Exact Path Integral for 3D Quantum Gravity

Nori Iizuka @ Osaka U.

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with ***A. Tanaka,*** ***and*** ***S. Terashima.***

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YKIS 2016, Kyoto

Can we calculate Quantum Gravity partition function?

$$Z_{gravity} = \int \mathcal{D}g_{\mu\nu} e^{-S_{gravity}}$$

Yes or No.

“Directly in the Bulk”

Can we calculate Quantum Gravity partition function?

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Yes or No.

In this talk, we try to calculate *pure quantum gravity* *partition function directly in 3D (2+1) bulk,*

$$Z_{gravity} = \int \mathcal{D}g_{\mu\nu} e^{-S_{gravity}}$$

$$S_{gravity} = \frac{1}{16\pi G_N} \int d^3x \sqrt{g} \left(R - 2\Lambda\right)$$

Why 3D pure Gravity?

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Why 3D pure Gravity?

$$S_{gravity} = \frac{1}{16\pi G_N} \int d^3x \sqrt{g}\,(R - 2\Lambda)$$

- Simplest!
- No gravitons
- Still black holes (Banados-Teitelboim-Zanelli or *BTZ BH* in short) exist, so interesting enough!

Why 3D pure Gravity?

$$S_{gravity} = \frac{1}{16\pi G_N}\int d^3x\sqrt{g}\,(R-2\Lambda)$$

- No string theory compactification (i.e., stringy derivation) to obtain pure 3D gravity
- This makes it difficult to construct concrete holographic setting for pure gravity

Why 3D pure Gravity?

(Banados-Teitelboim-Zanelli '92)

- BTZ BH sol'ns in pure gravity (*$8G_N$ = 1 unit*)

$$\boxed{ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(J(r)dt + d\phi)^2}$$

$$f(r) = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2} \qquad J(r) = -\frac{J}{2r^2}$$

- In order to have horizons; we need

$$M \geq 0$$

$$M \geq \frac{J}{\ell}$$

$$\left(r_H = \pm\ell\sqrt{M}\frac{\sqrt{1 \pm \sqrt{1 - \frac{J^2}{\ell^2 M^2}}}}{\sqrt{2}}\right)$$

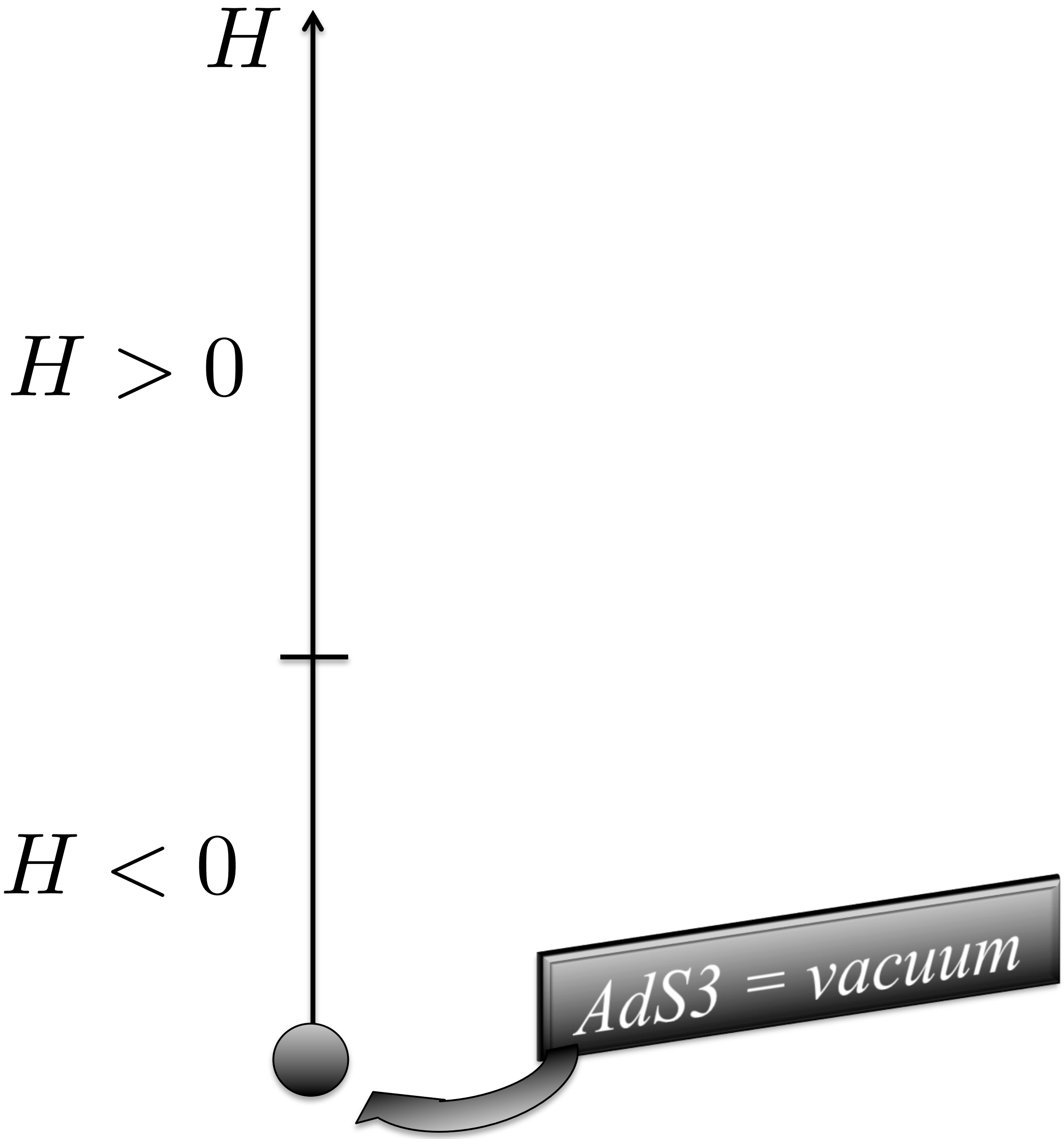
Why 3D pure Gravity?

