YKIS2016@YITP (2016/6/15)

Supersymmetry breaking and Nambu-Goldstone fermions in lattice models

Hosho Katsura
(Department of Physics, UTokyo)

Collaborators:
Yu Nakayama (IPMU → Rikkyo)
Noriaki Sannomiya (UTokyo)

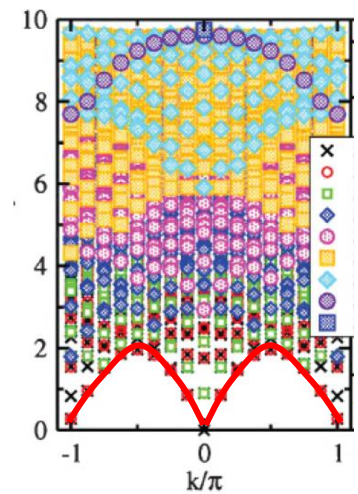
- arXiv:1606.03947

Susy and me

■ What I've been working on…

Condensed Matter & Statistical Physics

- Strongly correlated systems,
- Topological phases of matter,
- Quantum entanglement, …



VBS/CFT correspondence (2d AKLT ⇔ 1d Heisenberg)
J.Lou, S.Tanaka, H.K., N.Kawashima, *PRB* **84** (2011).

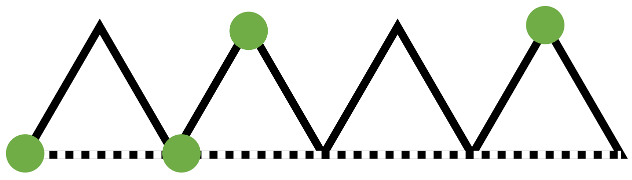
■ I'm not a high-energy physicist, or a string theorist, …
(at least for the moment). But…

■ My first paper (undergrad)

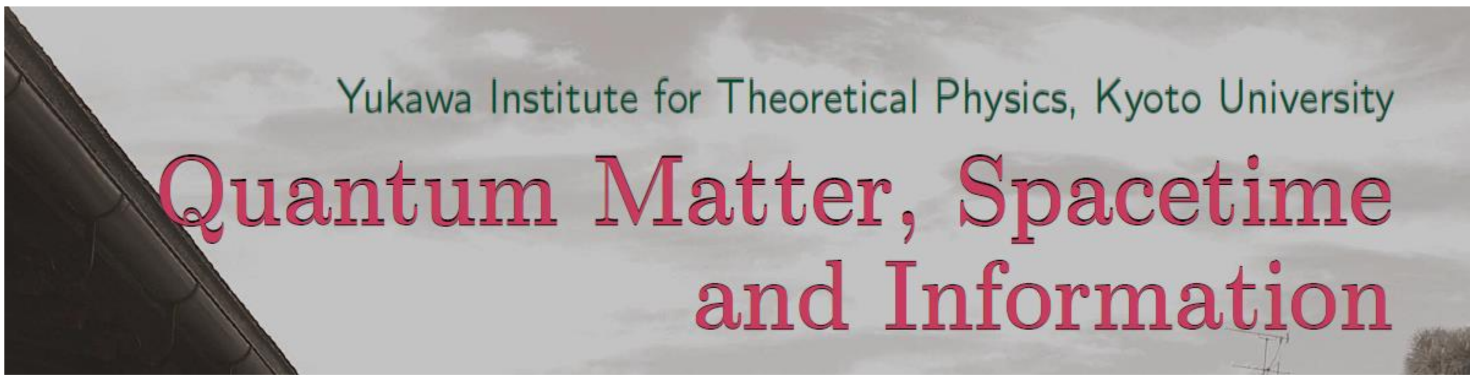
"Exact **supersymmetry** in the relativistic hydrogen atom in general dimensions" (arXiv:quant-ph/0410174),
H. Katsura & H. Aoki, *J. Math. Phys*. **47**, 032301 (2006).

Today's talk

- Many-body systems with built-in ***supersymmetry!***
  - Lattice-fermion model in 1 (spatial) dimension
  - Spontaneous supersymmetry breaking
  - Gapless excitation with linear dispersion



- Edge of the workshop



  - My talk contains some Information ($S\neq 0$)
  - The model lives in (1+1)-dim. flat Spacetime
  - Super-weird Quantum Matter, never synthesized …, cold atoms?

