Towards "*universal*" energy density functionals of non-relativistic spin-1/2 Fermi gases

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Outline

- Introduction
- Formalism
- Results and discussion
- Summary and future perspective

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Energy density of nonrelativistic spin-1/2 Fermi gases

<u>EDF</u>: $E[n]$ ➡ Application to DFT, TDDFT

However, its exact form is unknown except for free gases

$$E_{\mathrm{FG}}(n) = \frac{3k_{\mathrm{F}}^3}{10m} n \equiv \frac{3\left(3\pi^2\right)^{\frac{2}{3}}}{10m} n^{\frac{5}{3}}$$

Can we extend it to the interacting case?

$$f(n,V) \equiv \frac{E(n,V)}{E_{\mathrm{FG}}(n)}$$

<u>Partial wave expansion of the bare interaction V</u>

$$V = V_s(a_s, r_s, ..) + V_p\left(v_p, k_p, ..\right) + \cdots$$

a_s: s-wave scattering length
r_s: s-wave effective range
v_p: p-wave scattering volume
k_p: p-wave effective momentum

➡

<u>Approach from the dilute limit</u>

Homogenous matter EOS ($k_{\mathrm{F}} \to 0$)

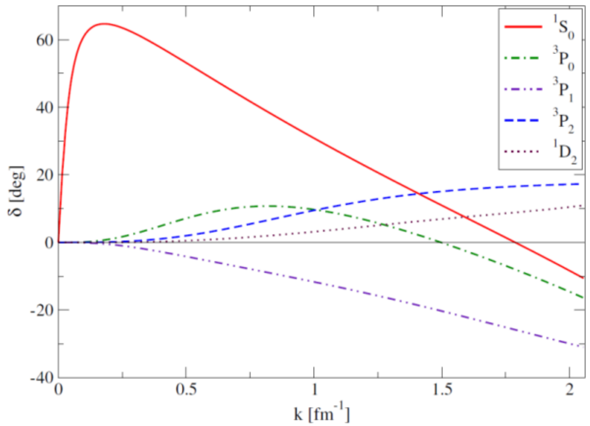
$$f(n,V) \equiv f(k_{\mathrm{F}} a_s, k_{\mathrm{F}} r_s, v_p k_{\mathrm{F}}^3, \ldots)$$

"Universal thermodynamics" and Bertsch parameter

Suppose the *s*-wave interaction is dominant and taking large scattering length limit, EDF does not depend on details of the interaction like a free gas.

Ya. B. Zel'dovich, JETP **38**, 1123 (1960).

<u>NN scattering phase shift</u>



A. Gezerlis, *et al*, arXiv : 1406.6109v2

$$k\cot\delta_k = -\frac{1}{a_s} + \frac{1}{2}k^2 r_s + \cdots$$

$$a_s = -18.5\text{ fm},\ r_s = 2.8\text{ fm}$$

<u>Universal thermodynamics in a unitary Fermi gas</u>

$$f(n,V) \to f(k_\mathrm{F} a_s \to \infty) \equiv \xi_\mathrm{B}$$

$$E(n,V) \to E(k_\mathrm{F} a \to \infty) \equiv \xi_\mathrm{B} E_\mathrm{FG}(n)$$

ξ_B: Bertsch parameter

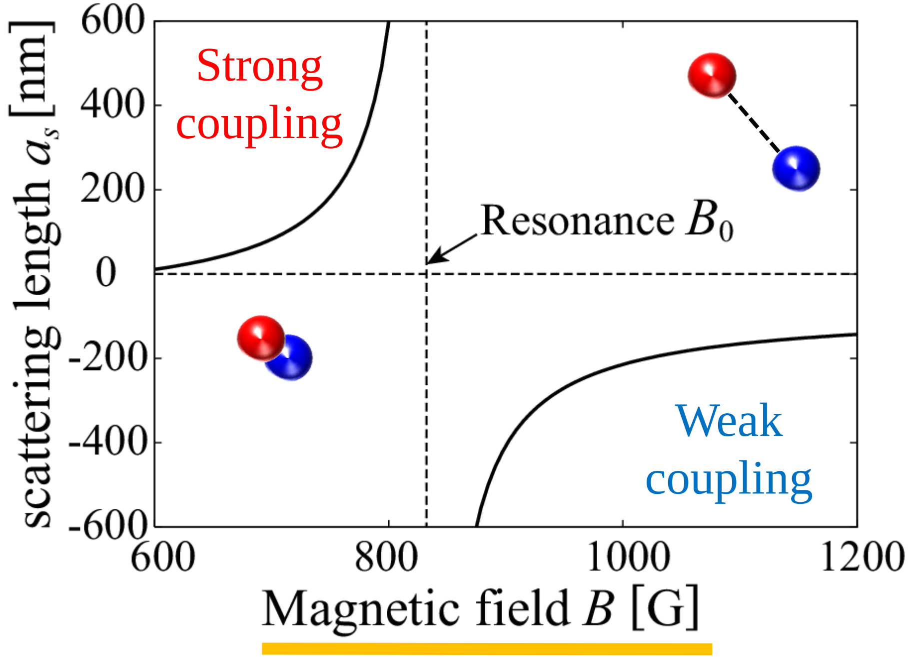
G. F. Bertsch,
Challenge problem
in many-body
physics (1999)

Tunable scattering length in ultracold Fermi gases

"Academic problem" can be now tested in cold atoms

Feshbach resonance in Fermi gases

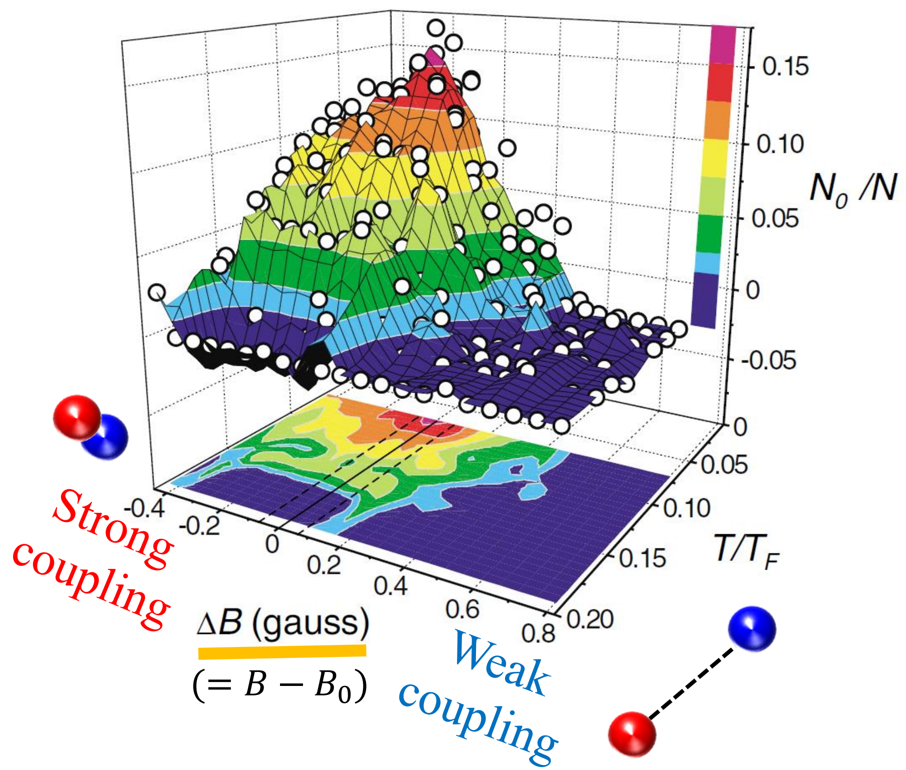
Scattering length a_s can be tuned by applying the external magnetic field



G. Zürn, *et al.*, PRL **110**, 135301 (2013).

Observation of BCS-BEC crossover

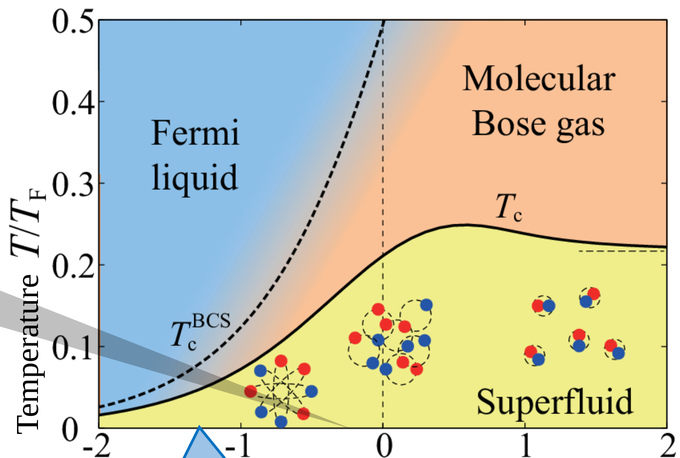
N_0: Condensate fraction



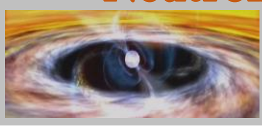
C. Regal, *et al.*, PRL **92**, 040403 (2004).

BEC-BCS crossover

Y. Ohashi, HT, and P. van Wyk, Prog. Part. Nucl. Phys. **111**, 103739 (2020).



Neutron matter



$a_s = -18.5\mathrm{fm}$

$(k_F a_s)^{-1} \simeq 0$

http://astro.riken.jp/wordpress/?page_id=1425

(Weak-coupling) BCS ← $(k_F a_s)^{-1}$ → BEC (Strong-coupling)

Energy density from EFT up to 4th order

$$E(k_F) = n\,\varepsilon_F \left[\frac{3}{5} + (g-1) \sum_{\nu=1}^{\infty} \mathcal{C}_\nu(k_F) \right]$$

g: spin multiplicity (g=2 for spin-1/2)

$$\mathcal{C}_1(k_F) = \frac{2}{3\pi} k_F a_s,$$

$$\mathcal{C}_2(k_F) = \frac{4}{35\pi^2}(11 - 2\ln 2)(k_F a_s)^2,$$

$$\mathcal{C}_3(k_F) = [0.0755732(0) + 0.0573879(0)\,(g-3)](k_F a_s)^3 + \frac{1}{10\pi}(k_F a_s)^2 k_F r_s + \frac{1}{5\pi}\frac{g+1}{g-1}(k_F a_p)^3,$$

$$\mathcal{C}_4(k_F) = -0.0425(1)\,(k_F a_s)^4 + 0.0644872(0)\,(k_F a_s)^3 k_F r_s + \gamma_4(k_F)\,(g-2)\,(k_F a_s)^4.$$

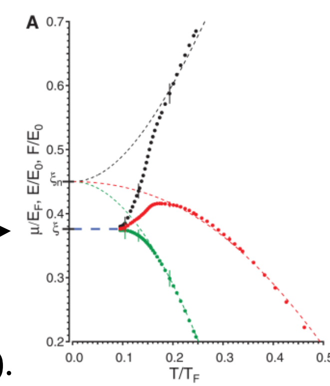
C. Wellenhofer, *et al.*, PRC **104**, 014003 (2021).

Unitarity limit $(k_F a)^{-1} = 0$

$$E(k_F) = \xi_B E_{FG}(k_F)$$

Determined experimentally from EOS

$\xi_B \simeq 0.37$ →



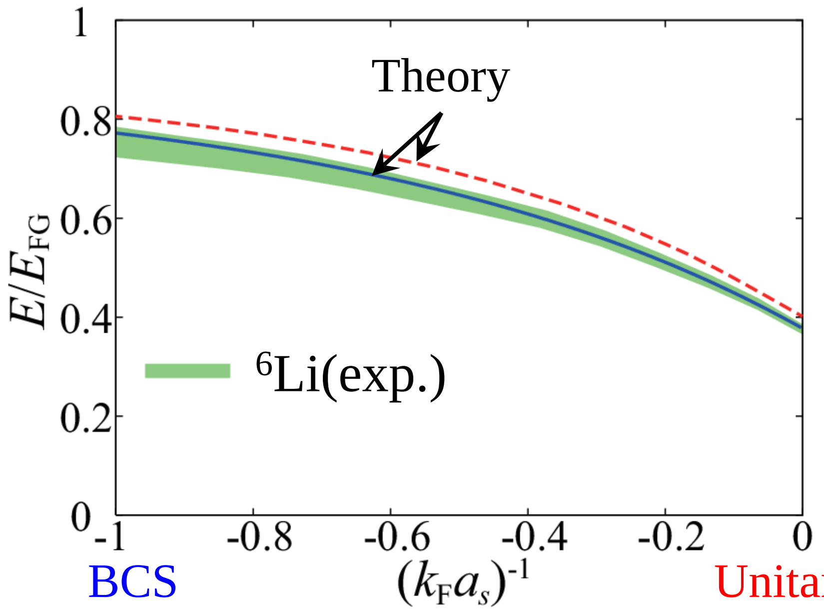
M. J. H. Ku, *et al.*, Science **335**, 3, (2012).