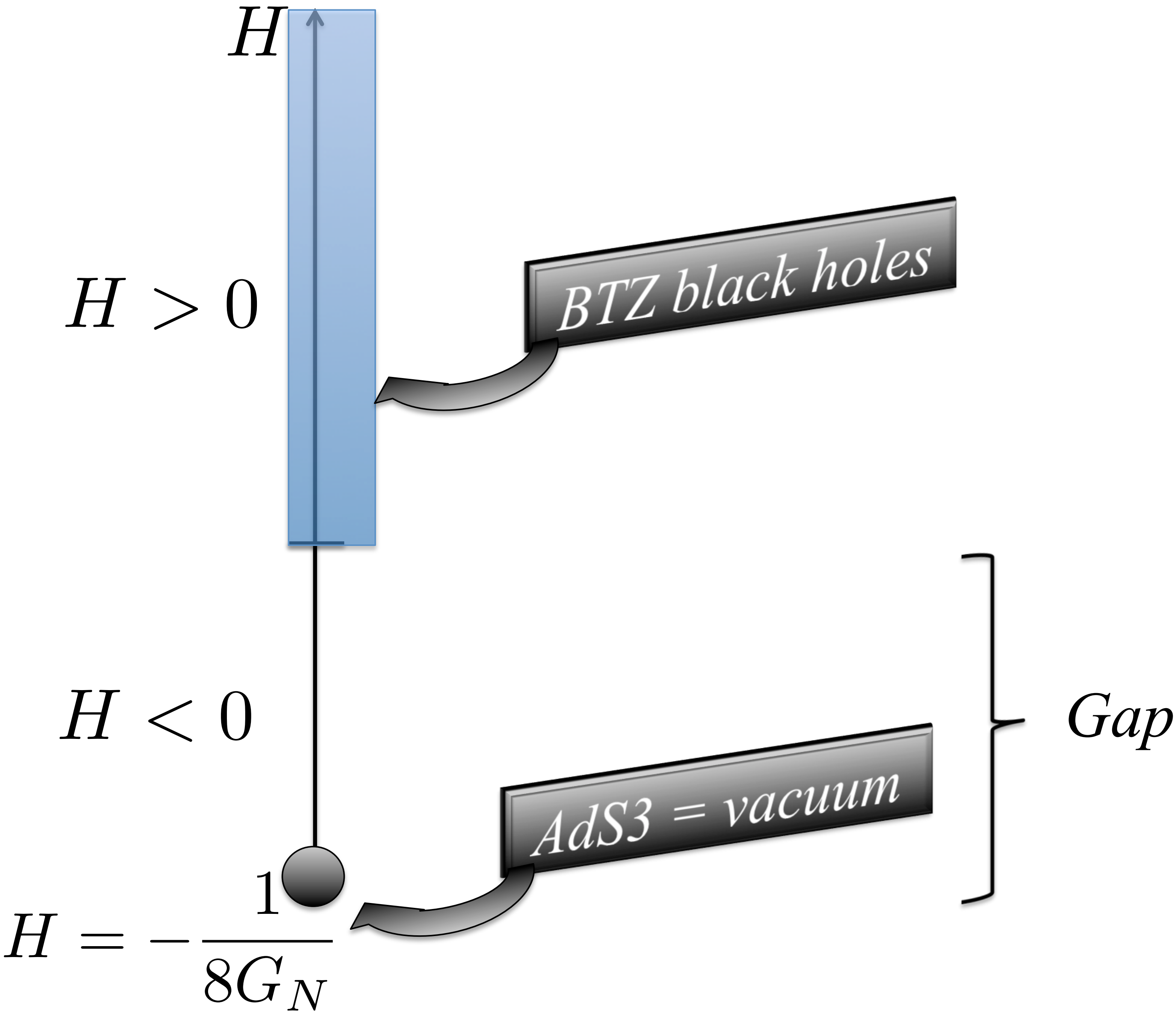
(Banados-Teitelboim-Zanelli '92)

- Consider $J=0$ for simplicity.

$$ds^2 = -(-M + \frac{r^2}{\ell^2})dt^2 + \frac{dr^2}{(-M + \frac{r^2}{\ell^2})} + r^2 d^2\phi$$

- BH exsits only $M \geq 0$
- If we set $M = -1$, then it becomes AdS_3 (no horizon, no sing., homogeneous, isotropic)
- AdS$_3$ vacuum is separated from the continuous black hole spectrum by a mass gap.


H
$H > 0$
$H < 0$
AdS3 = vacuum



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Gap
AdS3 = vacuum
$H = -\frac{1}{8G_N}$

“Directly in the Bulk”

Can we calculate Quantum Gravity partition function?

$$Z_{gravity} = \int \mathcal{D}g_{\mu\nu} e^{-S_{gravity}}$$

Our strategy

(NI – Tanaka - Terashima '15)

- 3D pure gravity (in the bulk)

Step 1

- 3D Chern-Simons theory (in the bulk)

Step 2

- 3D super-Chern-Simons theory (in the bulk)

Our strategy

(NI – Tanaka - Terashima '15)

- 3D super-Chern-Simons theory (in the bulk)
- Exact calculation is possible by localization

(with mild assumptions)

- The result for $c = 24$ is;

$$Z_{gravity} = J(q)J(\bar{q})$$

- This agrees with the extremal CFT partition function (Frenkel, Lepowsky, Meurman), predicted by Witten for 3D pure gravity

Plan of my talk:

- Introduction
- Quick 3D pure gravity overview
- Witten's proposal
- Our strategy in detail
- Localization and Exact results
- Discussions of "boundary fermions"

3D pure gravity overview

- Our theory = 3D pure gravity

gravity counter-term

$$S_{gravity} = \frac{1}{16\pi G_N} \int d^3x \sqrt{g} \, (R - 2\Lambda) + S_{GH} + S_c$$

Gibbons-Hawking boundary term

3D pure gravity overview

- Our theory = 3D pure gravity

gravity counter-term

$$S_{gravity} = \frac{1}{16\pi G_N}\int d^3x\sqrt{g}\,(R-2\Lambda) + S_{GH} + S_c$$

Gibbons-Hawking boundary term

- $\Lambda = -1/\ell^2$ (ℓ = AdS scale)

$$S_{GH} = \frac{1}{8\pi G_N}\int d^2x\sqrt{h}K$$

extrinsic curvature

$$S_c = -\frac{1}{8\pi G_N}\int d^2x\sqrt{h}$$

boundary metric

Step 1

3D pure gravity ⇔ 3D Chern-Simons

(Achucarro-Townsend '88, Witten '88)

- defining $SL(2,C)$ gauge fields as;

$$\left(\omega_\mu^a + \frac{i}{\ell} e_\mu^a\right) \frac{i}{2}\sigma_a dx^\mu = A\,, \left(\omega_\mu^a - \frac{i}{\ell} e_\mu^a\right) \frac{i}{2}\sigma_a dx^\mu = \bar{A}$$

Pauli matrix

Step 1

3D pure gravity ⇔ 3D Chern-Simons

(Achucarro-Townsend ’88, Witten ’88)

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Pauli matrix

- One can show, by defining $k \equiv \ell/4G_N$

$$S_{gravity} = \frac{ik}{4\pi}S_{CS}[A] - \frac{ik}{4\pi}S_{CS}[\bar{A}] \equiv S_{gauge}$$

where $$S_{CS}[A] = \int_M \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right)$$

Step 1

3D pure gravity ⇔ 3D Chern-Simons

(Achucarro-Townsend ’88, Witten ’88)

- Since action is decomposed into *holomorphic* part and *anti-holomorphic* part, regarding A and $\bar{A}$ are independent variables;

$$S_{gravity} = \frac{ik}{4\pi} S_{CS}[A] - \frac{ik}{4\pi} S_{CS}[\bar{A}] \equiv S_{gauge}$$

