Mirror Symmetry

- from 3d to 2d

2019年基研研究会：　素粒子論と数理物理学

－ 江口·Hanson 解の発見から40年 －

堀健太朗

研究をする喜び

アスペン

1996 Eguchi-H-Xiong

While computing topological string amplitudes, we noticed

mirror symmetry :

σ - model $\longleftrightarrow$ Landau-Ginzburg model

$X = \mathbb{CP}^{N-1}$

$Y = (\mathbb{C}^\times)^{N-1}$

size $= t$

$W = e^{-Y_1} + \cdots + e^{-Y_{N-1}} + e^{-t + Y_1 + \cdots + Y_{N-1}}$

c.f. 1992 Fendley-Intriligator S-matrix

1993 Batyrev quantum cohomology

1994 Givental J-function vs periods

mirror symmetry in 2d (2,2) QFTs

$$\mathcal{T} \longleftrightarrow \mathcal{T}^\vee$$

$\mathcal{T}$	$\mathcal{T}^\vee$
$Q_+ \ \bar{Q}_+ \ Q_- \ \bar{Q}_-$	$Q_+ \ \bar{Q}_+ \ \bar{Q}_- \ Q_-$
$U(1)_V$, $U(1)_A$	$U(1)_A$, $U(1)_V$
chiral, twisted chiral	twisted chiral, chiral
M_C , M_K	M_K , M_C
A-model, B-model	B-model, A-model

The mirror symmetry

σ - model $\longleftrightarrow$ Landau-Ginzburg model

$X = \mathbb{CP}^{N-1}$

size $= t$

$Y = (\mathbb{C}^\times)^{N-1}$

$W = e^{-Y_1} + \cdots + e^{-Y_{N-1}} + e^{-t + Y_1 + \cdots + Y_{N-1}}$

was later explained and generalized using

2d (2,2) gauged linear σ-model. 2000 H-Vafa

$\mathbb{CP}^{N-1} \leftsquigarrow$ $\mathbf{G} = U(1)$, $\mathbf{V} = \mathbb{C}(1)^{\oplus N}$

appears as effective target at Fayet-Iliopoulos $\zeta = \mathrm{Re}\, t > 0$

In EHX1996 we also studied topological string with Grassmannian target

$$G(k,N) = \{ \text{ subspace of } \mathbb{C}^N \text{ of dimension } k \ \}$$

$$\cong \frac{U(N)}{U(k)\times U(N-k)} \cong \frac{SL(N,\mathbb{C})}{P} \ ; \quad P = \left\{ \left(\begin{array}{c|c} k & * \\ \hline 0 & N-k \end{array} \right) \right\}$$

$$\dim G(k,N) = k(N-k)$$

$$\chi(G(k,N)) = \binom{N}{k}$$

c.f. $G(1,N) = \mathbb{CP}^{N-1}$

and "found" mirror symmetry :

σ - model $\longleftrightarrow$ Landau-Ginzburg model

$X = G(k, N)$

size = t

$$Y = (\mathbb{C}^\times)^{k(N-k)} \ni \left(e^{Y_{a,b}}\right)_{1 \le a \le k,\ 1 \le b \le N-k}$$

$$W = e^{-Y_{1,1}} + \sum_{a,b} e^{Y_{a,b}} \left(e^{-Y_{a+1,b}} + e^{-Y_{a,b+1}}\right) + e^{-t + Y_{k,N-k}}$$

generalizing the $k=1$ case.

It works for the topological string amplitudes **but** :

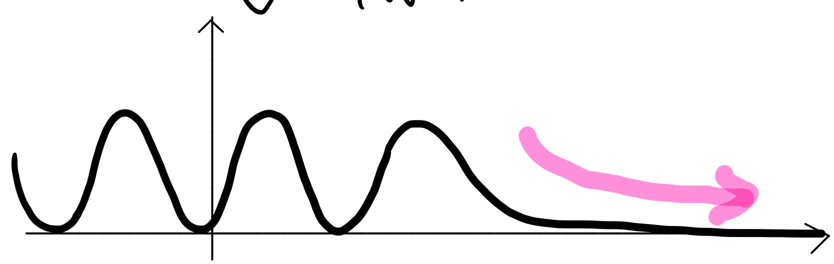
$G(k,N)$ σ - model :

① $G(k,N)$ is compact $\rightsquigarrow$ discrete spectrum

② $\chi = k(N-k)$ vacua with mass gap

③ $SU(N)$ symmetry $\rightsquigarrow$ $N-1$ Noether charges

$((\mathbb{C}^\times)^{k(N-k)}, W)$ LG-model :



② # Crit $(W) \leq k(N-k)$

① $<$: runaway potential

$\rightsquigarrow$ continuous spectrum

③ $\pi_1((\mathbb{C}^\times)^{k(N-k)}) = \mathbb{Z}^{\oplus k(N-k)}$ $\rightsquigarrow$ $k(N-k)$ topological charges

This is a problem!

In EHX1996 a remedy by partial compactification of $(\mathbb{C}^\times)^{k(N-k)}$

was proposed but no systematic way was found.

2005 Rietsch found a systematic way that solves ① , ② and

possibly also ③ :

$$(\mathbb{C}^\times)^{k(N-k)} \subset Y \subset \frac{SL(N,\mathbb{C})^\vee}{P^\vee}$$

$\vee$ ⋯ Langlands dual

We would like to understand

- How does W_{EHX} appear ?

- Whether/Why Rietsch's Y is correct?

  Why Langlands dual ?

