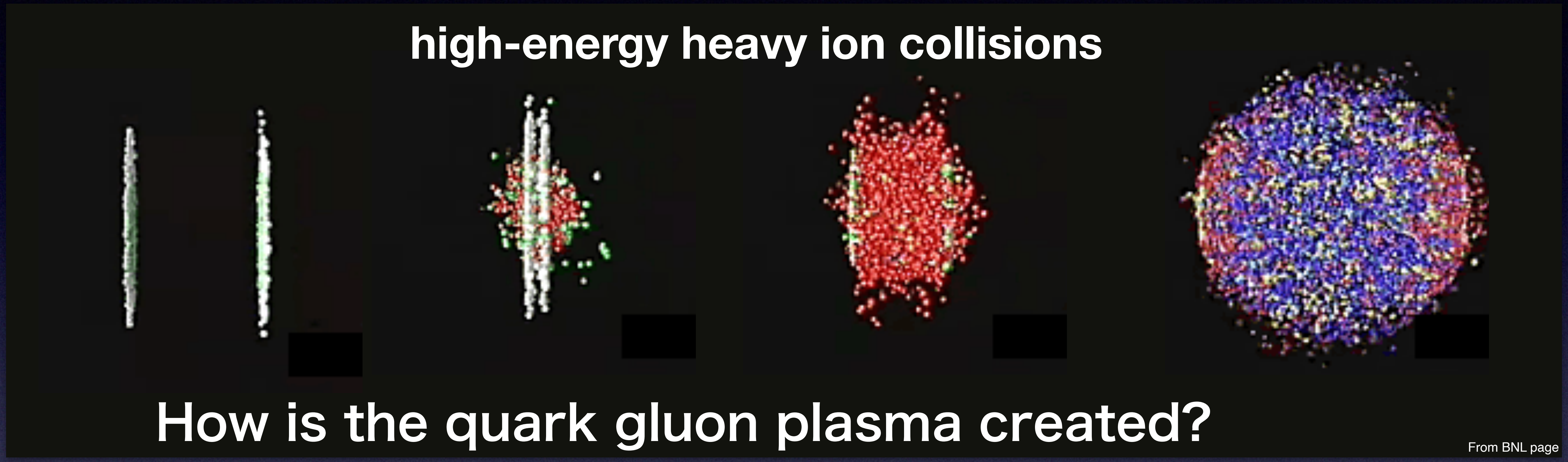
Hamiltonian lattice gauge theory and application to nonequilibrium and dense QCD

Yoshimasa Hidaka
(KEK)

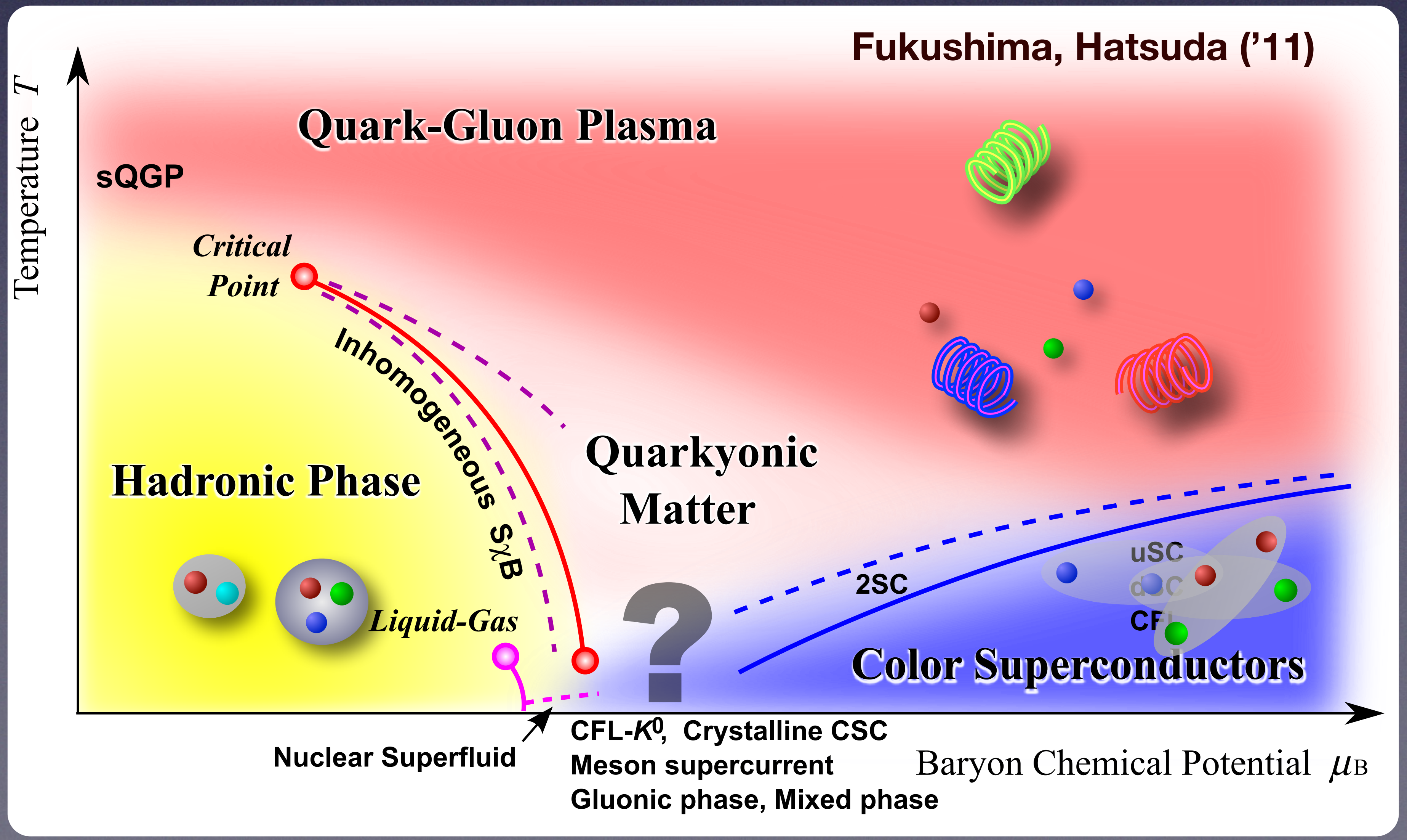
Based on
Hayata, YH, PRD 103 (2021) , 094502, JHEP 09 (2023) 123; JHEP 09 (2023) 126
Hayata, YH, Kikuchi PRD 104 (2021) 7, 074518
Hayata, YH, Nishimura, 2311.11643

Motivation
Big problem in QCD

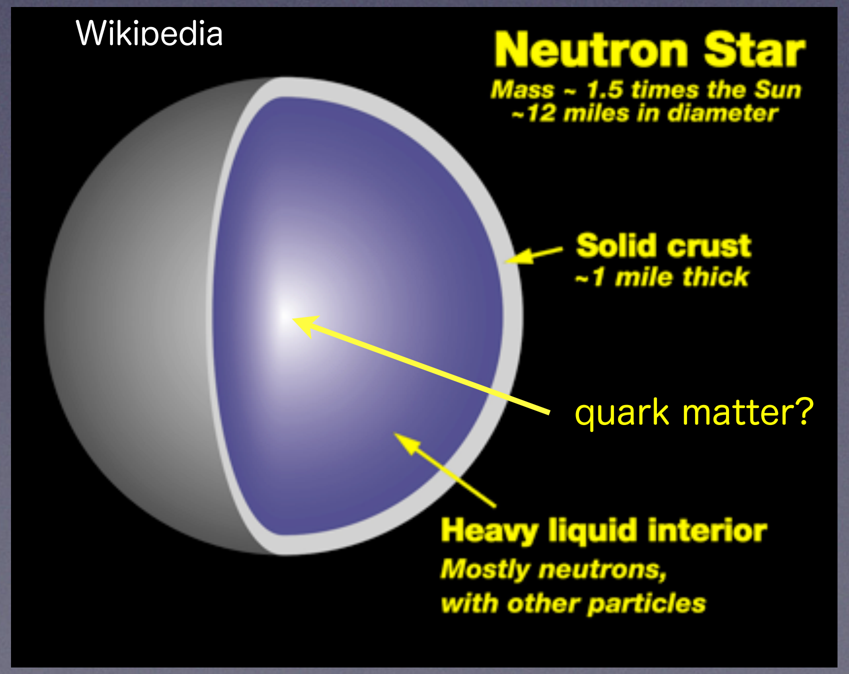
manybody dynamics of QCD



Dense QCD



What phases are realized in the interior of a neutron star?



Difficulty

Sign problem: Difficulties in first-principles calculations based on importance sampling

$$\langle O \rangle = \int \mathscr{D}A \, \det(D+m) e^{iS} O$$

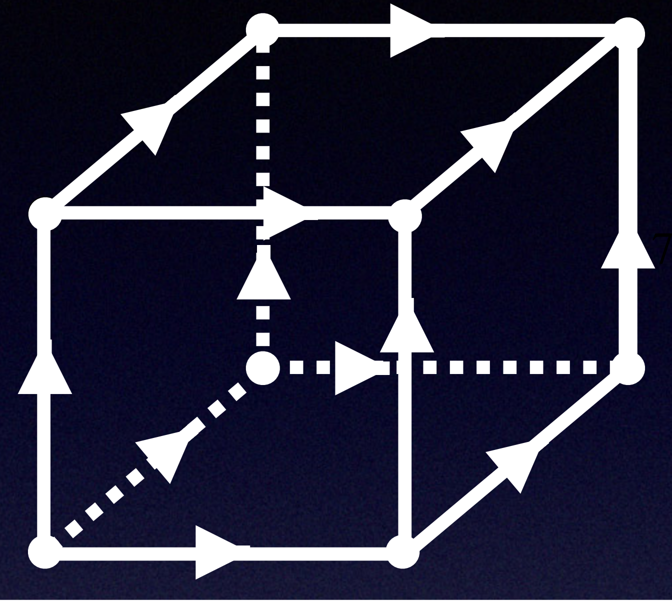
In real-time, finite-density problems, the weight is complex

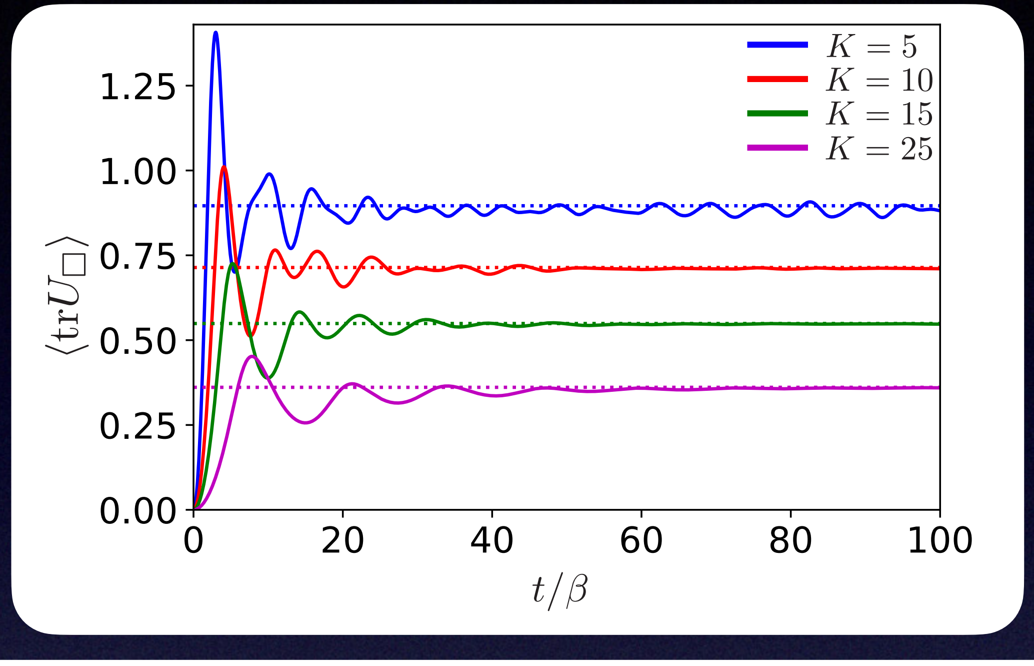
$$\not\approx \frac{1}{N} \sum_j O_j$$

Hamiltonian approach

Directly solve Schrodinger equation to avoid sign problem

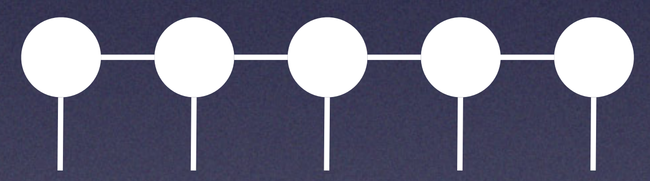
Smaller systems can be simulated directly



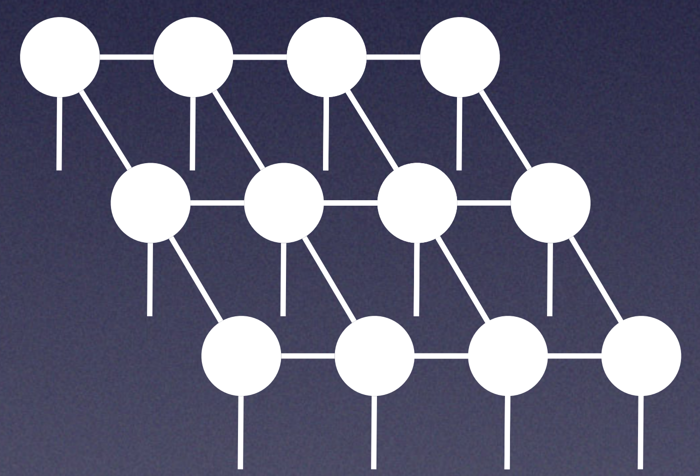


Tensor Networks

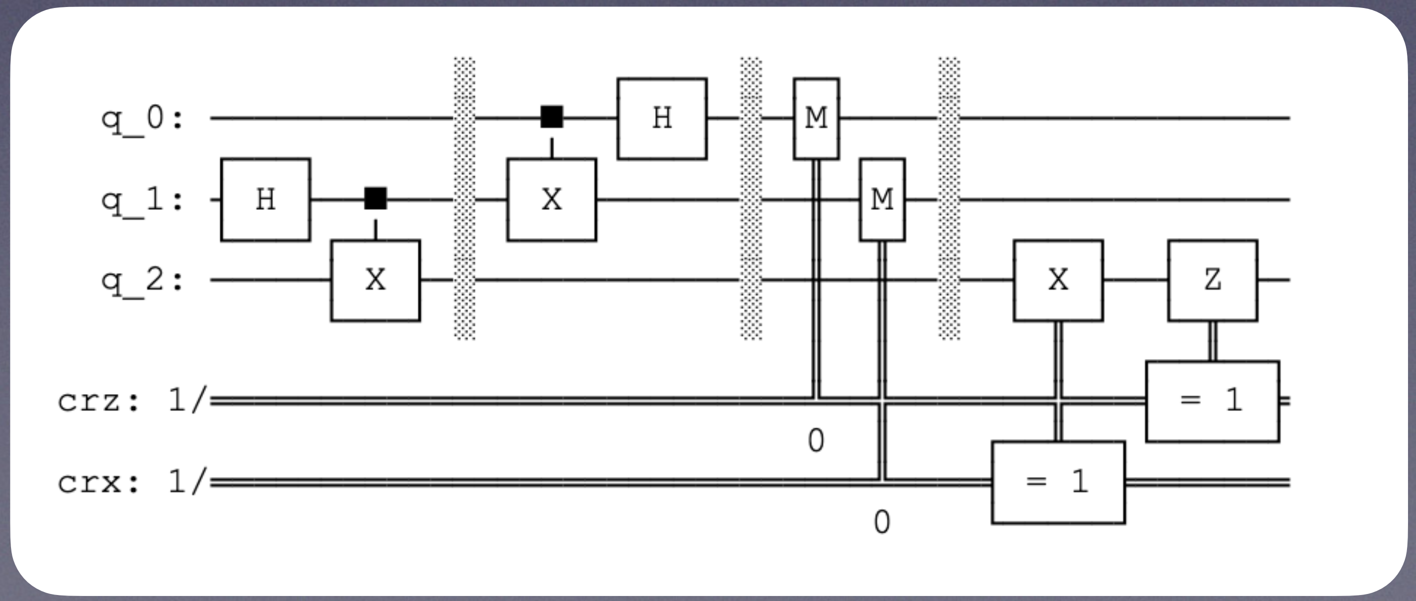
MPS



PEPS



Quantum simulation



Difficulty of Hamiltonian gauge theory

Infinite degrees of freedom

Link variable is continuous (regularization required)

$U \in \mathrm{SU}(N)$

continuous

What approximation is compatible with gauge symmetry?