- Then partition function “naturally” factorizes:

$$Z = Z_{hol} \times Z_{anti-hol}$$

Step 1

3D pure gravity ⇔ 3D Chern-Simons

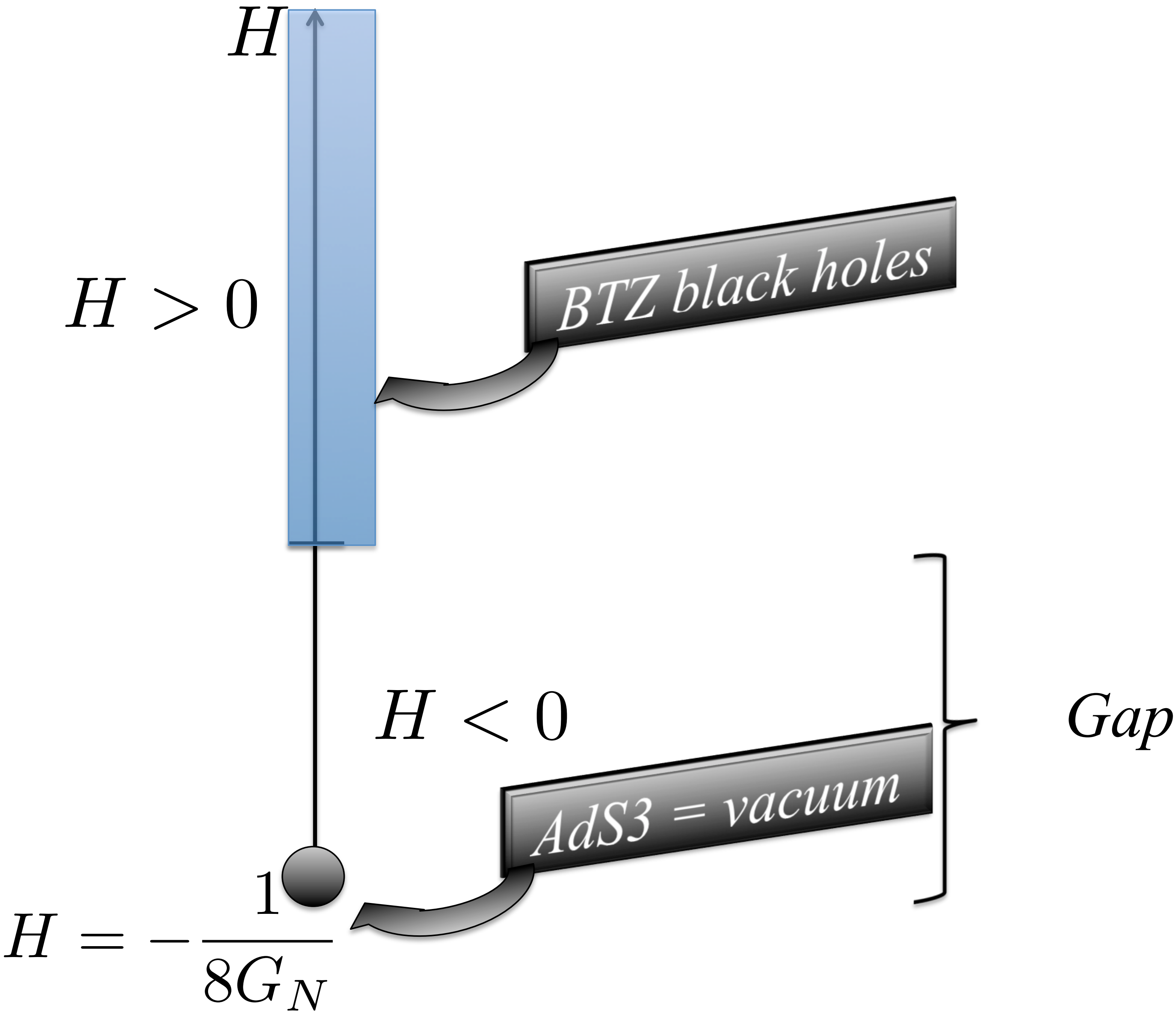
- **<u>A few more comments:</u>**
- Asym AdS_3 allows deformation of the metric; which is represented by the Virasoro algebra;
- Its central charge is obtained as (Brown - Henneaux '86)

$$c = \frac{3\ell}{2G_N} = 6k$$

- Using this, Cardy formula gives BTZ BH entropy in the large c limit (Strominger '97)

Witten’s proposal

(Witten ’06)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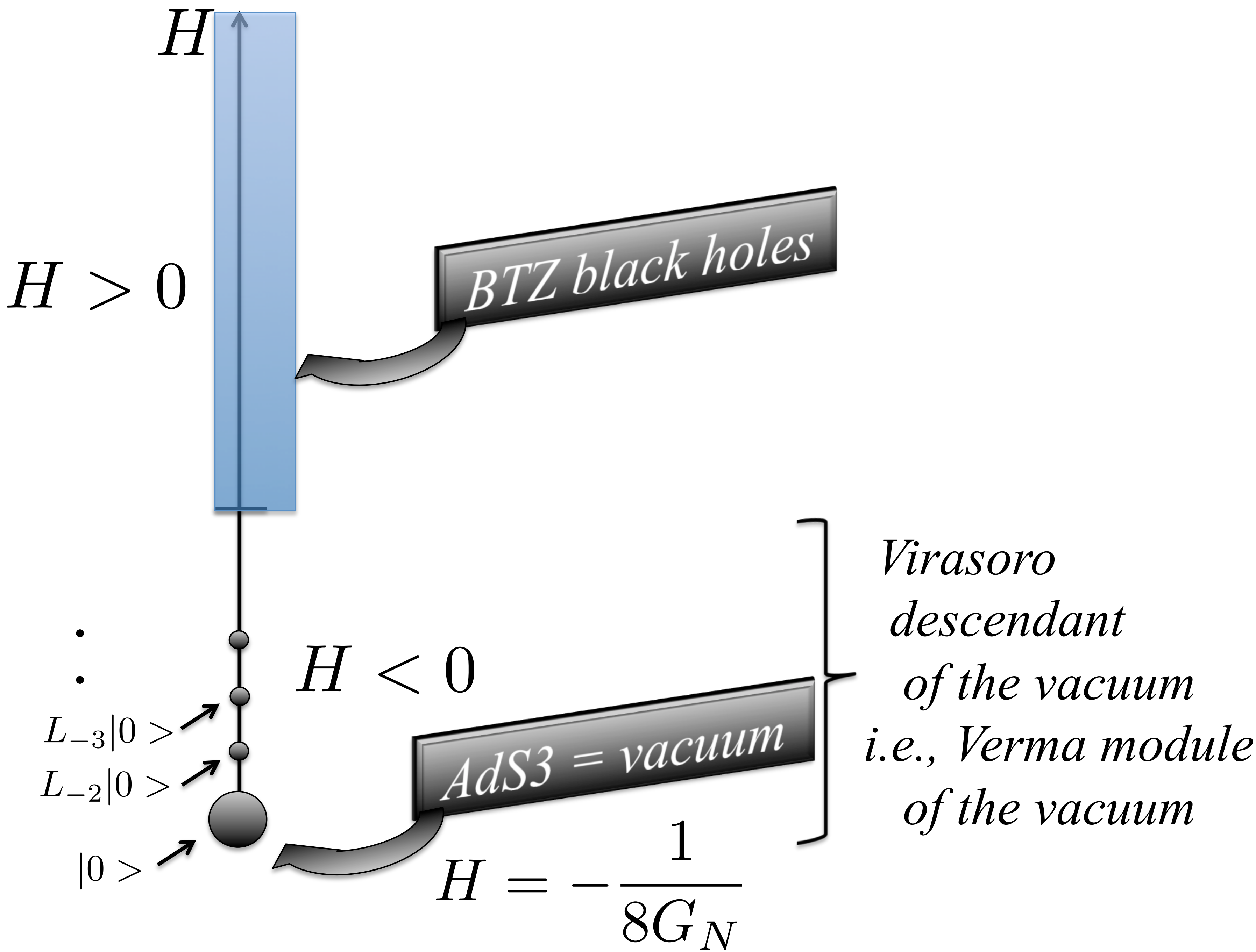