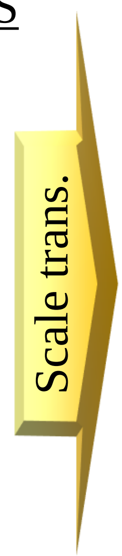
Unitary Fermi gas and neutron matter

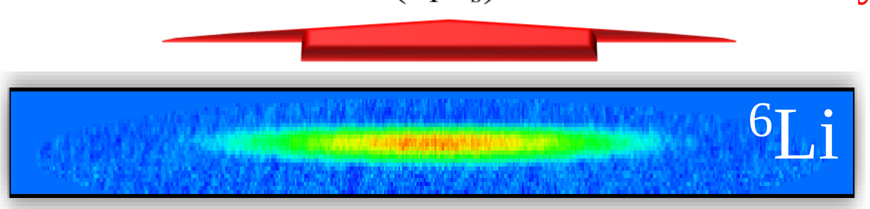
- The low-density neutron matter is also dominated by <u>the *s*-wave scattering</u> like an ultracold Fermi gas

M. Horikoshi, M. Koashi, HT, Y. Ohashi, and M. Kuwata-Gonokami, PRX, **7**, 041004 (2017).

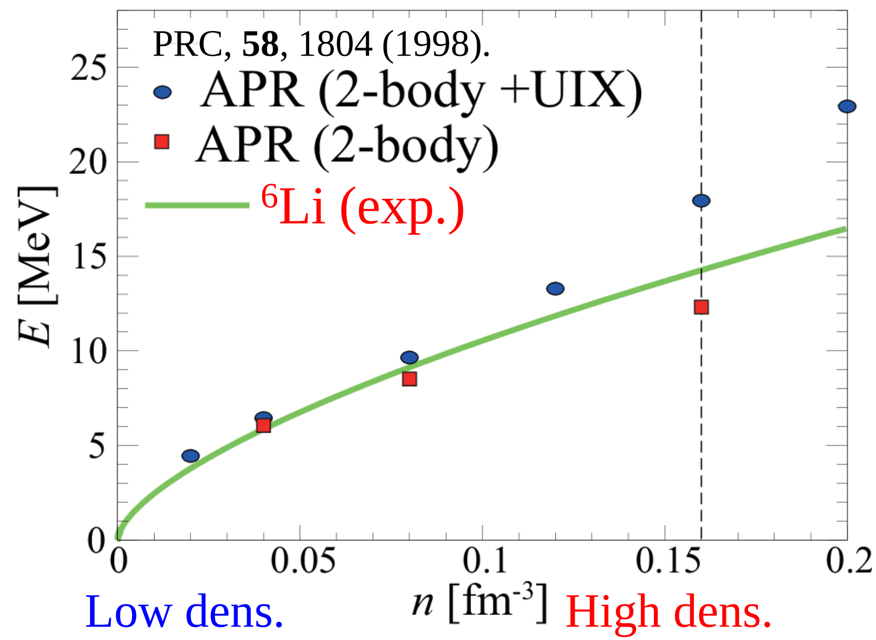
- <u>Precise measurement of cold atom EOS</u>







- <u>EOS of neutron matter and cold atom</u>



Agreement in the low density region

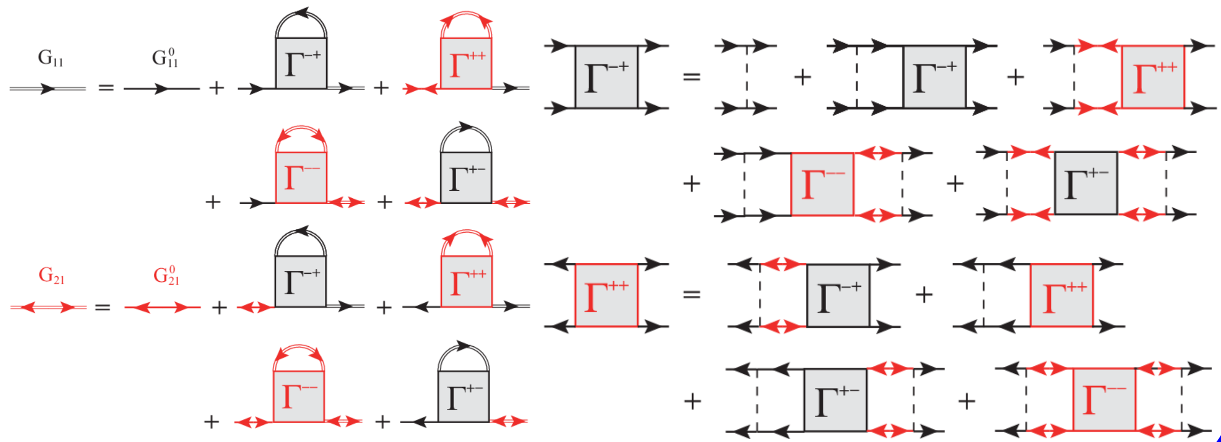
Unitary Fe

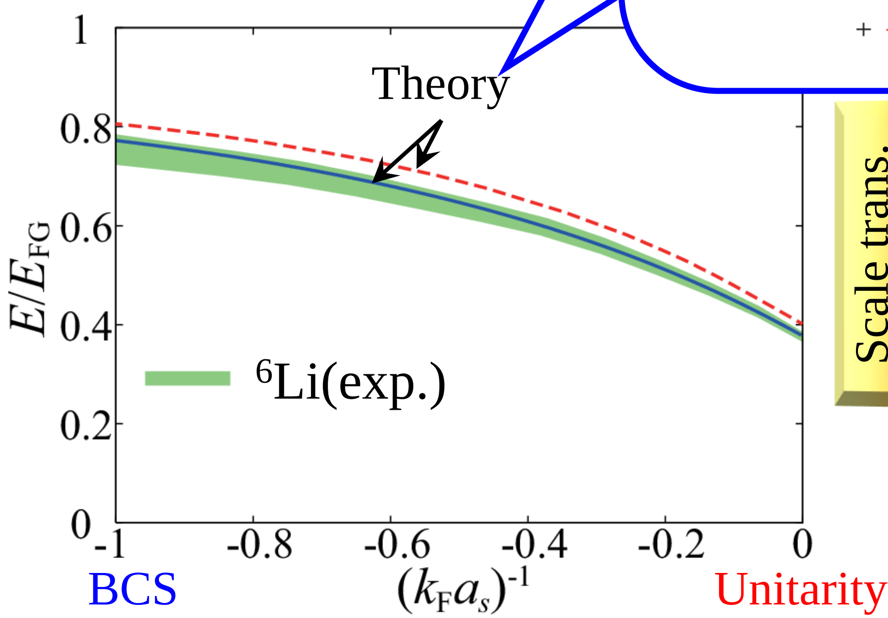
- The low-density
  the s-wave scatt

M. Horikoshi, M. Koashi,

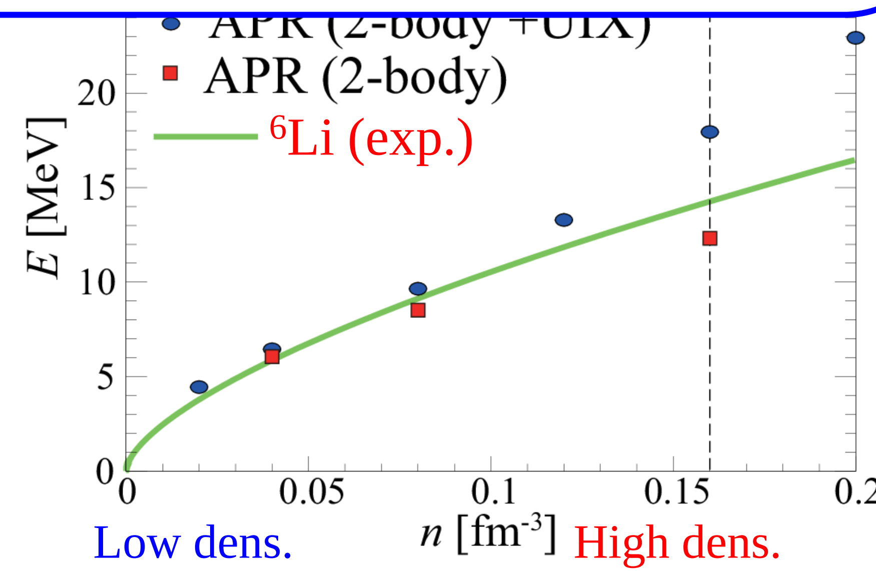
▸ Precise measurement of

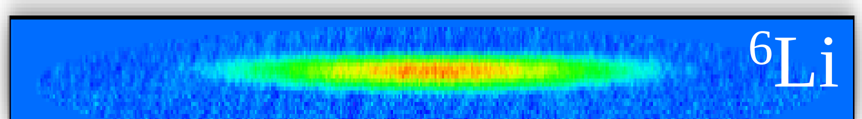
Many-body T-matrix approach: HT, *et al.*, PRA **95**, 043625 (2017).





Scale trans.





Agreement in the low density region

Density-induced BEC-BCS crossover

What happen when the effective range is not negligible?

(e.g., $k_{\mathrm{F}}r\sim 1$ becomes non-negligible in neutron matter at subnuclear density)

<u>Scattering phase shift</u>

$$k\cot\delta_s(k) = -\frac{1}{a} + \frac{1}{2}rk^2 - \mathcal{S}r^3k^4 + O(k^5)$$

a: scattering length
r: effective range
S: shape parameter

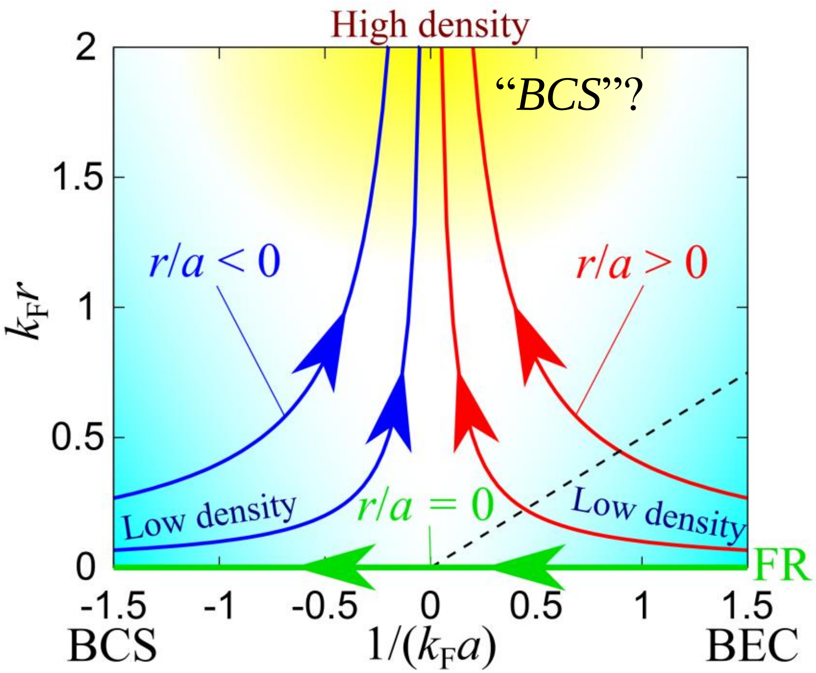
Cold atoms (zero-range interaction, $r/a = 0$)

Neutron matter (dineutron pairing) ($r/a < 0$)

Nuclear matter (deuteron pairng) ($r/a > 0$)

*In this talk, we consider only the positive *r*

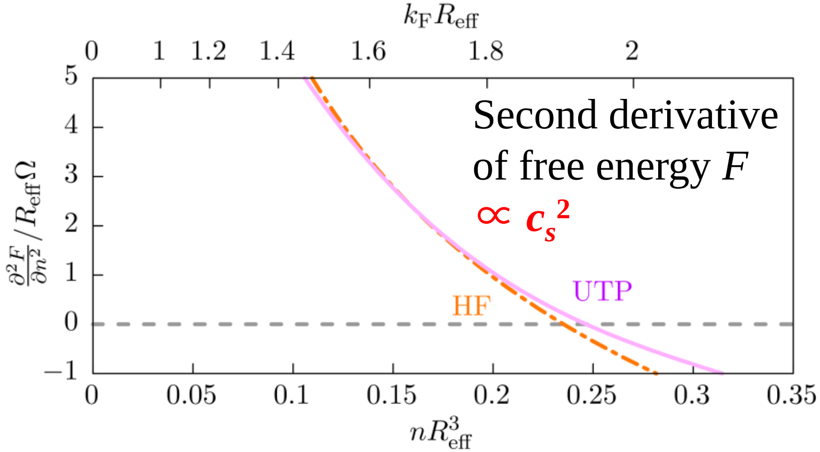
<u>Generalized crossover</u>



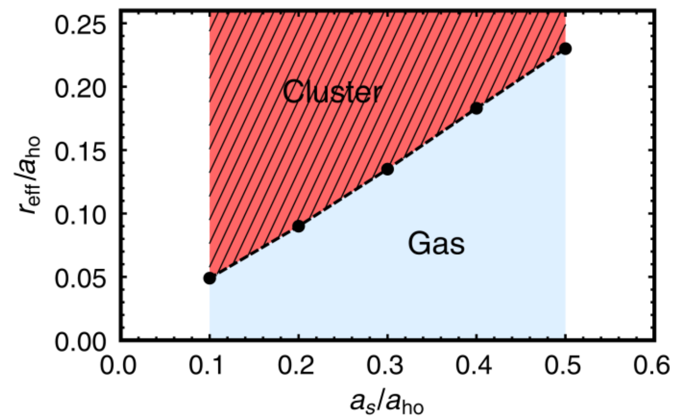
Dashed line: $\cot\delta_s(k = k_{\mathrm{F}}) = 0$
HT, JPSJ **88**, 093001 (2019).