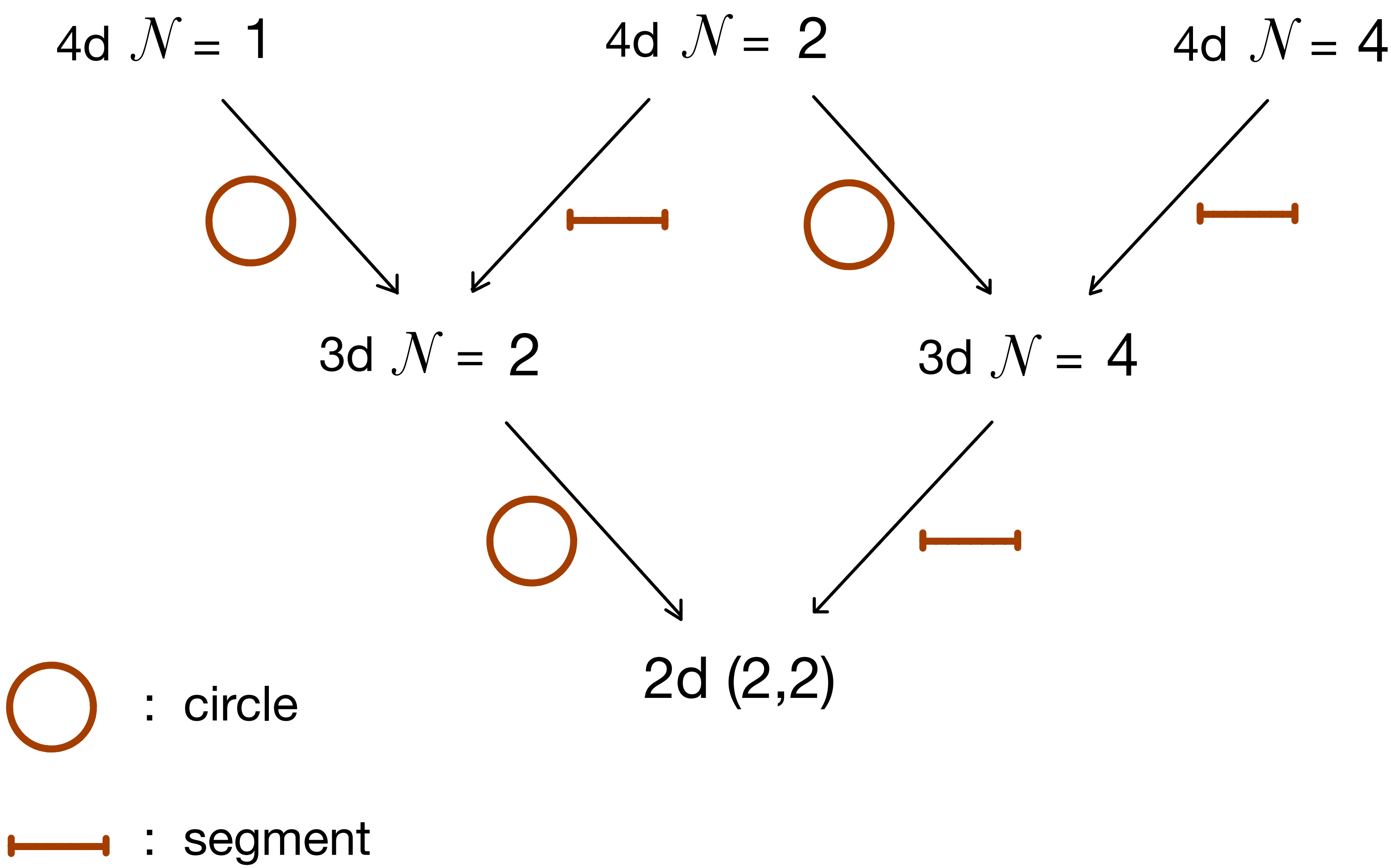
Gauged linear σ-model ?

$G(k, N) \rightsquigarrow \quad G = U(k), \; V = (\mathbb{C}^k)^{\oplus N}$

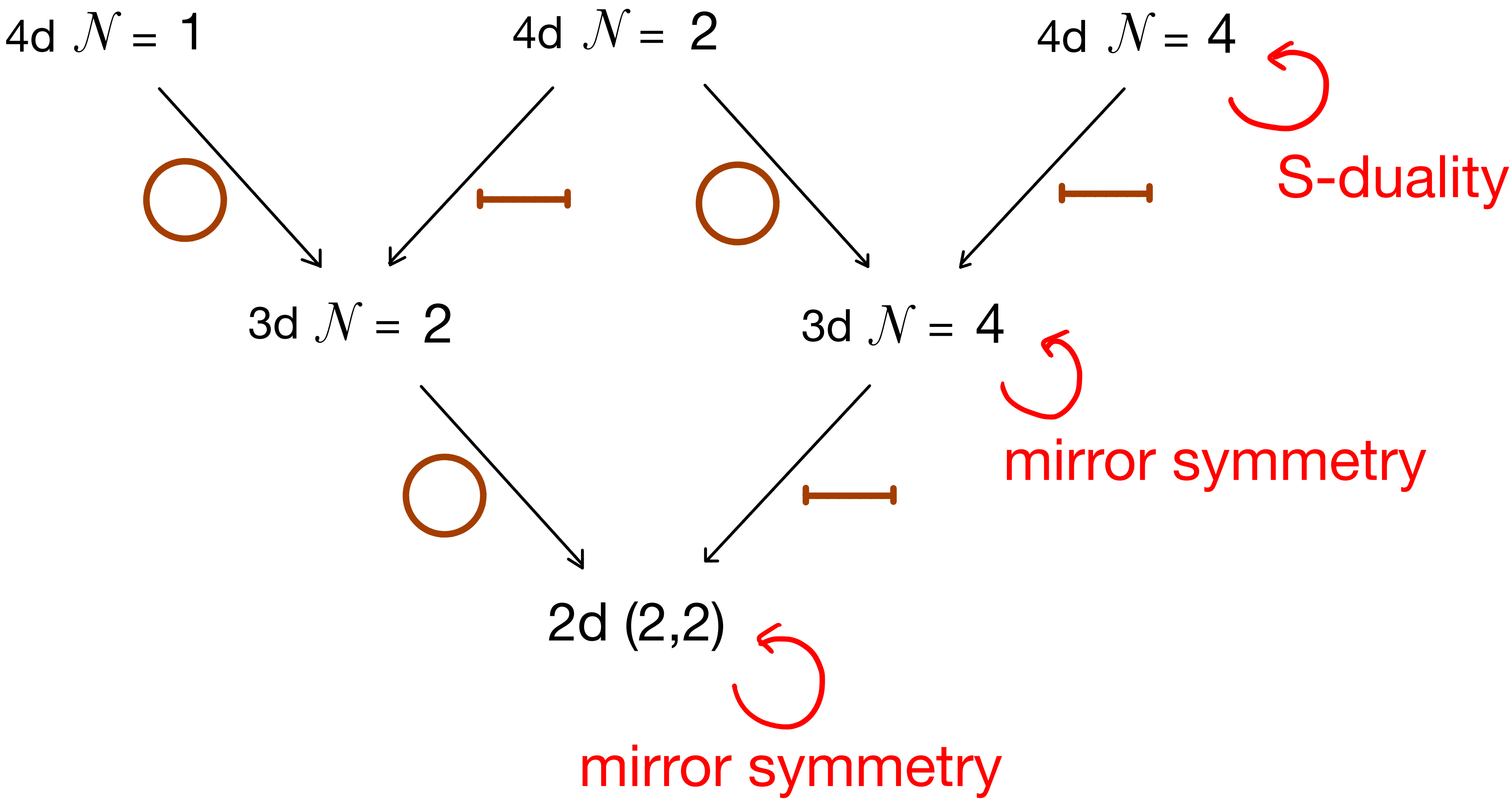
but the analysis of HV2000 does not straightforwardly apply to