Outline

1. Introduction & Motivation

- Supersymmetry and lattice models
- Extended Nicolai model

2. SUSY breaking in extended Nocolai model

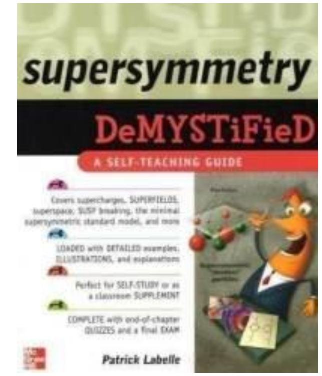
- Definition of SUSY breaking
- 1) Finite chain, 2) infinite chain

3. Nambu-Goldstone fermions

- 1. Variational result, 2. Numerical result
- Bosonization & RG analysis

4. Summary

Supersymmetry (SUSY QM) demystified



■ Algebraic structure

- Supercharges (Q & $Q^\dagger$) and fermion number (F)

$$Q^2 = 0, \quad (Q^\dagger)^2 = 0, \quad [F, Q^\dagger] = Q^\dagger, \quad [F, Q] = -Q.$$

- Hamiltonian

$$H = \{Q, Q^\dagger\} = QQ^\dagger + Q^\dagger Q$$

$$\langle \psi | H | \psi \rangle = \| Q | \psi \rangle \|^2 + \| Q^\dagger | \psi \rangle \|^2$$

- Conserved charges

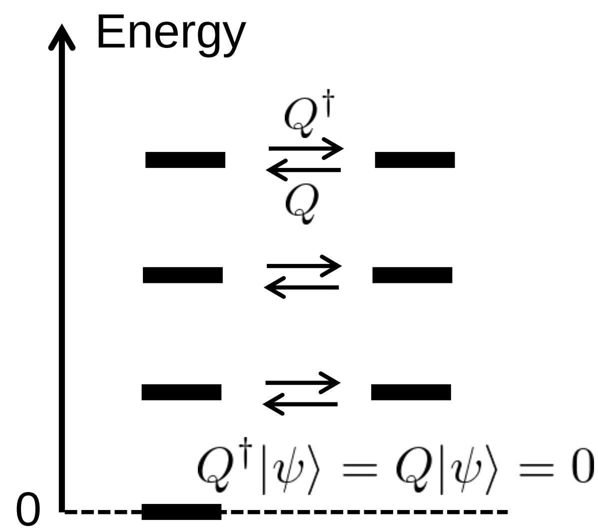
$$[H, Q] = [H, Q^\dagger] = [H, F] = 0$$

■ Spectrum of H

- $E \geqq 0$ for all states
- $E > 0$ states come in pairs

$$\{ |\psi\rangle, Q^\dagger |\psi\rangle \}, \; Q|\psi\rangle = 0$$

- $E = 0$ iff a state is a singlet (cohomology)
  SUSY breaking ⇔ No E=0 state



Elementary examples

■ Boson-fermion system

- Creation & annihilation operators (*b*: boson, *c*: fermion)

$$[b, b^\dagger] = 1, \quad \{c, c^\dagger\} = 1, \qquad [b, b] = \{c, c\} = 0$$

- Vacuum state $\quad b|\mathrm{vac}\rangle = c|\mathrm{vac}\rangle = 0$

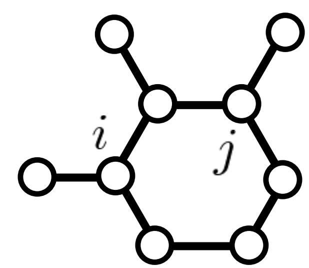
Total number of B and F!

|vac> is a SUSY singlet.

- Supercharges

$$Q = b^\dagger c, \quad Q^\dagger = c^\dagger b \quad \Rightarrow \quad \{Q, Q^\dagger\} = b^\dagger b + c^\dagger c$$

■ Lattice bosons and fermions



- Lattice sites: *i*, *j* = 1,2, …, *N*
- Creations & annihilations

$$[b_i, b_j^\dagger] = \delta_{i,j}, \quad \{c_i, c_j^\dagger\} = \delta_{i,j}, \quad [b_i, b_j] = \{c_i, c_j\} = 0.$$

(*b* and *f* are mutually commuting.)

- Vacuum state $\quad b_i|\mathrm{vac}\rangle = c_i|\mathrm{vac}\rangle = 0, \forall i$

Elementary examples (contd.)

■ Generalization

$$Q = \sum_j b_j^\dagger c_j , \quad Q^\dagger = \sum_j c_j^\dagger b_j$$

$$\Rightarrow \quad \{Q, Q^\dagger\} = \sum_j n_j^b + \sum_j n_j^f \qquad (\, n_j^b = b_j^\dagger b_j , \;\; n_j^f = c_j^\dagger c_j \,)$$

Total number of B and F! |vac> is a SUSY singlet.

■ SUSY in Bose-Fermi mixtures

Realization in cold-atom systems?
M. Snoek et al., PRL **95** (‘05); PRA **74** (‘06); G.S.Lozano et al., PRA **75** (‘07).