Mechanical instability in two-component Fermi gases with finite-range interaction

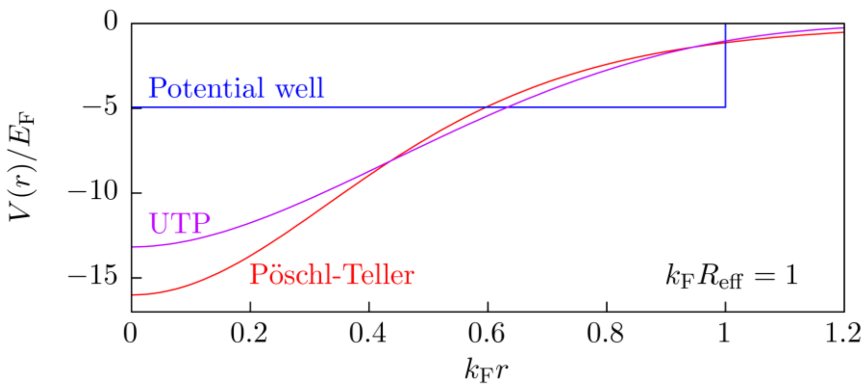
Collapse at divergent a_s
(DMC and HF calculation)
PRA **95**, 013633 (2017).

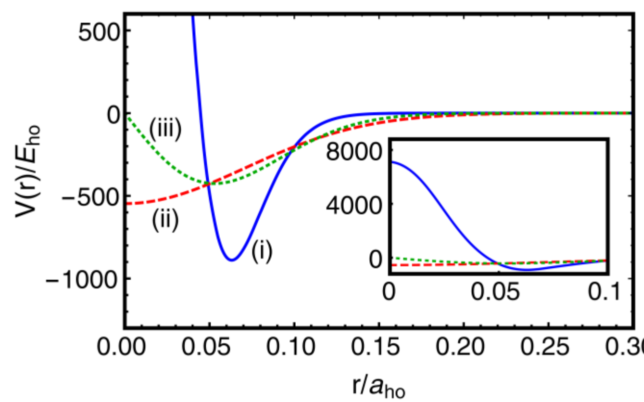


Cluster formation in trapped gases
(few-body calculation)
PRL **123**, 073401 (2019).



Interaction used in these work (in principle, nonzero shape parameter S…)
➡ unclear whether it is the effective-range correction or the effect beyond r_{eff}





Energy density and Tan's relation

How does the exact relation for the zero-range interaction can be modified?

Zero-range interaction ($r = 0$)

Tan's contact

$$n(\boldsymbol{k}) \xrightarrow[k\to\infty]{} \frac{C}{k^4}$$

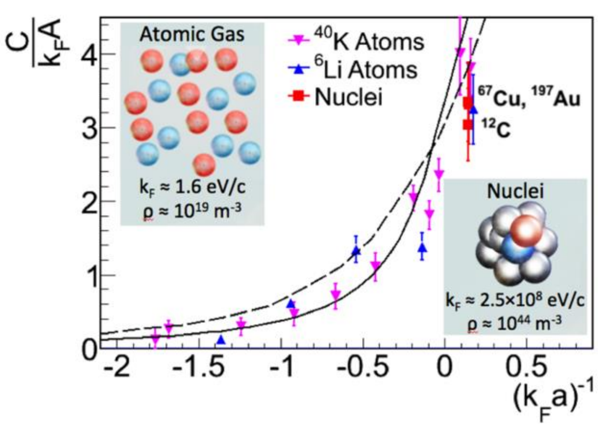
S. Tan, Ann. Phys. (NY) **323**, 2952, (2008).

Tan's energy relation

$$E = \sum_{k,\sigma} \frac{k^2}{2m}\left[n(\boldsymbol{k}) - \frac{C}{k^4}\right] + \frac{C}{4\pi m a}$$

=

?



O. Hen, *et al.*, PRC **92**, 045205 (2015)

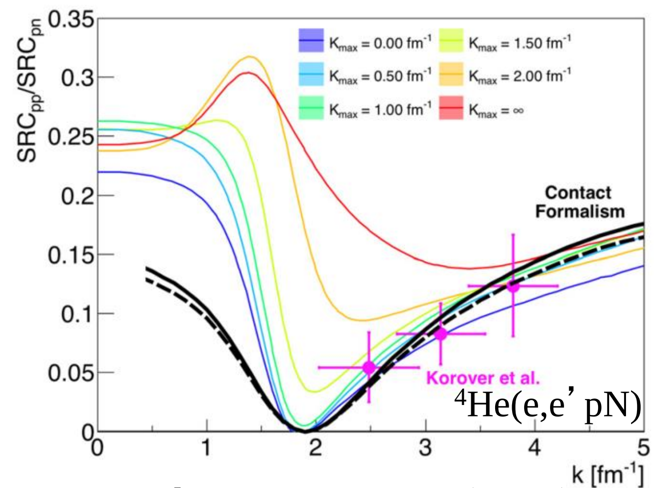
Nucleon-nucleon interaction ($r \neq 0$)

Nucleon momentum distribution

$$n_p(\boldsymbol{k}) = 2C_{pp}^{s=0}|\tilde{\varphi}_{pp}^{s=0}(\boldsymbol{k})|^2 + C_{pn}^{s=0}|\tilde{\varphi}_{pn}^{s=0}(\boldsymbol{k})|^2 + C_{pn}^{s=1}|\tilde{\varphi}_{pn}^{s=1}(\boldsymbol{k})|^2$$

The ratio of short-range correlated pp and pn pairs

$$\frac{SRC_{pp}}{SRC_{pn}}(k) = \frac{C_{pp}^{s=0}|\tilde{\varphi}_{pp}^{s=0}(k)|^2}{C_{pn}^{s=0}|\tilde{\varphi}_{pn}^{s=0}(k)|^2 + C_{pn}^{s=1}|\tilde{\varphi}_{pn}^{s=1}(k)|^2}$$



$k_F \sim 1.3$ fm^{-1}

R. Weiss, *et al.*, PLB **780**, 211 (2018).

In this talk…

- We discuss the ground-state properties in non-relativistic spin-1/2 Fermi gases with the finite-range interaction.
- Specifically, we focus on the *pure* effective-range corrections by using the interaction potential exactly reproducing the phase shift without higher-order coefficients.

What can we say with only s-wave **scattering length** and **effective range** in spin-1/2 fermionic systems?

$$k\cot\delta_s(k) = -\frac{1}{a} + \frac{1}{2}rk^2$$

$$f(n,V) \equiv f(k_{\mathrm{F}}a, k_{\mathrm{F}}r) \to E(n,V) = f(k_{\mathrm{F}}a, k_{\mathrm{F}}r)E_{\mathrm{FG}}(n)$$

In particular, we address:

1. Is the formation of droplets found due to the effective range correction?
2. What about the relation between Tan's contact and nuclear contact?

$$\frac{\partial^2(E/N)}{\partial n^2} \propto c_s^2 = 0\,? \qquad C = C_{\mathrm{nucl.}}?$$

HT and H. Liang, PRA **106**, 043308 (2022).

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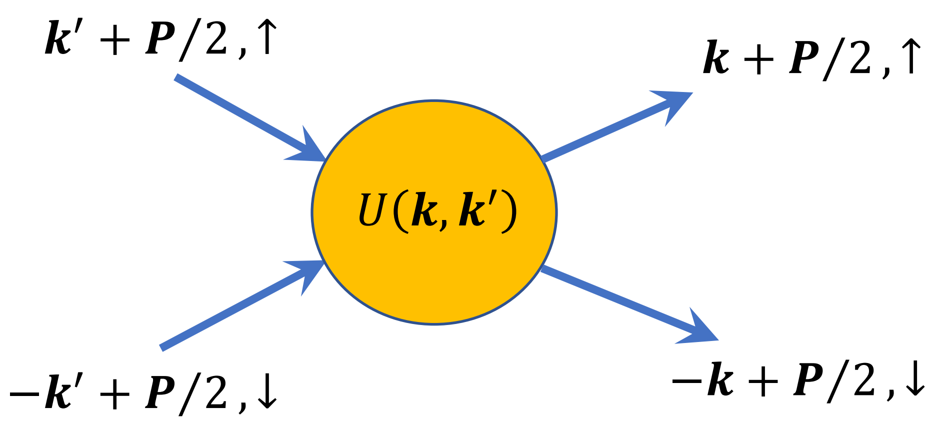
Hamiltonian

$$H = H_0 + V$$
$$= \sum_{\boldsymbol{k},\sigma} \xi_{\boldsymbol{k},\sigma} c^{\dagger}_{\boldsymbol{k},\sigma} c_{\boldsymbol{k},\sigma} + \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{P}} U(\boldsymbol{k},\boldsymbol{k}') c^{\dagger}_{\boldsymbol{k}+\boldsymbol{P}/2,\uparrow} c^{\dagger}_{-\boldsymbol{k}+\boldsymbol{P}/2,\downarrow} c_{-\boldsymbol{k}'+\boldsymbol{P}/2,\downarrow} c_{\boldsymbol{k}'+\boldsymbol{P}/2,\uparrow}$$

$\xi_{\boldsymbol{k},\sigma} = \frac{k^2}{2m} - \mu$: kinetic energy measured from the chemical potential μ

$c_{\boldsymbol{k},\sigma}$: annihilation operator of a fermion with momentum $\boldsymbol{k}$ and spin $\sigma = \uparrow, \downarrow$

$U(\boldsymbol{k},\boldsymbol{k}')$: non-local two-body interaction



$\boldsymbol{k}, \boldsymbol{k}'$: relative momenta

$\boldsymbol{P}$: c. o. m. momentum

Separable finite-range *s*-wave interaction

$$U(\boldsymbol{k},\boldsymbol{k}') = g\gamma_k\gamma_{k'}$$

Two-body *T*-matrix

$$T(\boldsymbol{k},\boldsymbol{k}';\omega) = U(\boldsymbol{k},\boldsymbol{k}') + \sum_{\boldsymbol{p}} U(\boldsymbol{k},\boldsymbol{p})\frac{1}{\omega_+ - 2\varepsilon_{\boldsymbol{p}}}T(\boldsymbol{p},\boldsymbol{k}';\omega)$$

Relation to the *s*-wave phase shift

$$-\frac{m}{4\pi}T(\boldsymbol{k},\boldsymbol{k}';2\varepsilon_{\boldsymbol{k}}) = \frac{1}{k\cot\delta_s(k) - ik}$$

$$k\cot\delta_s(k) = -\frac{1}{a} + \frac{1}{2}rk^2$$

Form factor

$$\gamma_k = \frac{1}{\sqrt{1+(k/\Lambda)^2}}$$

$$\Lambda = \frac{1}{r}\left[1+\sqrt{1-\frac{2r}{a}}\right]$$

$$g = \frac{4\pi a}{m}\frac{1}{1-a\Lambda}.$$

Two parameters can be expressed in terms of low-energy constants.

*Non-separability is out of scope in this work because it cannot be characterized by the effective range theory

Hartree-Fock-Bogoliubov theory

- Both density and pairing mean-fields are self-consistently treated
- Hartree-Fock term is NOT negligible in contrast to the zero-range interaction

$$
\begin{aligned}
H_{\mathrm{HFB}} = & \sum_{\boldsymbol{k},\sigma} \left[\xi_{\boldsymbol{k},\sigma} + \Sigma_\sigma(\boldsymbol{k})\right] c^\dagger_{\boldsymbol{k},\sigma} c_{\boldsymbol{k},\sigma} \\
& - \sum_{\boldsymbol{k}} \left[\Delta^*(\boldsymbol{k}) c_{-\boldsymbol{k},\downarrow} c_{\boldsymbol{k},\uparrow} + \Delta(\boldsymbol{k}) c^\dagger_{\boldsymbol{k},\uparrow} c^\dagger_{-\boldsymbol{k},\downarrow}\right] \\
& - \sum_{\boldsymbol{k},\boldsymbol{k}'} U(\boldsymbol{k},\boldsymbol{k}') \langle c^\dagger_{\boldsymbol{k},\uparrow} c^\dagger_{-\boldsymbol{k},\downarrow} \rangle \langle c_{-\boldsymbol{k}',\downarrow} c_{\boldsymbol{k}',\uparrow} \rangle \\
& - \sum_{\boldsymbol{p},\boldsymbol{p}'} U\left(\frac{\boldsymbol{p}-\boldsymbol{p}'}{2}, \frac{\boldsymbol{p}-\boldsymbol{p}'}{2}\right) \langle c^\dagger_{\boldsymbol{p},\uparrow} c_{\boldsymbol{p},\uparrow} \rangle \langle c^\dagger_{\boldsymbol{p}',\downarrow} c_{\boldsymbol{p}',\downarrow} \rangle
\end{aligned}
$$

<u>BCS-type pairing field</u>

$$\Delta(\boldsymbol{k}) = -\sum_{\boldsymbol{k}'} U(\boldsymbol{k},\boldsymbol{k}') \langle c_{-\boldsymbol{k}',\downarrow} c_{\boldsymbol{k}',\uparrow} \rangle$$

<u>Hartree-Fock-type self-energy</u>

$$\Sigma_\sigma(\boldsymbol{p}) = \sum_{\boldsymbol{p}'} U\left(\frac{\boldsymbol{p}-\boldsymbol{p}'}{2}, \frac{\boldsymbol{p}-\boldsymbol{p}'}{2}\right) \langle c^\dagger_{\boldsymbol{p}',\bar{\sigma}} c_{\boldsymbol{p}',\bar{\sigma}} \rangle$$

See also PRA **103**, 063306 (2021).

