Quasicrystal in QCD: Mixed Soliton of Pion and eta Meson

Zebin Qiu (Keio University)

Based on JHEP 05, 170 (2023), with Muneto Nitta
For "Condensed Matter Physics of QCD" 2024.3.20

Low Energy Dense QCD

Spontaneous Chiral Symmetry Breaking

$$U(1)_V \times SU(N_f)_L \times SU(N_f)_R \to U(1)_V \times SU(N_f)_V$$

Nambu Goldstone (NG) boson

$$\Sigma = \exp\left(iT^a \pi^a(x)/f_\pi\right) \in SU(N_f)$$

Chiral Perturbation Theory (ChPT):

$$\mathcal{L}_{\text{chiral}} = \frac{f_\pi^2}{4}\text{Tr}\left(\partial^\mu \Sigma \partial_\mu \Sigma\right) - \frac{b}{2}\text{Tr}\left[M(\Sigma - 1) + \text{h.c.}\right]$$

$N_f = 2$ case with approximately $m_u \approx m_d$

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \quad m_u \approx m_d \equiv m$$

Under Magnetic Field

Pions $\xrightarrow{\boldsymbol{B}}$ Chiral Soliton Lattice (CSL) / π^0 domain wall (DW)

D. T. Son and M. A. Stephanov, Phys. Rev. D 77, 014021 (2008).
T. Brauner and N. Yamamoto, JHEP 04, 132 (2017).

LLL: $n = 0$, Pion: Spin $s = 0$

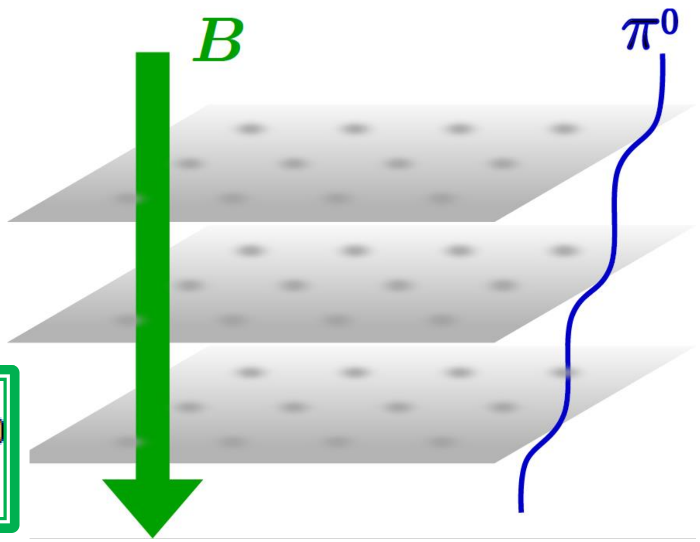
$$\varepsilon^2 = p_z^2 + \boxed{2\,|eB|\,(n + 1/2) + m^2 - 2seB}$$

$$\pi^{\pm} \xrightarrow{\boldsymbol{B}\uparrow} \text{“massive”},\ \Sigma \to \exp(i\tau^3\pi^3)$$

$\partial_{x,y} = 0$ to minimize the energy

$$\frac{f_\pi^2}{2}\left(\nabla\pi^0\right)^2 + m_\pi^2 f_\pi^2\left(1 - \cos\pi^0\right) \boxed{- \frac{\mu_* B}{4\pi^2}\partial_z\pi^0}$$

WZW term → z periodicity;
Sine-Gordon soliton



Homotopy

$$\pi_1\left(U\left(1\right)\right) = \mathbb{Z}$$

WZW Term & Triangle Anomaly

Coupling two U(1) via **triangle anomaly** to NG boson ϕ_i

$$\mathcal{L}_B = \frac{1}{8\pi^2}\epsilon^{\mu\nu\alpha\beta}\sum_i C_i \partial_\mu \phi_i A_\nu^B F_{\alpha\beta}$$