Large gauge redundancy

$$\dim \mathscr{H}_{\mathrm{phys}} \ll \dim \mathscr{H}_{\mathrm{total}}$$

need to solve Gauss law constraint

Outline

- **Formalism**
  - Kogut-Susskind Hamiltonian formalism

- **Application**
  - Confinement-deconfinement phase transition in mean field approximation (JHEP 09 (2023) 123)
  - Thermalization on a small lattice (Phys. Rev. D 103, 094502(2021))
  - QCD_2 at finite density (2311.11643)
  - Quantum scar (JHEP 09 (2023) 126)
  - Scrambling (Phys. Rev. D 104 (2021) 7, 074518)

- **Summary**

Kogut-Susskind Hamiltonian formalism

$SU(N)$ Gauge theory ($A_0 = 0$ gauge)

Commutation relation $[A_n^i(x), E_{mj}(x')] = i\delta_{nm}\delta_j^i\delta(x - x')$

Gauge field Electric field

Hamiltonian $H = \int d^3x \left(\frac{g^2}{2} E^2(x) + \frac{1}{2g^2} B^2(x) \right)$

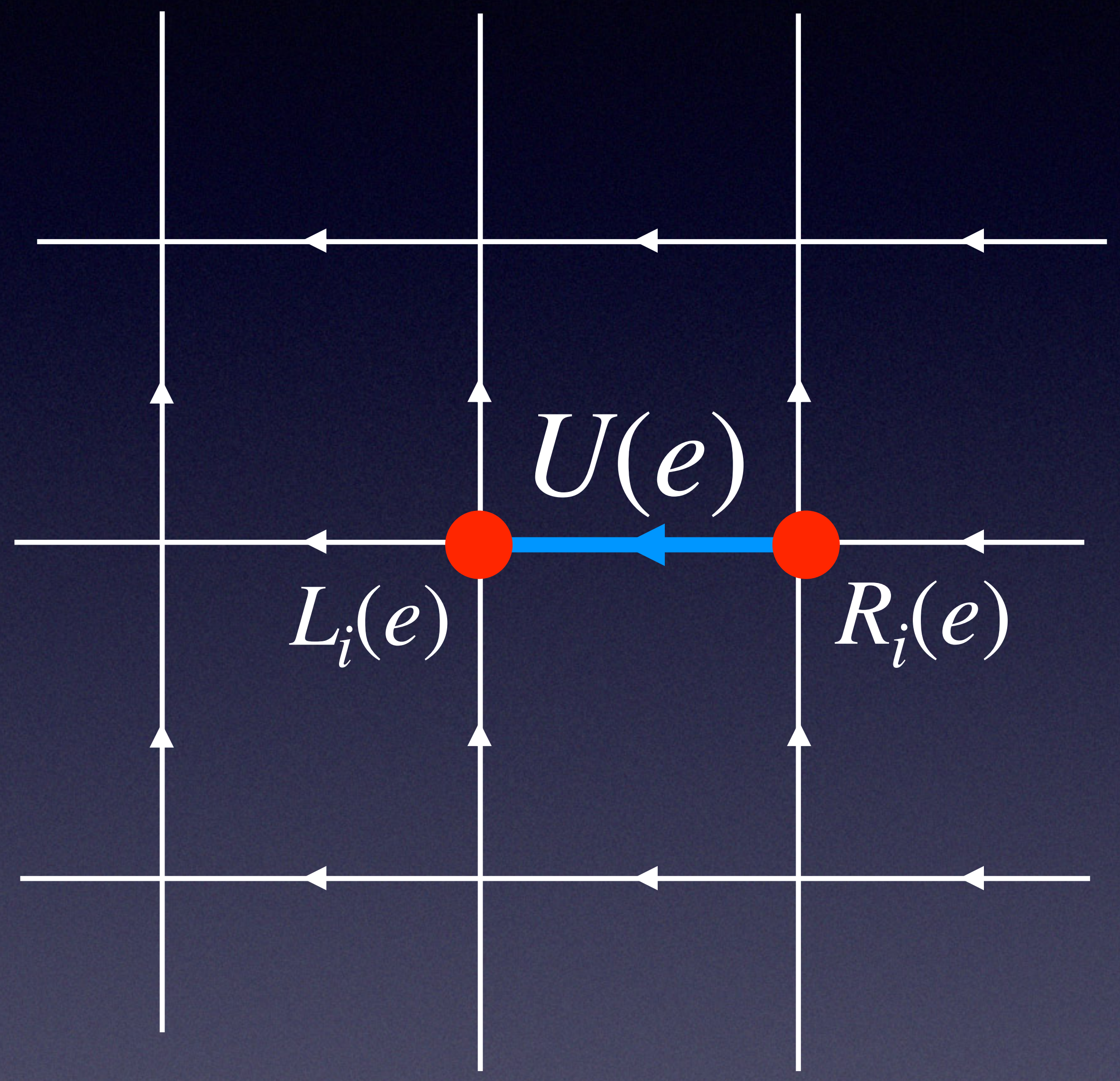
Magnetic field $B_l^i = \frac{1}{2}\epsilon_{lnm}(\partial_m A_n^i - \partial_m A_n^i + f_{jk}^i A_m^j A_n^k)$

Gauss law constraint $(\boldsymbol{D} \cdot \boldsymbol{E})^i \,|\, \Psi_{\text{phys}} \rangle = 0$

Kogut-Susskind Hamiltonian formalism

Kogut, Susskind, Phys. Rev. D 11, 395 (1975)

Time is continuous, space is discretized



$e^{i\int A} \to U(e)$:link variable $\in \mathrm{SU}(N)$ on edge e

$L_i(e), R_i(e)$:Left and right electric fields $\in \mathrm{su}(N)$

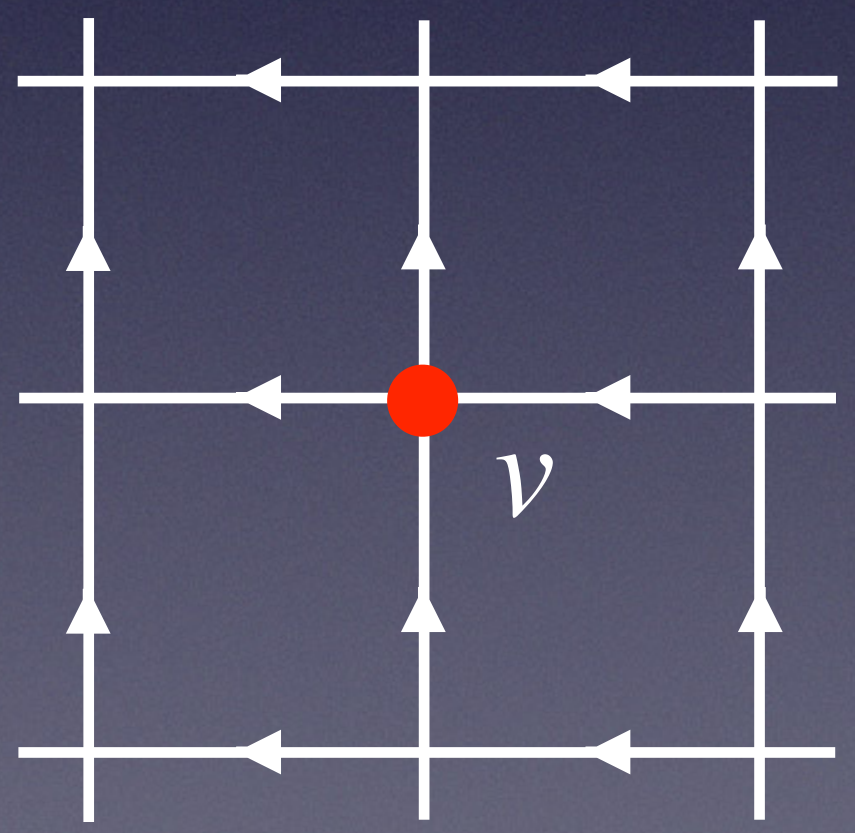
$L_i(e)$ and $R_i(e)$ are not independent

$$[U_{\mathrm{adj}}(e)]_i^j L_j(e) = R_i(e) \Rightarrow R_i^2(e) = L_i^2(e) =: E_i^2(e)$$

Commutation relation

$$[A^i_n(x), E_{mj}(x')] = i\delta_{nm}\delta^i_j\delta(x - x')$$

$$[R_i(e), U(e')] = U(e)T_i\delta_{e,e'}$$
$$[L_i(e), U(e')] = T_iU(e)\delta_{e,e'}$$
$$[L_i(e), L_j(e')] = -\mathrm{i}f^k_{ij}L_k(e)\delta_{e,e'}$$
$$[R_i(e), R_j(e')] = \mathrm{i}f^k_{ij}R_k(e)\delta_{e,e'}$$



Gauss law constraint $(D \cdot E)^i |\Psi_{\text{phys}}\rangle = 0$

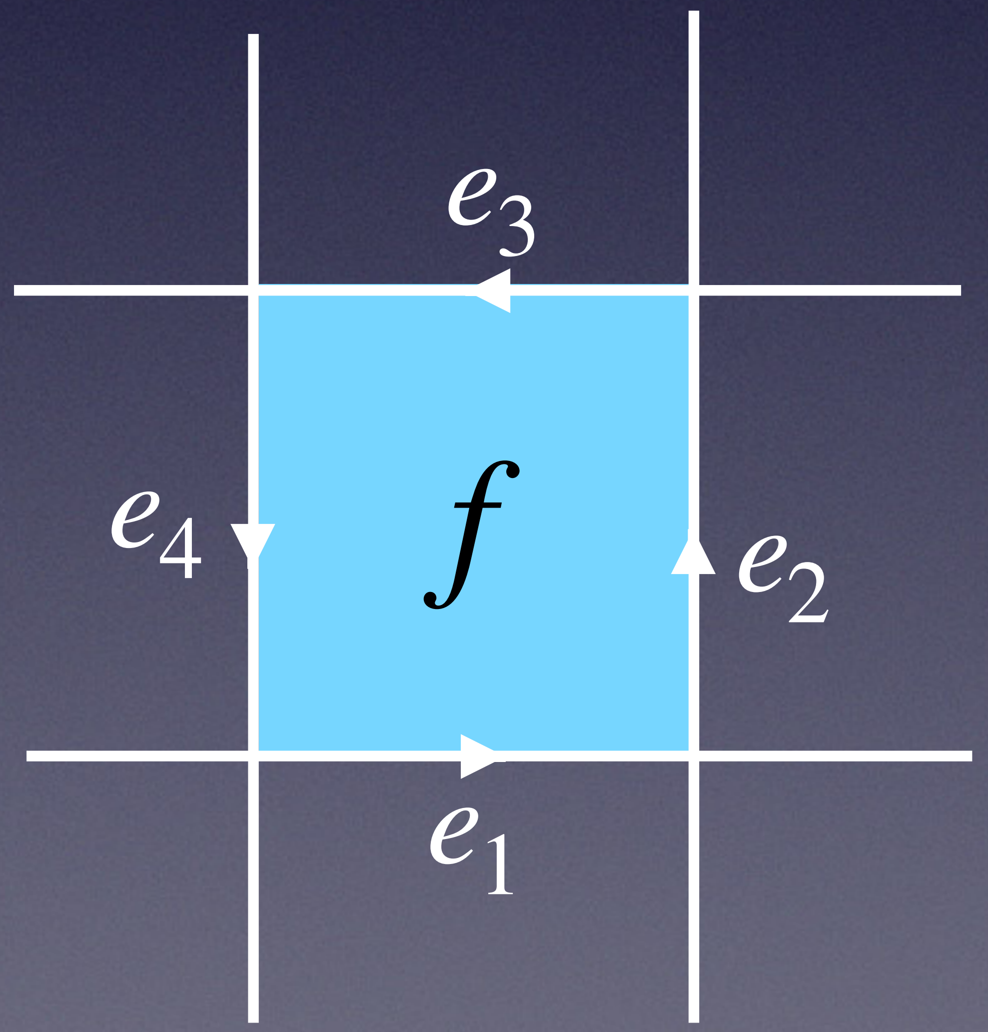
$$\left(\sum_{e \in C_1 | \mathrm{s}(e) = v} R_i(e) - \sum_{e \in C_1 | \mathrm{t}(e) = v} L_i(e) \right) |\Psi_{\text{phys}}\rangle = 0$$

C_1:**set of edges,** s,t: **source and target functions**

Hamiltonian

$$H = \frac{1}{2}\sum_{e\in C_1}(E(e))^2 - \frac{K}{2}\sum_{f\in C_2}(\mathrm{tr}U(f) + \mathrm{tr}U^\dagger(f))$$

C_2:set of faces



$$U(f) := U(e_4)U(e_3)U(e_2)U(e_1)$$

• Application

- Confinement-deconfinement phase transition in mean field approximation

- Thermalization on a small lattice

- QCD in (1+1) dimensions

Confinement-deconfinement phase transition in mean field approximation for $\mathrm{SU}(3)_k$ in (2+1) dimensions

k: cutoff parameter (q deformation)

Variational ansatz for wave function

Dusuel, Vidal, Phys. Rev. B 92 (2015) 12, 125150, Zache, González-Cuadra, Zoller, 2304.02527, Hayata, YH, JHEP 09 (2023) 126

$$|\Psi\rangle = \prod_{f\in\mathcal{F}} \sum_{a_f} \psi(a_f)\, \mathrm{tr}\, U_{a_f}(f)\, |0\rangle$$

We minimize the energy expectation value

open boundary condition, infinite volume limit

$$E = \min_{\psi} \langle\Psi|\, H\, |\Psi\rangle$$

We can calculate observables for given wave function

Hayata, YH, JHEP 09 (2023) 126

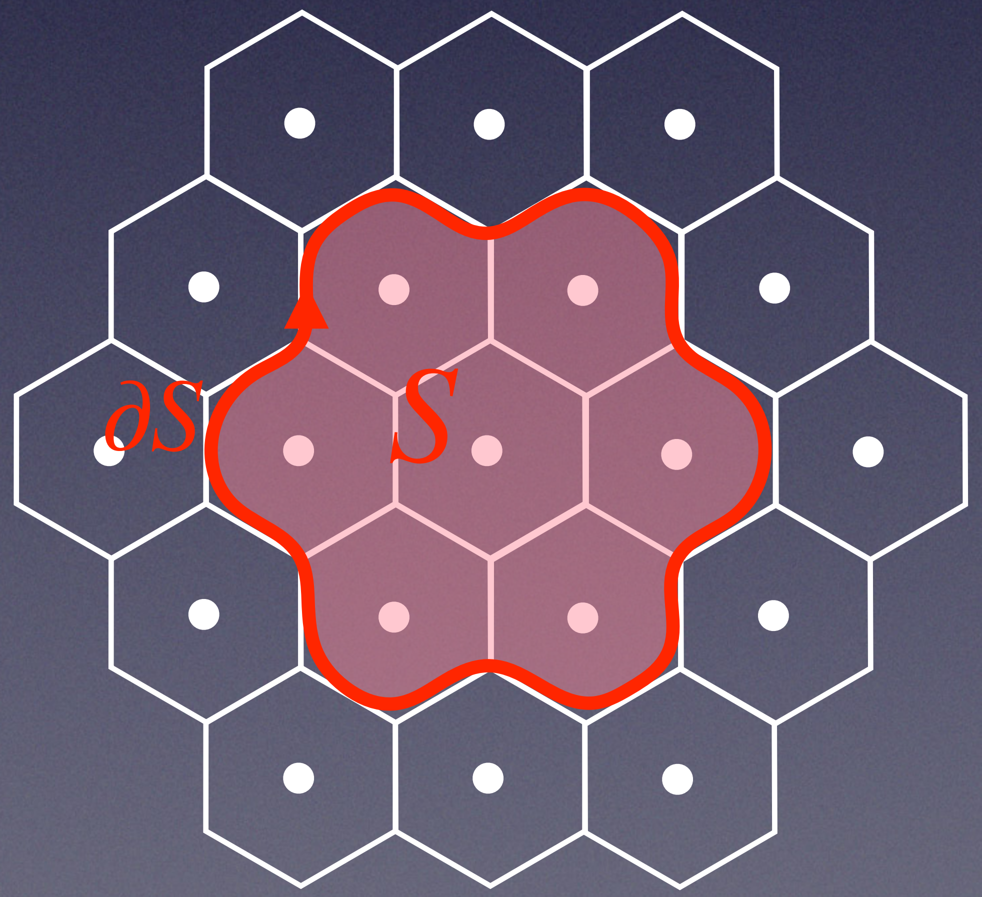
Energy density

$$h = \frac{1}{V}\langle H \rangle = \sum_{a,b,c} C_2(c) N^c_{\bar{a}b} \frac{d_c}{d_a d_b} |\psi(a)|^2 |\psi(b)|^2 - \frac{K}{2} \sum_{a,b} \psi^*(a) \left(N^a_{(1,0)b} + N^a_{(0,1)b} \right) \psi(b)$$

$C_2(a)$ Casimir invariant, d_a: quantum dimensions, N^c_{ab}: multiplicity

Wilson loop

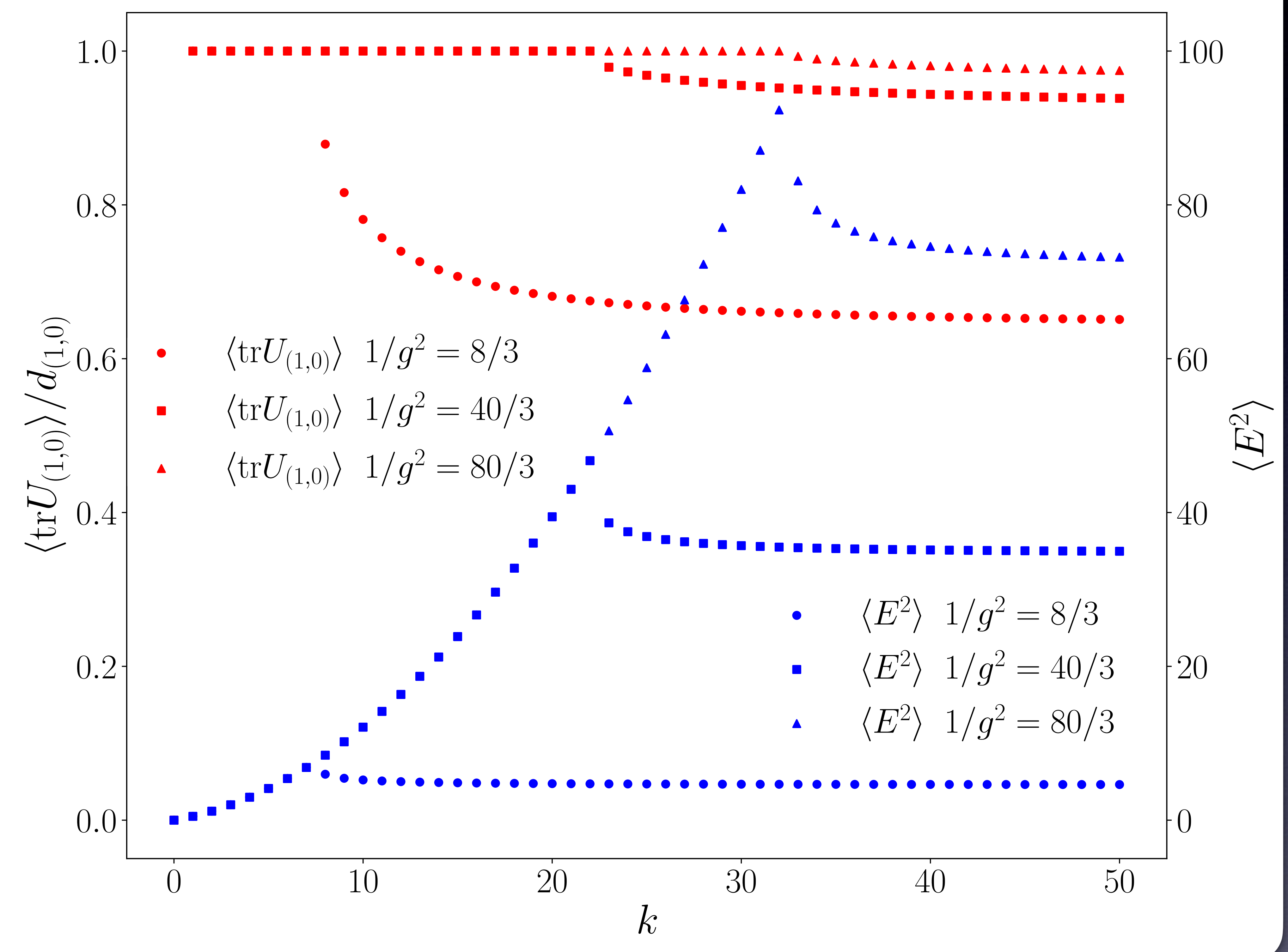
$$\langle \mathrm{tr}\, U_d(\partial S) \rangle = d_d \exp(-|S|\sigma_d)$$