non-Abelian gauge groups

Compactification



Compactification



2001 Aganagic-H-Karch-Tong

$$\begin{tikzcd} 3d\ \mathcal{N}=2 \arrow[dr, "\bigcirc"'] & & 3d\ \mathcal{N}=4 \arrow[ll, "\text{partial SUSY breaking}"'] \arrow[loop right] \\ & 2d\ (2,2) \arrow[loop right] & \end{tikzcd}$$

mirror symmetry

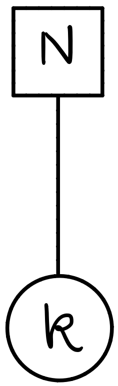
with Abelian G of

de Boer-H-Ooguri-Oz-Yin 1996

mirror symmetry

with Abelian G of HV2000

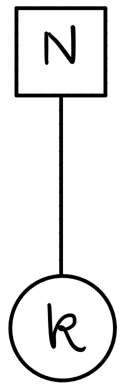
3d $\mathcal{N} = 4$ gauge theory



$$G = U(k)$$

$$H = \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^k) \otimes_{\mathbb{C}} \mathbb{H}$$

3d $\mathcal{N} = 4$ gauge theory

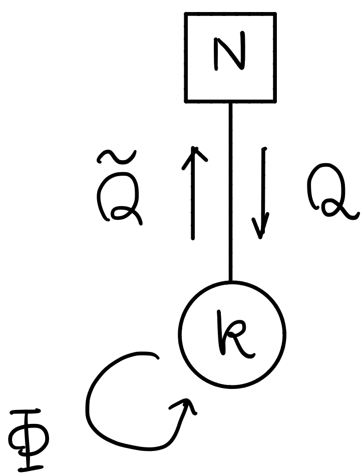


$$G = U(k)$$

$$H = \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^k) \otimes_{\mathbb{C}} \mathbb{H}$$

In 4 SUSY term :

$$G = U(k)$$

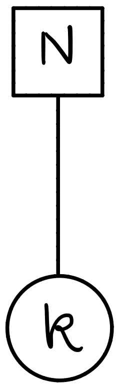


$$V = \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^k) \oplus \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^N)$$

$$\oplus\ \mathrm{End}(\mathbb{C}^k)$$

$$W = \mathrm{tr}_{\mathbb{C}^N} \tilde{Q} \Phi Q$$

3d $\mathcal{N} = 4$ gauge theory



Fayet-Iliopoulos $= (0,0,\xi)$ $\quad \xi > 0$:

$$\mathcal{M}_{\text{vac}} = \mathcal{M}_{\text{H}} = T^* G(k, N)$$

$\tilde{Q}$: fiber

Q : base

Breaking $\mathcal{N} = 4$ to $\mathcal{N} = 2$:

background scalar X to $U(1) \subset SU(2)_C \times SU(2)_H$ R-symmetry

$\rightsquigarrow$ mass $X, X, -2X$ to $Q, \tilde{Q}, \Phi$.

$\sigma \sim -X$: $\quad \mathcal{M}_{\text{vac}} = G(k, N)$

$\uparrow$

vector multiplet scalar

Then send $X \to \infty$ to isolate it.