D. T. Son, M. A. Stephanov, and A. R. Zhitnitsky, Phys. Rev. Lett. 86, 3955 (2001);

Baryon chemical potential μ and magnetic field $\boldsymbol{B} = B\hat{z}$

$$U(1)_B : A_\nu^B = (\mu, \mathbf{0}). \quad U(1)_{\mathrm{EM}} : A_\mu = (0, yB/2, -xB/2, 0)$$

ϕ_i be not only $\pi^{\pm,0}$ but also η mesons (for larger density)

$$\phi_3 = \frac{\pi_3}{f_\pi} : \quad \mathcal{L}_{\mathrm{WZW}} = \frac{\mu B}{4\pi^2}\partial_z \phi_3 \Rightarrow \mathrm{CSL}$$

$$\phi_0 \equiv \frac{\eta}{f_\eta} : \quad \mathcal{L}_\eta = \frac{\mu B}{12\pi^2}\partial_z \phi_0 \Rightarrow \eta\text{-CSL?}$$

D. T. Son and A. R. Zhitnitsky , Phys. Rev. D 70, 074018 (2004).

Separated CSLs are equivalent in math. But mixed η and π^0?

U(2) ChPT in Magnetic Field

$\pi^{\pm}$ decoupled, η and π^0 remained: $U = \exp\phi_0 \exp(i\tau_3\phi_3)$

Kinetic Anomaly

$$H = \frac{1}{2}\left[\alpha\left(\partial_z\phi_3\right)^2 + \left(\partial_z\phi_0\right)^2\right] - \frac{\gamma}{2\pi}\left(\partial_z\phi_3 + \frac{1}{3}\partial_z\phi_0\right)$$

$$+ \sin\beta\left(1 - \cos 2\phi_0\right) + \cos\beta\left(1 - \cos\phi_0\cos\phi_3\right)$$ Mass

Redefine parameters to have dimensionless quantities

$$\alpha \equiv \frac{f_\pi^2}{f_\eta^2}, \quad \beta \equiv \arctan\frac{a}{2mb}, \quad \gamma \equiv \frac{\mu B}{2\pi}\frac{\left[a^2 + (4mb)^2\right]^{1/4}}{f_\eta}$$

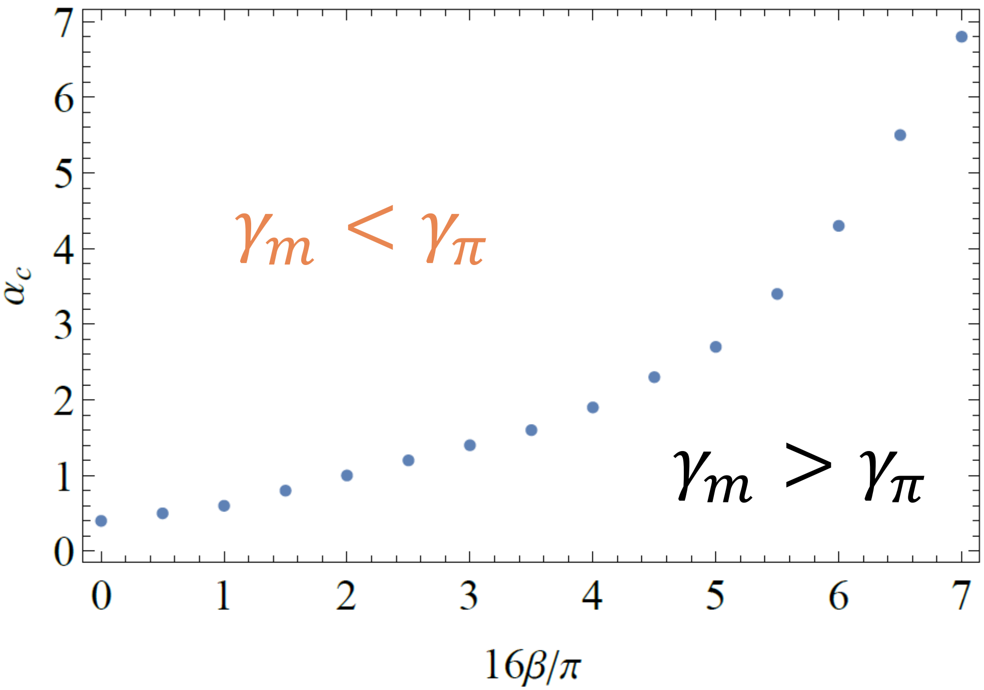
Boundary conditions to form Mixed Soliton Lattice

$$(\phi_3, \phi_0)\Big|_{z=0} = (0, 0), \quad (\phi_3, \phi_0)\Big|_{z=d} = (p\pi, q\pi), \quad \frac{p \pm q}{2} \in \mathbb{Z}$$