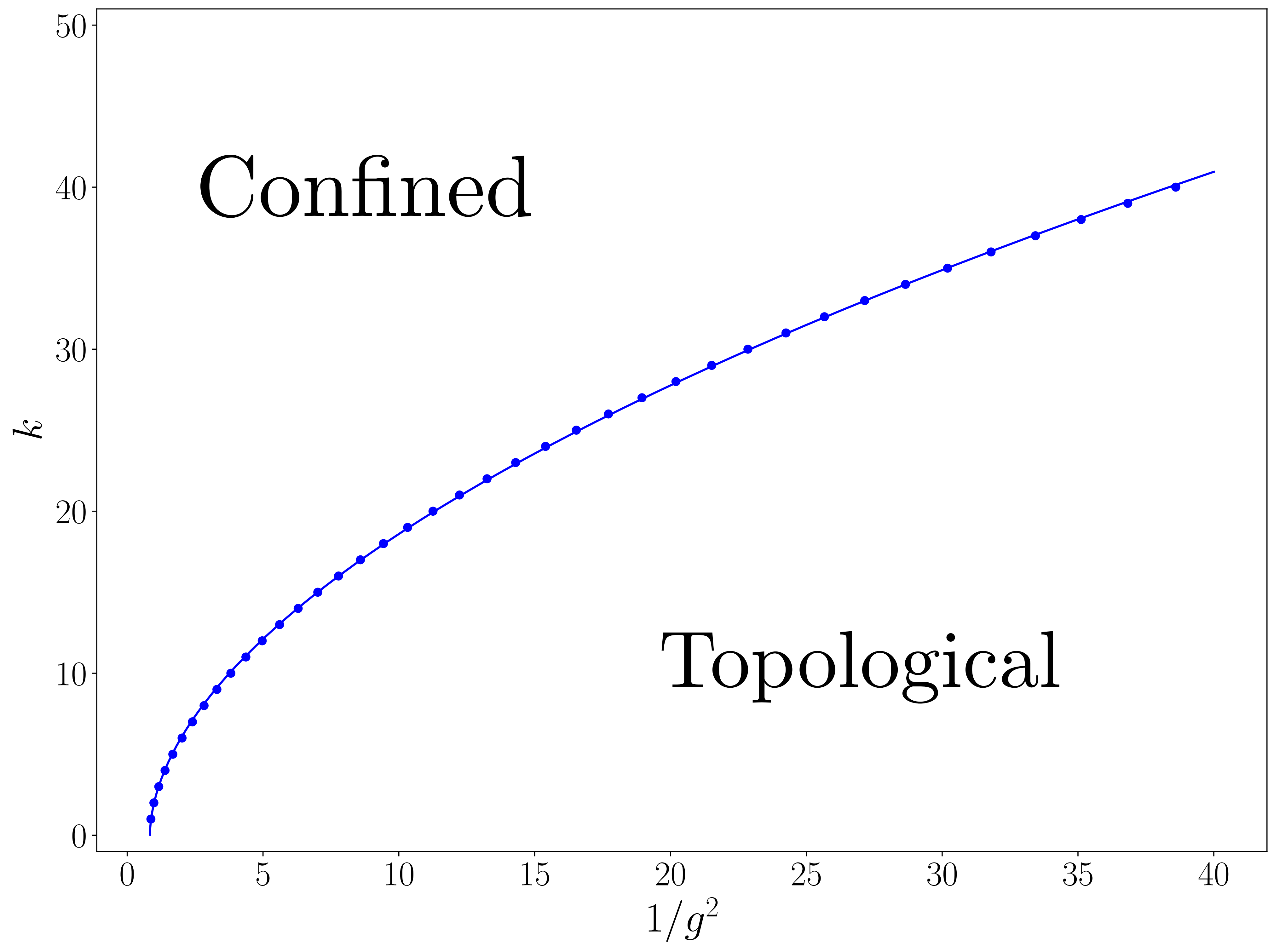


String tension $\sigma_d := \ln \dfrac{d_d}{\sum_{a,b} N^a_{db} \psi^*(a)\psi(b)}$

Numerical results

Numerical results





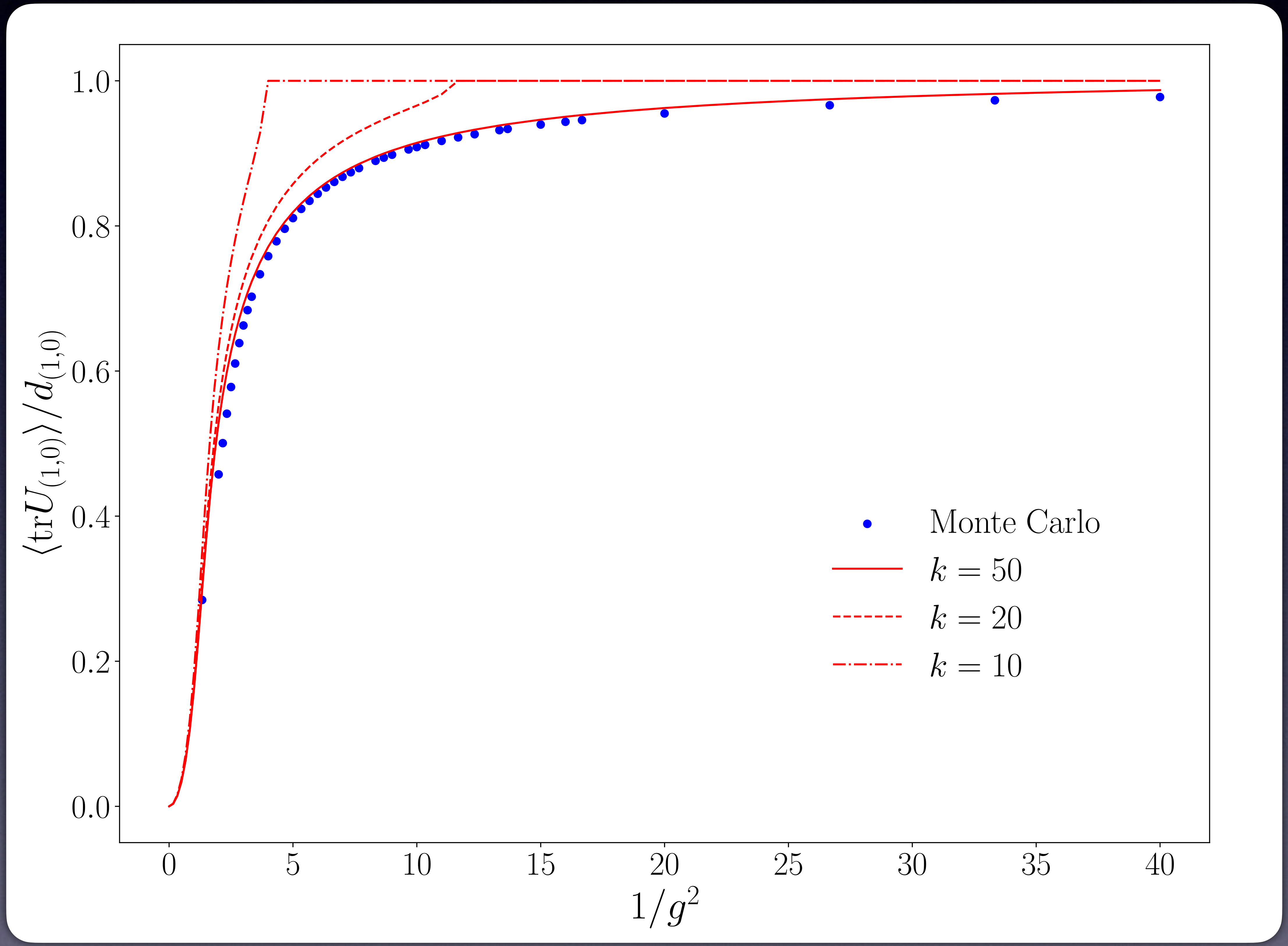
Phase transition occurs

Topological phase:

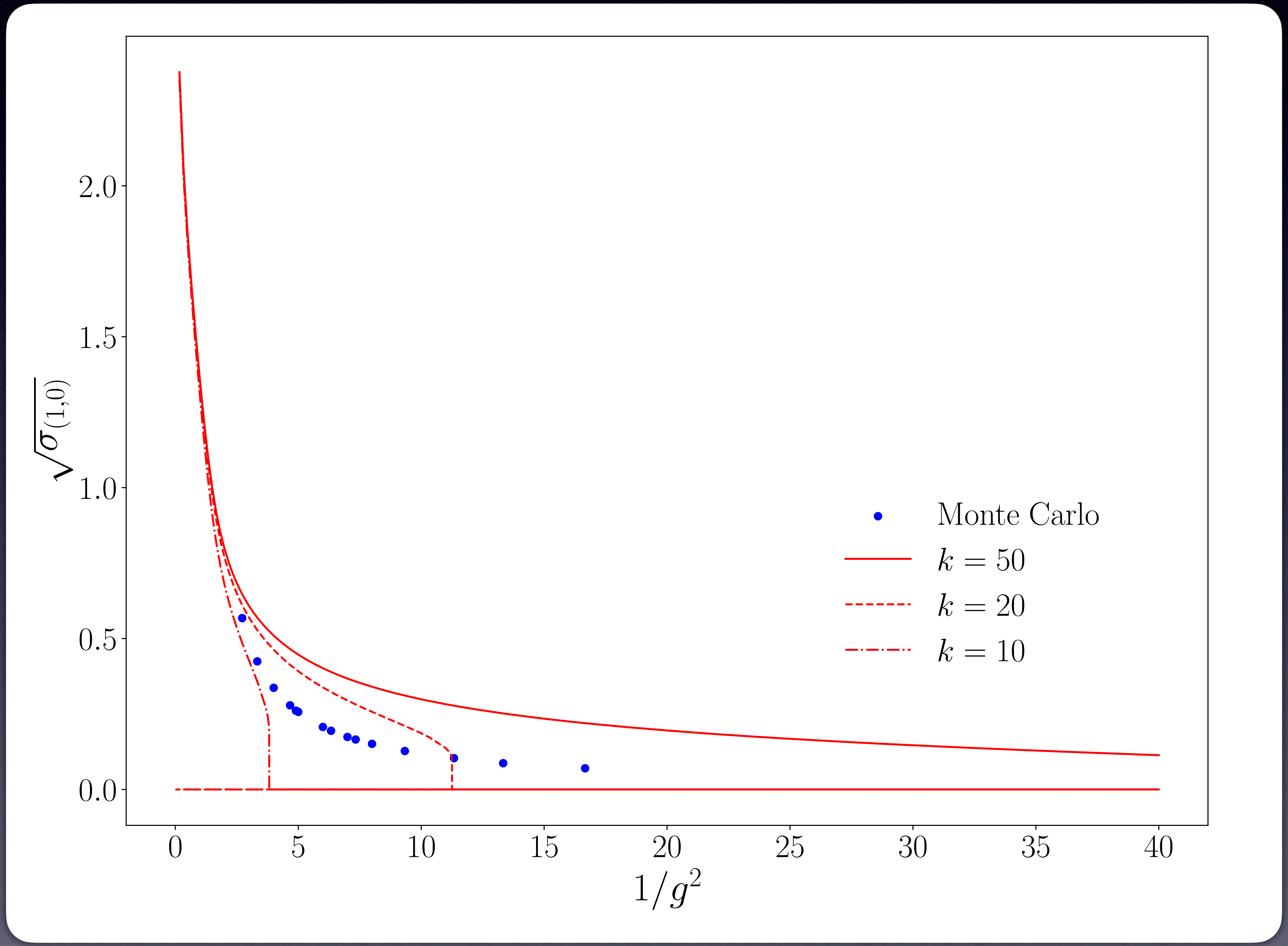
String-net condensation: $\psi(a) \sim d_a$ **where string tension vanishes**

Comparison with Monte-Carlo simulation

Plaquette (small Wilson loop)



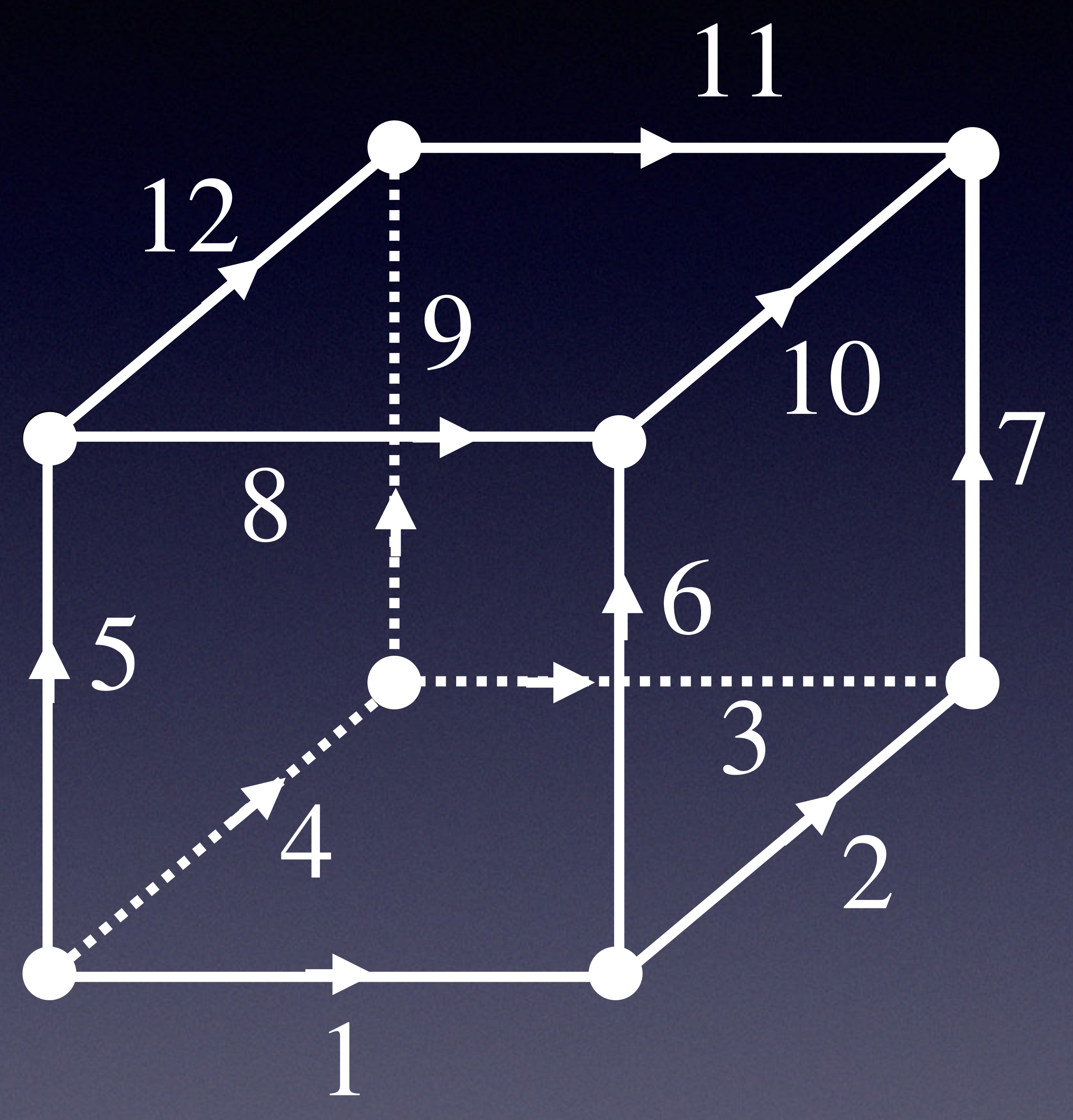
String tension



Good agreement for large k!

Thermalization on a small lattice

Small lattice system

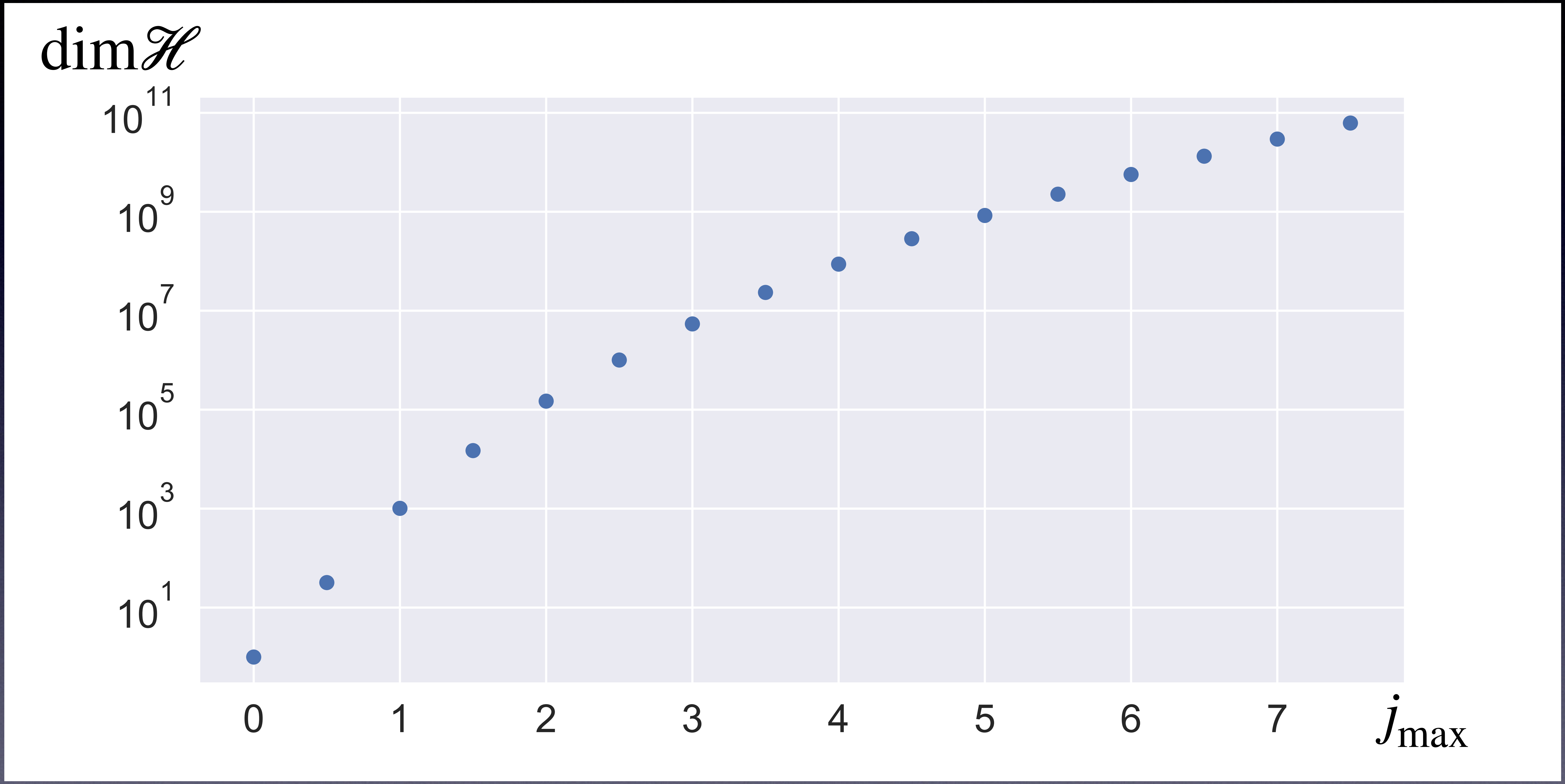


Basis

$$|j_1, \cdots, j_{12}\rangle = |j_1, j_2, j_6\rangle\, |j_2, j_3, j_7\rangle\, |j_3, j_4, j_8\rangle\, |j_1, j_4, j_5\rangle$$
$$|j_6, j_9, j_{10}\rangle\, |j_7, j_{10}, j_{11}\rangle\, |j_8, j_{11}, j_{12}\rangle\, |j_5, j_9, j_{12}\rangle$$