3d $\mathcal{N} = 4$ mirror symmetry

$$\mathcal{T} \longleftrightarrow \mathcal{T}^{\vee}$$

Q^{ui}_{α} $\qquad$ Q^{iu}_{α}

$SU(2)_C, \ SU(2)_H$ $\qquad$ $SU(2)_H, \ SU(2)_C$

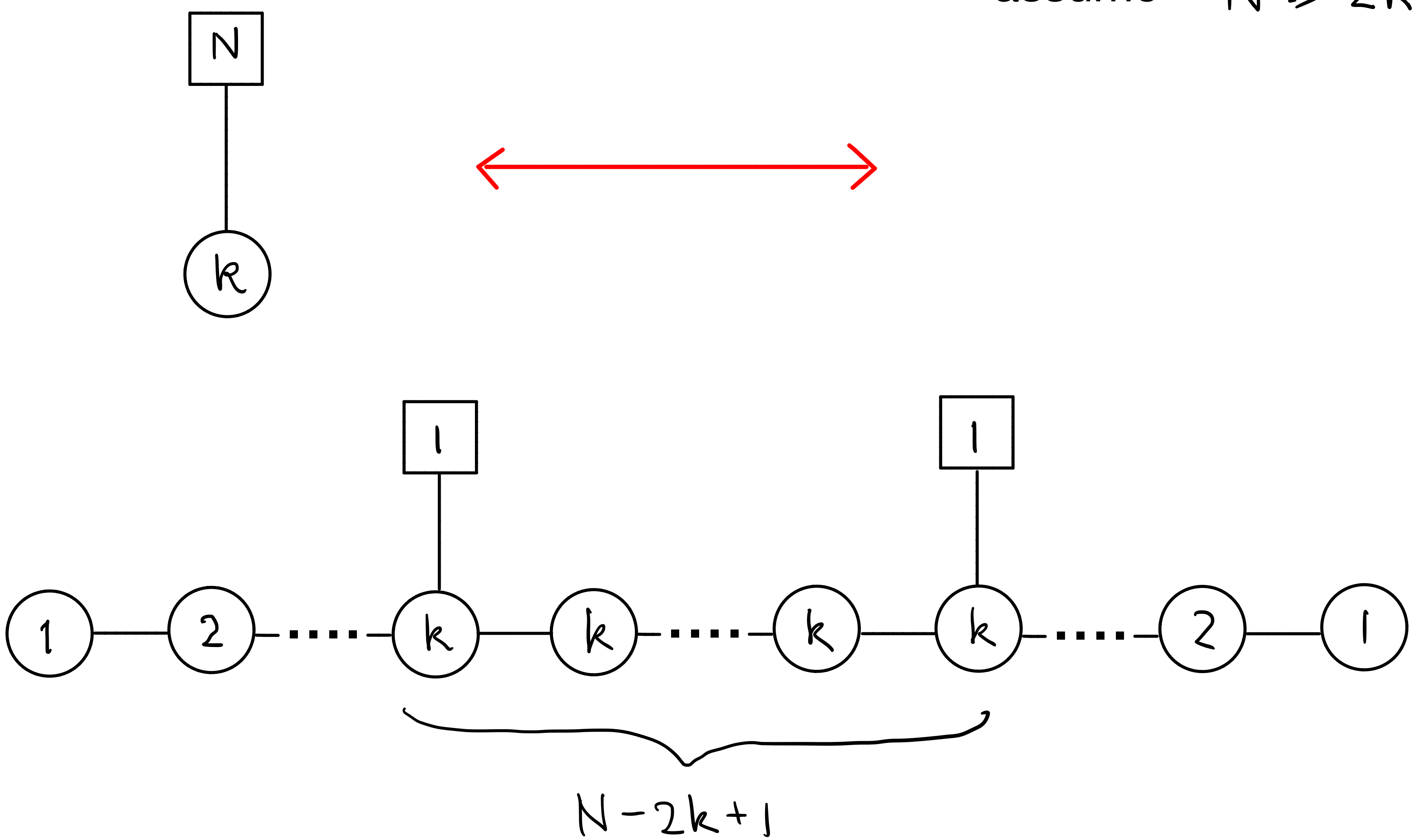
vector, hyper $\qquad$ hyper, vector

$M_C, \ M_H$ $\qquad$ $M_H, \ M_C$

mass, FI $\qquad$ FI, mass

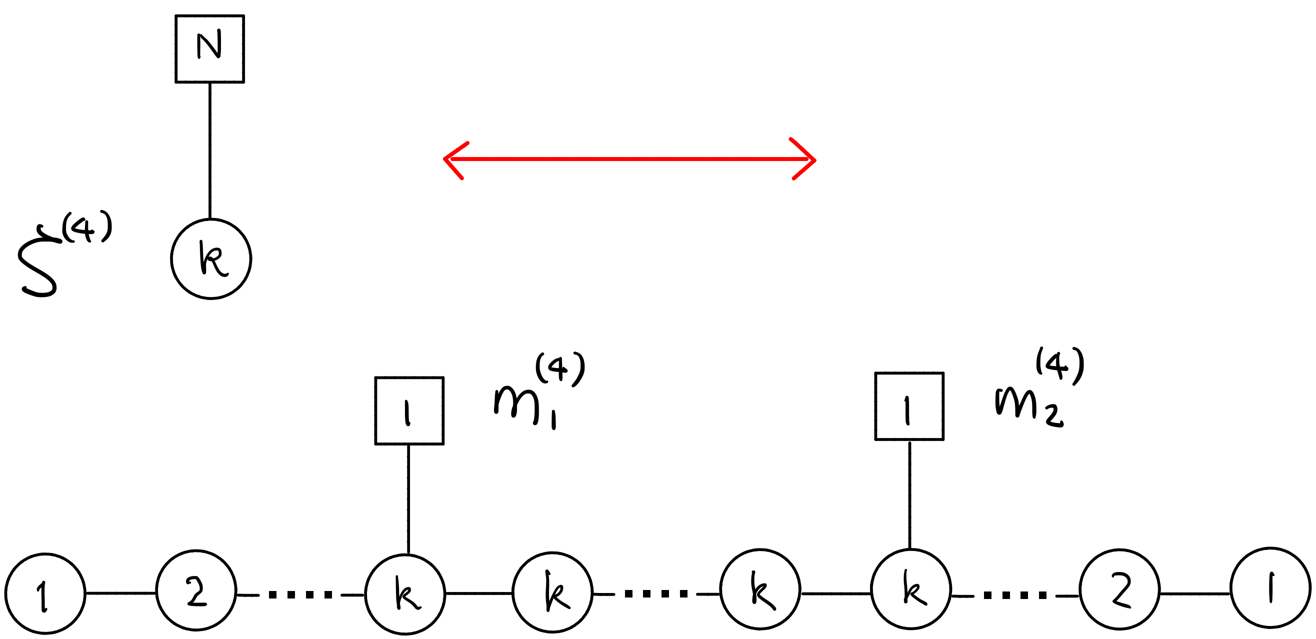