Mixed Soliton Lattice

Irrelevant $\gamma_\eta \gg \gamma_{\pi,m}$. Duel is between γ_π and γ_m.

(p,q)	Soliton Type	Critical magnetic field
(2,0)	π^0	γ_π
(0,2)	η	γ_η
(1,1)	Mixed	γ_m



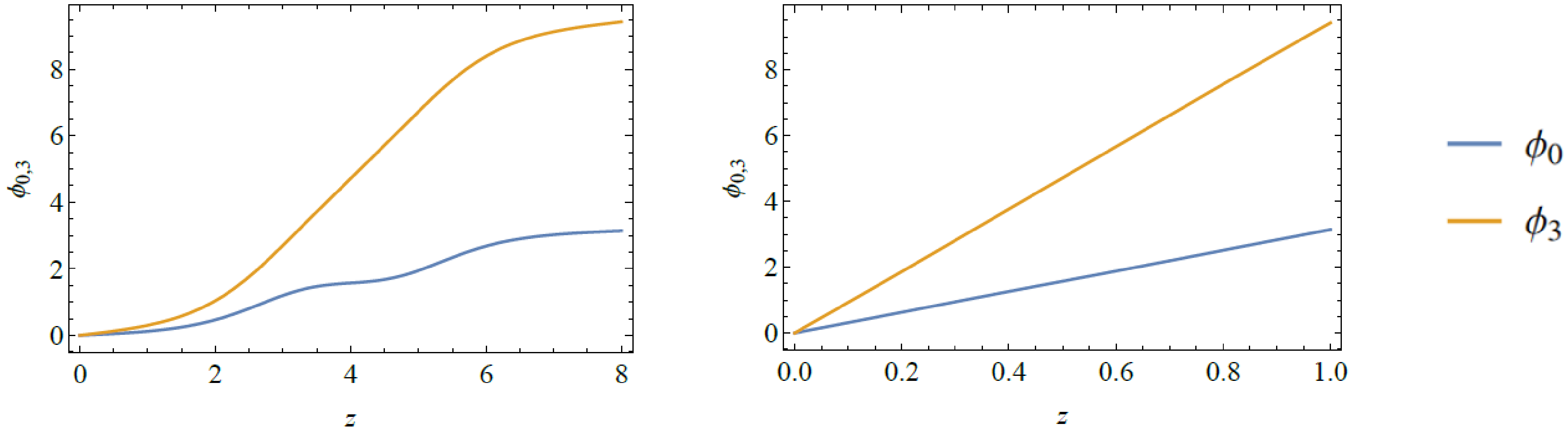
γ_m can be lowest! for $\alpha > \alpha_c(\beta)$

Strong Magnetic Field Limit

In $\gamma \to \infty$ limit, the ground state tends to take

$$r \equiv \frac{p}{q} = \frac{3}{\alpha}$$

Profiles: $\gamma \uparrow$, $d \downarrow$, $\phi_{3(0)} \to p(q)\pi z/d$; (almost linear)



Derivative large, period d (solved numerically) small, mass term subleading.