Naive cutoff $j_i \leq j_{\max} = k/2$

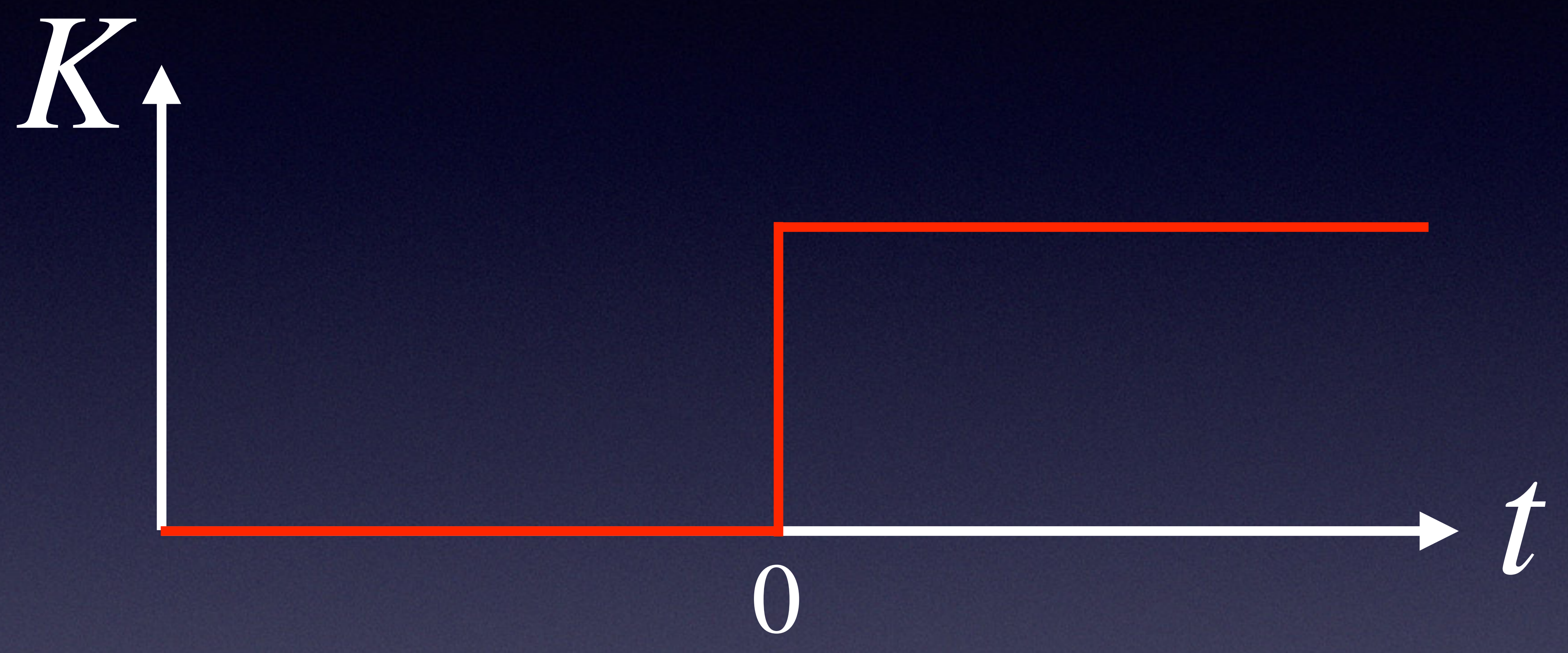
Dimension of Hilbert space



We employ $j_{\max} = 4:\ \dim\mathscr{H} = 87{,}426{,}119$

Setup

In order to mimic heavy ion collision experiments, the interaction quenching

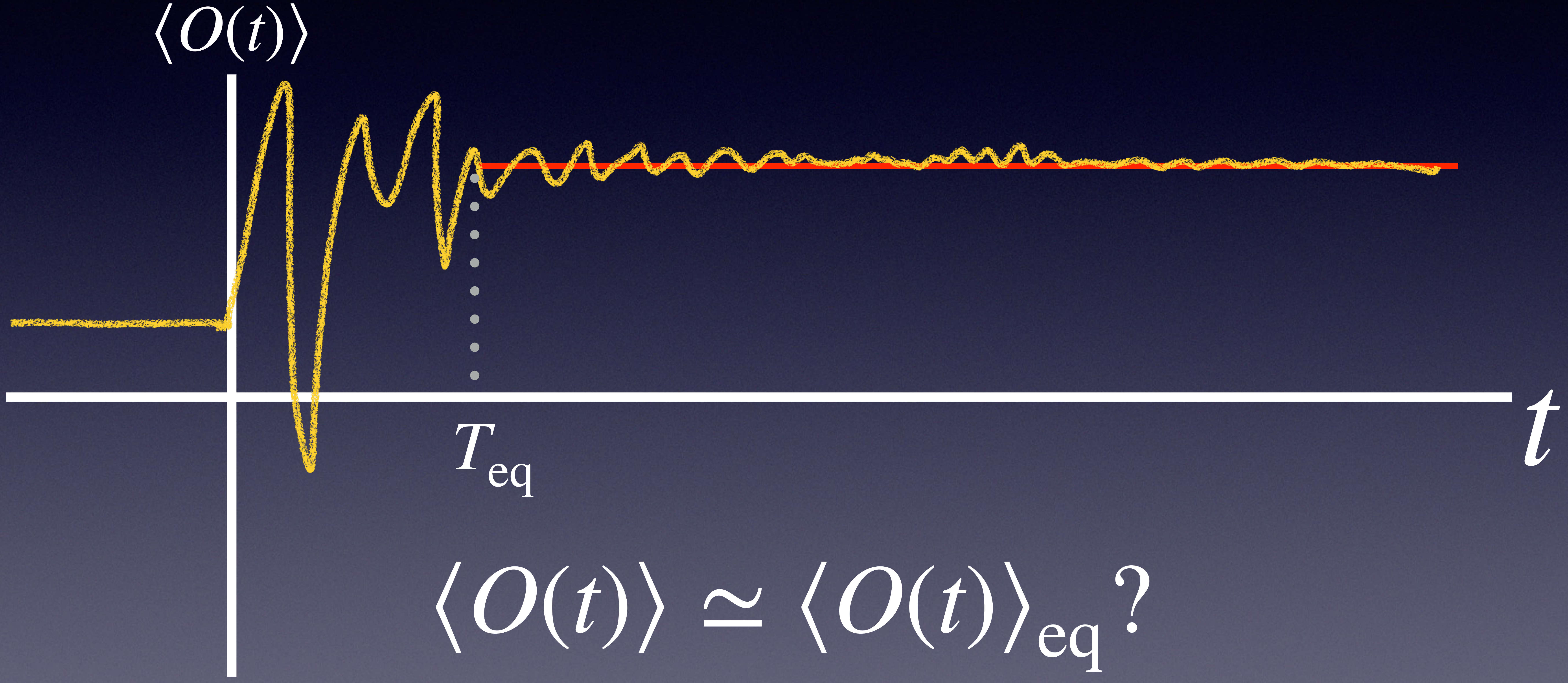


$$t < 0 \; |\mathrm{Vac}\rangle_{K=0}$$

$$t \geq 0 \; |\Psi(t)\rangle = e^{-\mathrm{i}Ht} |\mathrm{Vac}\rangle_{K=0}$$

Expected behavior

for an operator O $\quad \langle O(t) \rangle := \langle \Psi(t) | O | \Psi(t) \rangle$



$$\langle O(t) \rangle \simeq \langle O(t) \rangle_{\mathrm{eq}} ?$$

Temperature and Canonical Ensemble

Energy is fixed by an initial condition

$$E = \langle H \rangle = \langle \Psi(t) \,|\, H \,|\, \Psi(t) \rangle$$

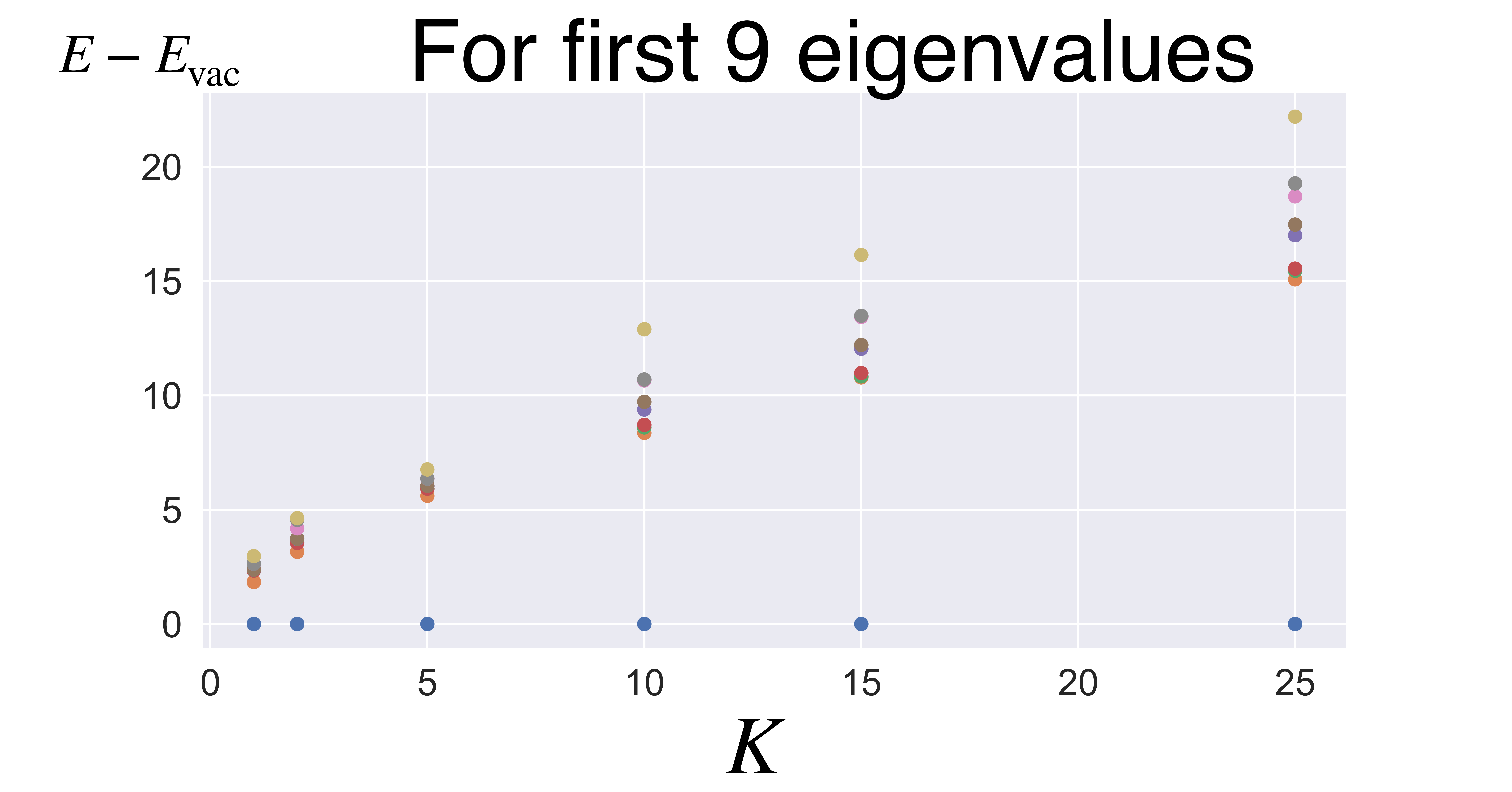
(Independent of time)

For a given energy,
a canonical distribution that reproduces
the expected value can be defined

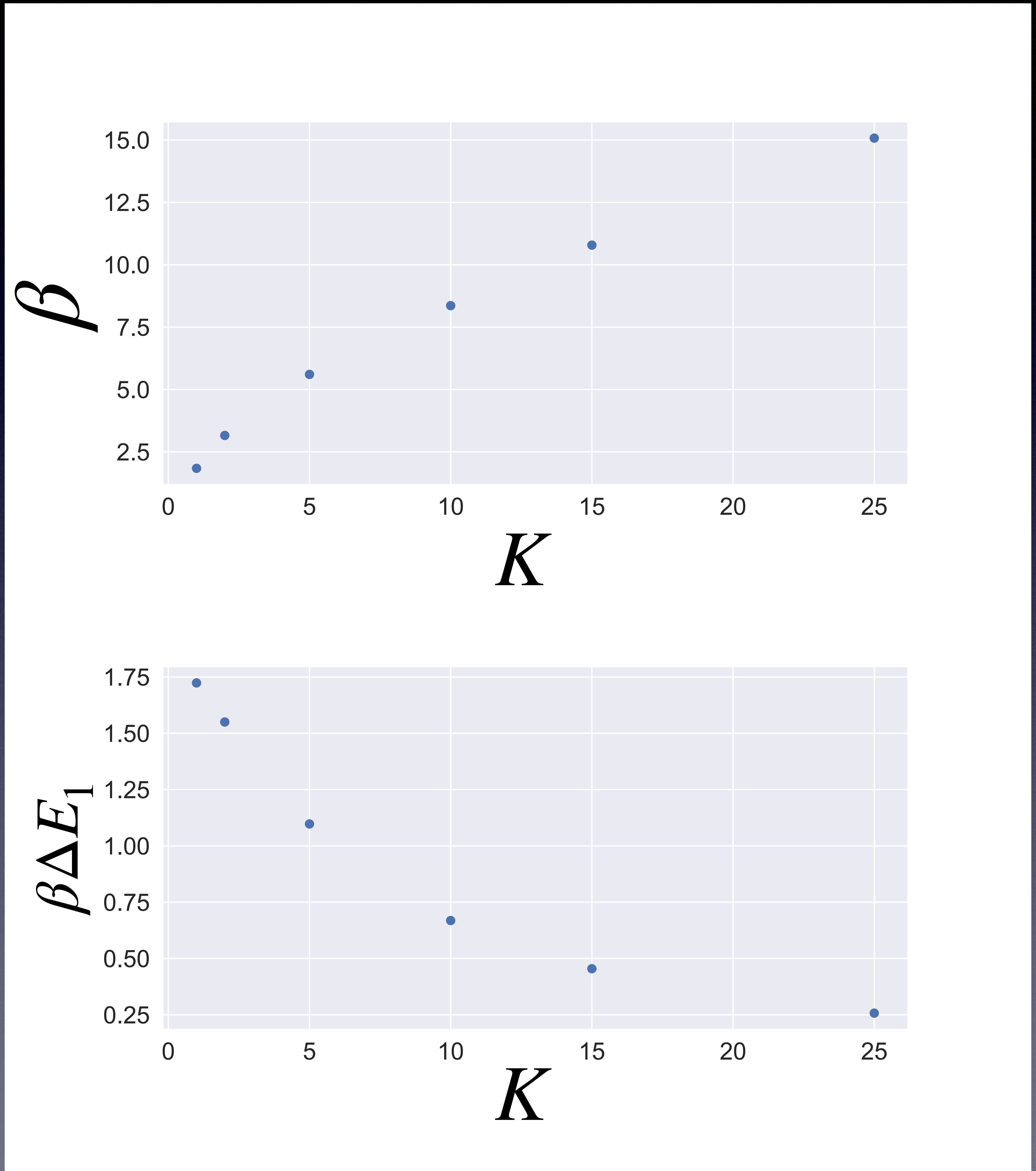
$$E = \langle H \rangle_{\mathrm{eq}} := \mathrm{tr}\rho_{\mathrm{eq}} H \quad \textbf{with} \quad \rho_{\mathrm{eq}} = \frac{e^{-\beta H}}{\mathrm{tr} e^{-\beta H}}$$