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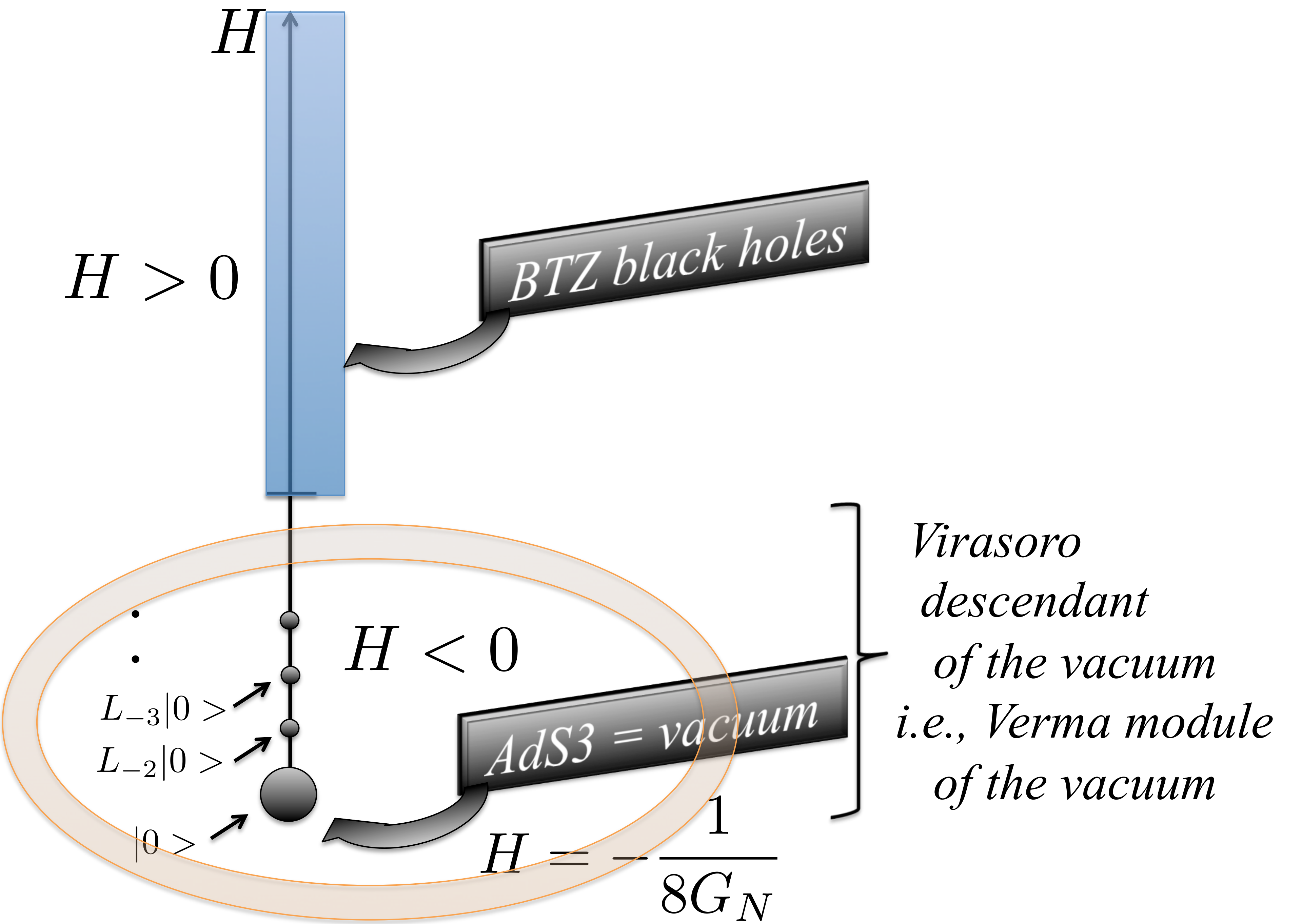
Virasoro descendant of the vacuum i.e., Verma module of the vacuum



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Virasoro descendant of the vacuum i.e., Verma module of the vacuum

Witten's proposal (Witten '06)

- Since

$$H_{vacuum} = -\frac{1}{8G_N} = -\frac{c}{12\ell} = -\frac{1}{\ell}\left(\frac{c}{24} + \frac{c}{24}\right)$$

- If the dual CFT exists, and assuming holomorphic factorization,

$$Z_{gravity} = Z_{hol}(q) \times Z_{anti-hol}(\bar{q})$$

- Then with $q \equiv e^{2\pi i\tau}$

$$(H_{hol} = L_0 - \frac{c}{24})$$

Witten's proposal (Witten '06)

- Its holomorphic partition function

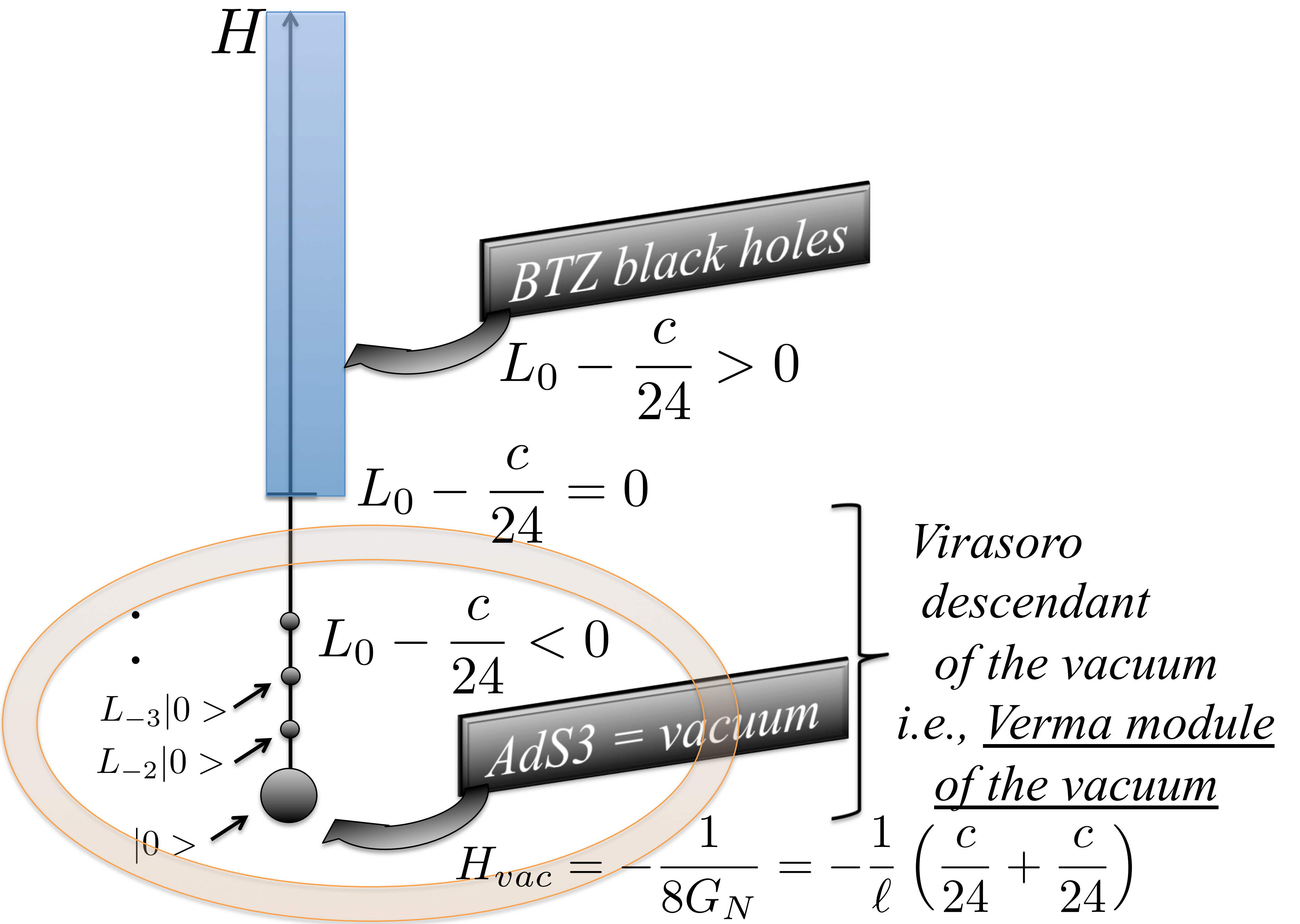
$$Z_{hol} = Tr\left(q^{L_0 - \frac{c}{24}}\right)$$