(Semi-) Analytical Proof

$\gamma \to \infty$, mass term becomes irrelevant:

$$H(\gamma \to \infty) \simeq \frac{1}{2}\left(\alpha\phi_3'^2 + \phi_0'^2\right) - \frac{\gamma}{2\pi}\left(\phi_3' + \frac{1}{3}\phi_0'\right) \equiv H_\infty .$$

Trick of total square finds us the minimum:

$$H_\infty = \frac{1}{2}\left[\left(\sqrt{\alpha}\phi_3' - \frac{\gamma}{2\sqrt{\alpha}\pi}\right)^2 + \left(\phi_0' - \frac{\gamma}{6\pi}\right)^2\right] - \frac{\gamma^2}{8\pi^2}\left(\frac{1}{\alpha} + \frac{1}{9}\right)$$

$$\geq -\frac{\gamma^2}{8\pi^2}\left(\frac{1}{\alpha} + \frac{1}{9}\right) \equiv E_{\min} .$$

Lattice period d_L and ratio p/q determined analytically

$$d_L = \frac{2\pi^2}{\gamma} \cdot p\alpha = \frac{2\pi^2}{\gamma} \cdot 3q \quad \Rightarrow \quad \frac{p}{q} = \frac{3}{\alpha},$$

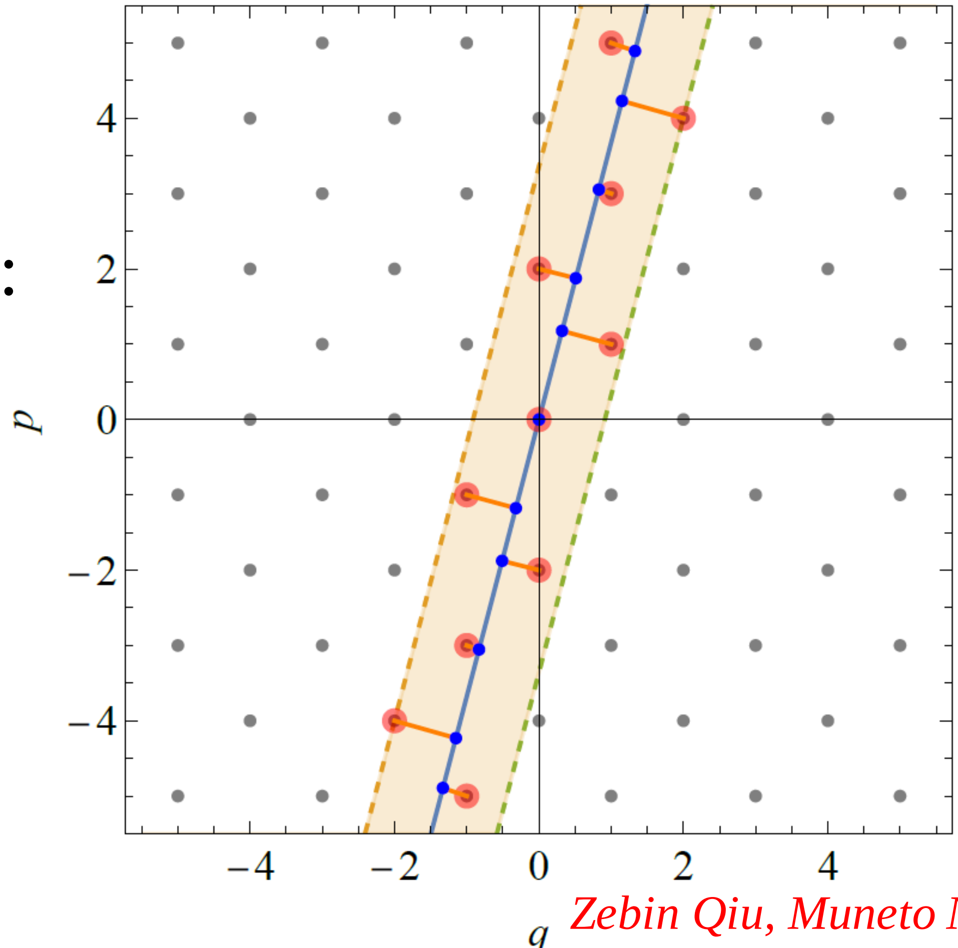
Quasicrystal in QCD

Grey dots: mixed soliton lattice of different *p*, *q*.

Blue solid line (example): $p/q = 3f_\eta^2/f_\pi^2$ (irrational)

Orange dots: *(p,q)* that adjacent to the blue line.