3d $\mathcal{N}=4$ mirror symmetry

assume $N \geqslant 2k$



Hanany-Witten 1996

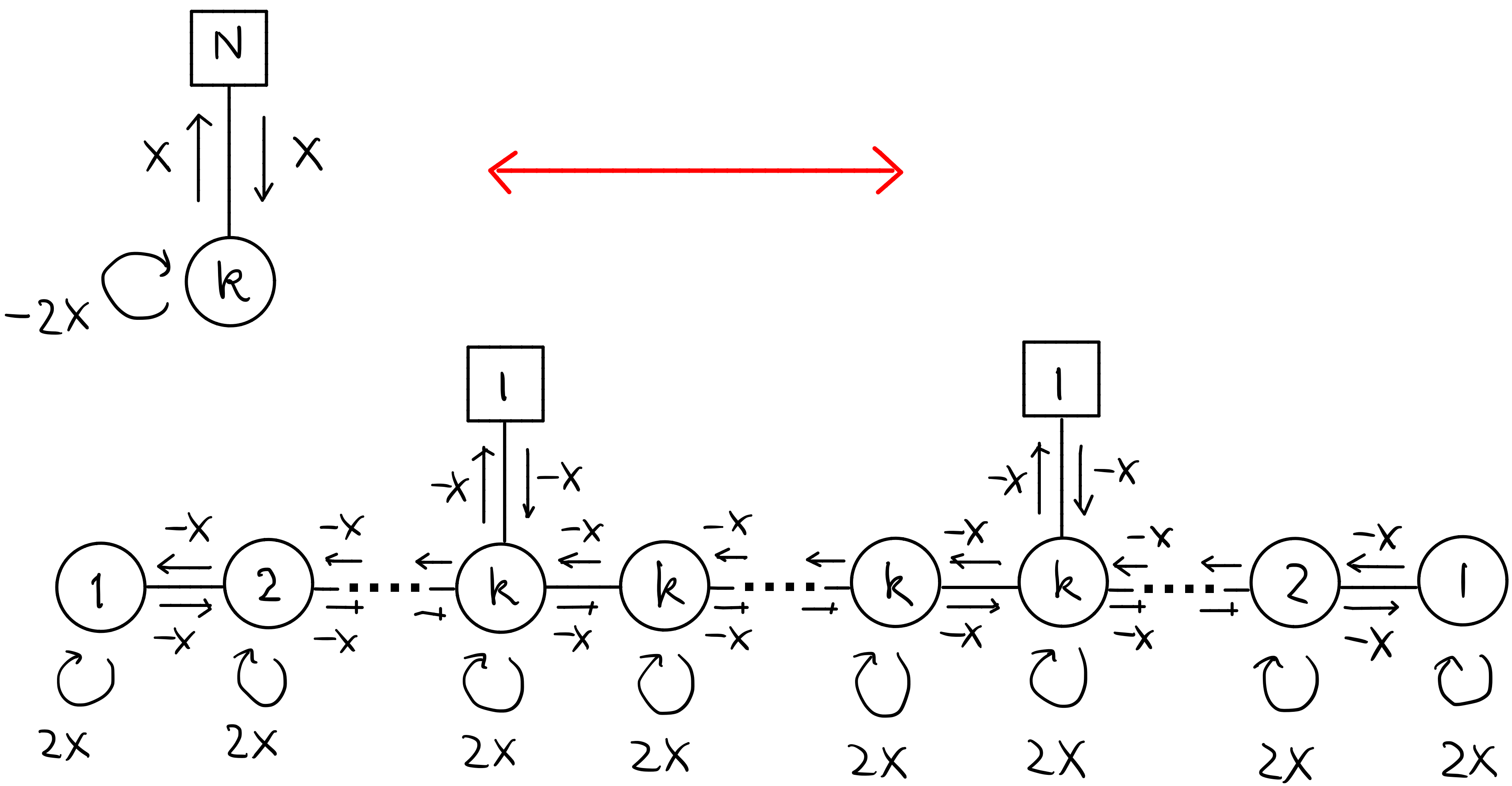
3d $\mathcal{N} = 4$ mirror symmetry



mirror map : $\zeta^{(4)} = m_2^{(4)} - m_1^{(4)}$

also $N-1$ masses $\longleftrightarrow$ $N-1$ FIs

$\mathcal{N} = 4 \rightarrow 2$ X- deformation : gives masses as



Will send $X \rightarrow \infty$ eventually.