- Yu-Yang model (*PRL* **100**, (‘08))

Hubbard-type model with equal hopping & equal-int. for any pair of sites.

$$H_{YY} = -\sum_{i \neq j} t_{i,j} (b_i^\dagger b_j + c_i^\dagger c_j) + \sum_{i,j} U_{i,j} (n_i^b n_j^b + n_i^f n_j^f + n_i^b n_j^f)$$

***H_{YY}* commutes with *Q & $Q^\dagger$*!** (*H_{YY} is not* $\{Q, Q^\dagger\}$)

Lattice models with built-in SUSY

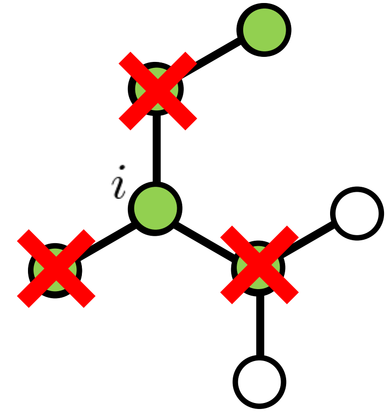
■ Fendley-Schoutens-de Boer model

PRL **90**, 120402 ('03); *PRL* **95**, 046403 ('05).

- Supercharge

Hard-core constraint

$$Q=\sum_i c_i P_{<i>} \qquad P_{<i>}=\prod_{j\ \mathrm{next\ to}\ i}(1-c_j^\dagger c_j)$$



- Hamiltonian

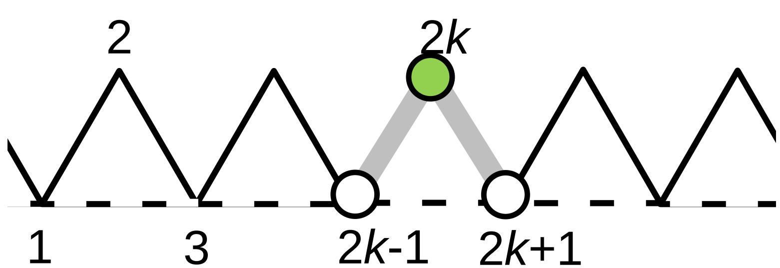
$$H=\{Q,Q^\dagger\}=\sum_i\sum_{j\ \mathrm{next\ to}\ i}P_{<i>}c_i^\dagger c_j P_{<j>}+\sum_i P_{<i>}$$

■ Nicolai model

"Supersymmetry and spin systems",
H. Nicolai, *JPA* **9**, 1497 ('76).

- Supercharge

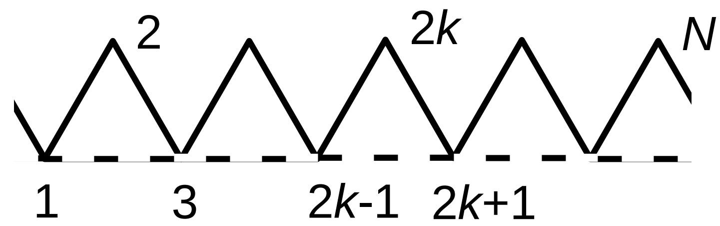
$$Q=\sum_k c_{2k-1}c_{2k}^\dagger c_{2k+1}$$



of E=0 states ~ exp (# of sites)

Both models tend to have massively degenerate E=0 states...

Extended Nicolai model



■ Definition

- Setting

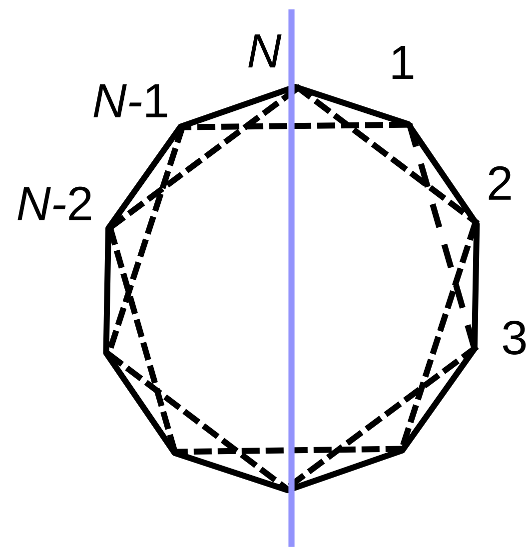
  1d lattice of length N (even). PBCs are imposed ($c_{N+1} = c_1$).

- New supercharge ($g > 0$)

  Linear term in c!

$$Q = \sum_k c_{2k-1} c_{2k}^\dagger c_{2k+1} + g \sum_k c_{2k-1}$$

  Nilpotent. Comprised solely of fermions.