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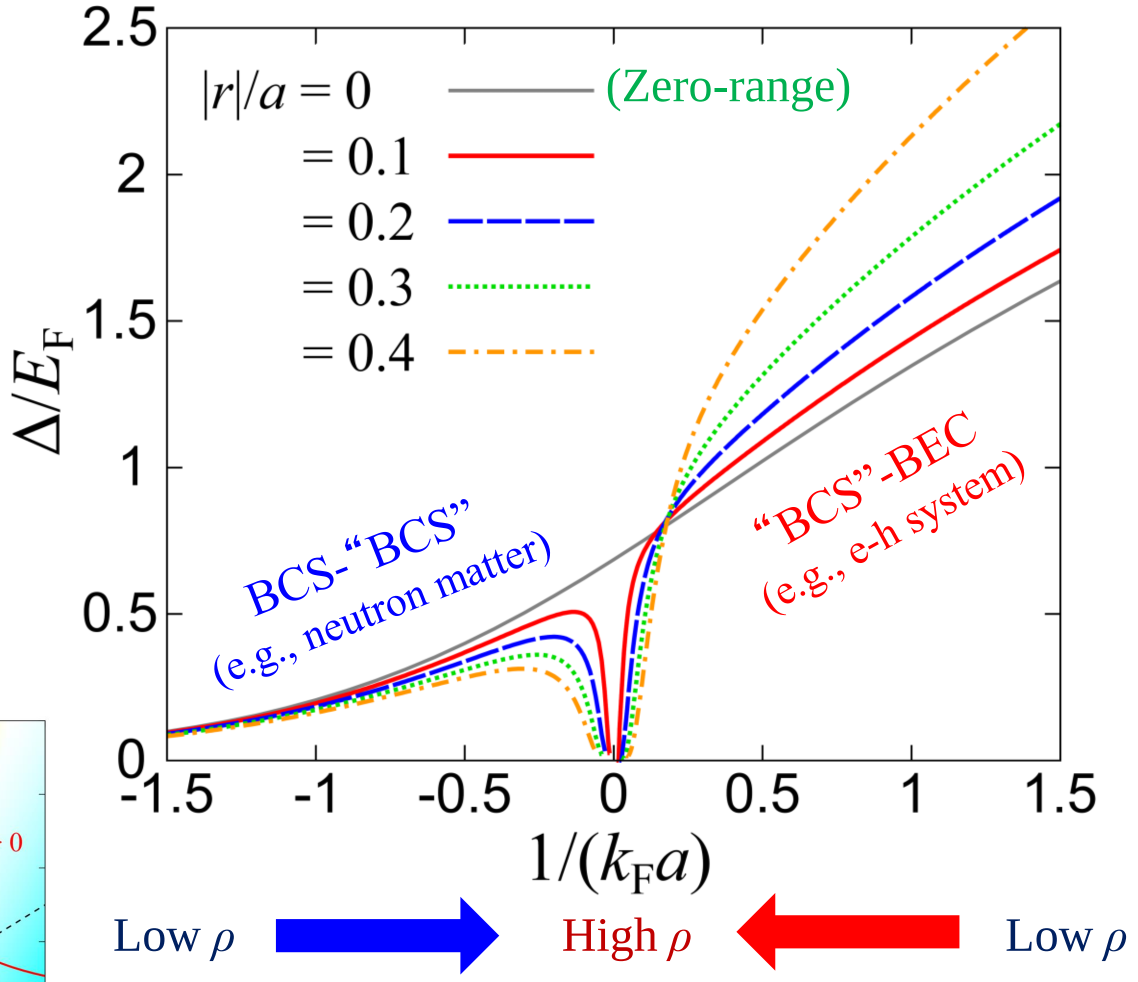
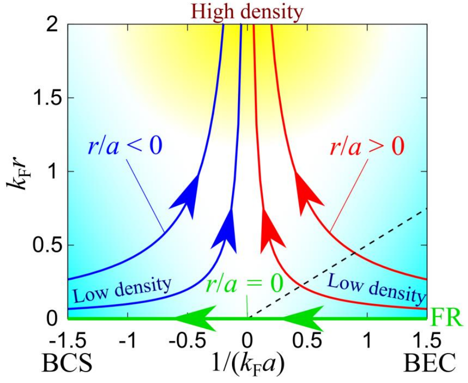
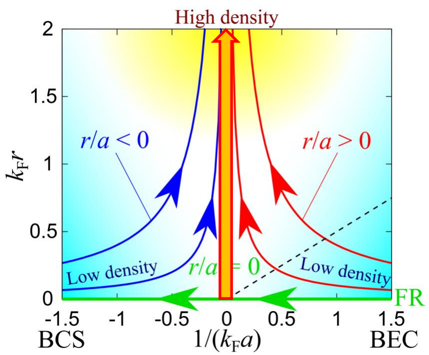
Superfluid order parameter

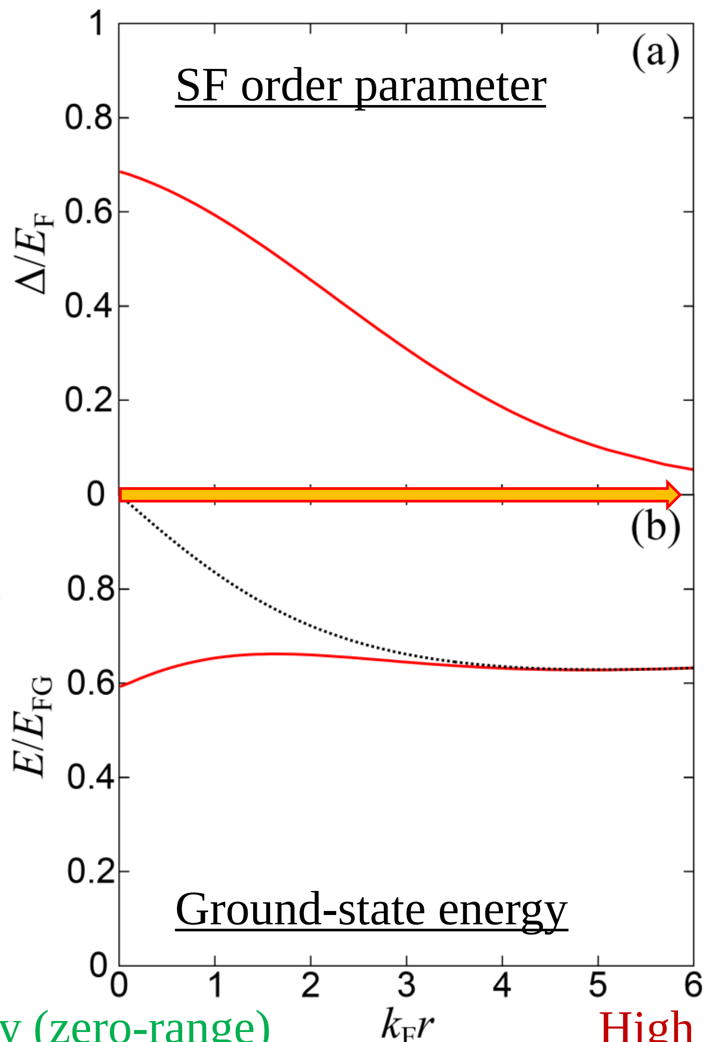




Effective-range dependence at $1/(k_F a) = 0$

*In the high-density regime, the value of $1/(k_F a)$ is not important





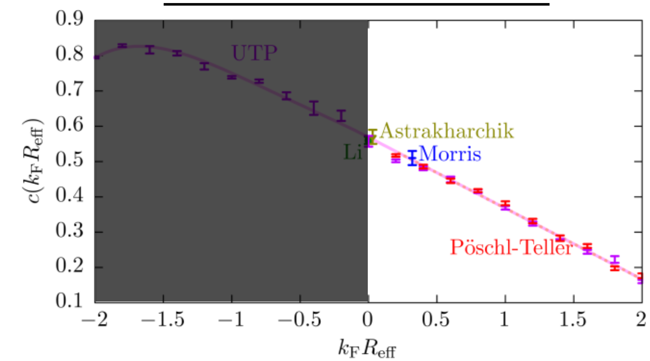
Unitarity (zero-range)

High density

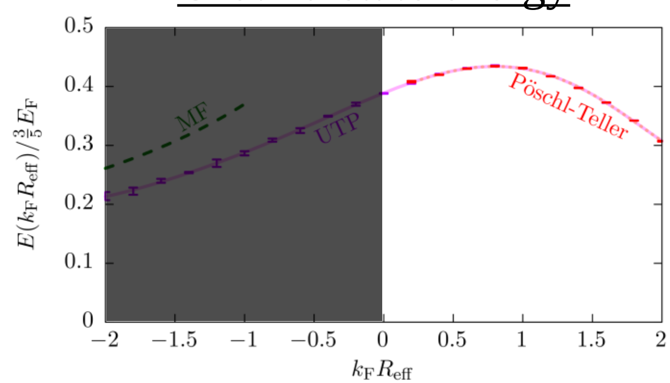
Qualitatively consistent with DMC

PRA **95**, 013633 (2017).

Condensate fraction

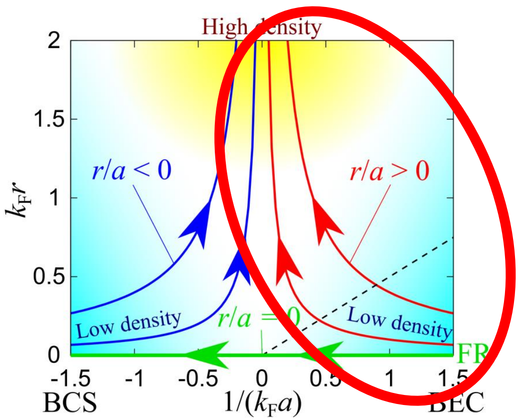


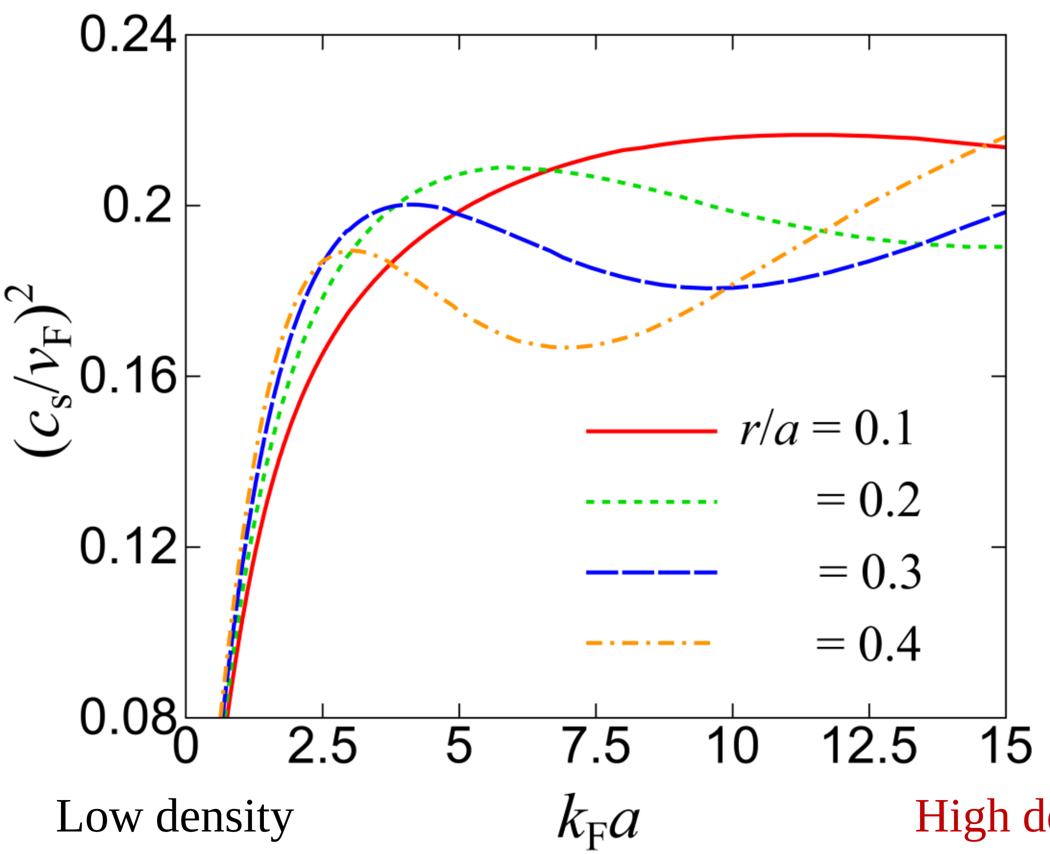
Ground-state energy



Sound velocity in the density-induced BEC-“BCS” crossover

No collapse in contrast to [PRA **95**, 013633 (2017)] reporting the collapse ($c_s = 0$).