should have the following forms:

$$Z_{hol}(q) = \underbrace{q^{-\frac{c}{24}} \prod_{n=2}^{\infty} \frac{1}{1-q^n}}_{\text{“Verma module of the vacuum”}} + \underbrace{\mathcal{O}(q)}_{\text{“Verma module from “BTZ BHs = primary states”}}$$

“Verma module of the vacuum”

“Verma module from “BTZ BHs = primary states”



H
BTZ black holes
$L_0 - \frac{c}{24} > 0$
$L_0 - \frac{c}{24} = 0$
$L_0 - \frac{c}{24} < 0$
$L_{-3}|0>$
$L_{-2}|0>$
$|0>$
AdS3 = vacuum
Virasoro descendant of the vacuum i.e., Verma module of the vacuum
$H_{vac} = -\frac{1}{8G_N} = -\frac{1}{\ell}\left(\frac{c}{24} + \frac{c}{24}\right)$

Witten's proposal (Witten '06)

- CS quantization: $\frac{c}{24} = \frac{k}{4} = \frac{\ell}{16G_N} \in Z$

holomorphic partition fn contains poles;

$$Z_{hol}(q) = \underbrace{q^{-\frac{c}{24}} \prod_{n=2}^{\infty} \frac{1}{1-q^n}}_{\text{“Verma module of the vacuum”}} + \underbrace{\mathcal{O}(q)}_{\text{“Verma module from “BTZ BHs as primary states”}}$$

All poles are uniquely fixed by the central charge

“Extremal CFT”

Witten's proposal (Witten '06)

- For $c = 24$ case, we have

$$Z_{hol} = Tr\left(q^{L_0 - \frac{c}{24}}\right) = \frac{1}{q} + 0 + \mathcal{O}(q)$$

- With the modular invariance, we have $J(q)$,

$$Z_{hol} = J(q) = j(q) - 744$$

$$= q^{-1} + 196884q + 21493760q^2 + \dots$$

$$q = e^{2\pi i\tau} \qquad\qquad \tau \to \frac{a\tau + b}{c\tau + d}$$

Witten's proposal (Witten '06)

- Similarly, for $c = 48$,

$$Z_{hol} = J(q)^2 - 393767$$

$$= q^{-2} + 1 + 42987520q + 40491909396q^2 + \cdots$$

Witten's proposal (Witten '06)

- For $c=24$ case, $J(q)$ function as a partition function is constructed by Frenkel, Lepowsky, and Meurman ('84, '86)

We would like to calculate the bulk quantum pure gravity partition function and "re-derive" these results (under mild assumptions)

Our strategy

(NI – Tanaka - Terashima '15)

- 3D pure gravity (in the bulk)

Step 1

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Step 2

- 3D super-Chern-Simons theory (in the bulk)

Step 1

3D pure gravity ⇔ 3D Chern-Simons

(Achucarro-Townsend '88, Witten '88)

- In step 1, we have; $SL(2,C)$ CS theory;

$$\boxed{S_{gravity} = \frac{ik}{4\pi} S_{CS}[A] - \frac{ik}{4\pi} S_{CS}[\bar{A}] \equiv S_{gauge}}$$

$$k \equiv \ell / 4G_N \qquad \left(\omega_\mu^a + \frac{i}{\ell} e_\mu^a\right) \frac{i}{2} \sigma_a dx^\mu = A\,,$$

$$S_{CS}[A] = \int_M \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right)$$

Step 2

We supersymmetrize CS theory

(NI – Tanaka - Terashima '15)

- By introducing auxiliary fields to construct 3D $\mathcal{N}=2$ vector multiplet,

$$V=(A,\sigma,D,\bar{\lambda},\lambda)$$

- we supersymmetrize the Chern-Simons action

$$S_{SCS}[V]=S_{CS}[A]+\int d^3x\sqrt{g}\ \mathrm{Tr}\Big(-\bar{\lambda}\lambda+2D\sigma\Big).$$

- Note that additional fermions and bosons are only auxiliary fields

Step 2

We supersymmetrize CS theory

(NI – Tanaka - Terashima ’15)

- Therefore we expect

$$\int \mathcal{D}A\, e^{-S_{CS}[A]} \approx \int \mathcal{D}V\, e^{-S_{SCS}[V]}$$

to holds.

Our strategy

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- 3D pure gravity (in the bulk)

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- 3D super-Chern-Simons theory (in the bulk)

In step 1,
we have discussed action equivalence,
i.e., classical equivalence.
But we have not discussed
the path-integral <u>measure</u> yet.

Let’s discuss it, since we are doing
quantum theory rather than
classical theory

We need to define the measure
for quantum gravity in terms of the
Chern-Simons theory,
such that gravity path-integral can be
calculated in terms of the Chern-
Simons theory path-integral.

The Chern-Simons theory lives on three-dimensional Euclidian space which we call ‘*M*’.
Since the Chern-Simons theory is topological, it depends on only the <u>topology of *M*</u> *and*
the <u>boundary metric of *M*</u>

Since we solve the quantum gravity in asym AdS b.c., and asym Euclidean AdS is parametrized by the torus, with its complex moduli τ, M should have a boundary torus with moduli τ.

Note that in gravity, radial rescaling changes the size of torus, so <u>the size of torus cannot be a parameter</u> for the boundary.

Note that different space-time topology
in gravity side should be mapped into
different topology for M
in the Chern-Simons side.
And in the gauge theory,
one regards the Chern-Simons theory
on different topology M
as a *different* theory.

So for the correspondence to work,
we decompose the metric path-integral
into each "sector"
distinguished by the topology of M,
and then we *need to* sum over
different topology for M,
where all of them should have
a boundary torus with τ.

Furthermore, in the CS theory, boundary conditions related by the modular transformation for τ, are regarded as giving different theory.

Therefore we need to sum over different b.c. for the CS theory, where all of the b.c. with complex structure τ are related by the modular transformation.