Numerical results

Energy eigenvalues for $j_{\max} = 4$



K-dependence of temperature



The first excitation energy

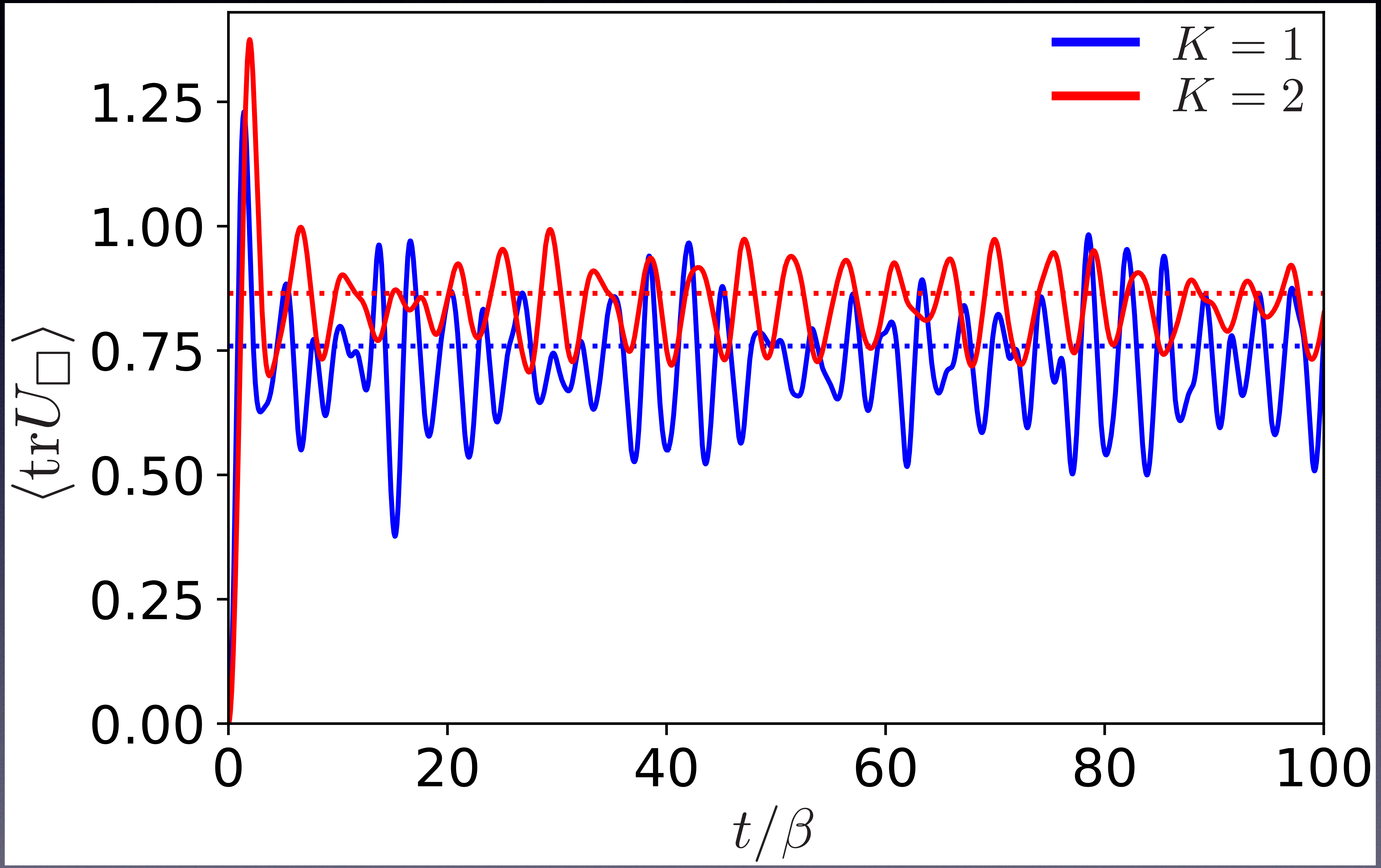
$$\Delta E_1 : E_1 - E_0$$

Typical energy scale

$\beta\Delta E_1 > 1$ **Low T**

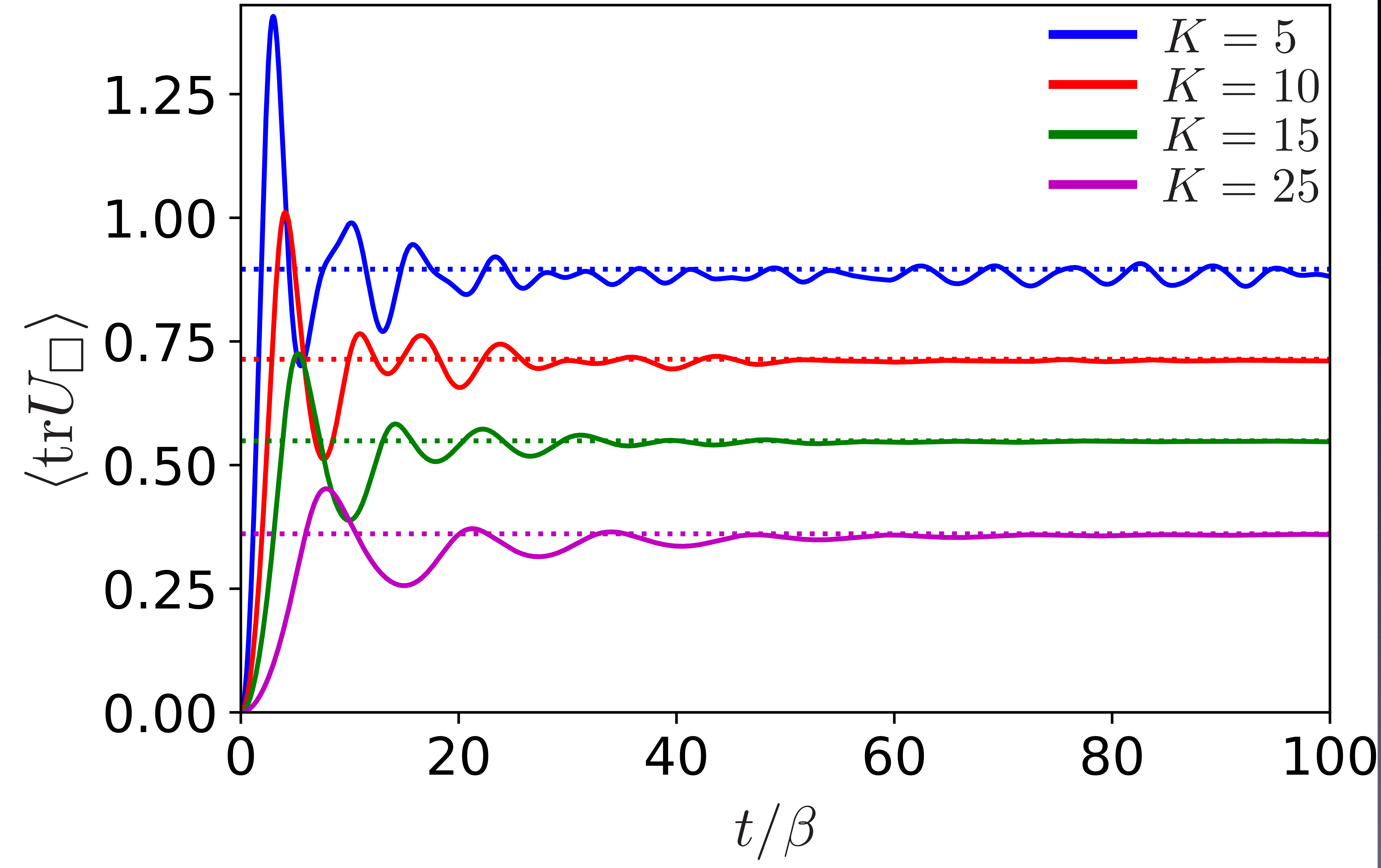
$\beta\Delta E_1 < 1$ **Hight T**

Expected value of Wilson loop
Strong coupling (low T)



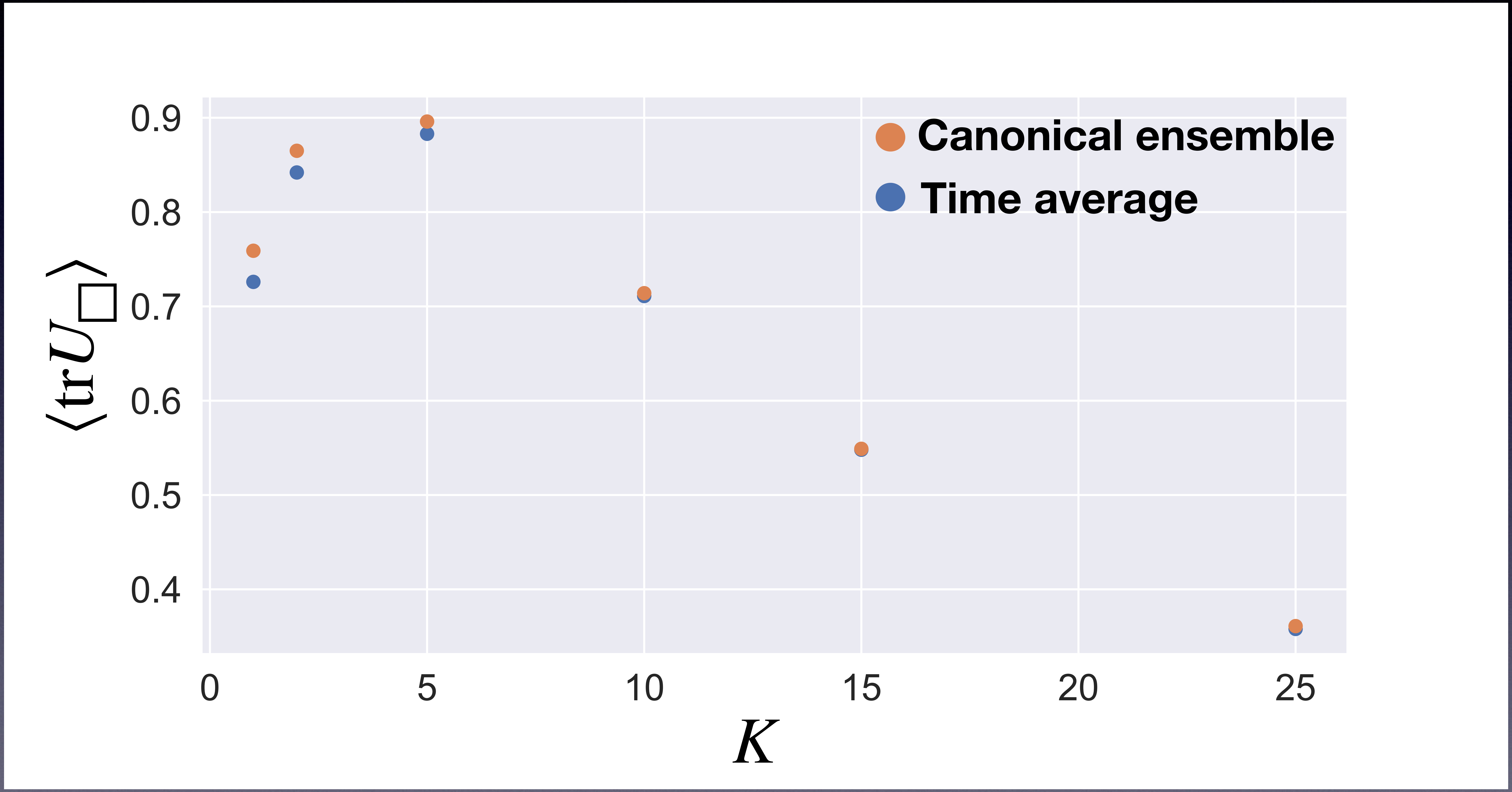
Fluctuations are not small.

Expected value of Wilson loop
Weak coupling (high T)



Steady state observed

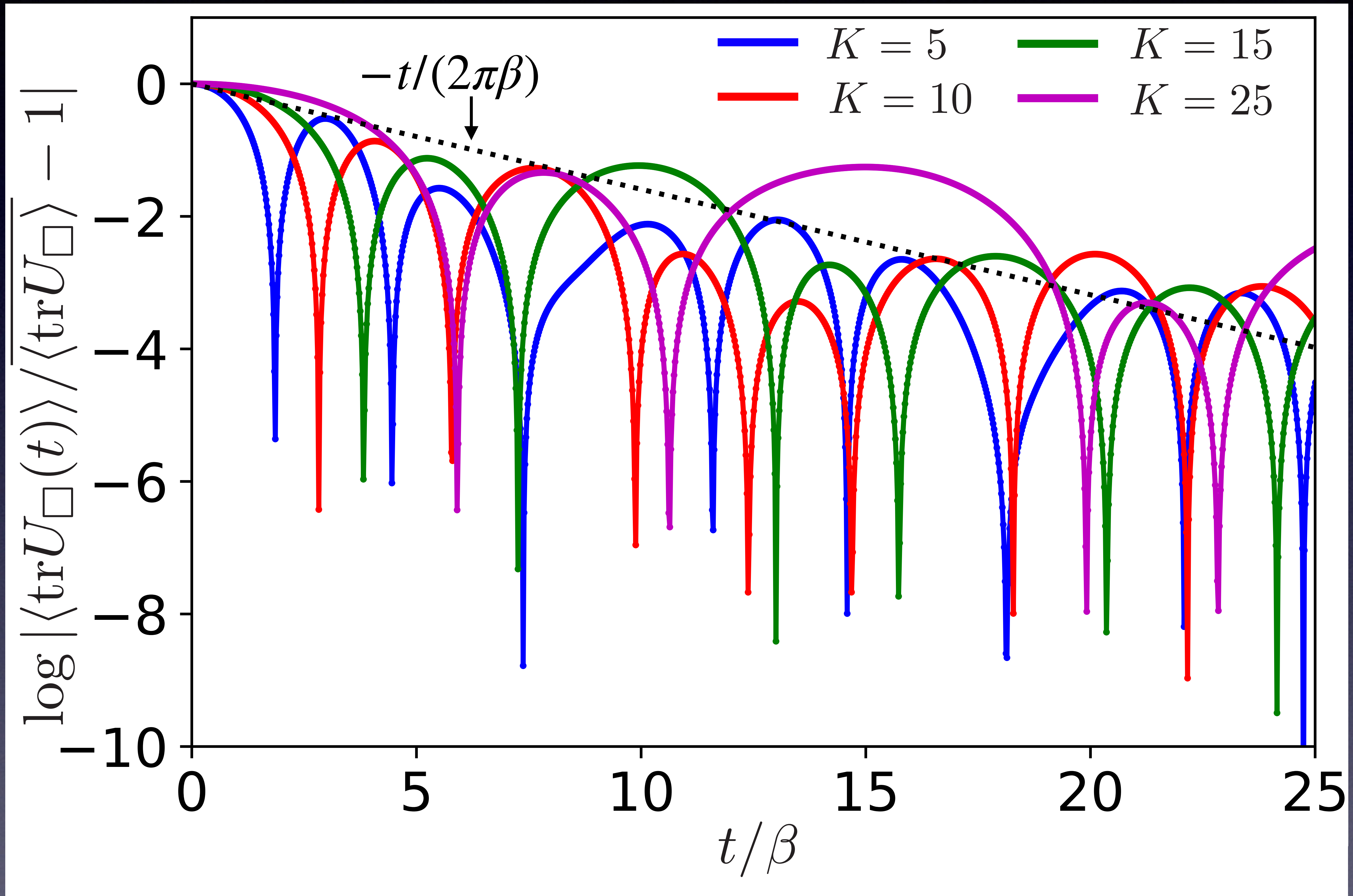
Long-time average vs canonical ensemble



Difference is less than 1% for $K > 5$

Relaxation to equilibrium

$$\langle \mathrm{tr} U_\Box(t) \rangle - \overline{\langle \mathrm{tr} U_\Box \rangle} \sim e^{-t/\tau_{\mathrm{eq}}}$$

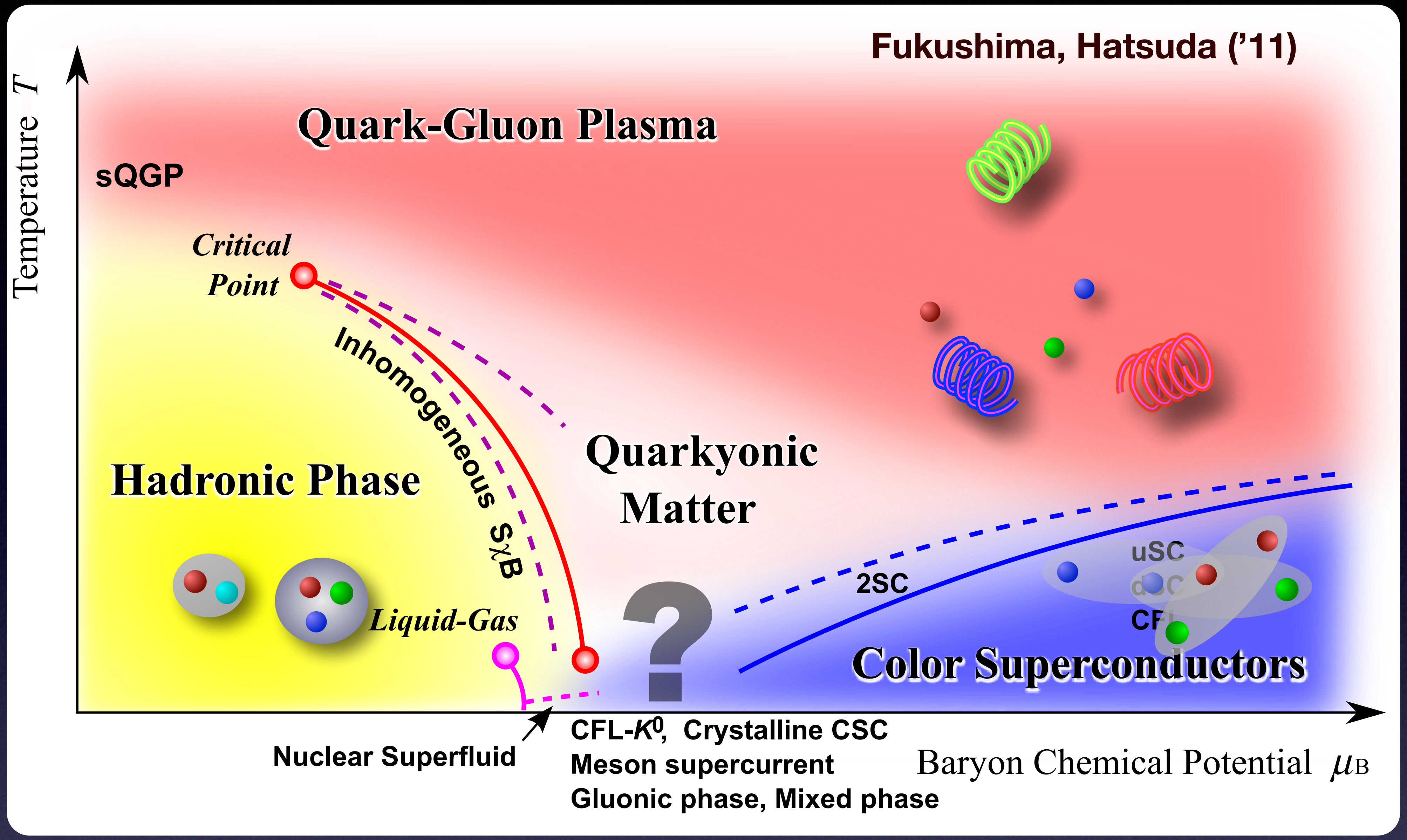


Close to Boltzmann time $2\pi\beta$.

Goldstein, Hara, Tasaki, New J. Phys. 17 (2015) 045002

QCD_2 at finite density

QCD at finite density



- What is the equation of state for QCD at finite density?
- How does the quark distribution function change when transitioning from baryonic matter to quark matter?
  - What kind of phase is realized? An inhomogenous phase?