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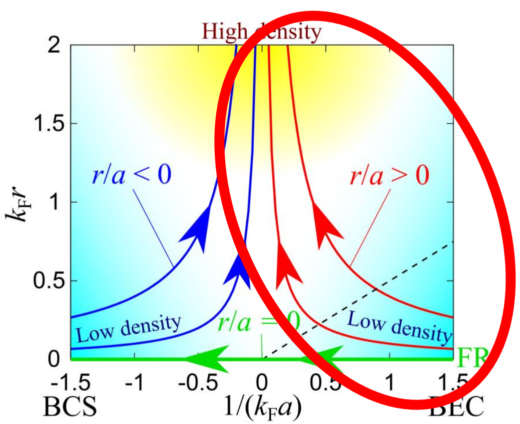
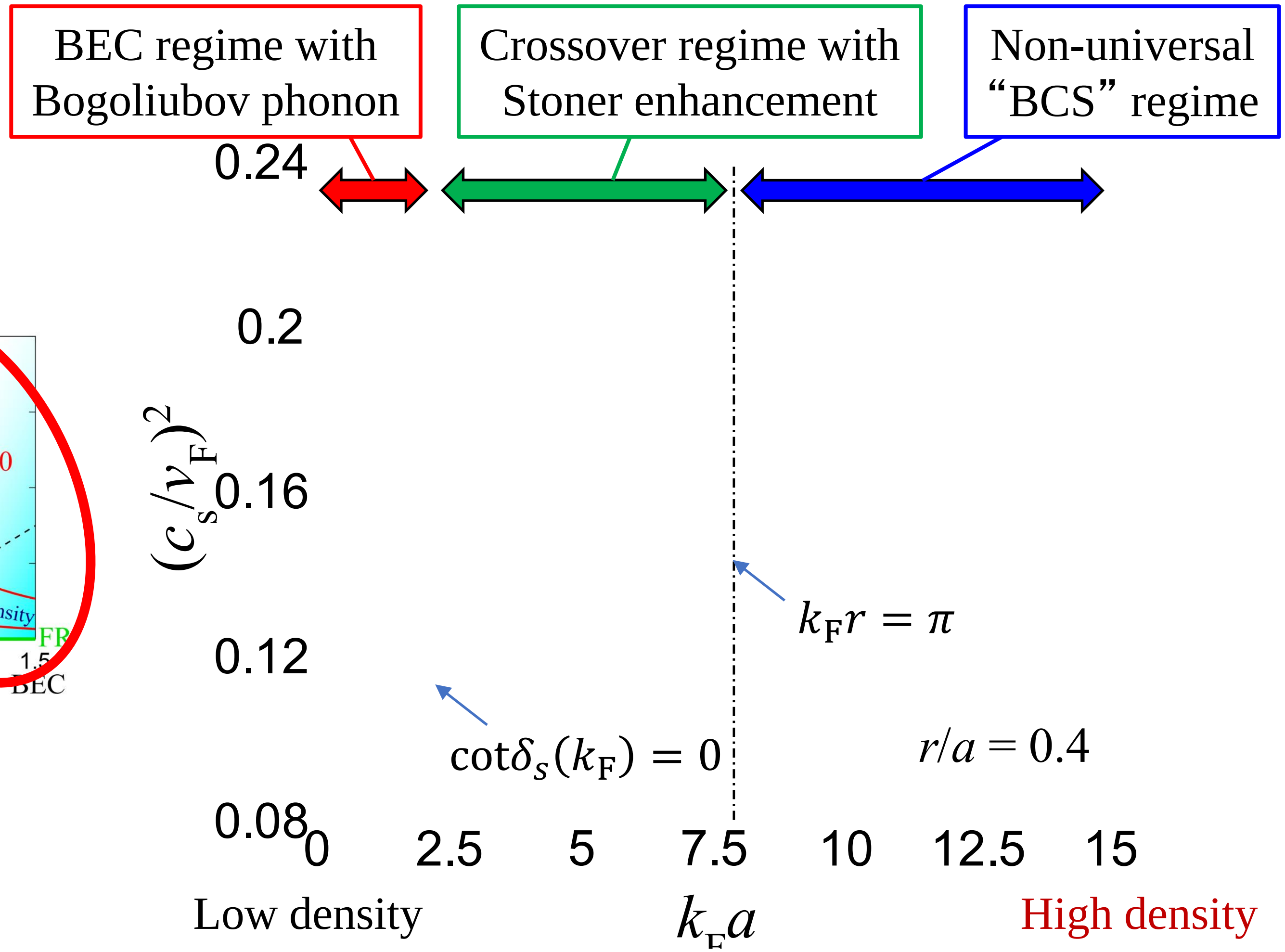
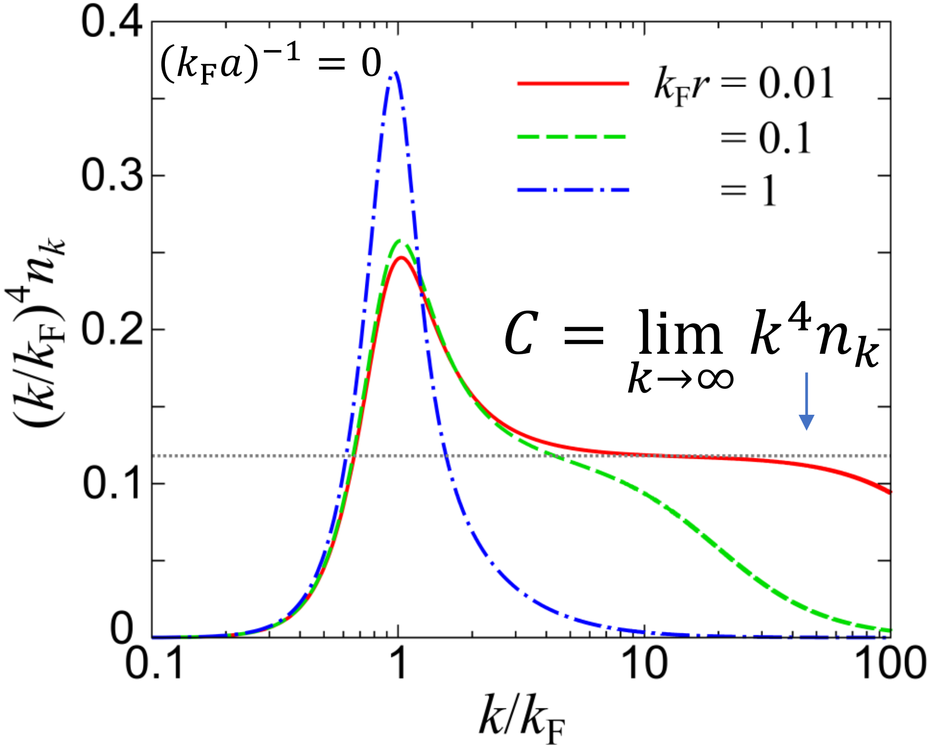




High-momentum tails

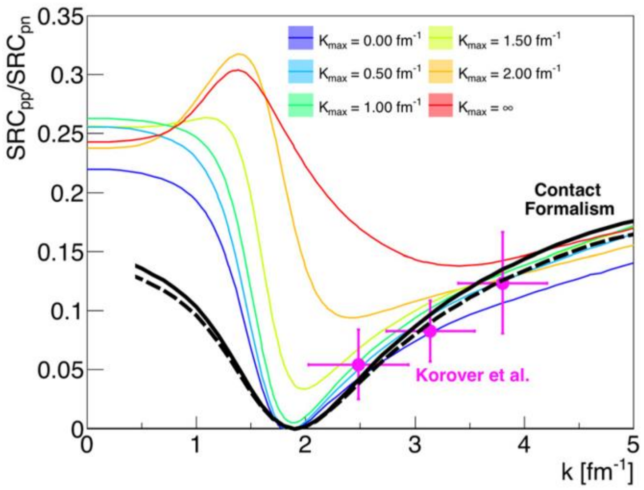
More detailed investigations on the relation between Tan's contact and nuclear contact seem to be needed (e.g., repulsive core and so on).



The high momentum tails should be suppressed by the finite-range correction

Nuclear contact

$r \simeq 1.76$ fm, $k_F \sim 1.3$ fm^{-1}, $k_F r \simeq 2.3$



R. Weiss, *et al*., Phys. Lett. B **780**, 211 (2018).

Are they different?

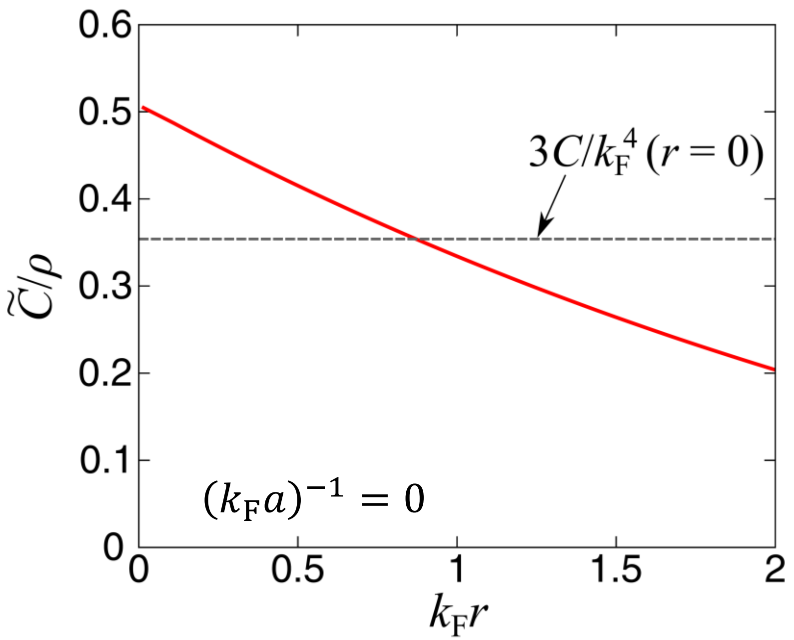
Nuclear-contact-like parameter

Tan's contact and nuclear contact are slightly different in terms of relevant energy scales

<u>Definition of the nuclear contact</u>

$$\frac{\int_{k_F}^{\infty} n(\boldsymbol{k})d\boldsymbol{k}}{\int_{0}^{\infty} n(\boldsymbol{k})d\boldsymbol{k}} = \frac{C_{nn}^{s=0} + C_{pp}^{s=0} + C_{pn}^{s=0} + C_{pn}^{s=1}}{A/2}$$

R. Weiss, *et al.*, Phys. Lett. B **780**, 211 (2018).

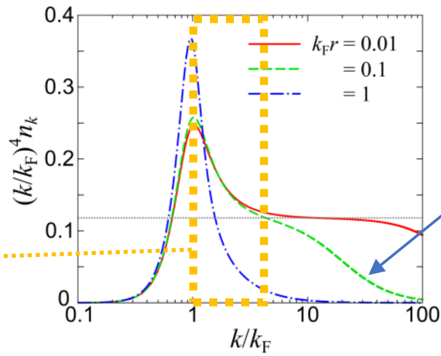


<u>Nuclear contact-like parameter</u>

$$\tilde{C} = 2\sum_{\boldsymbol{k}} \theta(k - k_F) n_{\boldsymbol{k}}$$

(assumption)

$n_{\boldsymbol{k}} \simeq \frac{C}{k^4}$ at $k \geqslant k_F$ ➡ $\frac{\tilde{C}}{\rho} \simeq \frac{2}{(2\pi)^3\rho}\int_{k_F}^{\infty} 4\pi k^2 dk \frac{C}{k^4} = \frac{3C}{k_F^4}$



Main contribution to the difference comes from near the Fermi momentum (many-body scale)

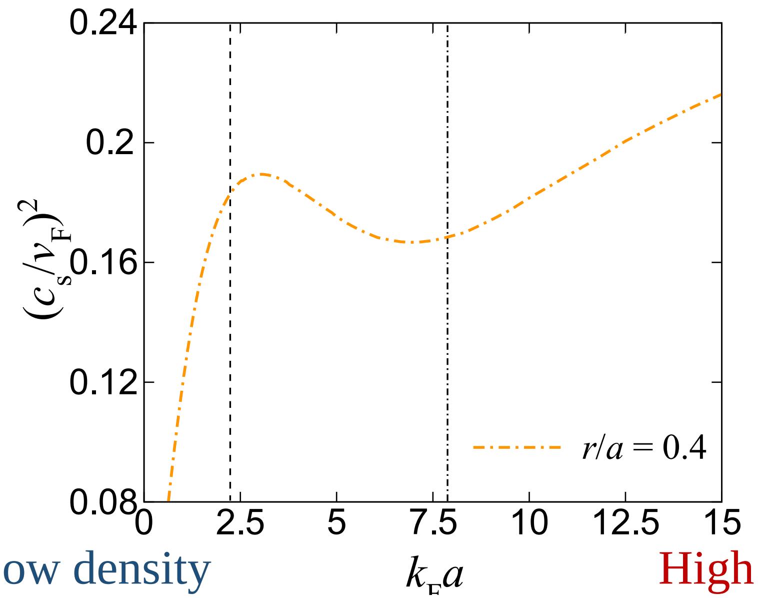
Outline

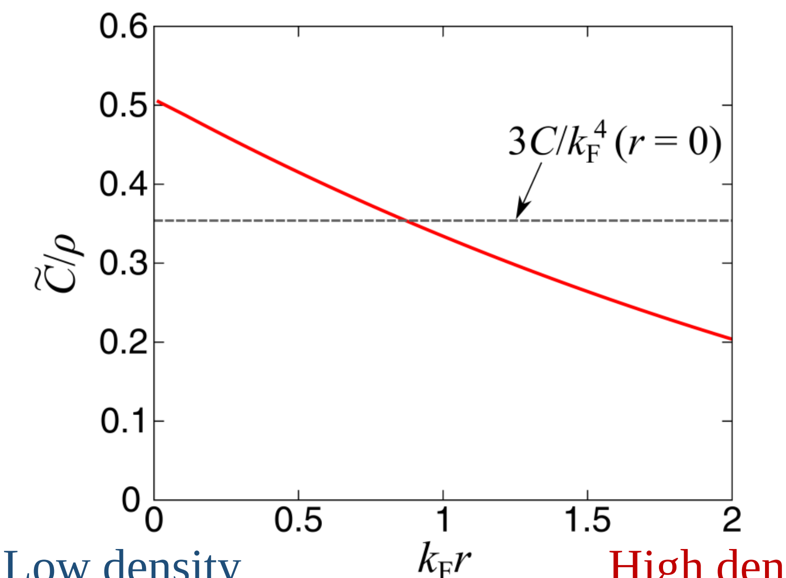
- Introduction
- Formalism
- Results and discussion
- Summary and future perspective

Summary

Paper: HT and H. Liang, PRA **106**, 043308 (2022).

