Plane partitions and BPS state counting in diverse dimensions

To the memory of Prof. Tohru Eguchi

素粒子論と数理物理学 ー 江口-Hanson 解の発見から 40 年 ー

2019 年 10 月 19 日

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Joint works with Eguchi-san

1. W(infinity) algebra in two-dimensional black hole

with S.-K.Yang, hep-th/9209122 (Cambridge)

2. Topological strings, flat coordinates and gravitational descendants

with Y. Yamada,S.-K.Yang, hep-th/9302048 (Cambridge)

3. Toda lattice hierarchy and the topological description of the c = 1 string theory

hep-th/9404056 (Hiroshima)

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4. Five-dimensional gauge theories and local mirror symmetry

hep-th/0005008 (Hiroshima)

5. Topological strings and Nekrasov's formulas

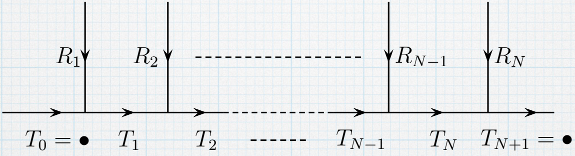
hep-th/0310235 (Nagoya)

6. Geometric transitions, Chern-Simons gauge theory and Veneziano type amplitudes

hep-th/0312234 (Nagoya)

Topological string and plane partition

Veneziano type amplitudes as a building block of Nekrasov partition function (q=t).



$$K^{SU(N)}_{\{R_i\}} = \prod_{i=1}^{N} \dim_q R_i \cdot \langle 0 | \prod_{k=1}^{N} V_-^{[R_k^t]} V_+^{[R_k]} Q_k^{L_0} | 0 \rangle$$

$$V_\pm^{[R]}(q) := V_\pm(x_i = q^{\mu_i^R - i + 1/2}), \qquad V_\pm(x_i) := \exp\left(\sum_n \frac{p_n(x_i)}{n} \alpha_\pm \right)$$

Inspired by Okounkov-Reshetikhin-Vafa (hep-th/0309208)

Topological string and plane partition

Counting of plane partitions by discrete time evolution of the Young diagram

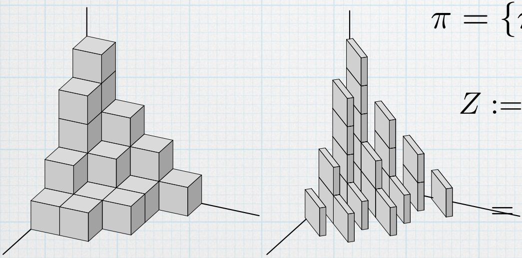


Figure 3: A 3d partition and its diagonal slices

$$\pi = \{\pi_{i,j}\} \qquad \pi_{i,j} \geq \pi_{i+1,j}, \pi_{i,j+1}$$

$$Z := \sum_{\pi} q^{|\pi|}$$

$$= \left\langle \prod_{n>0} V_+(q^{n-\frac{1}{2}}) \prod_{n>0} V_-(q^{-n-\frac{1}{2}}) \right\rangle$$

$$V_+(z)V_-(w) = \left(1 - \frac{z}{w}\right)^{-1} V_-(w)V_+(z)$$

from hep-th/0309208

Topological string and plane partition

Generating function of the numbers of plane partitions

$$\mathcal{M}_3(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} \frac{q^k}{(1-q^k)^2} \right)$$

MacMahon function and the Hodge integral

$$Z_{\mathrm{top}} = \exp\left(\frac{\chi}{2} \sum_g g_s^{2g-2} \int_{\overline{\mathcal{M}}_g} c_{g-1}^3(\mathcal{H}) \right) = \mathcal{M}_3(q)^{\frac{\chi}{2}}$$

$$q := e^{-g_s}$$

Derivation of topological vertex (hep-th/0309208)

Refined MacMahon function

MacMahon function of degree d

$$\mathcal{M}_d(q) = \text{P.E.}[f_d(q)], \qquad f_d(q) = \frac{q}{(1-q)^{d-1}}$$