QCD_2

Properties of (1+1) dimensions

- **Gauge fields are nondynamical**
- **Hilbert space is finite dimensional in Open Boundary Condition(OBC)**

(dimensionless)QCD_2 Hamiltonian

$$J = \frac{ag_0}{2}, w = \frac{1}{2g_0 a}, m = m_0/g_0$$

We use $g_0 = 1$ unit

$$H/g_0 = J \sum_{n=1}^{N-1} E_i^2(n)$$

Electric field term

$$+w \sum_{n=1}^{N-1} \left(\chi^\dagger(n+1)U(n)\chi(n) + \chi^\dagger(n)U^\dagger(n)\chi(n+1) \right)$$

Hopping term

$$+m \sum_{n=1}^{N} (-1)^n \chi^\dagger(n)\chi(n)$$

Mass term

Elimination of Link variables U

Sala, Shi, Kühn, Bañuls, Demler, Cirac, Phys. Rev. D 98, 034505 (2018)
Atas, Zhang, Lewis, Jahanpour, Haase , Muschik , Nature Commun. 12, 6499 (2021)

$$\Theta\chi(n)\Theta^\dagger := U(n-1)U(n-2)\cdots U(1)\chi(n)$$

$$\Theta H\Theta^\dagger = J\sum_{n=1}^{N-1}\Big(\sum_{m=1}^{n}\chi^\dagger(m)T_i\chi(m)\Big)^2$$ **Electric fields term**

$$+w\sum_{n=1}^{N-1}\Big(\chi^\dagger(n+1)\chi(n)+\chi^\dagger(n)\chi(n+1)\Big)$$

Hopping term

$$+m\sum_{n=1}^{N}(-1)^n\chi^\dagger(n)\chi(n)$$ **mass term**

As a variational ansatz of wave function

- We employ a matrix product state

$$|\psi\rangle = \sum_{\{n_i\}} |n_1\rangle \cdots |n_N\rangle \mathrm{tr} M_1^{n_1} \cdots M_N^{n_N}$$

$[M_i^{n_i}]_{ij}$ **:** $D \times D$ **matrix**

- Optimize the wave function by density matrix renormalization group technique

$$E = \min_{\psi} \langle \psi | H | \psi \rangle$$

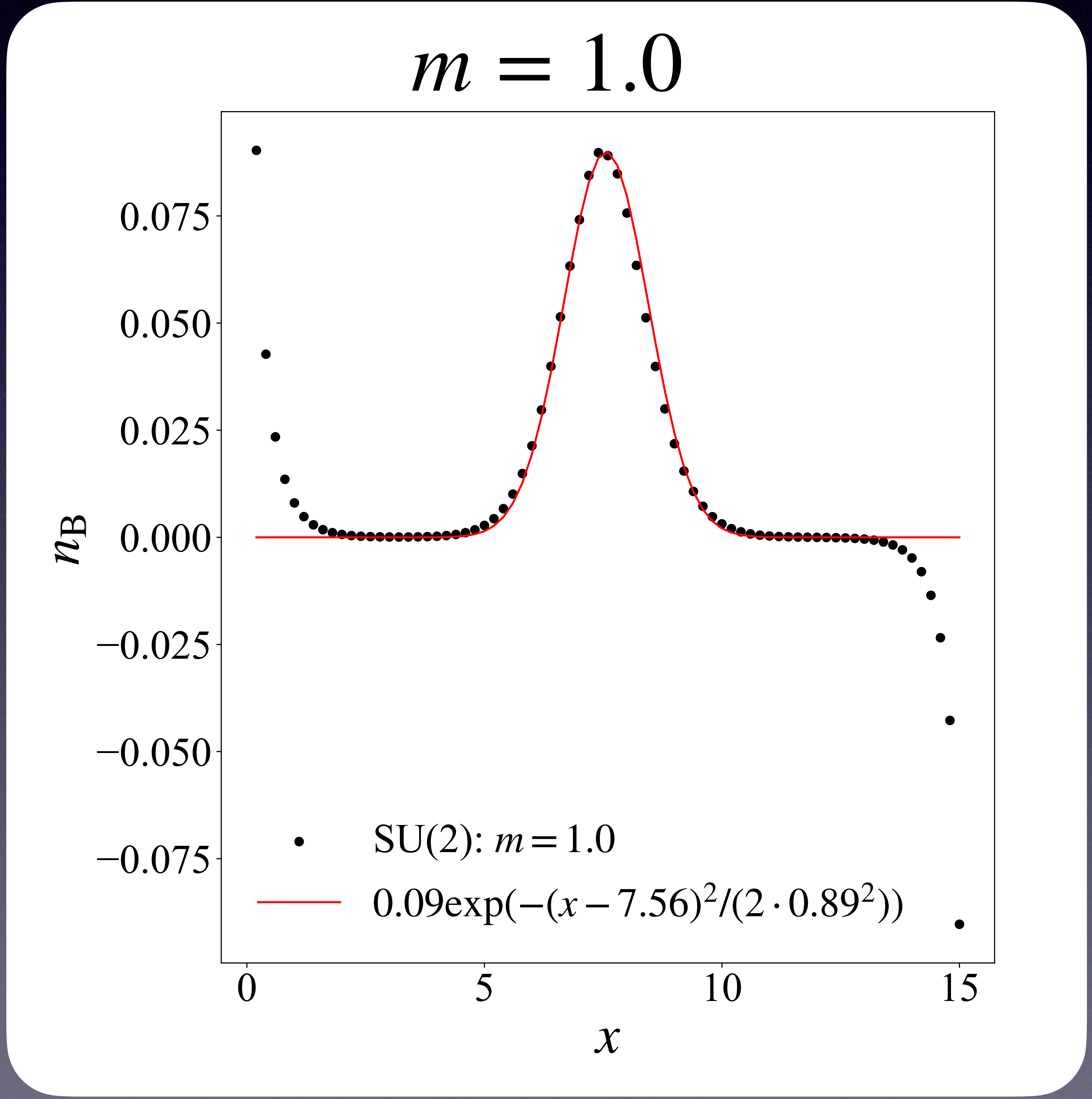
We employ iTensor

Numerical results

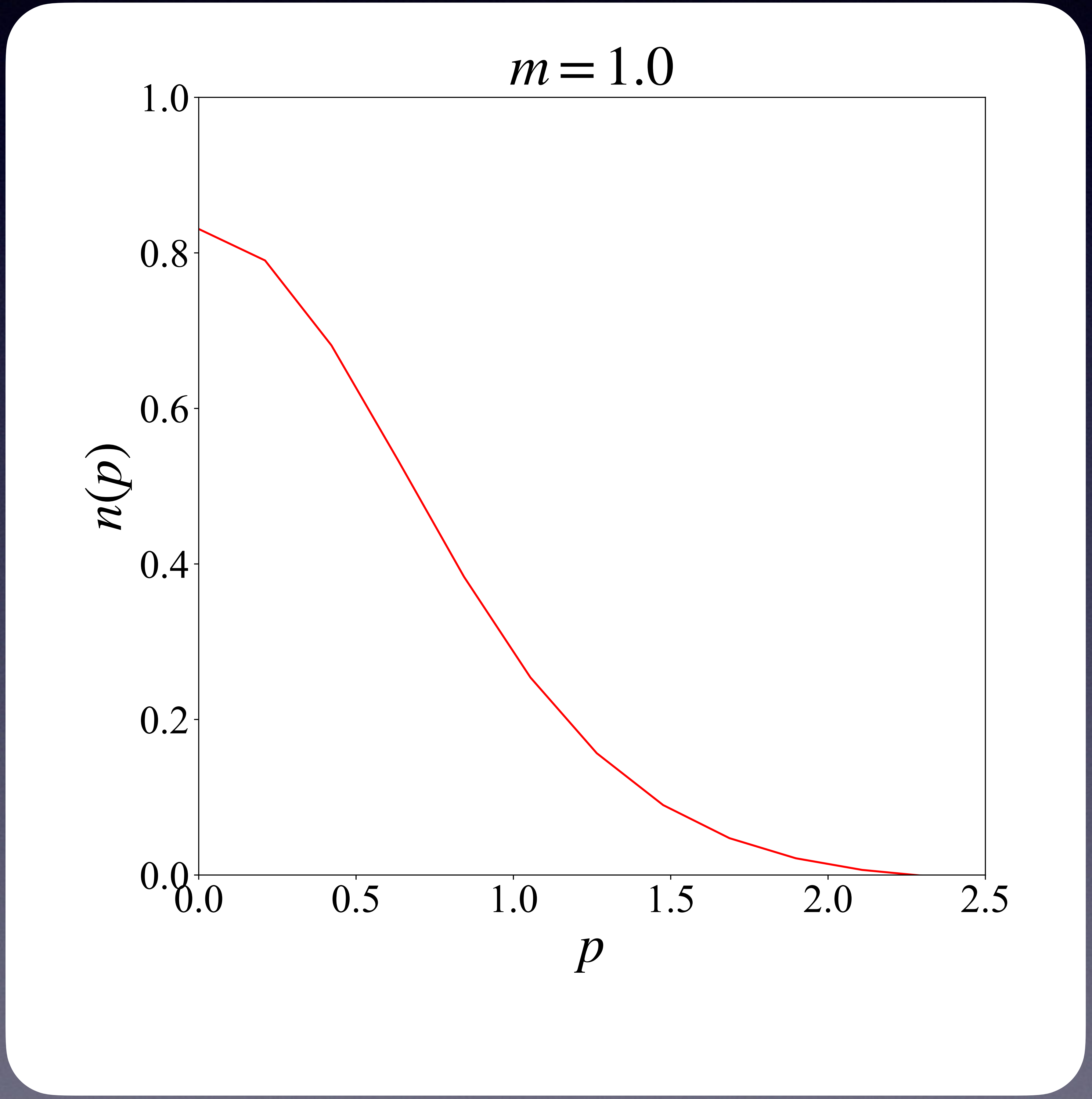
Color SU(2), 1 flavor, vacuum

single baryon state $\dim\mathscr{H} = 2^{300}$ $J = 1/20$ $w = 5$ volume $V = 15$