- Hamiltonian $\quad H = \{Q, Q^\dagger\}$

■ Symmetries

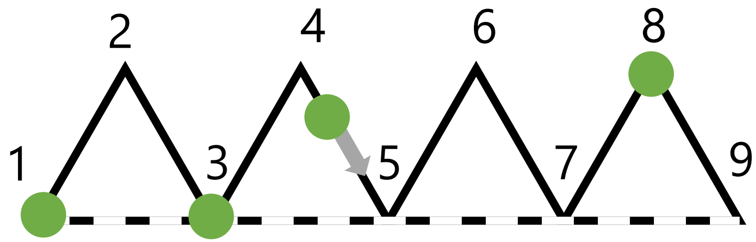
- SUSY $\quad [H, Q] = [H, Q^\dagger] = 0,$
- U(1) $\quad [H, F] = 0, \quad F \sum_j c_j^\dagger c_j$
- Translation $\quad [T, Q] = [T, Q^\dagger] = 0, \quad T : c_j \to c_{j+2}$
- Reflection $\quad [U, Q] = [U, Q^\dagger] = 0, \quad U : c_j \to -(-1)^j\, c_{N-j}$

Hamiltonian (explicit expression)

$$H = H_{\rm hop} + H_{\rm charge} + H_{\rm pair} + \frac{N}{2} g^2$$

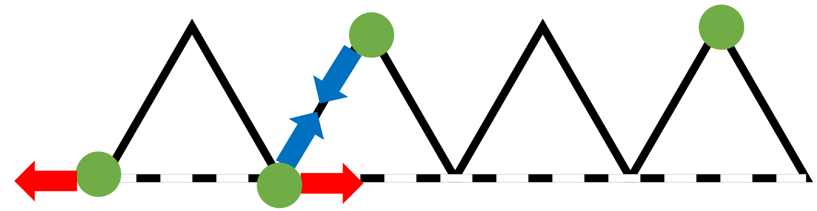
1. Hopping term

$$H_{\rm hop} = g \sum_{j=1}^{N} (-1)^j (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j)$$



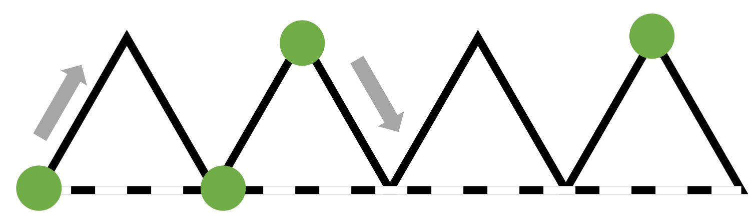
2. Charge-charge int.

$$H_{\rm charge} = -\sum_{j=1}^{N} n_i n_{i+1} + \sum_{k=1}^{N/2} (n_{2k} + n_{2k-1} n_{2k+1})$$



3. Pair hopping

$$H_{\rm pair} = \sum_{k=1}^{N/2} (c_{2k}^\dagger c_{2k+3}^\dagger c_{2k-1} c_{2k+2} + {\rm H.c.})$$



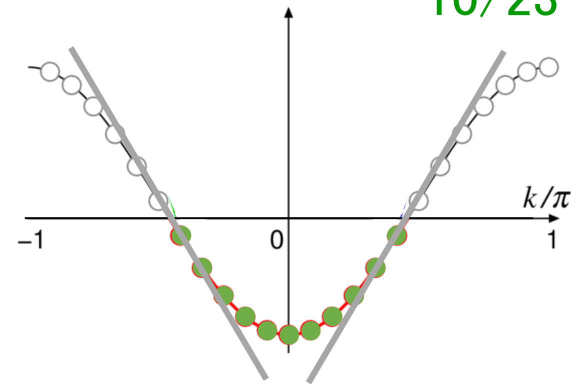
Very complicated and seems intractable…

cf) Original Nicolai model (g = 0): $H_{\rm Nic} = H_{\rm charge} + H_{\rm pair}$

Large-*g* limit

- Free fermions

  In the large-*g* limit, $H \sim g^2 N/2 + H_{\rm hop}$.
  (The (many-body) g.s. energy of $H_{\rm hop}$) ∝ gN



• SUSY is broken (No E=0 states)

• Gapless excitations
  Dirac fermions in continuum limit

Nambu-Goldstone theorem?

Also the case for finite g?

Results

- SUSY breaking
  1) Finite chain: SUSY is broken for any $g > 0$.
  2) Infinite chain: SUSY is broken when $g > 4/\pi = 1.2732...$

- NG fermions
  Rigorous result
    Existence of low-lying states with $E(p) \leqq$ (const.) $|p|$.
  Analytical & numerical result
    Effective field theory ~ c=1 CFT with TL parameter close to 1.