Unfortunately d=4 fails to count solid partitions

Plethystic exponential : Character of symmetric algebra

$$\text{P.E.}[F(t_1, t_2, \cdots, t_\ell)] = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} F(t_1^k, t_2^k, \cdots, t_\ell^k)\right).$$

$$F(t_1, t_2, \cdots, t_\ell) = \sum_{n_1, \cdots, n_\ell \in \mathbb{Z}} a_{n_1 \cdots n_\ell} t_1^{n_1} \cdots t_\ell^{n_\ell}$$

$$\text{P.E.}[F(t_1, t_2, \cdots, t_\ell)] = \prod_{n_1, \cdots, n_\ell \in \mathbb{Z}} (1 - t_1^{n_1} \cdots t_\ell^{n_\ell})^{-a_{n_1 \cdots n_\ell}}$$

Refined MacMahon function

Let us introduce the following "refinement"

$$\mathcal{M}_d^{(k)}\left[\begin{array}{c}\vec{t}\\ \vec{q}\end{array}\right](\mathfrak{q}) = \mathrm{P.E.}\left[f_d^{(k)}(q_1,\cdots,q_{d+k},t_1,\cdots,t_k,\mathfrak{q})\right]$$

$$f_d^{(k)} = \frac{\mathfrak{q}[t_1]\cdots[t_k]}{[q_1]\cdots[q_{d+k-1}]}, \qquad [X] := X^{\frac{1}{2}} - X^{-\frac{1}{2}}$$

We will see the generating functions of BPS state counting are expressed in terms of refined MacMahon functions

Generalized ADHM equations

ADHM description of 4d Yang-Mills Instanton

ADHM = BPS condition for D0-D4 system

Introduce two vector spaces = "Chan-Paton bundles"

$$\dim_{\mathbb{C}} N = n \qquad \dim_{\mathbb{C}} K = k$$

for D4 brane   for D0 brane

$$B_{1,2} \in \mathrm{Hom}_{\mathbb{C}}(K, K) \qquad I, J^{\dagger} \in \mathrm{Hom}_{\mathbb{C}}(N, K)$$

$$\mu_{\mathbb{C}} = [B_1, B_2] + IJ = 0$$ F term condition

D-term condition can be traded with the stability condition

Generalized ADHM equations

One can consider BPS condition for D0-D6 and D0-D8 systems

D0-D6 BPS condition

$$B_{1,2,3},\ \ Y \in \mathrm{Hom}_{\mathbb{C}}(K,K) \qquad I = J^{\dagger} \in \mathrm{Hom}_{\mathbb{C}}(N,K)$$

$$\mu_{\mathbb{C}} = [B_i, B_j] + \frac{1}{2}\epsilon_{ijk}\left[B_k^{\dagger}, Y\right] = 0, \qquad \mu_B = Y \cdot I = 0$$

D0-D8 BPS condition (only for Calabi-Yau)

$$B_{1,2,3,4} \in \mathrm{Hom}_{\mathbb{C}}(K,K) \qquad I = J^{\dagger} \in \mathrm{Hom}_{\mathbb{C}}(N,K)$$

$$\mu_{\mathbb{C}} = [B_a, B_b] + \frac{1}{2}\Omega_{abcd}\left[B_c^{\dagger}, B_d^{\dagger}\right] = 0$$

I is required for imposing the stability condition

Generalized ADHM equations

(Virtual) Dimensions of the moduli space

$$\{GL(k,\mathbb{C})\} \longrightarrow \{(B_i, I)\} \longrightarrow \{\mathrm{ADHM}\}$$

symmetry Variables Constraints

$$\dim_{\mathbb{C}} \mathcal{M}^{4D}_{\mathrm{ADHM}} = 2nk, \qquad \dim_{\mathbb{C}} \mathcal{M}^{6D}_{\mathrm{ADHM}} = 0, \qquad \dim_{\mathbb{C}} \mathcal{M}^{8D}_{\mathrm{ADHM}} = nk,$$

Only $\mathcal{M}^{4D}_{\mathrm{ADHM}}$ has regular tangent space

$\mathcal{M}^{8D}_{\mathrm{ADHM}}$ cannot be hyperKaehler

The action of $(q_1, \cdots, q_d)$ on $\mathbb{C}^d$ and the maximal torus of $U(n)$: $(e^{a_1}, \cdots, e^{a_n})$ induce the topic action on $\mathcal{M}^{2dD}_{\mathrm{ADHM}}$

Equivariant characters

Fixed points of the toric action are labelled by partitions (d=2), plane partitions (d=3) and solid partitions (d=4)

At each fixed point the (virtual) tangent space of $\mathcal{M}^{2dD}_{\mathrm{ADHM}}$ is decomposed into the rep. space of the toric action