All of these gives sequence of transformation as

$$Z_{gravity} = \int \mathcal{D}g_{\mu\nu} e^{-S_{gravity}} \to \sum \int \mathcal{D}g_{\mu\nu}^{sector} e^{-S_{gravity}}$$

$$\to \left(\sum \int \mathcal{D}A e^{-\frac{ik}{4\pi} S_{CS}[A]} \right) \left(\sum \int \mathcal{D}\bar{A} e^{+\frac{ik}{4\pi} S_{CS}[\bar{A}]} \right)$$

$$\to \left(\sum \int \mathcal{D}V e^{-\frac{ik}{4\pi} S_{SCS}[V]} \right) \left(\sum \int \mathcal{D}\bar{V} e^{+\frac{ik}{4\pi} S_{SCS}[\bar{V}]} \right)$$

$$= Z_{hol} \times Z_{anti-hol}$$

where, $$Z_{hol} = \sum \int \mathcal{D}V e^{-\frac{ik}{4\pi} S_{SCS}[V]}$$

Furthermore, we need to evaluate

$$Z_{hol} = \sum_{\substack{\text{all topology having the} \\ \text{boundary torus w./ } \tau \text{ and} \\ \text{its modular transformation}}} \int_M \mathcal{D}V e^{-\frac{ik}{4\pi} S_{SCS}[V]}$$

How can we calculate this quantity?

Localization

- One can show that

$$Z[t] = \int_M \mathcal{D}V e^{-\frac{ik}{4\pi} S_{SCS}[V] + tS_{SYM}}$$

is actually t - independent since S_{SYM} is Q-exact

$$S_{SYM} = \int_M \mathrm{Tr}\Big(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \sigma D^\mu \sigma + \frac{1}{2}(D + \frac{\sigma}{l})^2$$

$$+ \frac{i}{4} \bar{\lambda} \gamma^\mu D_\mu \lambda + \frac{i}{4} \lambda \gamma^\mu D_\mu \bar{\lambda} + \frac{i}{2} \bar{\lambda}[\sigma, \lambda] - \frac{1}{4l} \bar{\lambda}\lambda\Big)$$

Localization

- Setting $t \to \infty$, only $S_{SYM} = 0$ configuration is expected to contribute to the path-integral for V

$$Z[t] = \int_M \mathcal{D}V e^{-\frac{ik}{4\pi} S_{SCS}[V] + tS_{SYM}}$$

- Then if only $F_{\mu\nu} = 0$ contributes, by writing this in terms of the metric, we use that this eq is nothing but **the Einstein equation!** (Witten '88)

So we have,

$$Z_{hol} = \sum_{\substack{\cancel{\text{all topology}}\text{ having the} \\ \text{boundary torus w./ } \tau \text{ and} \\ \text{its modular transformation}}} \int_M \mathcal{D}V e^{-\frac{ik}{4\pi} S_{SCS}[V]}$$

Only solutions of Einstein eqs contributes, and it is known that all of the solutions of the Einstein eq in 3D have <u>solid torus topology</u>

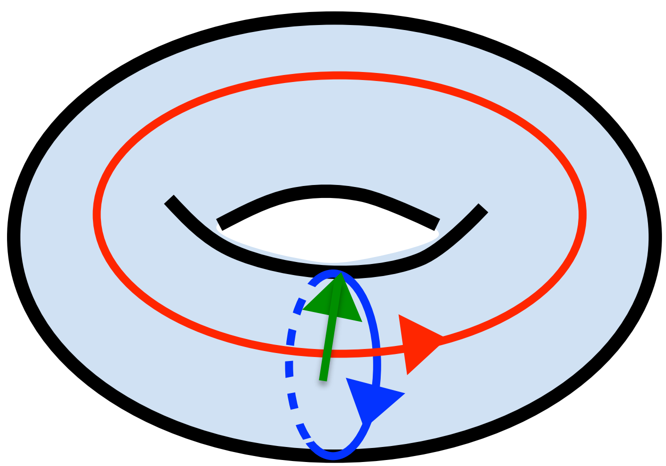
So we have,

$$Z_{hol} = \sum_{\substack{\boxed{\text{solid torus only}} \text{ having the} \\ \text{boundary torus w./ } \tau \text{ and} \\ \text{its modular transformation}}} \int_M \mathcal{D}V e^{-\frac{ik}{4\pi} S_{SCS}[V]}$$

Only solutions of Einstein eqs contributes, and it is known that all of the solutions of the Einstein eq in 3D have <u>solid torus topology</u>

(Dijkgraaf-Maldacena-Moore-Verlinde '00, Maloney-Witten '07, Manschot-Moore '07)

Solid torus

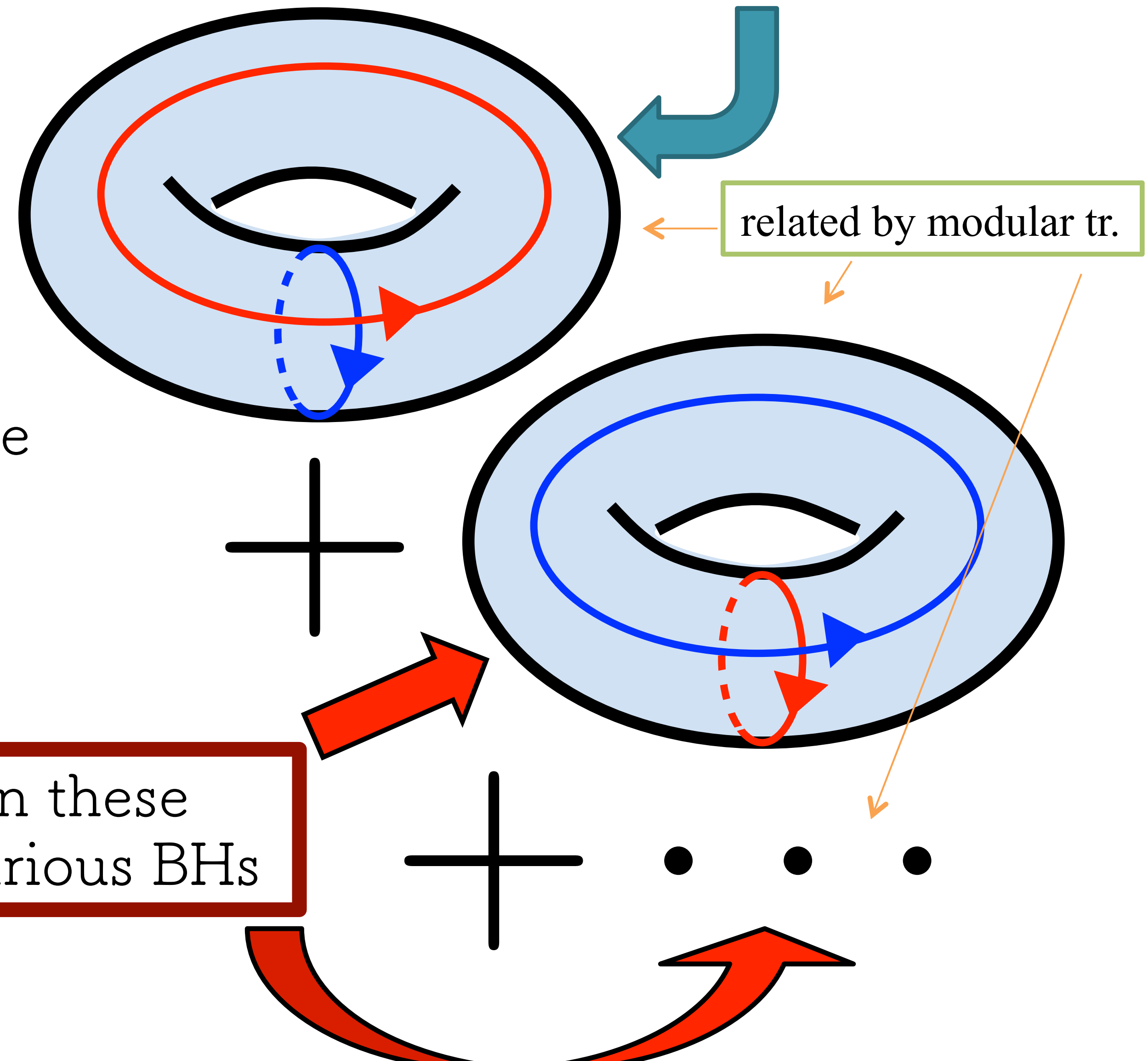


: Non-Contractable circle
: Contractable circle
: Boundary torus
: radial direction (emergent direction)