LHS

Set $\zeta^{(4)} = N X + \zeta^{(2)}$

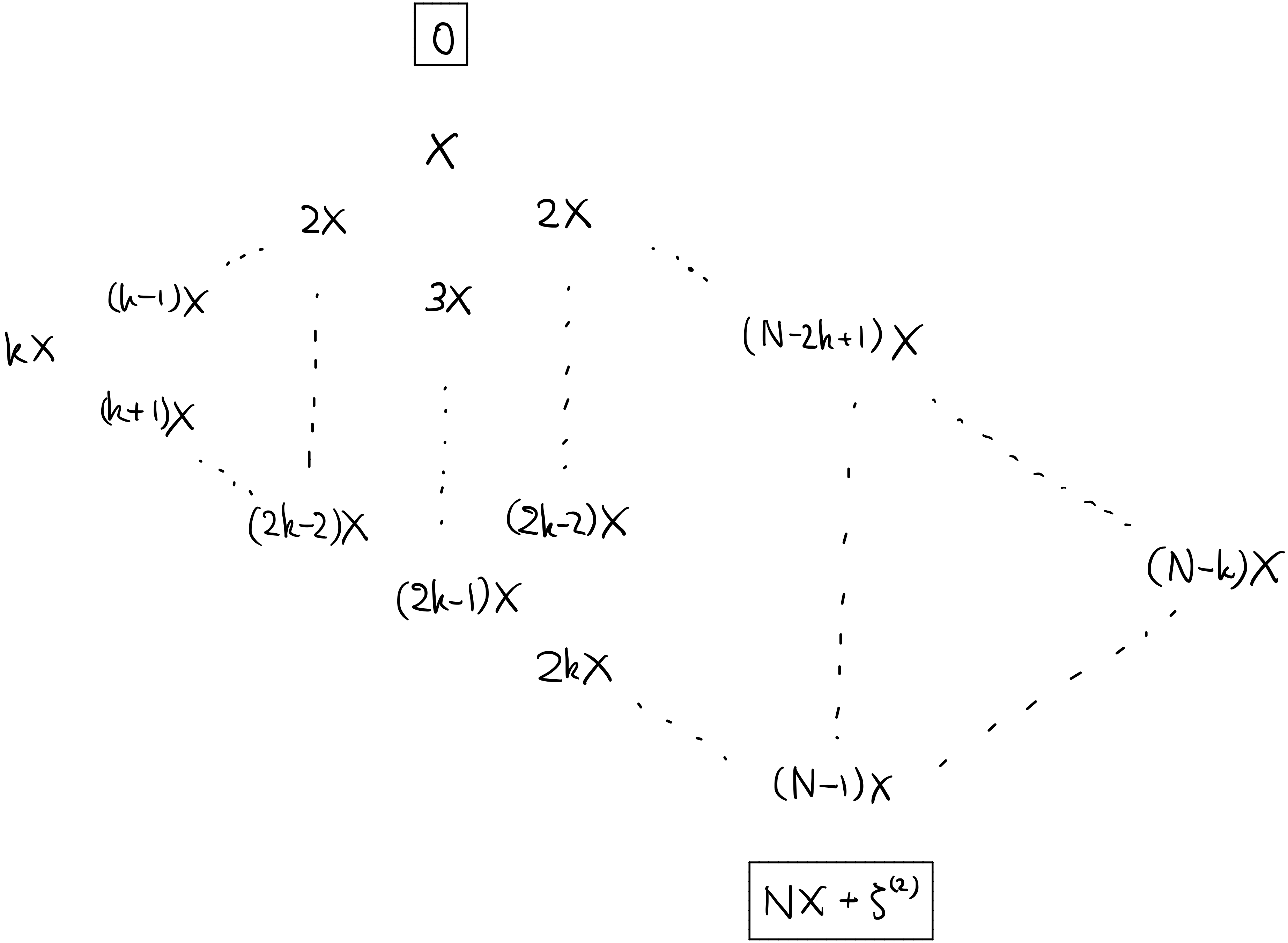
and look near $\sigma \sim -X$

$$\rightsquigarrow \quad \mathcal{M}_{\text{vac}} = G(k, N)$$

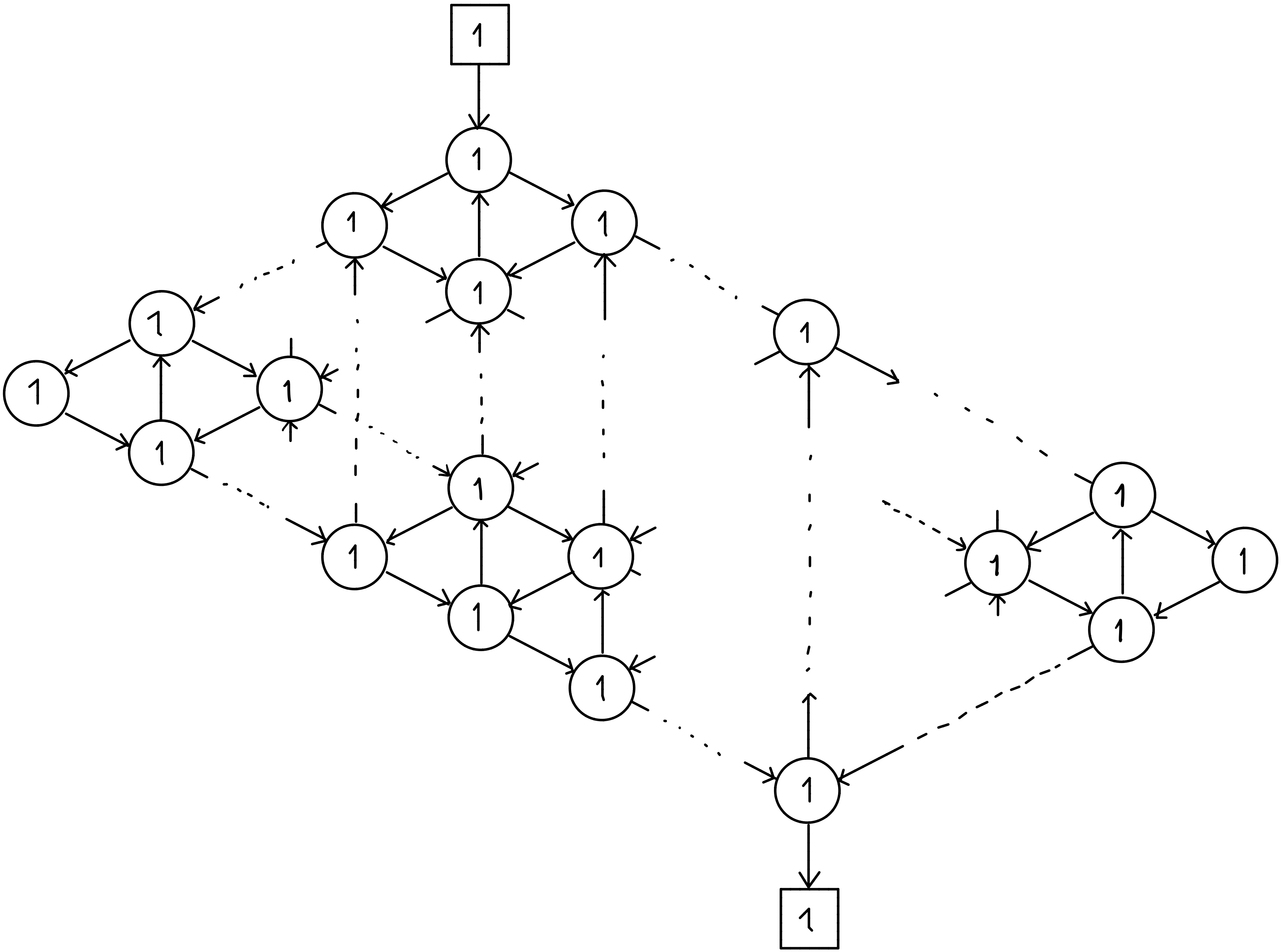
size = $\zeta^{(2)}$

$\longleftrightarrow$ RHS = ?

RHS : Set $m_1^{(4)} = 0$, $m_2^{(4)} = NX + \zeta^{(2)}$ and look near $\sigma \sim$



RHS : Light fields are



Compactification on $S^1_R = \mathbb{R}/2\pi R\,\mathbb{Z}$ $\qquad \zeta_{3d} := \zeta^{(2)}$

$$\zeta_{2d} = 2\pi R\, \zeta_{3d} \qquad \leftarrow \text{match at } \mu = \frac{1}{2\pi R}$$