▪We have discussed the ground-state properties of non-relativistic spin-1/2 Fermi gases with the finite range interaction.
▪What can we say with only two parameters (a, r)? Our answer is that the sound velocity exhibits a non-monotonic behavior but no collapse associated with the finite-range correction.
▪We elaborated the relation between Tan's contact and the nuclear contact. While Tan's contact characterizes short-range correlations beyond many-body scale, the nuclear contact captures the pair-correlations around the Fermi momentum.





Low density | High density | Low density | High density

Future work: Beyond the HFB calculation. Extension of Tan's energy relations to nuclear system and its applications to constructing true "*universal*" EDF.

Appendix

Large effective-range limit

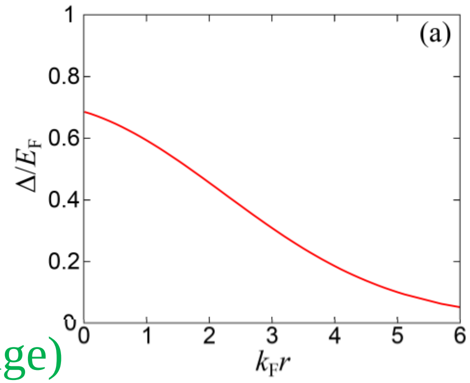
$$U(\boldsymbol{k},\boldsymbol{k}) = g\frac{1}{1+(k/\Lambda)^2} \qquad \Lambda = \frac{2}{r} \quad (a^{-1}=0)$$ "Lorentzian form"

$$U(\boldsymbol{k},\boldsymbol{k}) \to -\frac{4\pi^2}{m}\delta(k) \quad (r\to\infty)$$

- Vanishing HF self-energy

$$\Sigma_{r\to\infty}(\boldsymbol{k}) = g\sum_{\boldsymbol{k}'}\gamma^2_{\frac{|\boldsymbol{k}-\boldsymbol{k}'|}{2}}\theta\left(-\frac{k'^2}{2m}+\mu-\Sigma(\boldsymbol{k}')\right) = 0$$

- Vanishing SF order parameter



Unitarity (zero-range) "BCS"

Large effective range limit
= Weak-coupling limit

No droplet formation

Equations for physical quantities

Number density

$$\rho = \sum_{\boldsymbol{k}} \left[1 - \frac{\xi_{\boldsymbol{k}} + \Sigma(\boldsymbol{k})}{E_{\boldsymbol{k}}} \right]$$

HF self-energy

$$\Sigma(\boldsymbol{k}) = g \sum_{\boldsymbol{k}'} \gamma^2_{\frac{|\boldsymbol{k}-\boldsymbol{k}'|}{2}} n_{\boldsymbol{k}'}$$

Gap equation

$$1 = -g \sum_{\boldsymbol{k}} \frac{\gamma_k^2}{2E_{\boldsymbol{k}}}$$

Quasiparticle dispersion

$$E_{\boldsymbol{k}} = \sqrt{\{\xi_{\boldsymbol{k}} + \Sigma(\boldsymbol{k})\}^2 + |\Delta(\boldsymbol{k})|^2}$$

Pairing gap

$$\Delta(\boldsymbol{k}) = -\gamma_k g \sum_{\boldsymbol{k}'} \gamma_{k'} \langle c_{-\boldsymbol{k}',\downarrow} c_{\boldsymbol{k}',\uparrow} \rangle \equiv \Delta \gamma_k$$

Ground-state energy

$$E = \sum_{\boldsymbol{k}} [\xi_{\boldsymbol{k}} + \Sigma(\boldsymbol{k}) - E_{\boldsymbol{k}}] - \frac{|\Delta|^2}{g} - \sum_{\boldsymbol{k}} \Sigma(\boldsymbol{k}) n_{\boldsymbol{k}}$$

Sound velocity

$$c_s = \sqrt{\frac{1}{m\rho\kappa}}$$

Compressibility

$$\kappa = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial \mu} \right)$$

Beyond effective-range theory?

Yamaguchi interaction

Phys. Rev. **95**, 1628 (1954).

$$U(k,k') = g\,\frac{1}{1+(k/\Lambda)^2}\,\frac{1}{1+(k'/\Lambda)^2}$$

$$-\frac{m}{4\pi}\left[-\frac{1}{a}+\frac{1}{2}rk^2-\mathcal{S}r^3k^4-ik\right]$$

$$=\left[1+\left(\frac{k}{\Lambda_\mathrm{Y}}\right)^2\right]^2\left[\frac{1}{g_\mathrm{Y}}-\frac{m\Lambda_\mathrm{Y}^3}{8\pi}\frac{1}{(k+i\Lambda_\mathrm{Y})^2}\right]$$

Nonzero up to S

Gaussian interaction

$$U(k,k') = g e^{-k^2/\Lambda^2} e^{-k'^2/\Lambda^2}$$

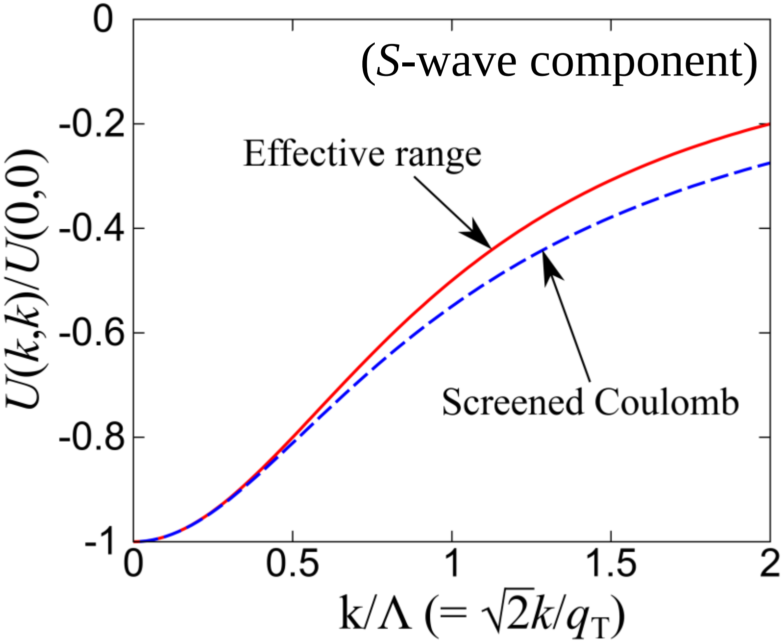
$$-\frac{m}{4\pi}\left[-\frac{1}{a}+\frac{1}{2}rk^2-\mathcal{S}r^3k^4-ik\right]+O(k^5)$$

$$=e^{\frac{2k^2}{\Lambda_\mathrm{G}^2}}\left[\frac{1}{\lambda}+\frac{m}{4\pi^2}\left\{\Lambda_\mathrm{G}\sqrt{\frac{\pi}{2}}+i\pi k e^{-\frac{2k^2}{\Lambda_\mathrm{G}^2}}\right\}\right]$$

Nonzero for all order (but $e^{\#k^2}$)

Screened Coulomb interaction

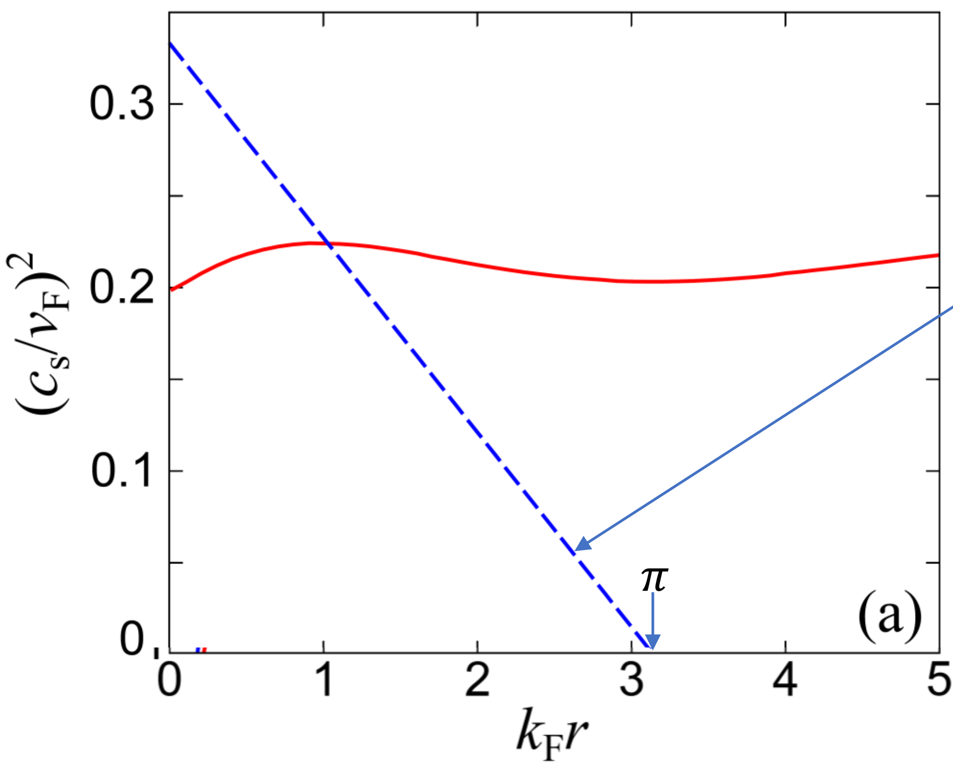
$$V_\mathrm{C}(\boldsymbol{q}) = -\frac{4\pi e^2}{q^2+q_\mathrm{T}^2} \equiv \frac{4\pi e^2}{(\boldsymbol{k}-\boldsymbol{k}')^2+q_\mathrm{T}^2}$$



Do we encounter the mechanical collapse?

—No, at least within the effective-range corrections.