Beige band: possibly degenerate ground states, forming Quasicrystal!



Zebin Qiu, Muneto Nitta, JHEP 05, 170 (2023)

*validity (for future study) $\mu B \ll \Lambda^3, \quad \Lambda \simeq 4\pi f_{\pi,\eta}; \quad \gamma \ll 5.6\pi^2$

Remark: Rotational Counterpart

In parallel to WZW term, anomalous rotational ($\boldsymbol{\Omega} = \Omega \hat{z}$) Lagrangian is the coupling $j^5_{\rm CVE}$ to axial rotation.

$$\mathcal{L}^{\Omega}_{\eta} = \frac{\mu^2 \Omega}{6\pi^2} \partial_z \phi_0, \quad \mathcal{L}^{\Omega}_{\pi} = \frac{\mu \mu_I \Omega}{2\pi^2} \partial_z \phi_3$$

X.-G. Huang, K. Nishimura and N. Yamamoto, JHEP 02 (2018) 069

We consider no isospin checmial potential $\mu_I = 0$. Recap:

	$\boldsymbol{\eta}$	$\boldsymbol{\pi^0}$
Rotation $\boldsymbol{\Omega}$	$(\partial_z \phi_0) \mu^2 \Omega / 6\pi^2$	0
Magnetic field $\boldsymbol{B}$	$(\partial_z \phi_0) \mu B / 12\pi^2$	$(\partial_z \phi_3) \mu B / 4\pi^2$

In rotational case, the net winding number must be from η only. But π^0 can have local "up and down"

Non-Abelion CSL

η-CSL can split into "dimer", i.e., "Up" $\oplus$ "Down"

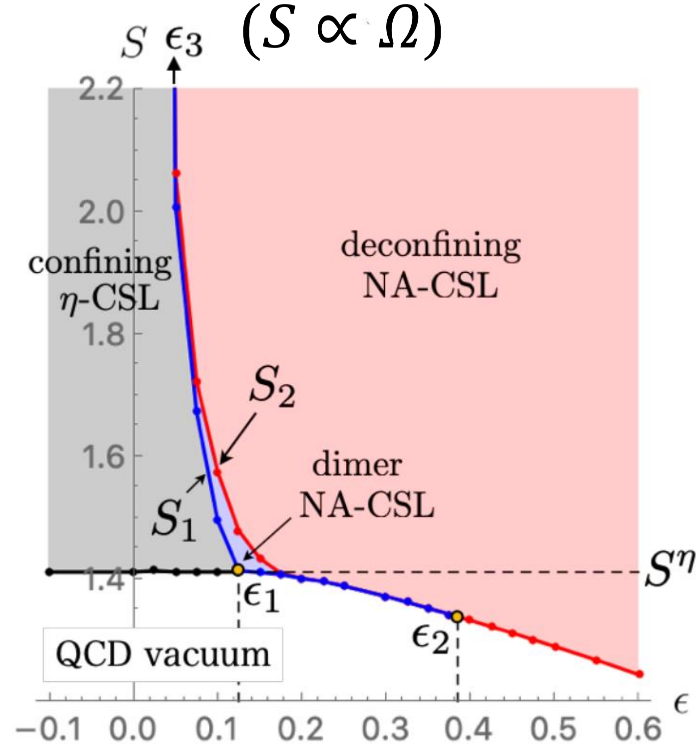
$$(\phi_3, \phi_0)\Big|_{z=0} = (0,0)\,;$$

$$(\phi_3, \phi_0)\Big|_{z=l} = \begin{cases} (0, 2\pi) & \eta\text{-CSL} \\ (\pi, \pi) & \text{"Up" branch} \\ (-\pi, \pi) & \text{"Down" branch} \end{cases}$$

Each branch is a "mixed soliton"
The interaction between branches
1. attractive: η-CSL; 2. null: dimer; 3. repulsive: NA-CSL

$$\mu^2 \Omega_c \sim 1.3 \cdot N_c 2\pi^2 f_\eta^2 m_\eta$$

$(S \propto \Omega)$



$(\epsilon = 1 - f_\pi^2/f_\eta^2 \sim 0.3)$

M. Eto, K. Nishimura and M. Nitta, JHEP 08 (2022) 305

Conclusion

- Depending on decay constants and effective masses, the **mixed soliton lattice** of η and π^0 can have lower energy / critical magnetic field than separate ones.