This is the equivariant charactor and it gives the weight of the localization computation of path integral

Localization = sum over the fixed points

Topological partition function

$$Z_{\rm top}(q_i;\mathfrak{q}) := \langle {\rm P.E.}[\chi_\pi(q_i)]\rangle = {\rm P.E.}[F(q_i;\mathfrak{q})]$$

$$\langle {\rm P.E.}[\chi_\pi(q_i)]\rangle = \sum_\pi \mathfrak{q}^{|\pi|}{\rm P.E.}[\chi_\pi(q_i)]$$ Localization

$$Z^{6D}_{{\rm top},U(1)} = \mathcal{M}_3^{(3)}\begin{bmatrix} & \hbar q_1^{-1} & \hbar q_2^{-1} & \hbar q_3^{-1} & \\ q_1 & q_2 & q_3 & \hbar^{-\frac{1}{2}}\mathfrak{q} & \hbar^{-\frac{1}{2}}\mathfrak{q}^{-1} \end{bmatrix} \quad (1)$$

Nekrasov (2008), Okounkov (2015)

M theoretic ! $\quad q_4 := \hbar^{-\frac{1}{2}}\mathfrak{q}, \qquad q_5 := \hbar^{-\frac{1}{2}}\mathfrak{q}^{-1}$

$$q_1q_2q_3q_4q_5 = 1$$

The coupling const is on an equal footing with q_i

Topological partition function

$$Z^{6D}_{\mathrm{top},U(n)} = \mathcal{M}_3^{(4)} \left[\begin{array}{cccccc} & \hbar q_1^{-1} & \hbar q_2^{-1} & \hbar q_3^{-1} & \hbar^{-n} & \\ q_1 & q_2 & q_3 & \hbar & \hbar^{-\frac{1}{2}}\mathfrak{q} & \hbar^{\frac{1}{2}}\mathfrak{q} \end{array} \right] (1)$$

Awata-H.K. (2009)

It does NOT depend on the Coulomb moduli $(a_1, \cdots, a_n)$

$$Z^{8D}_{\mathrm{top},U(1),adj} = \mathcal{M}_3^{(4)} \left[\begin{array}{cccccc} & q_1 q_2 & q_2 q_3 & q_3 q_1 & \mu & \\ q_1 & q_2 & q_3 & q_4 & \mu^{\frac{1}{2}}\mathfrak{q} & \mu^{\frac{1}{2}}\mathfrak{q}^{-1} \end{array} \right] (1)$$

Nekrasov (2017)

It is NOT $\mathcal{M}_4^{(k)}$ but $\mathcal{M}_3^{(4)}$ which appears

$$Z^{8D}_{\mathrm{top},U(1),adj} \to \mathcal{M}_3(\mathfrak{q})^{\frac{m(\epsilon_1+\epsilon_2)(\epsilon_2+\epsilon_3)(\epsilon_3+\epsilon_1)}{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}}.$$

BPS/VOA correspondence ??

$$\sum \mathfrak{q}^{|\pi|}(\text{Infinite Product}) = \text{Infinite Product}$$

"Super-Integrability" ?

Exchange relation of VOA

$$g_{ij}(z,w)V_+^{(i)}(z)V_-^{(j)}(w) = g_{ji}(w,z)V_-^{(j)}(w)V_+^{(i)}(z)$$

E.g. $g(z,w) = (z-qw)(z-t^{-1}w)(z-tq^{-1}w)$

Ding-Iohara-Miki algebra

VEV of vertex operators, screening operators
—> Infinite product

BPS/VOA correspondence ??

M theory vertex $V(\lambda,\mu,\nu) = \sum_{\pi\to(\lambda,\mu,\nu)} (-\mathbf{q})^{|\pi|}\hat{\mathbf{a}}(\chi_\pi)$

is proposed for computing $Z^{6D}_{\rm top}$ Nekrasov-Okounkov (2014)

In a particular limit

$$q_1, q_3 \to 0,\ \ (|q_1| << |q_3|); \qquad q_2 \to \infty; \qquad \hbar = \text{fixed}$$

$V_{\lambda,\mu,\nu}$ reduces to the refined TV (intertwiner of DIM)

Awata-Feigin-Shiraishi (2012)

Is there any underlying VOA for M theory vertex?

Concerning $Z^{8D}_{\rm top}$

Is there any VOA acting on the space of solid partitions?