Comparison with zero-momentum Hartree approximation

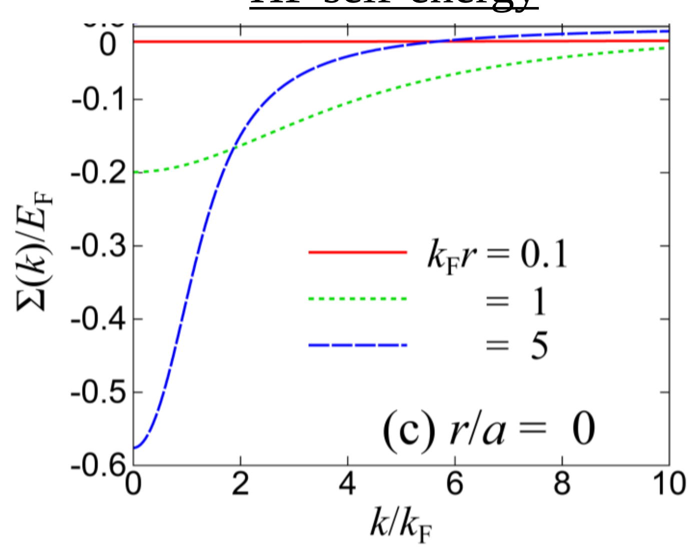


$$\Sigma_H = U(\mathbf{0},\mathbf{0})\rho/2 \equiv g\rho/2$$

PRA **97**, 013601 (2018).

$$c_{s,H}^2 = \frac{1}{m\rho\kappa_0}\left[1 - \frac{2}{\pi}\frac{1}{\Lambda/k_F - (k_F a)^{-1}}\right]$$

HF self-energy

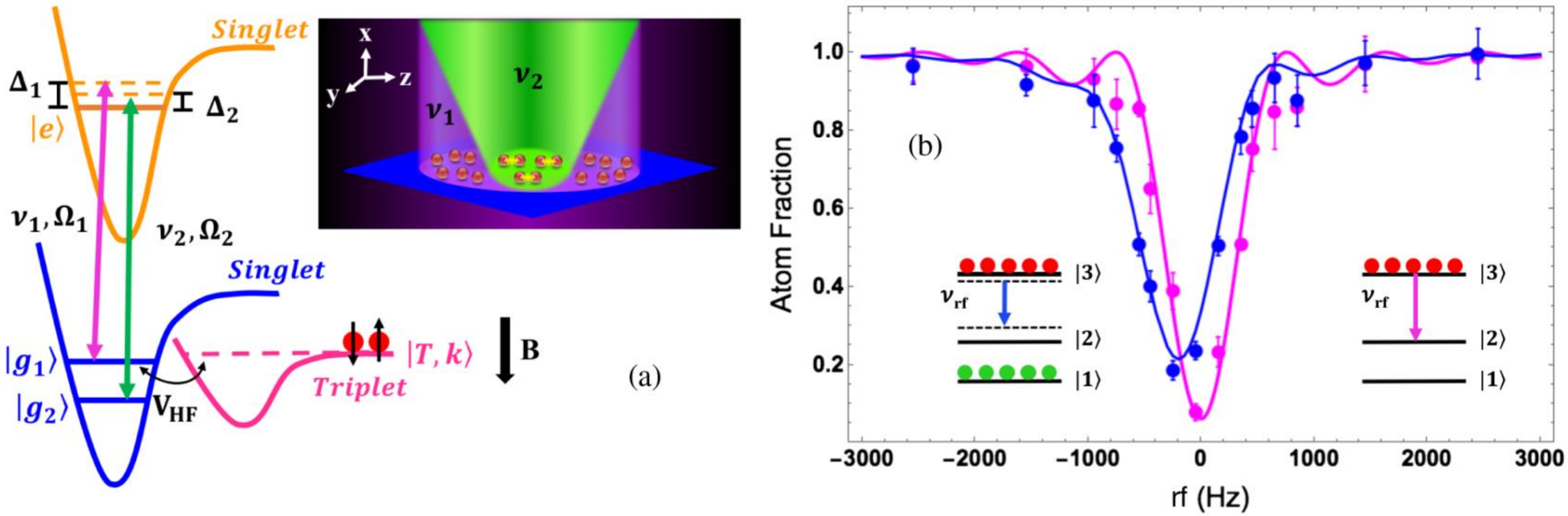


Stoner enhancement of the compressibility κ

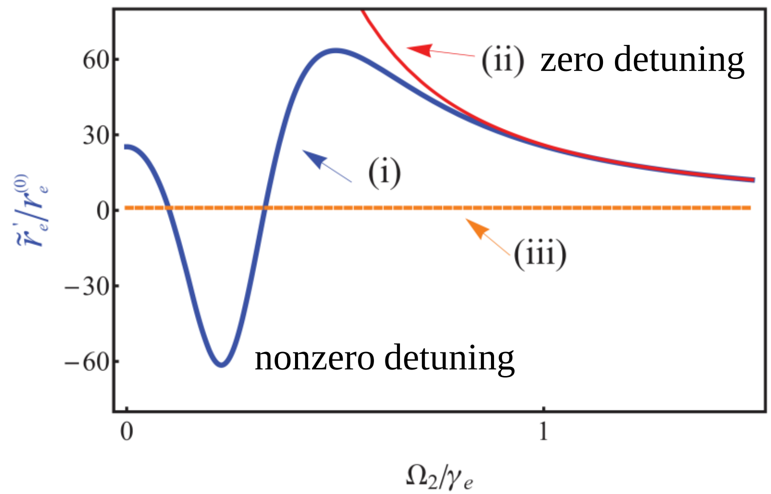
➡ Suppression of $c_s = \frac{1}{\sqrt{\rho m \kappa}}$

*However, the collapse occurs in PRA **95**, 013633 (2017) at $k_F r \simeq 2$ because of higher coefficients in $\delta_s(k)$.

Optical control of the effective range in cold atoms



PRL **116**, 075301 (2016), PRL **121**, 163404 (2018), PRL **122**, 040405 (2019).



<u>Rabi-frequency dependence of controllable effective range</u>

PRA **86**, 063625 (2012).

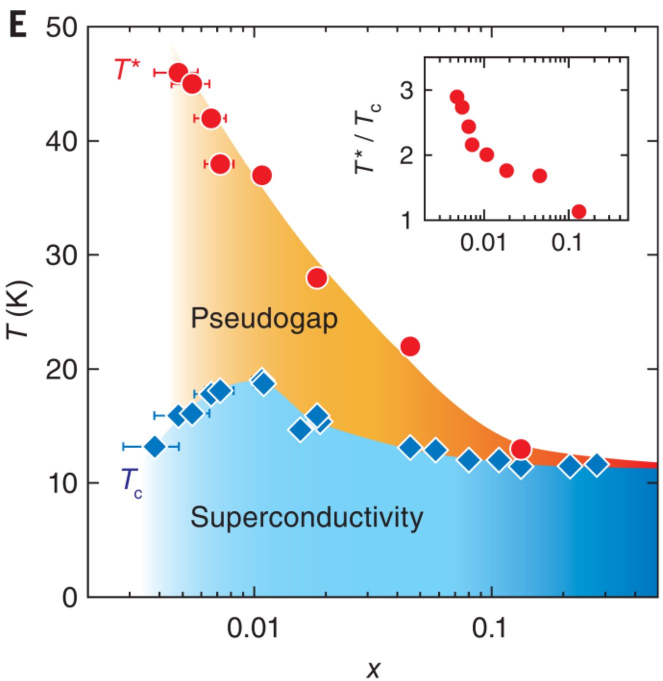
➡ A system with both large scattering length and effective range (compared to higher order coefficients) can be realized in future experiments.

BEC-BCS crossover in condensed matter systems

Scattering length (interaction) CANNOT be tuned ➡ Density can be tuned

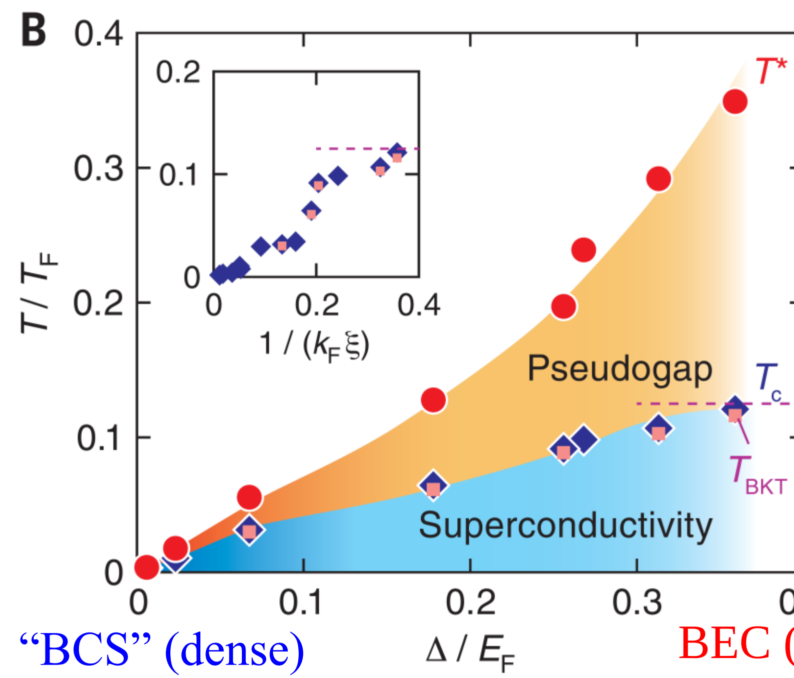
BEC-BCS crossover in Li_xZrNCl (lithium-intercalated layered nitride)

Temperature



Carrier dope (density)

Temperature scaled by T_F



"BCS" (dense) BEC (dilute)

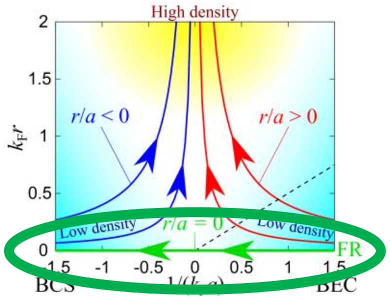
Pairing gap scaled by E_F

Y. Nakagawa, *et al.*, Science **372**, 6538 (2021).

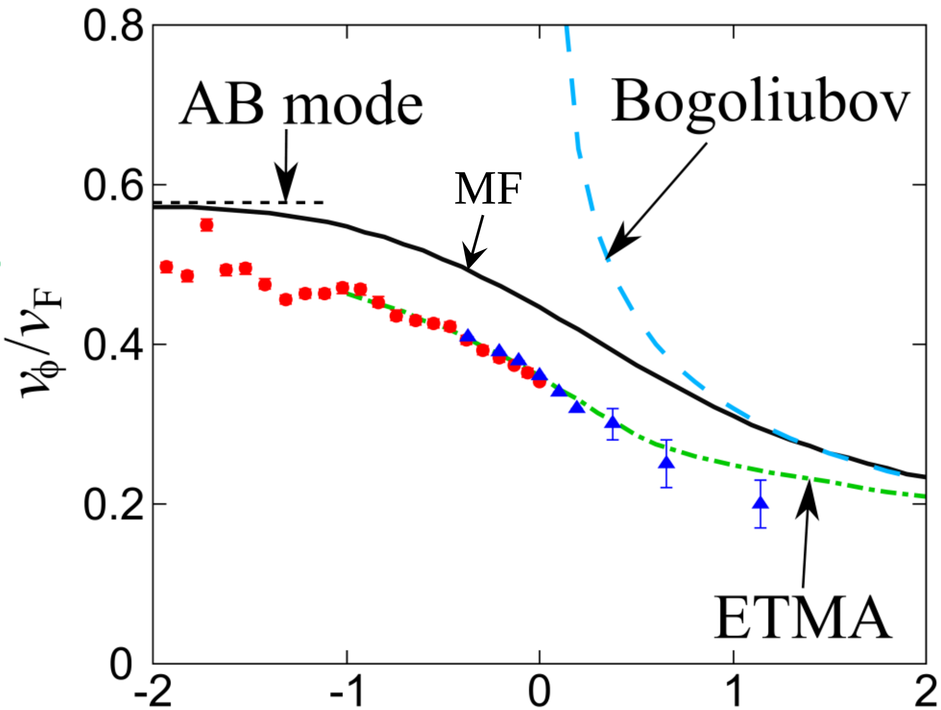
Others: FeSe [PNAS **111**, 16309 (2014).], Organic SC [PRX **12**, 011016 (2022).], Excitons in bilayer graphene [Science **375**, 6577 (2022).], etc…

Sound velocity throughout the BCS-BEC crossover in "*ultracold Fermi gases*"

- Smooth change from the Anderson-Bogoliubov (AB) mode of Fermi superfluids to the Bogoliubov phonon of molecular bosons



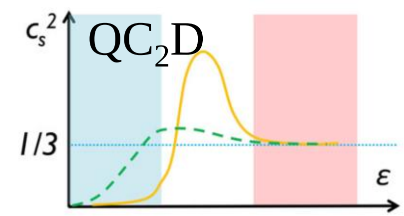
Zero range



(Weak-coupling) $(k_F a_s)^{-1}$ (Strong-coupling)

Monotonic, and no peak in the case with contact-type interaction

How about the sound velocity in the density-induced crossover?

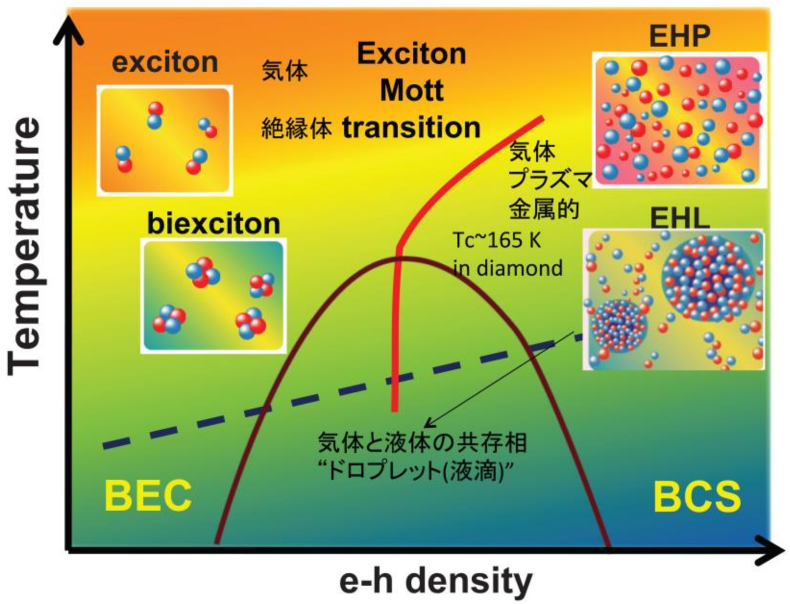


PRD **105**, 076001 (2022).