(NI – Tanaka - Terashima ’15)

CS theory living on this is equivalent to AdS space

$Z =$



: Euclidean time

: Spatial circle

CS theories living on these are equivalent to various BHs

Using localization technique

- Asym AdS = Dirichlet b.c.
- Metric maps => gauge boson *A*
- SUSY Dirichlet b.c. at the boundary torus is given by;

(Sugishita - Terashima '13)

$$A_\varphi \to a_\varphi \,, \qquad A_{t_E} \to a_{t_E} \,,$$

$$\sigma \to 0 \,, \qquad \lambda \to e^{-i(\varphi - t_E)} \gamma^\theta \bar{\lambda}$$

Using localization technique

- The classical contribution & the one-loop determinant gives exact (non-perturbative) answer:

(Sugishita - Terashima '13)

$$Z_{(c,d)} = \int_{\text{b.c.}} DV e^{-\frac{ik}{4\pi} S_{SCS}[V] + tS_{SYM}} = Z_{classical} \times Z_{one-loop}$$

$$Z_{classical} = e^{ik\pi Tr(a_\varphi a_{t_E})}$$

Zeta-function regularization

$$Z_{one-loop} = \prod_{m \in Z} \Big(m - \alpha(a_\varphi) \Big) = e^{i\pi\alpha(a_\varphi)} - e^{-i\pi\alpha(a_\varphi)},$$

Exact results

(NI – Tanaka - Terashima ’15)

- For the non-rotating BTZ black hole, b.c. from the metric gives:

$$a_\varphi = \frac{1}{2i\beta}\sigma_3 = \frac{1}{2\tau}\sigma_3\,, \qquad a_{t_E} = \frac{1}{2}\sigma_3$$

- With this, Z is

$$Z_{(c=1,d=0)} = e^{\frac{1}{4}(k+2)\frac{2\pi i}{\tau}} - e^{\frac{1}{4}(k-2)\frac{2\pi i}{\tau}}$$

Exact results

(NI – Tanaka - Terashima '15)

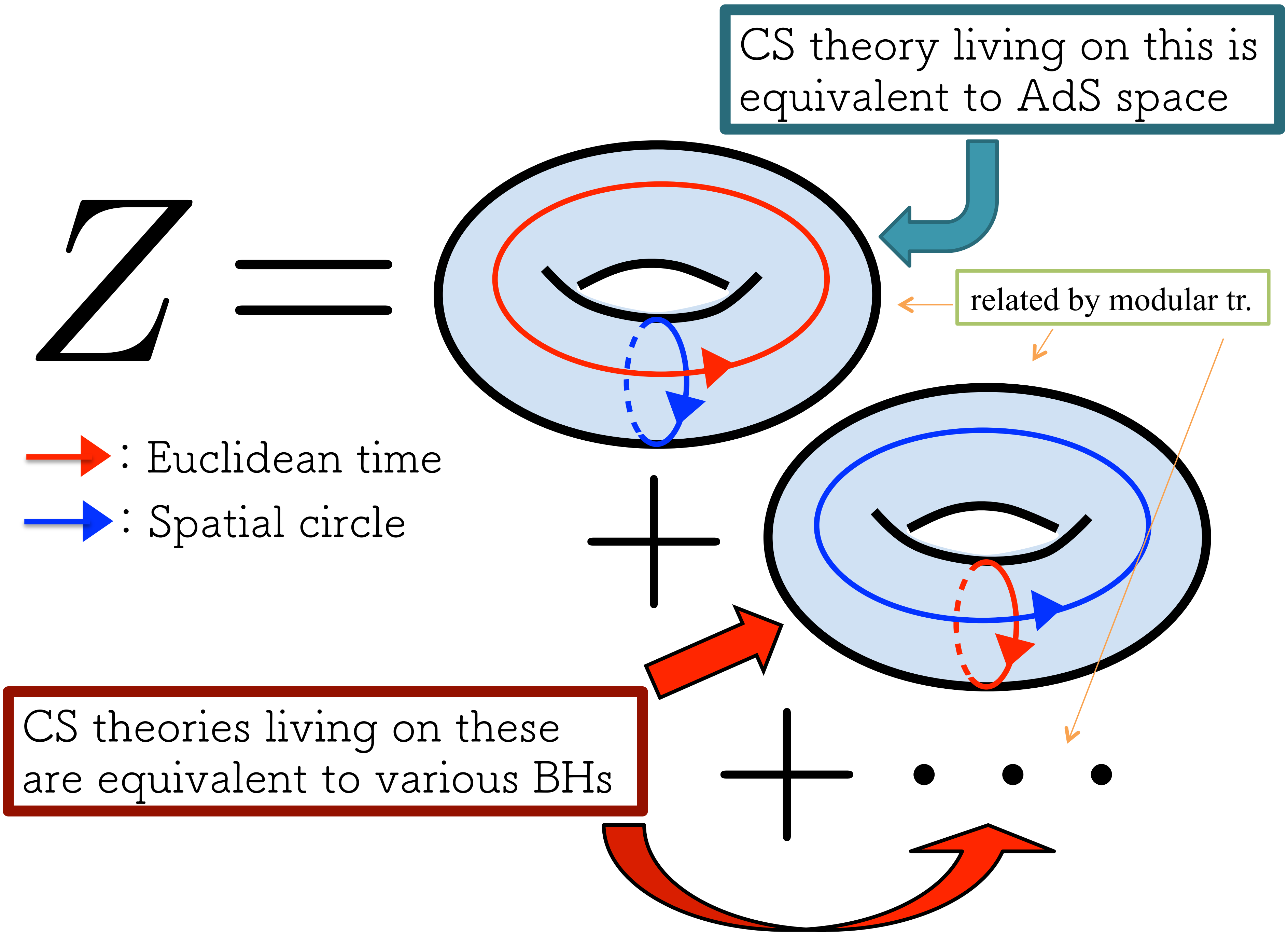
- Non-rotating BTZ is related to the thermal AdS with by the modular transf. with *c=1, d=0,*

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

- For all generic case are then given by

$$Z_{(c,d)} = e^{-2\pi i \frac{1}{4}(k+2)\frac{a\tau+b}{c\tau+d}} - e^{-2\pi i \frac{1}{4}(k-2)\frac{a\tau+b}{c\tau+d}}$$

$\Rightarrow$ we need to sum over *(c, d)*


CS theory living on this is equivalent to AdS space
$Z =$
$+$
$+ \cdots$
related by modular tr.
: Euclidean time
: Spatial circle
CS theories living on these are equivalent to various BHs

Exact results

(NI – Tanaka - Terashima ’15)

- sum over *(c, d):* there is a good way to perform such summation with appropriate regularization, called the “Rademacher sum”
- Taking such regularization we obtain:

Exact results

(NI – Tanaka - Terashima '15)