Baryon number density



Quark distribution function

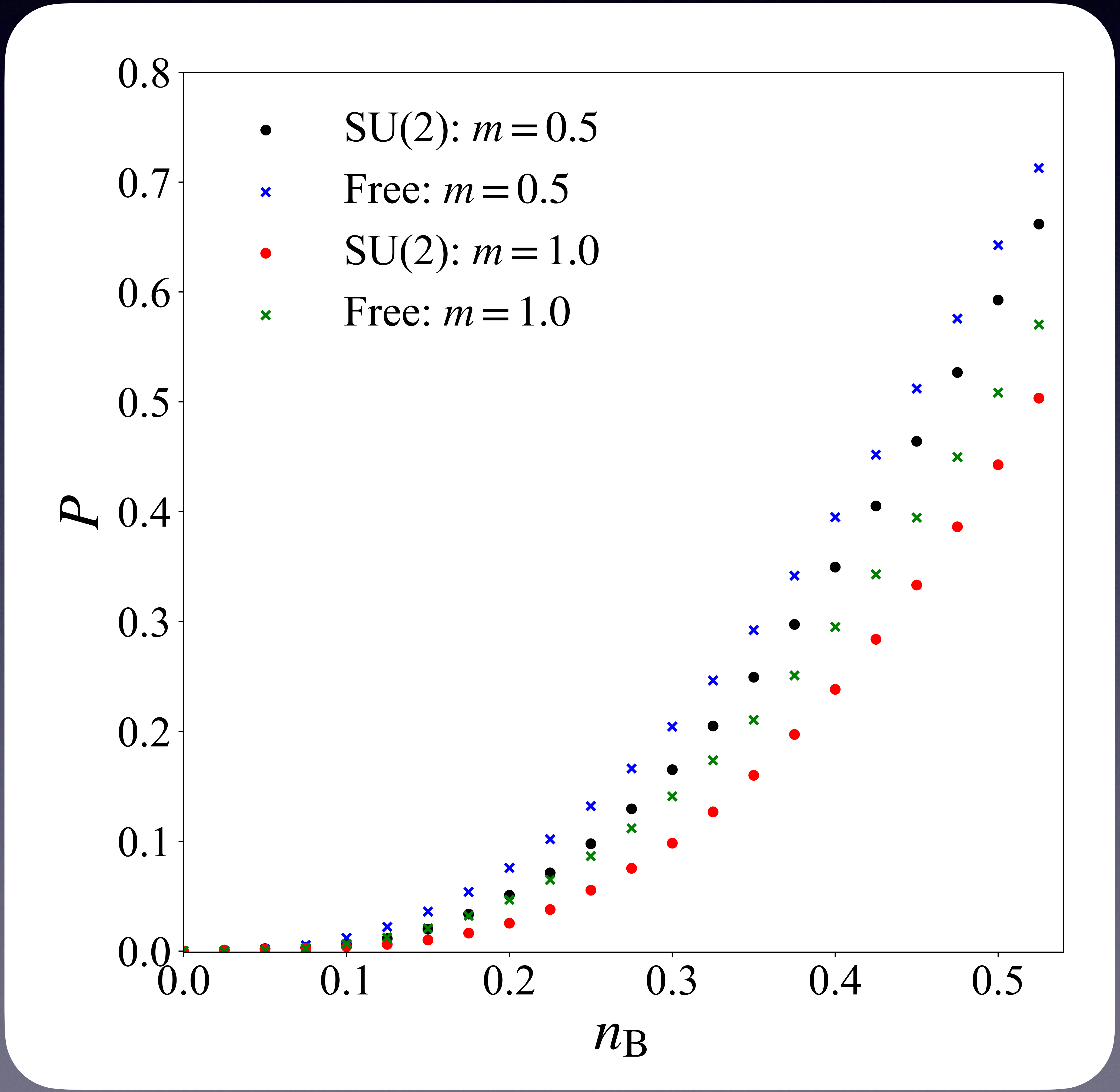


Baryon size ~ 1

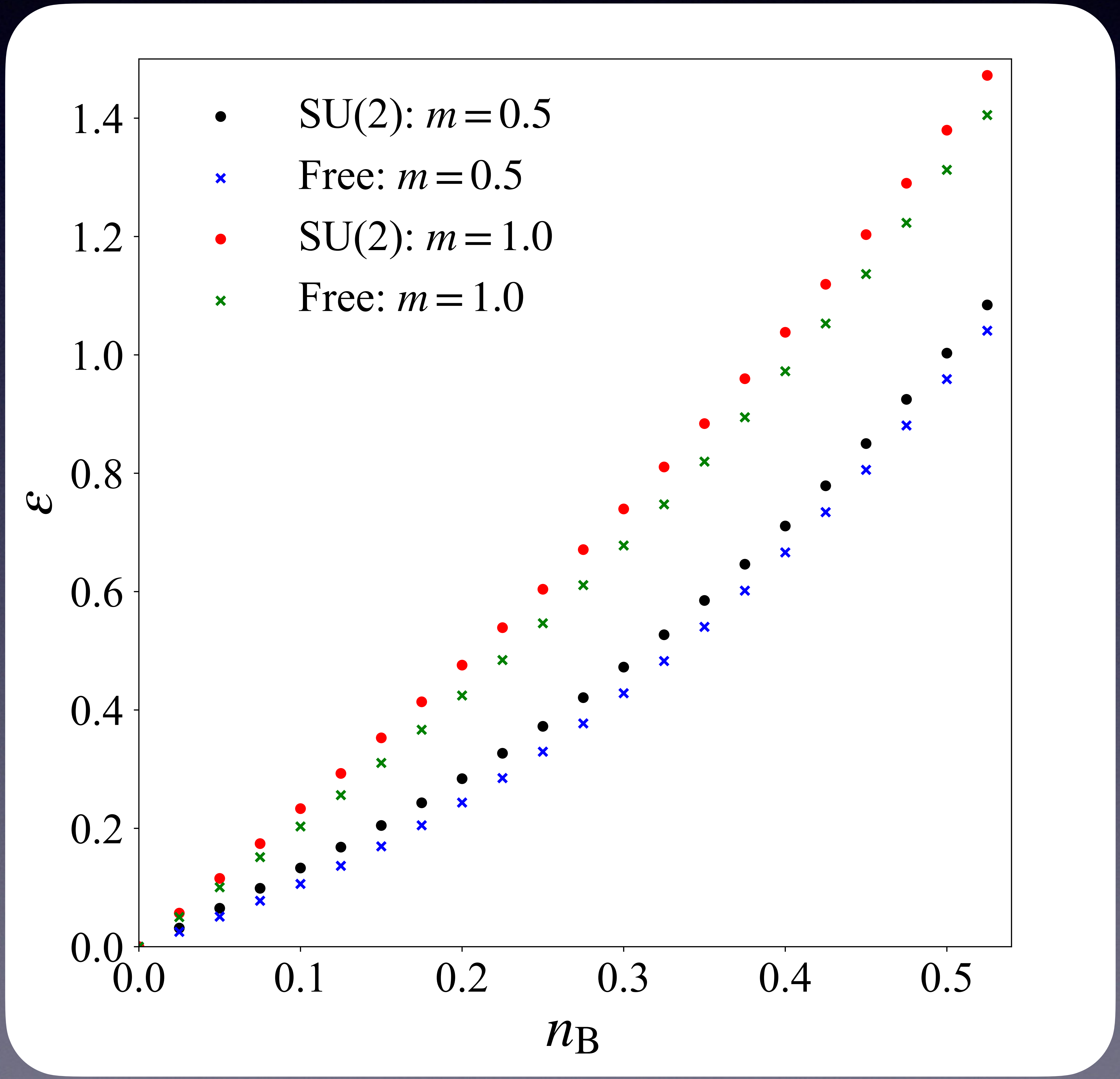
Color SU(2), 1 flavor, vacuum

$$J = 1/8 \quad w = 2 \quad V = 40 \quad \dim\mathscr{H} = 2^{320}$$

Pressure



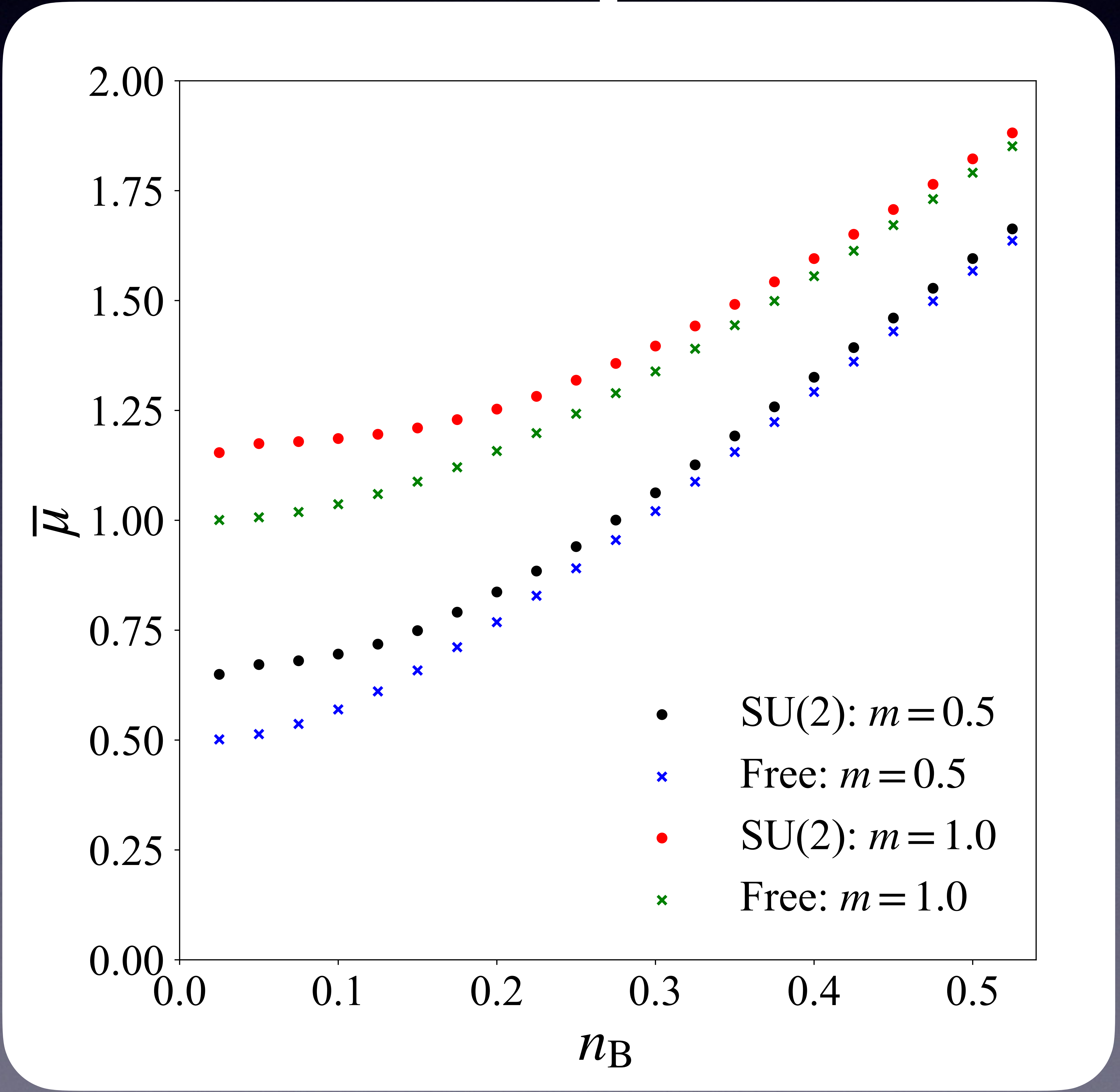
Energy density



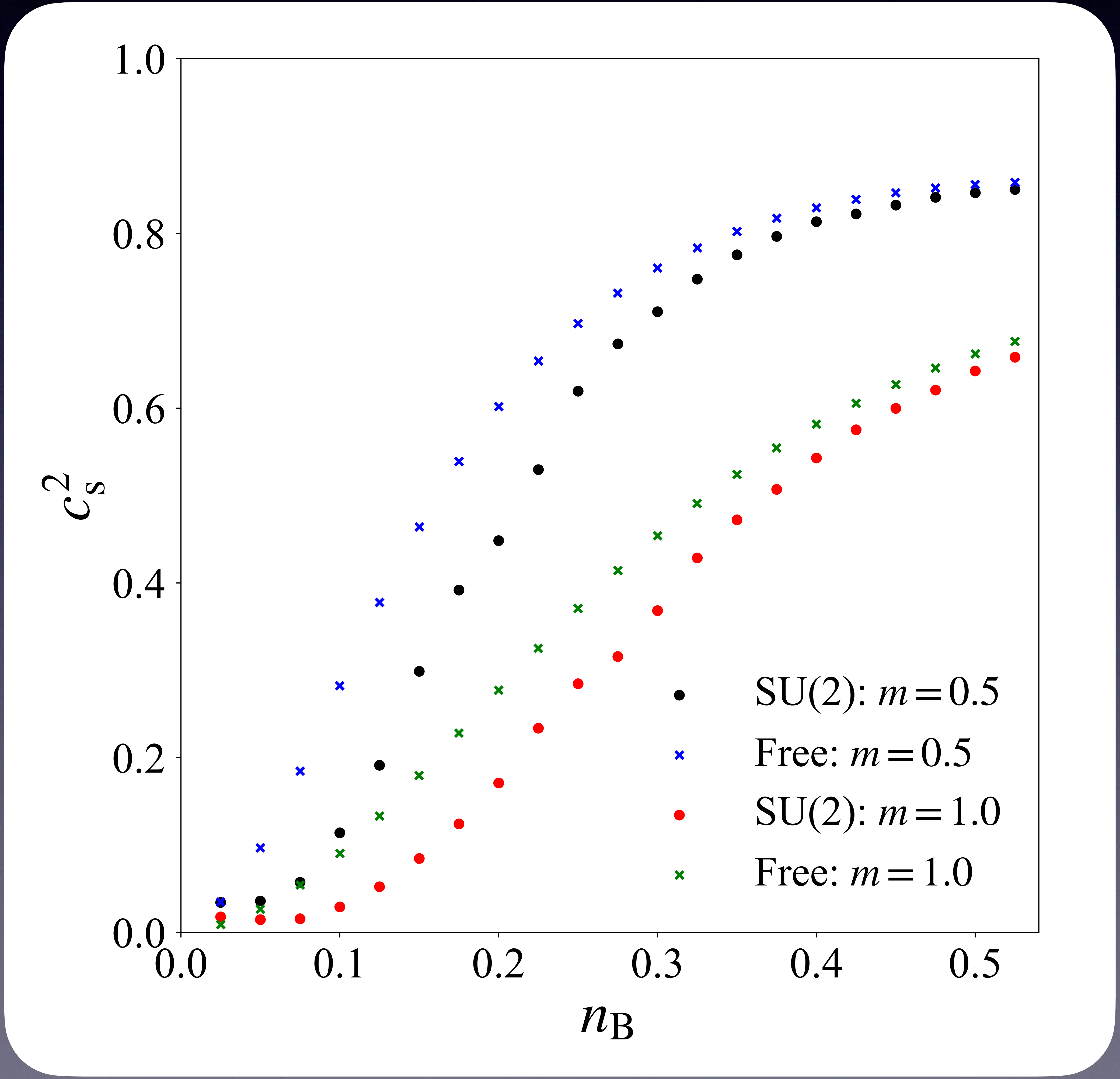
Color SU(2), 1 flavor, vacuum

$J = 1/8 \;\; w = 2 \;\; V = 40 \;\; \dim\mathscr{H} = 2^{320}$

Chemical potential

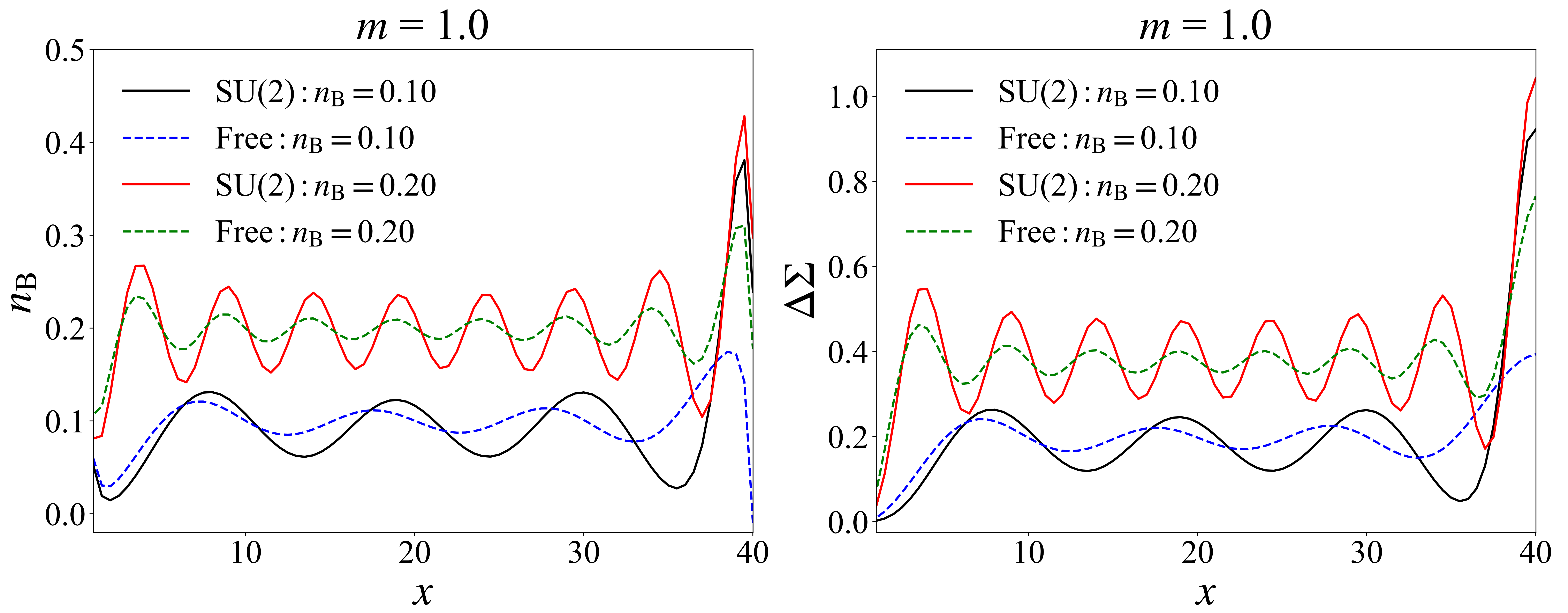


Sound velocity



Inhomogeneous phase

$J = 1/8 \;\; w = 2 \;\; V = 40 \;\; \dim\mathscr{H} = 2^{320}$

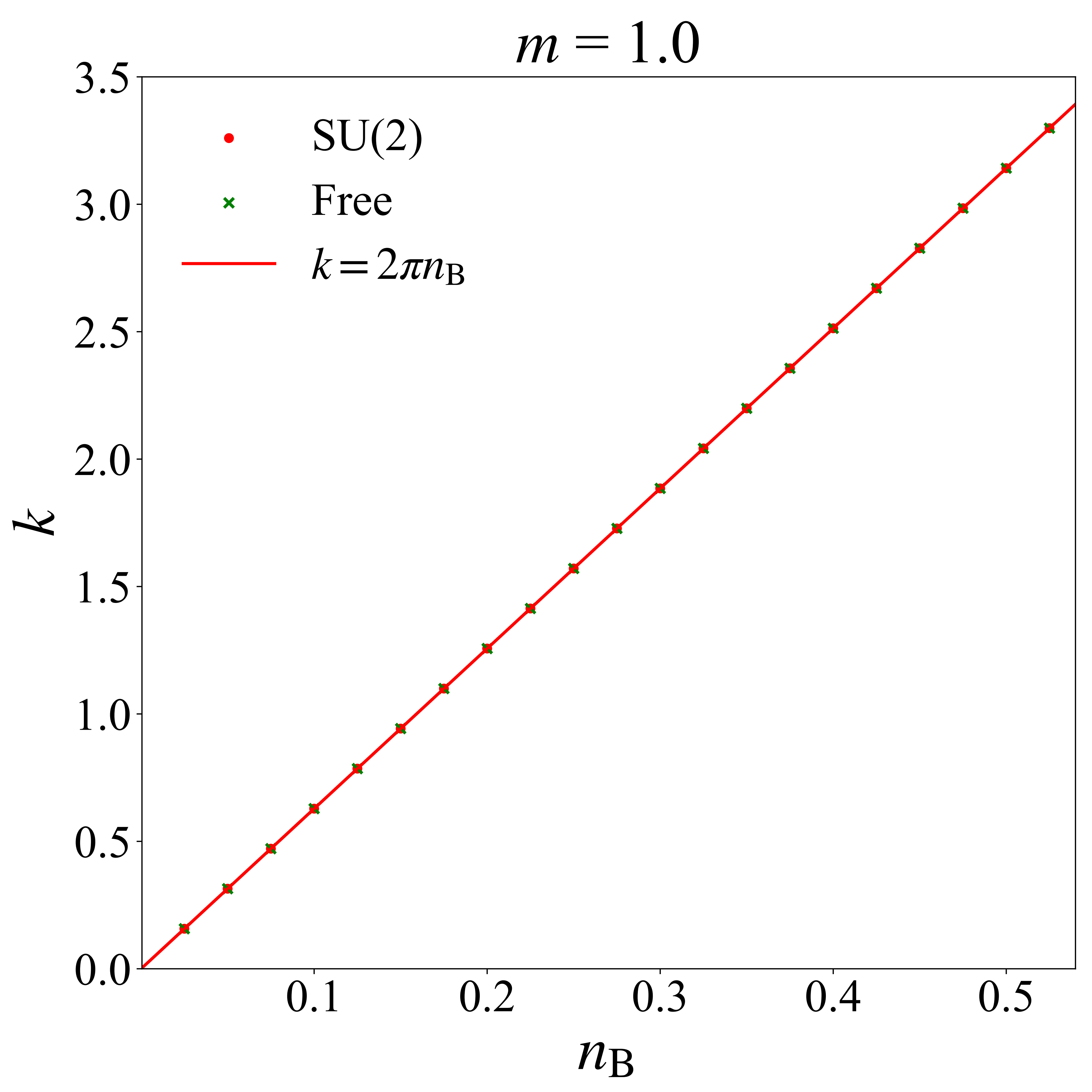


$$\Delta\Sigma = \langle \bar{\psi}\psi(x) \rangle - \langle \bar{\psi}\psi(x) \rangle_{\mu=0}$$

Wave number dependence

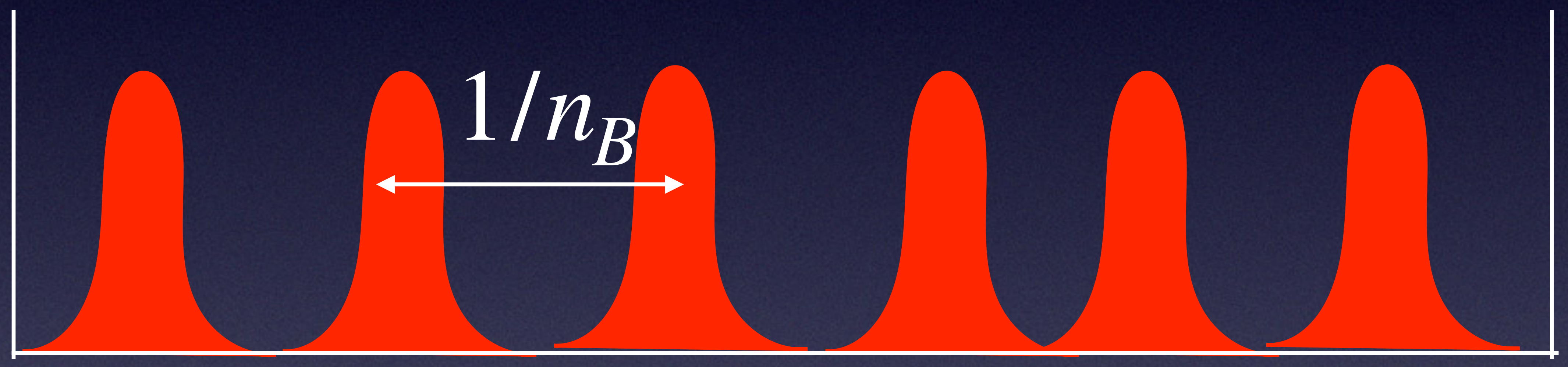
$J = 1/8 \;\; w = 2 \;\; V = 40 \;\; \dim\mathscr{H} = 2^{320}$