ETMA: H. Tajima, *et al.*, PRA, **95**, 043625 (2017).
Cold atom expr.: M. Horikoshi, *et al.*, PRX, **7**, 041004 (2017).
S. Hoinka, *et al.*, Nat. Phys. **13**, 943 (2017).

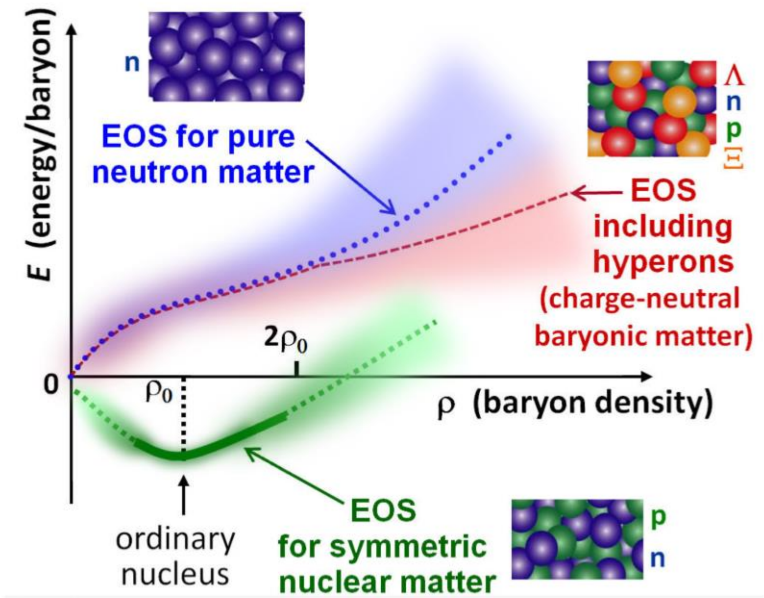
Large effective range limit is really BCS? (high density)

Electron-hole system



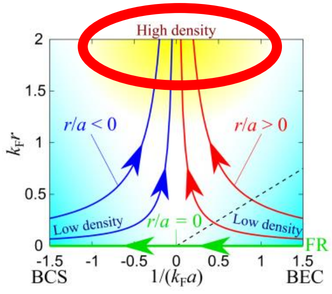
From Prof. J. Omachi's slide
新学術「中性子星核物質」スクール

Nuclear matter



H. Tamura, JPS Conf. Proc. **1**, 011003 (2014).

large range



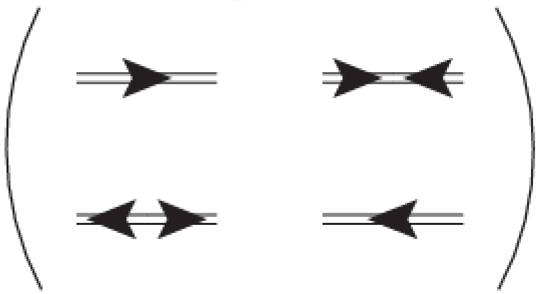
Mechanical collapse to the droplet phase?

(note that these are four-component mixtures)

Matrix Green's function

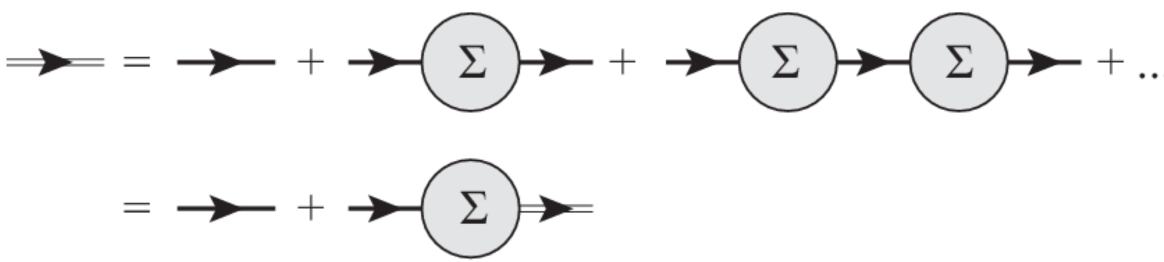
$$\hat{G}(\boldsymbol{p},\tau) = -\left\langle T_\tau \left[\Psi_{\boldsymbol{p}}(\tau)\Psi_{\boldsymbol{p}}^\dagger(0)\right]\right\rangle$$
$$= -\begin{pmatrix} \left\langle T_\tau \left[c_{\boldsymbol{p},\uparrow}(\tau)c_{\boldsymbol{p},\uparrow}^\dagger(0)\right]\right\rangle & \left\langle T_\tau \left[c_{\boldsymbol{p},\uparrow}(\tau)c_{-\boldsymbol{p},\downarrow}(0)\right]\right\rangle \\ \left\langle T_\tau \left[c_{-\boldsymbol{p},\downarrow}^\dagger(\tau)c_{\boldsymbol{p},\uparrow}^\dagger(0)\right]\right\rangle & \left\langle T_\tau \left[c_{-\boldsymbol{p},\downarrow}^\dagger(\tau)c_{-\boldsymbol{p},\downarrow}(0)\right]\right\rangle \end{pmatrix}$$

Scalar representation



Dyson's equation with the matrix self-energy $\hat{\boldsymbol{\Sigma}}(\boldsymbol{p})$

$$\hat{G}(p) = \hat{G}^0(p) + \hat{G}^0(p)\hat{\Sigma}(p)\hat{G}^0(p) + \hat{G}^0(p)\hat{\Sigma}(p)\hat{G}^0(p)\hat{\Sigma}(p)\hat{G}^0(p) + \cdots$$
$$= \hat{G}^0(p) + \hat{G}^0(p)\hat{\Sigma}(p)\hat{G}(p)$$



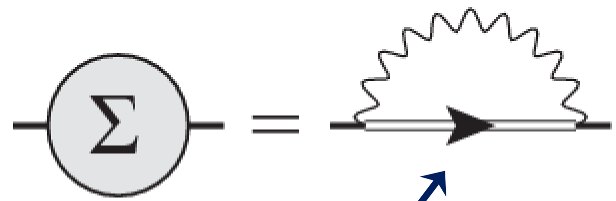
BCS-Gor'kov Green's function

$$\hat{G}^0(p) = \left[i\omega_n - (\varepsilon_{\boldsymbol{p}} - \mu)\tau_3 + \Delta\tau_1 + h\right]^{-1} \qquad \hat{G}_\sigma(p) = \int_0^\beta d\tau \hat{G}_\sigma(\boldsymbol{p},\tau)e^{i\omega_n\tau} \qquad ※p = (\boldsymbol{p}, i\omega_m)$$

Extended T-matrix approximation

ETMA self-energy $\hat{\Sigma}(p)$

$$\hat{\Sigma}(p) = -T\sum_{q}\sum_{i,j=\pm}\Gamma^{ij}(q)\tau_i\hat{G}(p+q)\tau_j$$

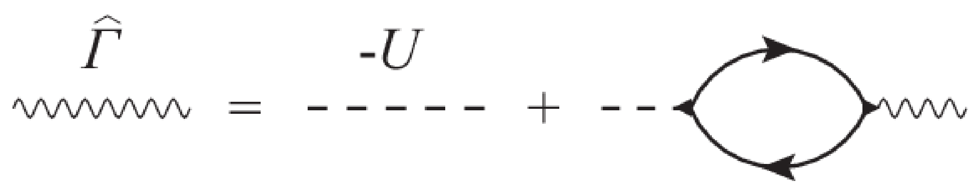


*Ordinary TMA uses G^0 (non-interacting one) here

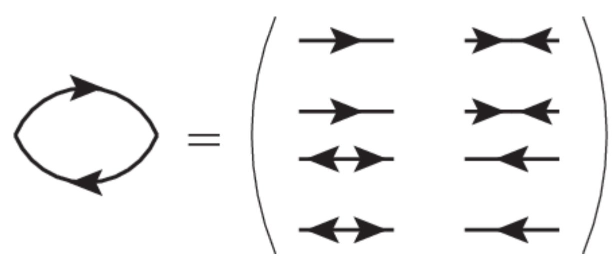
Particle-Particle scattering matrix $\hat{\Gamma}(q)$

$$\hat{\Gamma}(q) = -U\left[\hat{1} + U\hat{\Pi}(q)\right]^{-1} = \begin{pmatrix}\Gamma^{-+}(q) & \Gamma^{--}(q)\\ \Gamma^{++}(q) & \Gamma^{+-}(q)\end{pmatrix}$$

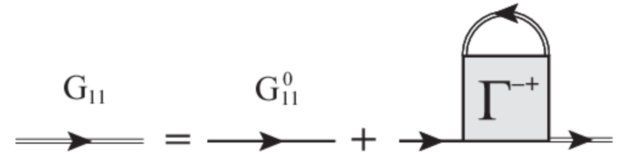
Pairing fluctuations



Matrix pair propagator $\hat{\Pi}(\mathbf{q})$

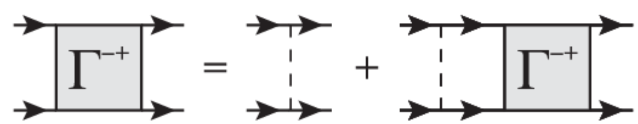


Diagonal Green's function



$$G_{11} = G^0_{11} + G^0_{11}\Sigma_{11}G_{11}$$

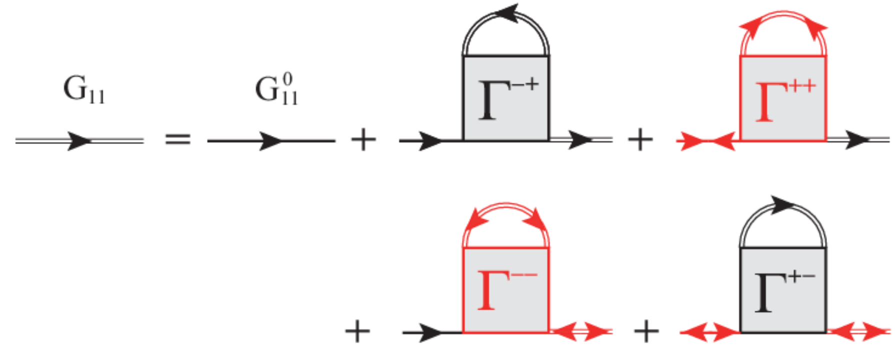
Diagonal scattering matrix



$$\Gamma^{-+} = \frac{-U}{1 + U\Pi^{-+}}$$

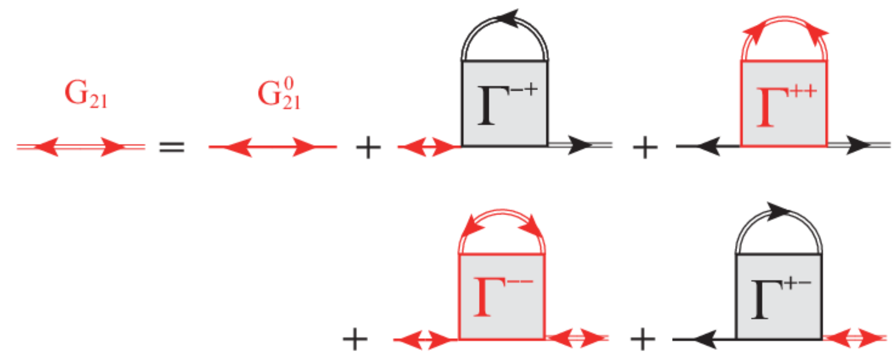
Diagonal Green's function

$$G_{11} = G_{11}^0 + G_{11}^0\Sigma_{11}G_{11} + G_{12}^0\Sigma_{21}G_{11} + G_{11}^0\Sigma_{12}G_{21} + G_{12}^0\Sigma_{22}G_{21}$$



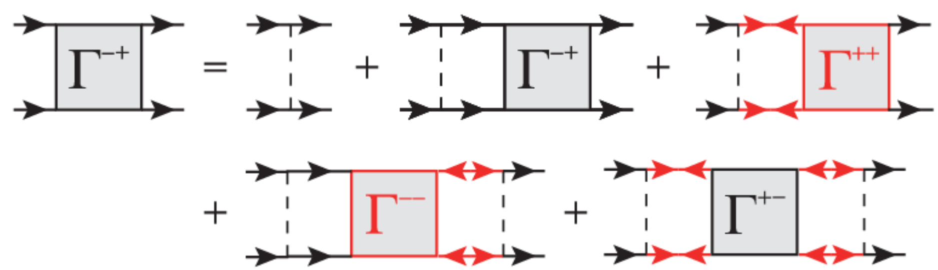
Anomalous Green's function

$$G_{21} = G_{21}^0 + G_{21}^0\Sigma_{11}G_{11} + G_{22}^0\Sigma_{21}G_{11} + G_{21}^0\Sigma_{12}G_{21} + G_{22}^0\Sigma_{22}G_{21}$$



Diagonal scattering matrix

$$\Gamma^{-+} = \frac{-U - U^2\Pi^{+-}}{(1 + U\Pi^{-+})(1 + U\Pi^{+-}) - U^2\Pi^{++}\Pi^{--}}$$



Anomalous scattering matrix

$$\Gamma^{++} = \frac{-U^2\Pi^{++}}{(1 + U\Pi^{-+})(1 + U\Pi^{+-}) - U^2\Pi^{++}\Pi^{--}}$$