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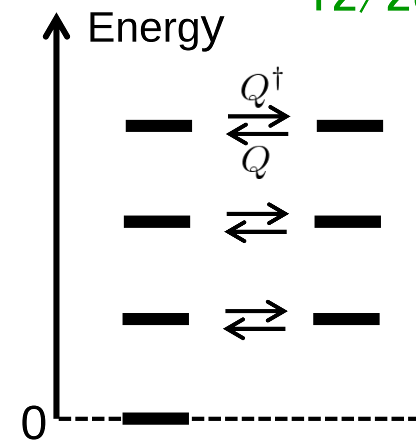
SUSY breaking



$$E(V) \underset{V\to\infty}{\longrightarrow} 0$$

■ Naïve definition

SUSY is unbroken ⇔ E=0 state exists
SUSY is broken ⇔ No E= 0 state

Subtle issue... (Witten, *NPB* **202** ('82))
"SUSY may be broken in any finite volume yet restored in the infinite-volume limit."

■ More precise definition

- (Normalized) ground state: ψ_0
- Ground-state energy density: $\epsilon_0 := \frac{1}{V}\langle\psi_0|H|\psi_0\rangle$
  V= (# of sites) for lattice systems

SUSY is said to be spontaneously broken if the ground-state energy density (energy per site) is strictly positive.

Applies to both finite and infinite-volume systems!

SUSY breaking in finite Nicolai chains

Theorem 1

Consider the extended Nicolai model on a finite chain of length *N*. If *g* > 0, then SUSY is spontaneously broken.

Proof

- Local operator s.t. $\{Q, O_k\} = g$ well-defined when *g*>0

$$O_k = c^\dagger_{2k-1}\left[1 - \frac{1}{g}(c^\dagger_{2k}c_{2k+1} + c_{2k-3}c^\dagger_{2k-2}) + \frac{2}{g^2}c_{2k-3}c^\dagger_{2k-2}c^\dagger_{2k}c_{2k+1}\right]$$

- Proof by contradiction

Suppose ψ_0 is an *E*=0 ground state.
Then we have

$$\langle\psi_0|\{Q, O_k\}|\psi_0\rangle = \langle\psi_0|QO_k + O_kQ|\psi_0\rangle = 0$$

But this leads to ψ_0 =0. **Contradiction. No *E*=0 state!**

→ (g.s. energy/*N*) > 0 for any finite *N*.

SUSY breaking in the infinite Nicolai chain

Theorem 2

Consider the extended Nicolai model on the infinite chain.
If $g > 4/\pi$, then SUSY is spontaneously broken.

Proof

- Lower bound for g.s. energy

$$H = \frac{N}{2} g^2 + H_{\mathrm{hop}} + H_{\mathrm{Nic}}$$

Original Nicolai (g=0)

Since H_{Nic} is positive semi-definite, the g.s. energy of H is bounded from below by $E_0 \geq N g^2/2 + E_0^{\mathrm{hop}}$.

- G.s. energy of H_{hop} (Free-fermion chain)

$$E_0^{\mathrm{hop}} = -\frac{2g}{\tan(\pi/N)} \geq -\frac{2g}{\pi} N \quad \Rightarrow \quad \boxed{\frac{E_0}{N} \geq \frac{g}{2}\left(g - \frac{4}{\pi}\right)}$$

NOTE) The condition $g > 4/\pi$ may not be optimal…

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Nambu-Goldstone (NG) fermions?

■ NG bosons in non-relativistic systems

Effective Lagrangian approach, counting rules, …

Watanabe-Murayama, *PRL* **108** (‘12); Hidaka, *PRL* **110**, (’13).

■ Analogy (Blaizot-Hidaka-Satow, *PRA* **92** (‘15))

- Ferromagnet (Type B)

$$\frac{1}{V}\langle [S^+, S^-]\rangle_0 = 2m^z$$

Quadratic dispersion

- SUSY system

$$\frac{1}{V}\langle \{Q, Q^\dagger\}\rangle_0 = \epsilon_0$$

Quadratic dispersion?
Fermionic excitation?

■ Examples

- Yu-Yang model (Bose-Fermi mixture)

  YES $\omega \propto p^2$

  SUSY spin-wave states

  Fermionic!