2d $G(k,N)$ σ-model has dynamical scale Λ so that

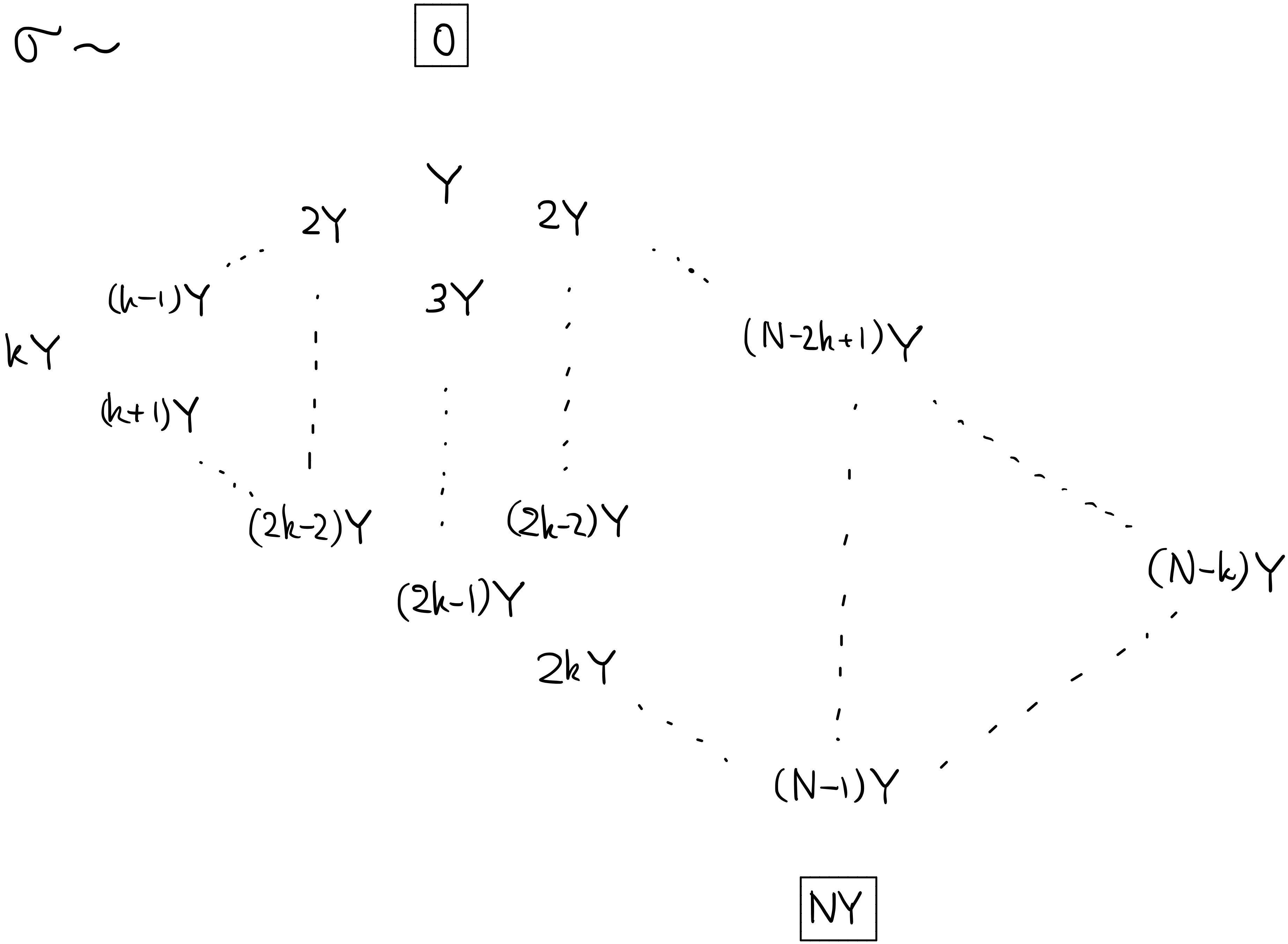
$$\zeta_{2d}(\mu) = N \log(\mu/\Lambda)$$

Thus $\quad \zeta^{(2)} = \zeta_{3d} = \dfrac{N}{2\pi R} \log\left(\dfrac{1}{2\pi R \Lambda}\right) = NY$

$$Y := \frac{1}{2\pi R} \log\left(\frac{1}{2\pi R\Lambda}\right) \to \infty \quad \text{as} \quad R \to 0$$

Again, we look at $\sigma = \sigma_{2d} \sim$

$\sigma \sim$



Integrating out matter fields including Kaluza-Klein modes, we obtain

$$\Delta \widetilde{W} = -\sum_{i,n} \left(m_i + \frac{in}{R} \right) \left(\log \left(m_i + \frac{in}{R} \right) - 1 \right)$$

$$m_i = \langle Q_i, \sigma \rangle + \beta_i Y, \qquad \beta_i = \begin{cases} 1 & \text{for } \swarrow, \searrow, \downarrow \\ -2 & \text{for } \uparrow \end{cases}$$

2d vector multiplet scalar (shifted)

$$\frac{\partial \Delta \widetilde{W}}{\partial \sigma_a} = -\sum_{i,n} Q_i^a \log \left(m_i + \frac{in}{R} \right)$$

$$= -\sum_i Q_i^a \log \left(e^{\pi R m_i} - e^{-\pi R m_i} \right)$$

$$= \sum_i Q_i^a \, e^{-\operatorname{sgn}\beta_i \, 2\pi R m_i} + \underbrace{\cdots}_{\text{cancels with classical term and X-massed contributions}}$$

$$\widetilde{W}_{eff} = -\sum_i \frac{1}{\mathrm{sgn}\,\beta_i \cdot 2\pi R}\, e^{-\mathrm{sgn}\,\beta_i \cdot 2\pi R\, m_i}$$

$$m_i = \langle Q_i, \sigma\rangle + \beta_i Y, \qquad Y = \frac{1}{2\pi R}\log\left(\frac{1}{2\pi R\Lambda}\right)$$

$$= -\sum_i \frac{(2\pi R\Lambda)^{|\beta_i|}}{\mathrm{sgn}\,\beta_i \cdot 2\pi R}\, e^{-\mathrm{sgn}\,\beta_i \cdot 2\pi R \langle Q_i, \sigma\rangle}$$

$$\frac{(2\pi R\Lambda)^{|\beta_i|}}{\mathrm{sgn}\,\beta_i \cdot 2\pi R} \xrightarrow{R\to 0} \begin{cases} \Lambda & \beta_i = 1 \\ 0 & \beta_i = -2 \end{cases}$$