- The resultant holomorphic partition function:

$$Z_{hol}[q] \equiv Z_{(0,1)}(\tau) + \sum_{\substack{c>0,\\(c,d)=1}} \Big(Z_{c,d}(\tau) - Z_{c,d}(\infty) \Big)$$

$$= R^{(-k_{eff}/4)}(q) - R^{(-k_{eff}/4+1)}(q)\,,$$

- where

$$q \equiv e^{2\pi i\tau} \qquad k_{eff} \equiv k + \underline{2}$$

Quantum shift

$$c = 6k_{eff}$$

Exact results

(NI – Tanaka - Terashima '15)

- And $R^{(m)}(q)$ is given by:

$$R^{(m)}(q) \equiv e^{2\pi i m\tau} + \sum_{\substack{c>0,\\(c,d)=1}} \left(e^{2\pi i m \frac{a\tau+b}{c\tau+d}} - e^{2\pi i m \frac{a}{c}} \right)$$

$$= q^m + (\text{const.}) + \sum_{n=1}^{\infty} c(m,n) q^n$$

$$c(m,n) \equiv \sum_{\substack{c>0,\\(c,d)=1,\\ d \bmod c}} e^{2\pi i (m\frac{a}{c}+n\frac{d}{c})} \sum_{\nu=0}^{\infty} \frac{\left(\frac{2\pi}{c}\right)^{2\nu+2}}{\nu!(\nu+1)!} (-m)^{\nu+1} n^{\nu}$$

Exact results

(NI – Tanaka - Terashima '15)

- For c = *24* ($c = 6k_{eff}$), we obtain

$$Z_{hol}(q) = R^{(m=-1)}(q) - R^{(m=0)}(q)$$

$$= J(q) + \text{const.}$$

- Except for the *const.* term which we can choose to be any value we want by regularization, we obtain *J(q)* function!

Summary:

(NI – Tanaka - Terashima '15)

- Exact calculation is possible by localization technique, and the results for $c = 24$ is;

$$Z_{gravity} = J(q)J(\bar{q})$$

- This agrees with the extremal CFT partition function of Frenkel, Lepowsky, and Meurman, predicted by Witten for 3D pure gravity!
- Furthermore, localization gives "justification" of semi-classical approximation

Boundary fermions discussions

Boundary fermion

(NI – Honda - Tanaka - Terashima '15)

- Dirichlet b.c. (for AdS_3) and SUSY yields b.c. for fermion as

$$\lambda\Big|_{\rm bdry} = e^{-i(\varphi - t_E)}\sigma^3\bar{\lambda}\Big|_{\rm bdry},$$

- This b.c. is incompatible with the gaugino "mass term" in

$$S_{SCS}[V] = S_{CS}[A] + \int d^3x\sqrt{g}\ {\rm Tr}\Big(-\bar{\lambda}\lambda + 2D\sigma\Big).$$

Boundary fermion

(NI – Honda - Tanaka - Terashima '15)

- This means that SCS's mass term for gaugino vanishes at the boundary and therefore there are "boundary localized fermions".
- We "guessed" its contribution to partition function as;

$$Z_{\text{B-fermion}}(q) = \prod_{n=1}^{\infty}(1-q^n)$$

- And we define $$\frac{Z_{hol}(q)}{Z_{\text{B-fermion}}(q)} \equiv Z_{\text{gravity}}^{\text{large }k}(q)$$

"bulk pure gravity" partition function

(NI – Honda - Tanaka - Terashima '15)

- We can show that;

$$
Z^{\text{large } k}_{\text{gravity}}(q) = \underbrace{q^{-\frac{k_{\text{eff}}}{4}} \prod_{n=2}^{\infty} \frac{1}{1-q^n}}_{\textit{"Verma module of the vacuum"}} + \underbrace{\sum_{\Delta=1}^{\infty} c_{\Delta}^{(k_{\text{eff}})} q^{\Delta} \prod_{n=1}^{\infty} \frac{1}{1-q^n}}_{\textit{"Verma module from "BTZ BHs as primary states"}}
$$

"bulk pure gravity" partition function

(NI – Honda - Tanaka - Terashima '15)

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$$Z^{\text{large } k}_{\text{gravity}}(q)$$

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“bulk pure gravity” partition function

(NI – Honda - Tanaka - Terashima ’15)

- With

$$\boxed{c_\Delta^{(k_{\rm eff})} = c\left(-\frac{k_{\rm eff}}{4}, \Delta\right) - c\left(-\frac{k_{\rm eff}}{4} + 1, \Delta\right)}$$

- This is always positive intergers: satisfying Cardy formula:

$$\boxed{\lim_{k\to\infty} \log c_\Delta^{(k_{eff})} = 2\pi\sqrt{k_{eff}\Delta} = 2\pi\sqrt{\frac{c_Q\Delta}{6}}}$$

$$c^{(4)}_{\Delta=1} = 196884, \quad c^{(4)}_{\Delta=2} = 21493760,$$

$$c^{(8)}_{\Delta=1} = 42790636, \quad c^{(8)}_{\Delta=2} = 40470415636,$$

$$c^{(12)}_{\Delta=1} = 2549912390, \quad c^{(12)}_{\Delta=2} = 1271557790892,$$

Summary 1:

(NI – Tanaka - Terashima '15)

- Exact calculation is possible by localization technique, and the results for $c = 24$ is;

$$Z_{gravity} = J(q)J(\bar{q})$$

- This agrees with the extremal CFT partition function of Frenkel, Lepowsky, and Meurman, predicted by Witten for 3D pure gravity.
- Furthermore, localization gives "justification" of semi-classical approximation

Summary 2:

(NI – Tanaka - Terashima ’15)

- The resultant partition function for the holomorphic part is

$$Z_{hol}[q] = R^{(-k_{eff}/4)}(q) - R^{(-k_{eff}/4+1)}(q)\,,$$

$$R^{(m)}(q) = q^m + (\mathrm{const.}) + \sum_{n=1}^{\infty} c(m,n) q^n$$

$$c(m,n) \equiv \sum_{\substack{c>0,\\ (c,d)=1,\\ d \, mod \, c}} e^{2\pi i (m\frac{a}{c} + n\frac{d}{c})} \sum_{\nu=0}^{\infty} \frac{\left(\frac{2\pi}{c}\right)^{2\nu+2}}{\nu!(\nu+1)!} (-m)^{\nu+1} n^{\nu}$$

Summary 3:

(NI – Honda - Tanaka - Terashima '15)

- Taking into account "boundary fermions"

$$Z^{\text{large } k}_{\text{gravity}}(q) \ \left(\equiv \frac{Z_{hol}(q)}{Z_{\text{B-fermion}}(q)}\right)$$

Satisfying Cardy formula for BH

$$= \underbrace{q^{-\frac{k_{\text{eff}}}{4}} \prod_{n=2}^{\infty} \frac{1}{1-q^n}}_{\text{“Verma module of the vacuum”}} + \underbrace{\sum_{\Delta=1}^{\infty} c_{\Delta}^{(k_{\text{eff}})} q^{\Delta} \prod_{n=1}^{\infty} \frac{1}{1-q^n}}_{\text{“Verma module from “BTZ BHs as primary states”}}$$

"Verma module of the vacuum"

"Verma module from "BTZ BHs as primary states"

Summary 4:

(NI – Honda - Tanaka - Terashima '15)

- All of these results can be generalized to higher spin gravity theory w/ *SL(N,C)*:
- There, instead of Virasoro algebra, we have a good expression for the partition function in terms of characters for the vacuum and primaries in 2D unitary CFT with W_N symmetry.
- Again exhibiting Cardy formula in the large central charge limit.