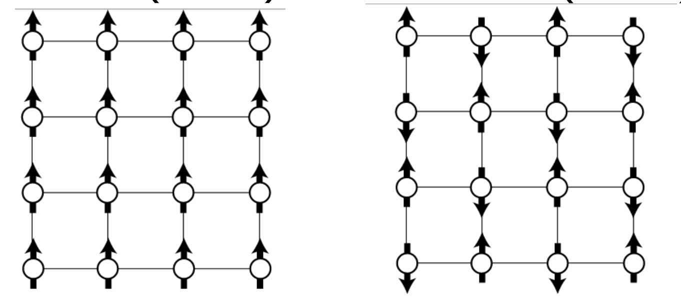
$$|\psi_k\rangle = \sum_j e^{-\mathbf{i}kj}\, c_j^\dagger b_j \,|\psi_0\rangle$$

- Extended Nicolai model

  NO! $\omega \propto p$

  Low-lying fermionic states with $\omega \leqq$ (const.)$|p|$ exist.

Warm-up: Heisenberg model



Hamiltonian $H = J \sum_{\langle i,j \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j$

■ Ferromagnetic case

- Fully polarized ground state: $| \Uparrow \rangle$
- Spin wave $\sum_k e^{i\boldsymbol{p}\cdot\boldsymbol{R}_k} S_k^- | \Uparrow \rangle$, Exact eigenstate! ➡ $\omega \propto p^2$

■ Antiferromagnetic case (Horsch-von der Linden, *ZPB*, **72** ('88))

- Neel state: not even an eigenstate!
- Bijl-Feynman ansatz

  Exact ground state: ψ_0

  Fourier component of spins: $S_{\boldsymbol{p}}^\alpha = \sum_k e^{i\boldsymbol{p}\cdot\boldsymbol{R}_k} S_k^\alpha \quad (\alpha = z, \pm)$

$$|\psi_p\rangle = \frac{S_{\boldsymbol{p}}^\alpha |\psi_0\rangle}{\| S_{\boldsymbol{p}}^\alpha |\psi_0\rangle \|} \quad ➡ \quad \epsilon_{\mathrm{var}}(\boldsymbol{p}) = \frac{1}{2} \frac{\langle [S_{-\boldsymbol{p}}^\alpha, [H, S_{\boldsymbol{p}}^\alpha]] \rangle_0}{\langle S_{-\boldsymbol{p}}^\alpha S_{\boldsymbol{p}}^\alpha \rangle_0}$$

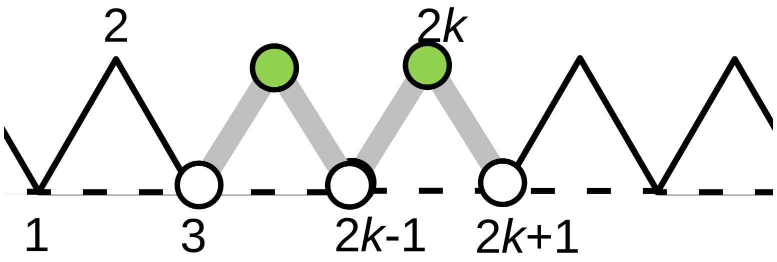
Linear around $\boldsymbol{q} = (\pi, \pi, \ldots)$

Variational argument

- SUSY "spin waves" in extended Nicolai model
  - Local supercharge

$$q_k = \frac{g}{2}(c_{2k-1} + c_{2k+1}) + c_{2k-1}c_{2k}^{\dagger}c_{2k+1} \qquad (Q = \sum_k q_k)$$

$$\{q_k, q_\ell^{\dagger}\} = \begin{cases} \text{nonzero} & |k-\ell| \le 1 \\ 0 & \text{otherwise} \end{cases}$$



  - Fourier components

$$Q_p^{\dagger} = \sum_k e^{\mathbf{i}pk}\, q_k^{\dagger} \qquad (Q_0^{\dagger} = Q^{\dagger})$$

  - Ansatz (ψ_0: SUSY broken g.s.)

$$|\psi_p\rangle = \frac{(Q_p + Q_p^{\dagger})|\psi_0\rangle}{\|(Q_p + Q_p^{\dagger})|\psi_0\rangle\|} \quad \Rightarrow \quad \epsilon_{\text{var}}(p) = \frac{\langle\, [Q_p, [H, Q_p^{\dagger}]\,]\,\rangle_0}{\langle\, \{Q_p, Q_p^{\dagger}\}\,\rangle_0}$$

$[H, Q_p^{\dagger}]$ is a sum of local operators. ($[H,\, Q_p] = [Q^{\dagger},\, \{Q,\, Q_p^{\dagger}\}]$)
But, $[Q_p, [H, Q_p^{\dagger}]]$ may not be so because $[q_k, q_\ell^{\dagger}] \neq 0$ for all *k*, *l*.

Variational argument (contd.)