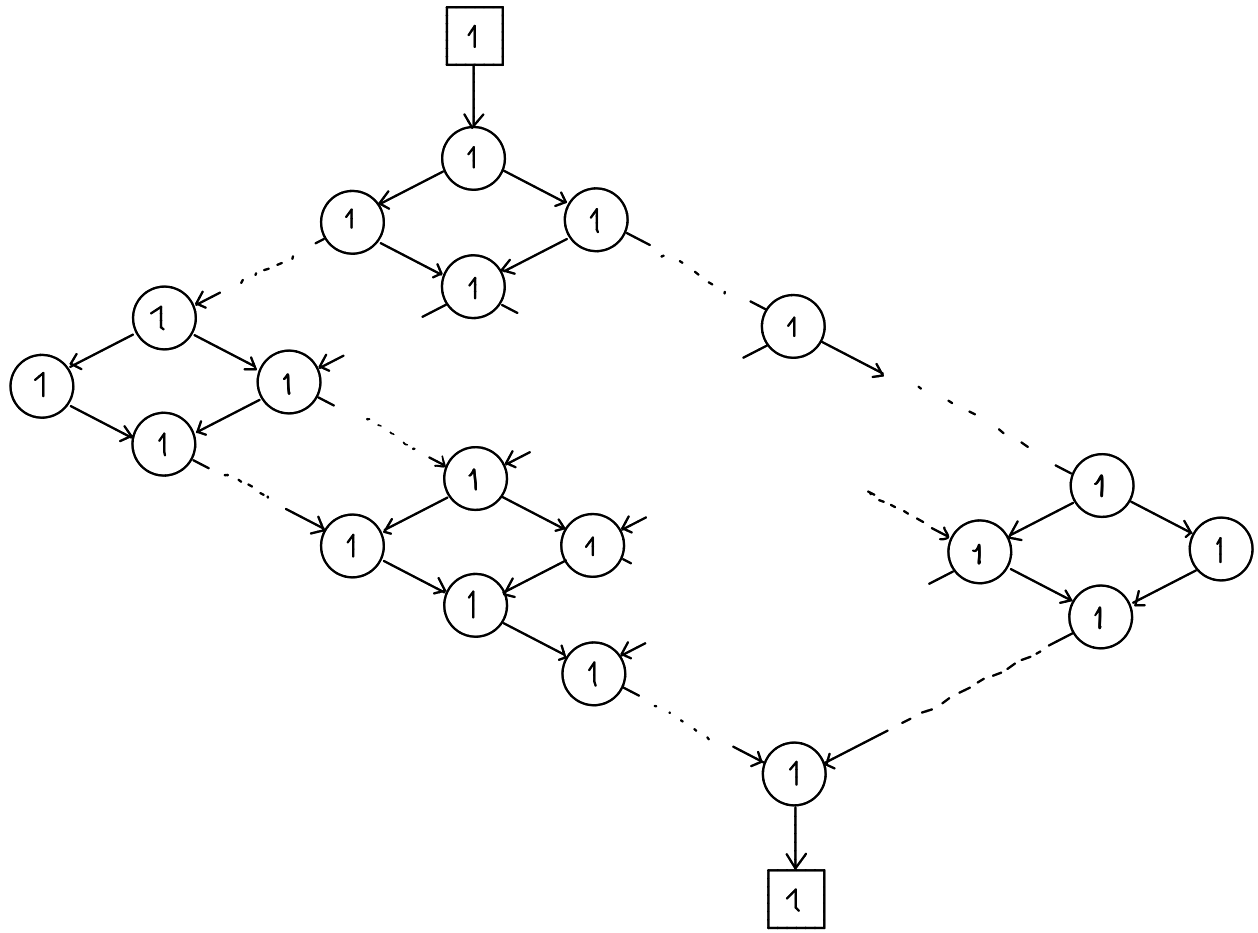
Note also $\sigma = \sigma_{3d} + i A_2$; $\quad A_2 \equiv A_2 + \frac{1}{R}$

$$\Theta := 2\pi R\, \sigma \equiv \Theta + 2\pi i$$

Thus as $R \to 0$,

$$\widetilde{W}_{eff} \to -\sum_{\beta_i = 1} \Lambda\, e^{-\langle Q_i, \Theta\rangle}$$

$\beta_i = 1$ fields are all but $\uparrow$'s, i.e.



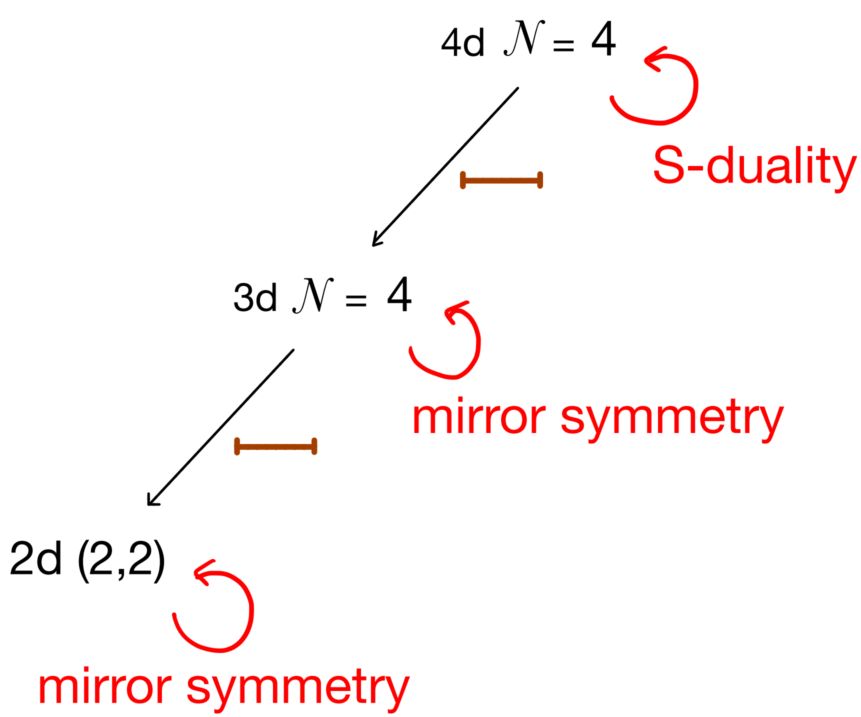
$$\widetilde{W}_{eff} = -\sum_{\beta_i=1} \Lambda\, e^{-\langle Q_i, \Theta \rangle}$$ is nothing but W_{EHX} !

However, we obtained a model without partial compactification.

We must have made some mistake somewhere....

Also, relation to Langlands duality (if any) is not clear.

Perhaps, it is better to consider compactification on segment.



It is indeed an active area of research...

江口さん、 これまで導いて下さり

ありがとうございます。

私自身もそのような存在になれるよう

がんばりたいと思います。