Wave number dependence



Hadronic picture

If hadron interactions are repulsive



distance $1/n_B \Rightarrow k = 2\pi n_B$

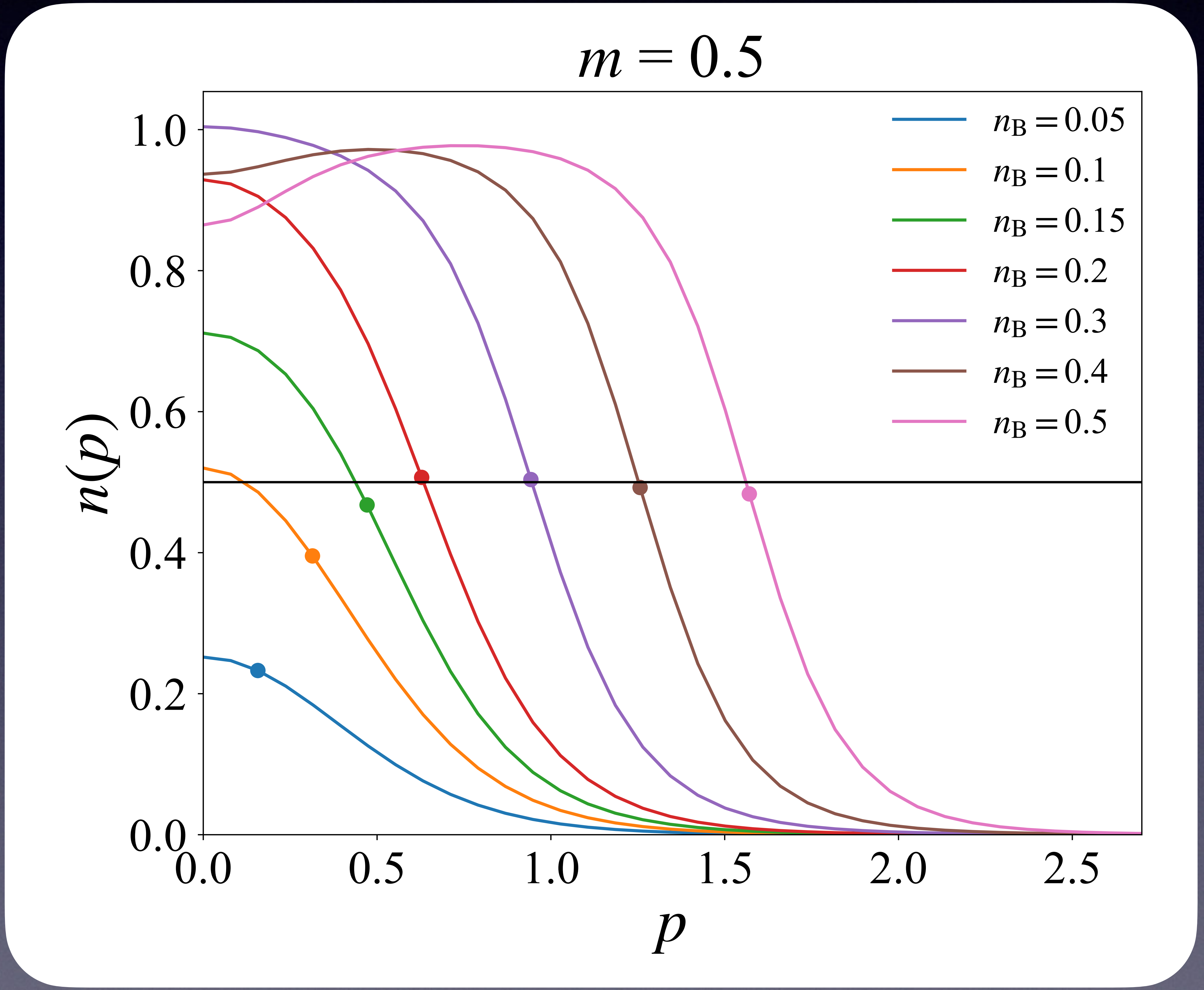
Quark picture

If interactions between quarks
Fermi surface is unstable

$\Rightarrow$density wave $k = 2p_{\rm F} = 2\pi n_{\rm B}$

Quark distribution function

$J = 1/8 \;\; w = 2 \;\; V = 60 \quad \dim\mathscr{H} = 2^{480}$



- **Low density**
  No Fermi sea

- **High density**
  Fermi-sea
  +density wave pairing

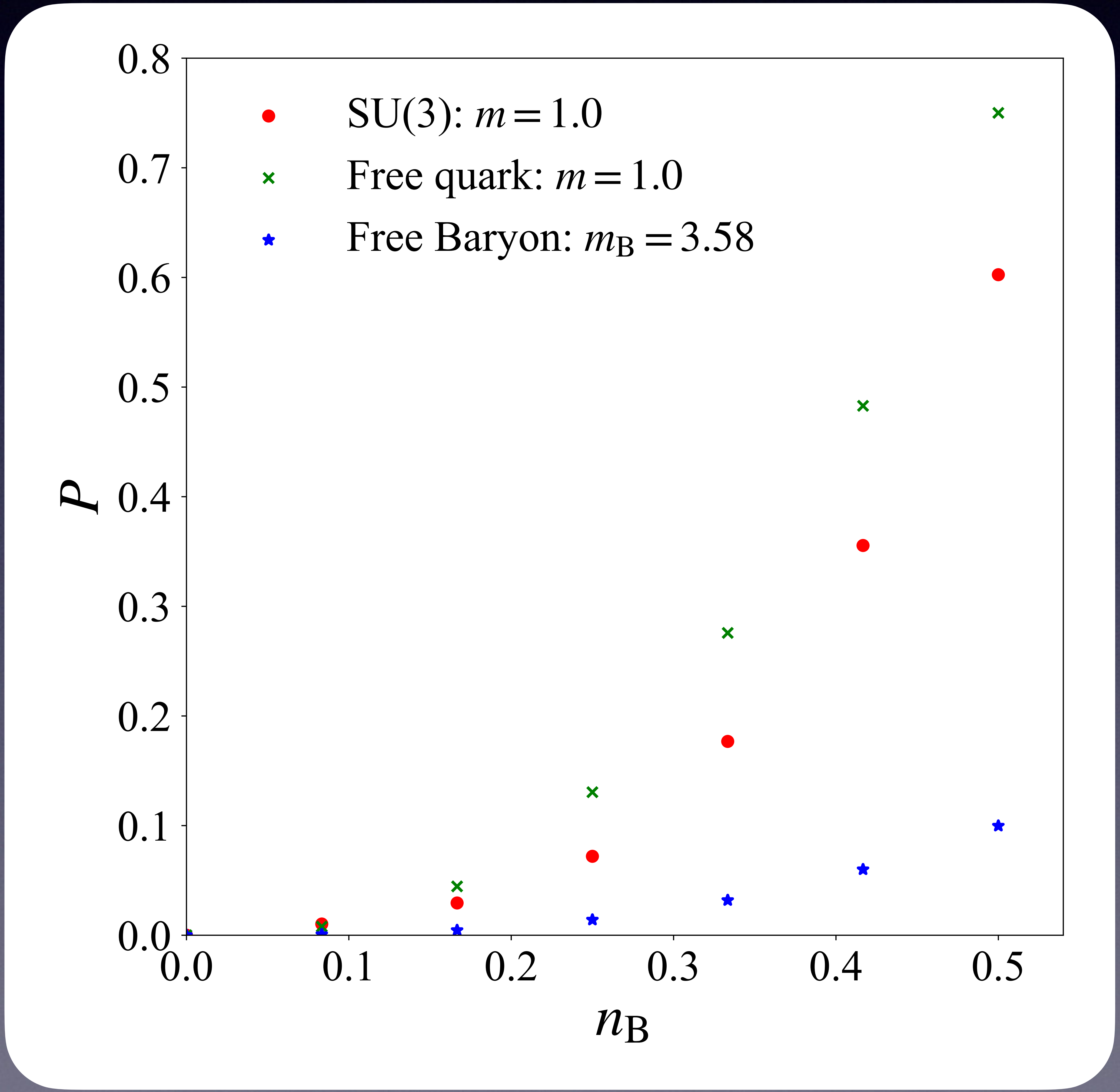
baryon quark transition around $n_B \sim 0.2$

SU(3) QCD with $N_f = 1$

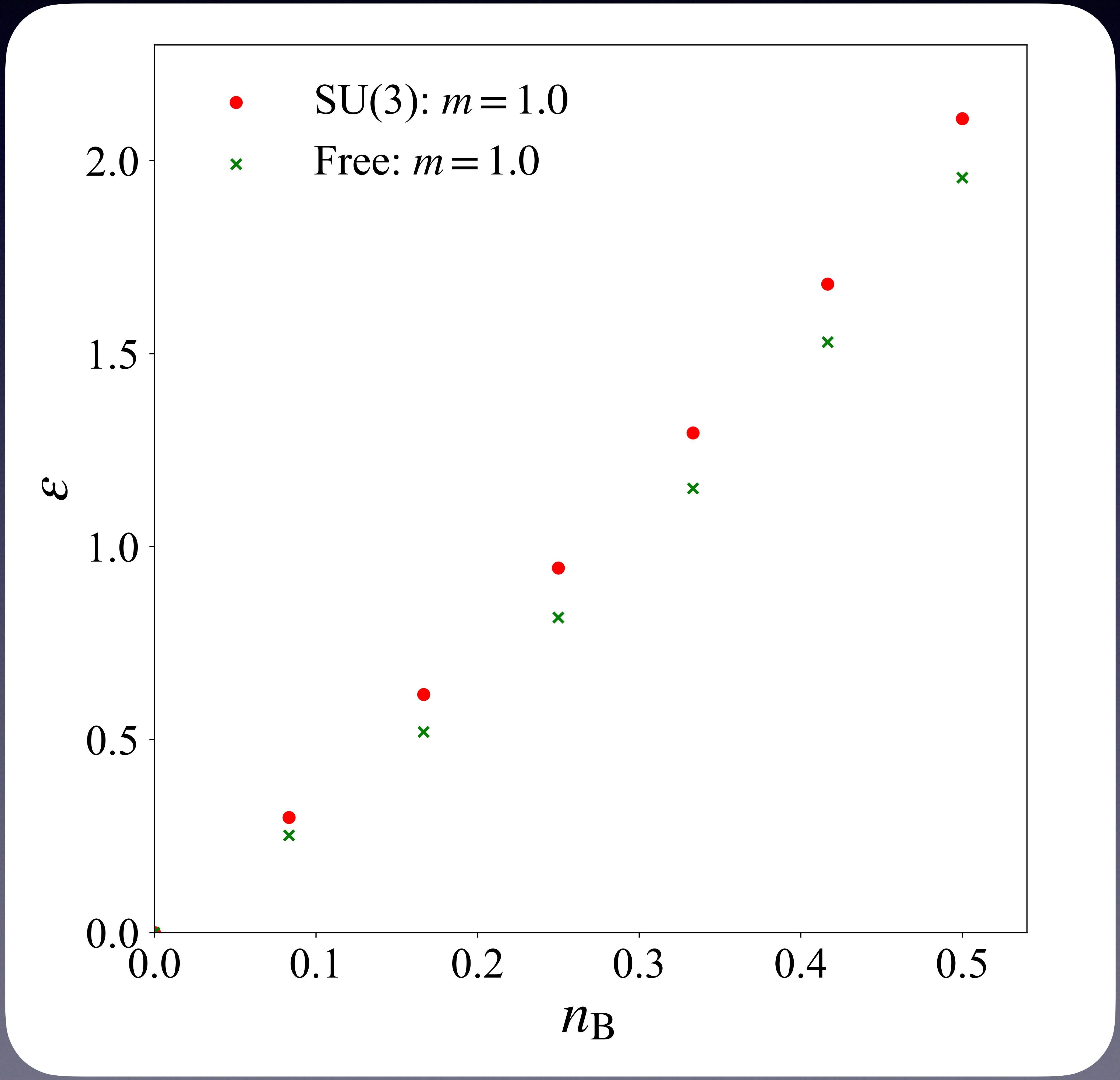
Color SU(3), 1 flavor, vacuum

$J = 1/8 \;\; w = 2 \;\; V = 12 \;\; \dim\mathscr{H} = 2^{144}$

Pressure

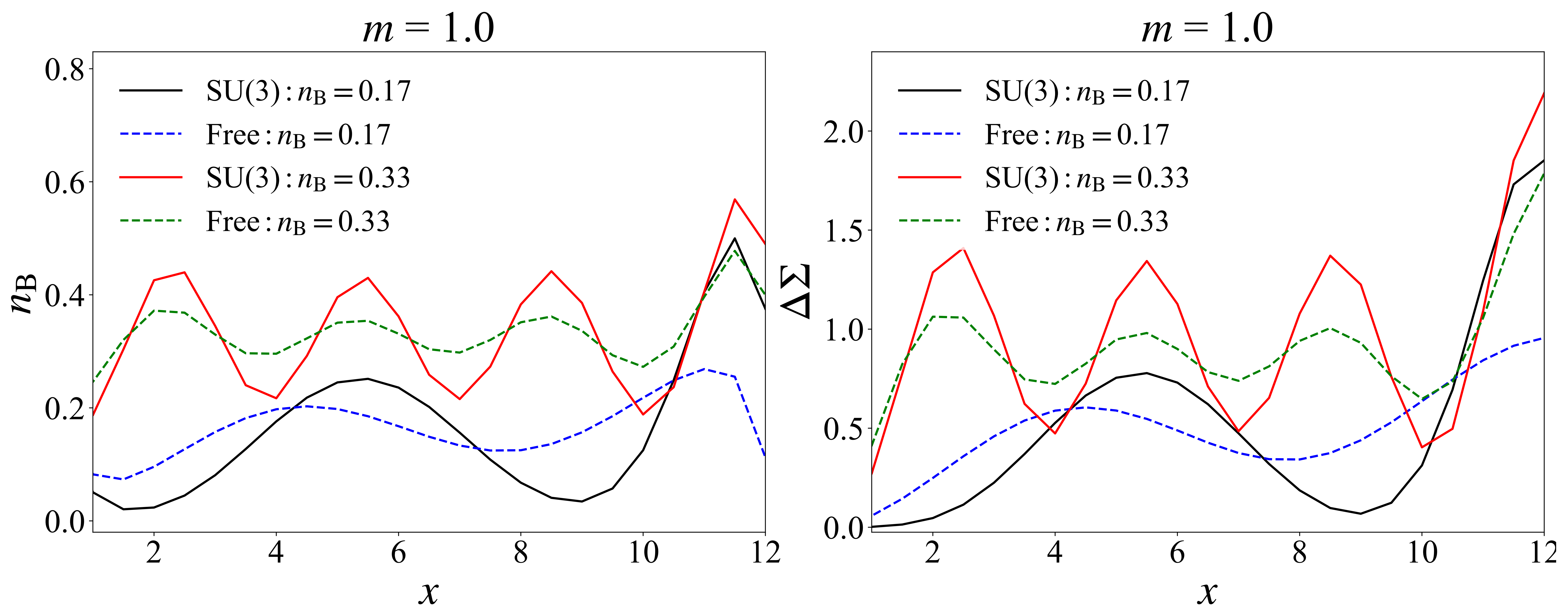


Energy density



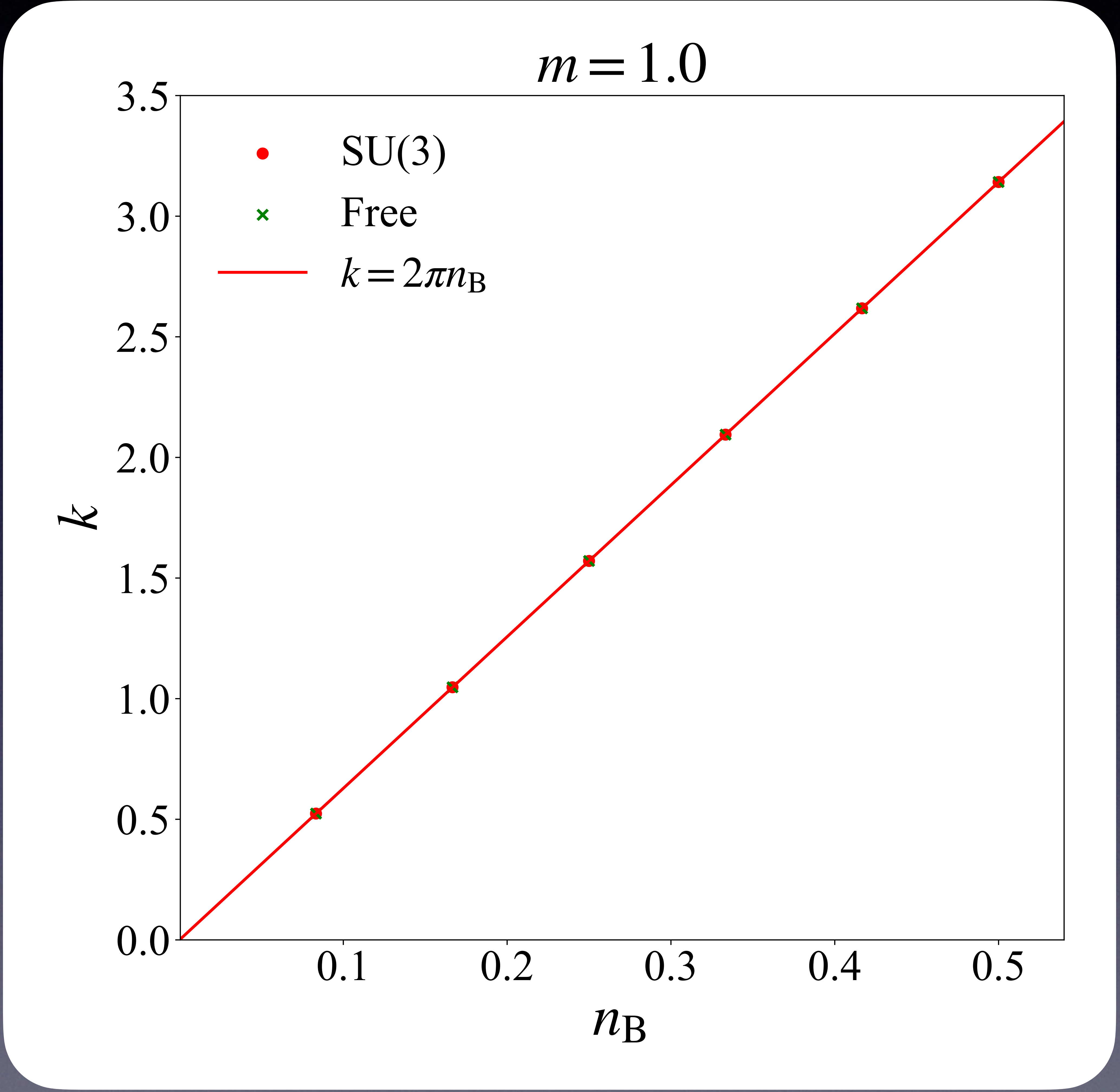
Inhomogenous phase

$J = 1/8 \;\; w = 2 \;\; V = 12 \;\; \dim\mathscr{H} = 2^{144}$

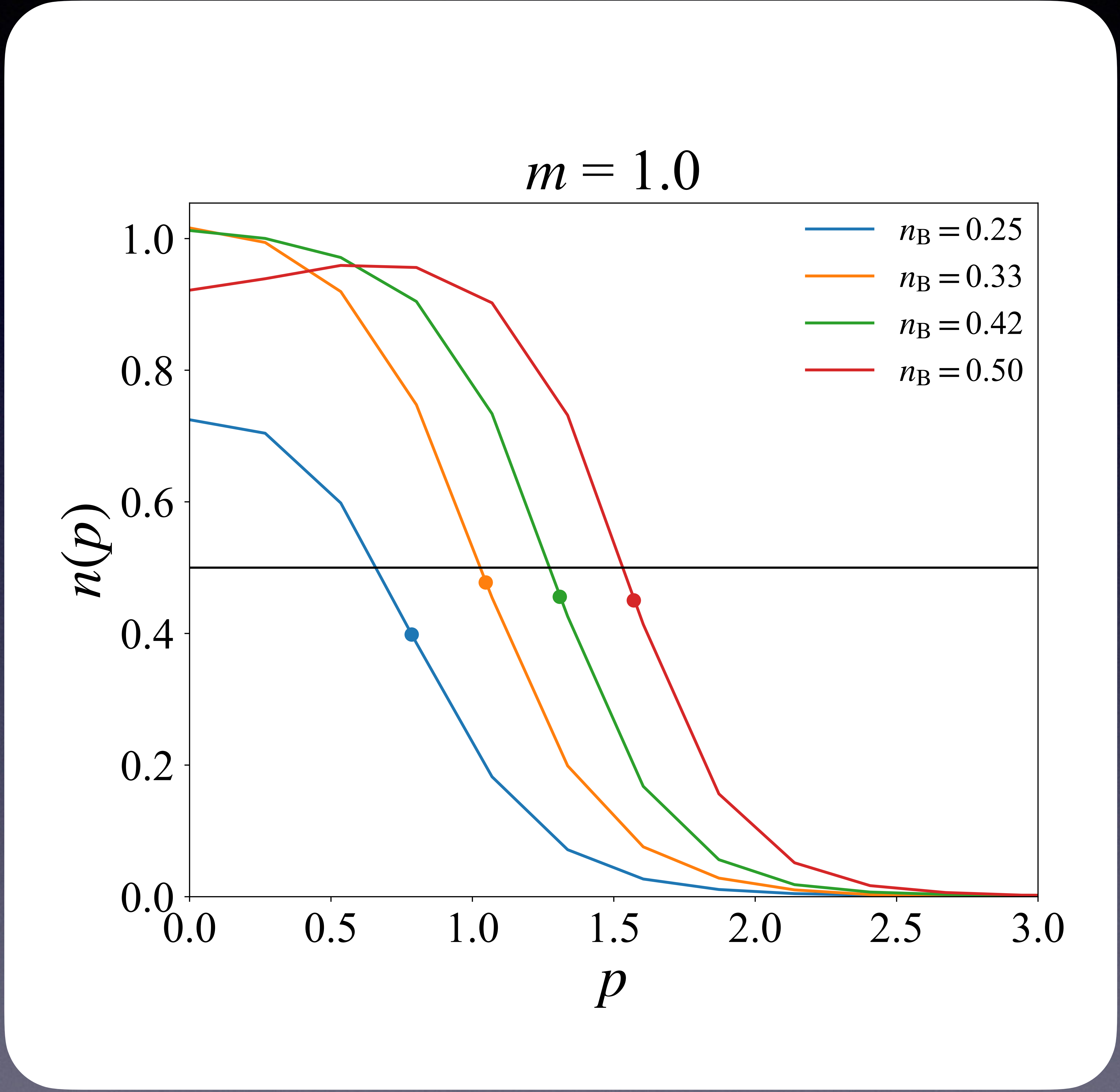


$$\Delta\Sigma = \langle \bar{\psi}\psi(x) \rangle - \langle \bar{\psi}\psi(x) \rangle_{\mu=0}$$

Wave number dependence



Quark distribution



Baryon quark transition around $n_B = 0.3$?

Summary

- **Formalism**

  Kogut-Susskind Hamiltonian formalism

- **Application**

  $\mathrm{SU}(3)_k$ **gauge theory in** $(2+1)$ **dimensions**

  Confinement-topological phase transition

  Thermalization of Yang-Mills theory in (3+1)-dimensional small systems

  Relaxation time of thermalization

  $\tau_{\mathrm{eq}} \sim 2\pi/T$ Boltzmann time

  QCD_2 **at finite density**

  baryon quark transition, inhomogeneous phase