- Useful inequality (Pitaevskii-Stringari, *JLTP* **85** ('91))

$$|\langle\psi|\,[\,A^\dagger,B\,]\,|\psi\rangle|^2 \le \langle\psi|\,\{A^\dagger,A\}\,|\psi\rangle\,\langle\psi|\,\{B^\dagger,B\}\,|\psi\rangle$$

Holds for any state *ψ* and any operators *A*, *B*.

Local!

$$\Rightarrow \quad \left|\langle[Q_p,[H,Q_p^\dagger]\,]\rangle_0\right|^2 \le \langle\{Q_p,Q_p^\dagger\}\rangle_0\,\langle\{[Q_p,\,H],\,[H,\,Q_p^\dagger]\}\rangle_0$$

- Upper bound for the lowest dispersion

$$\epsilon_{\rm var}(p)^2 \le \frac{\langle\,\{\,[Q_p,H],[H,Q_p^\dagger]\,\}\,\rangle_0}{\langle\,\{Q_p,Q_p^\dagger\}\,\rangle_0} = \frac{f_{\rm n}(p)}{f_{\rm d}(p)}$$

For $|p|$<<1,

$$\Rightarrow \quad \boxed{\epsilon(p) \le ({\rm Const.}) \times |p|}$$

$f_n(p)$: 1. Local, 2. $f_n(-p) = f_n(p)$, 3. $f_n(0) = 0$
$f_d(p)$: 1. Local, 2. $f_d(-p) = f_d(p)$, 3. $f_d(0) = E_0$

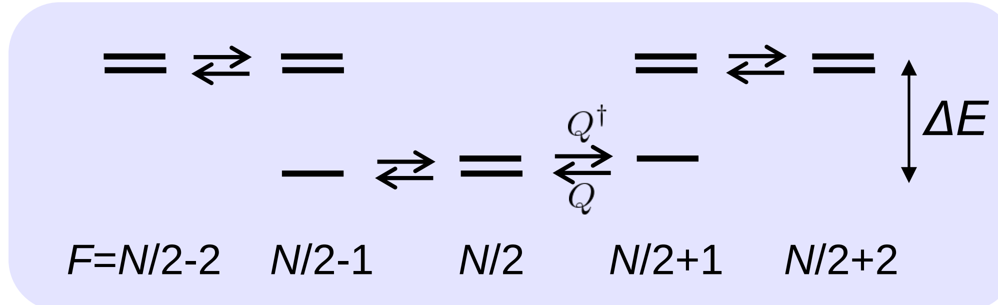
$E_0 > 0$ if SUSY is broken! $\quad \langle\{Q_p,Q_p^\dagger\}\rangle_0 \sim \langle\{Q,Q^\dagger\}\rangle_0$

NOTE) U(1), translation, reflection symmetries have been used.
Implicit assumption: The g.s. multiplicity is finite in the infinite-*N* limit.

Numerical results

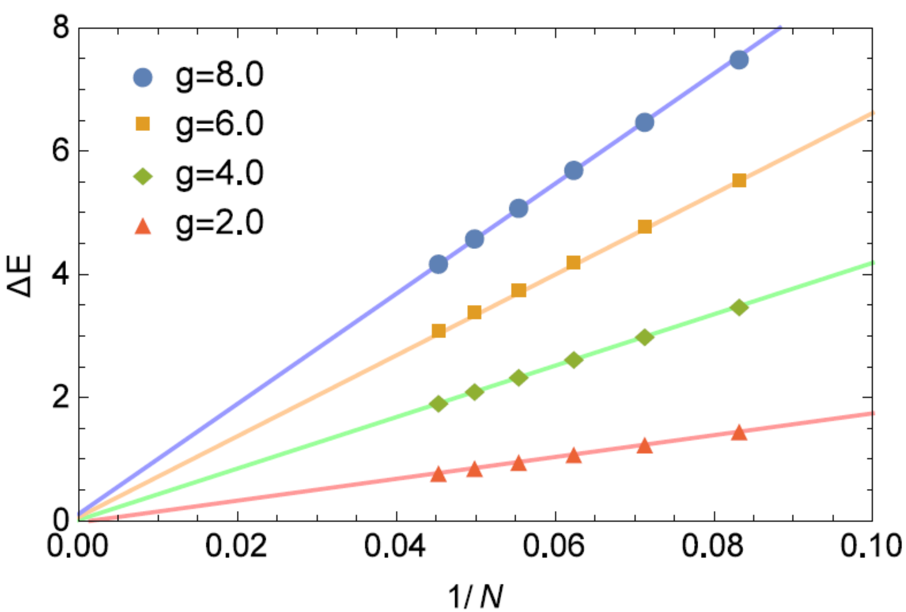
- Exact diagonalization

  N= 12, 14, …, 22
  4 ground states
  8 first excited states
  (Independent of g & N)



• 1st excitation energy

$$\epsilon = v|p|$$



• Central charge

Finite-size scaling
Blote-Cardy-Nightingale, *PRL* **56** (‘86)

$$\frac{E_0}{N} = e_\infty + \frac{\pi v c}{3N^2} + O\left(\frac{1}{N^3}\right)$$

g	2.0	4.0	6.0	8.0
c	0.9705	1.008	1.020	1.025

Gapless linear dispersion!
Described by c=1 CFT!