- In strong magnetic field / density limit, the ground state ratio *p/q* approaches $3f_\eta^2/f_\pi^2$ which is generally irrational. The result is a **mesonic quasicrystal**.

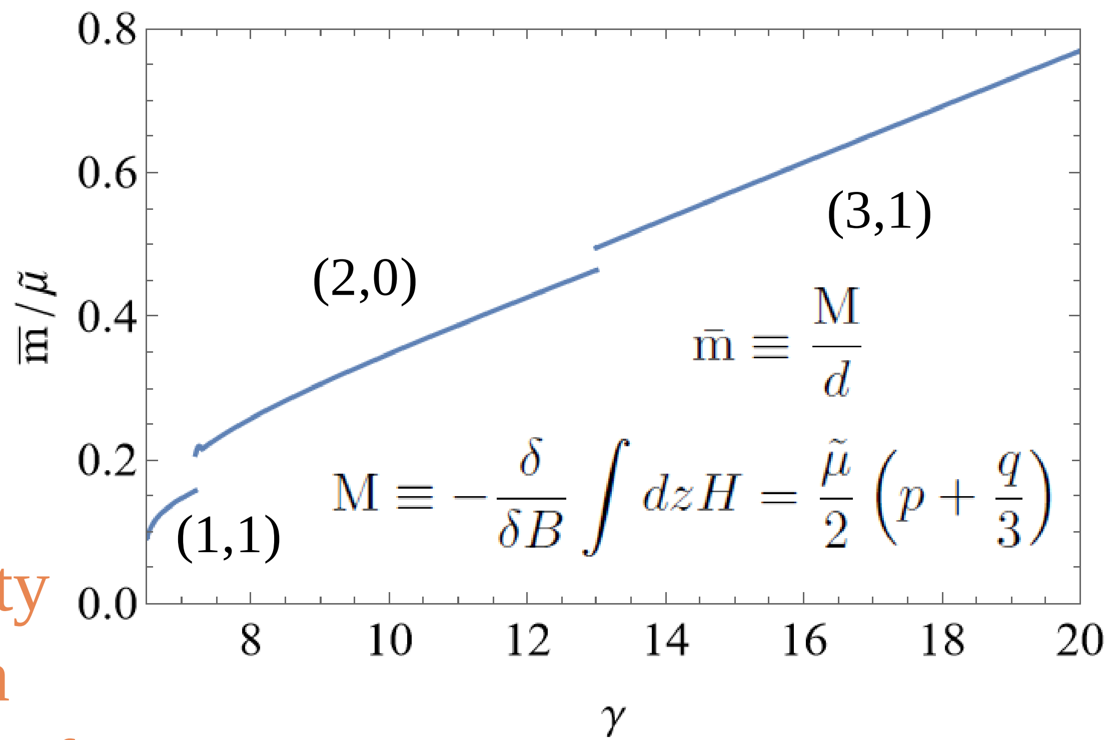
- In a realistic (yet strong) magnetic field, the density for eta meson to become physically relevant could be lower than thought, via mixing with other mesons.

Outlook: Intermediate *B*

Example: up to p+q=4, under $\alpha = 0.7$ and $\beta = \pi/16$: competitive configurations are (1,1), (2,0), and (3,1) with critical $\gamma_m = 6.5$, $\gamma_\pi = 6.7$ and $\gamma_{31} = 7.3$.

Ground State

$$= \begin{cases} (1,1) & \gamma \in [6.4,\ 7.2) \\ (2,0) & \gamma \in [7.2,\ 13.0) \\ (3,1) & \gamma \in (13.0,\ \ldots) \end{cases}$$



Magnetic moment density quantifies the alternation as a piece-wise function of μB