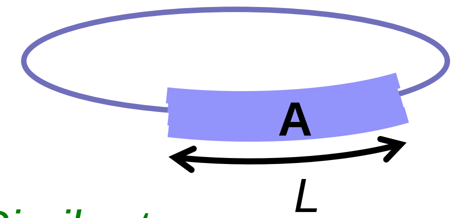
Tomonaga-Luttinger liquid parameter

c=1 CFTs are further specified by Tomonaga-Luttinger (TL) parameters ***K***. (Or equivalently, boson compactification radius.)

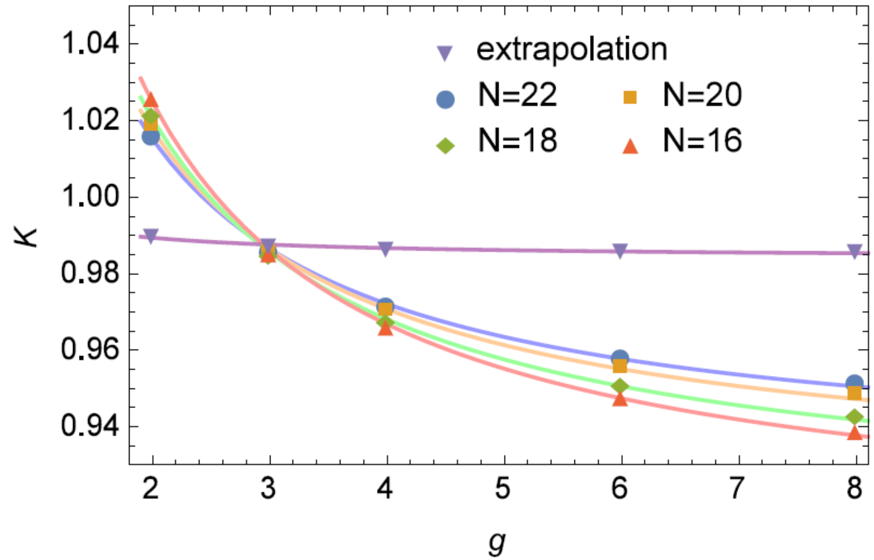
- Number fluctuation

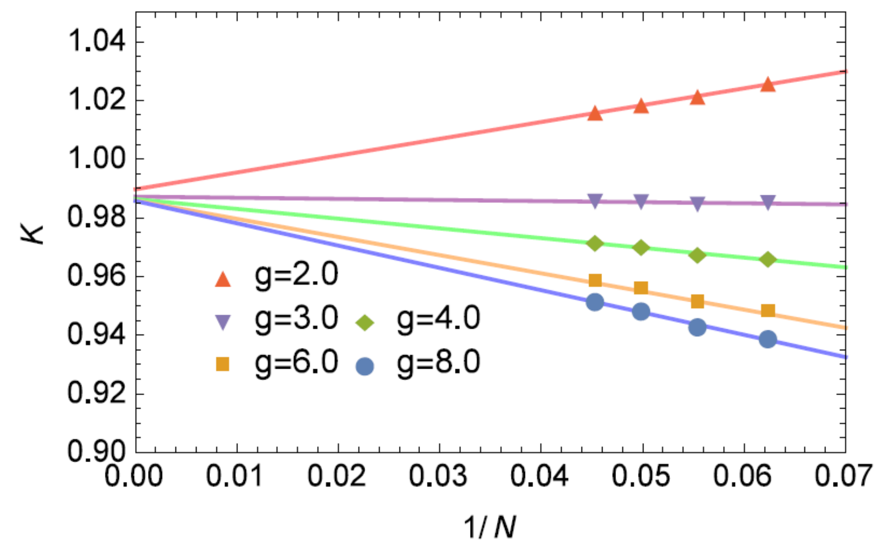
Song, Rachel et al., *PRB* **59** (‘12)

$$\langle\,(N_A-\langle N_A\rangle_0)^2\,\rangle_0=\frac{K}{\pi^2}\log\left(\frac{\sin(\pi L/N)}{\sinh(\pi\alpha/N)}\right)$$



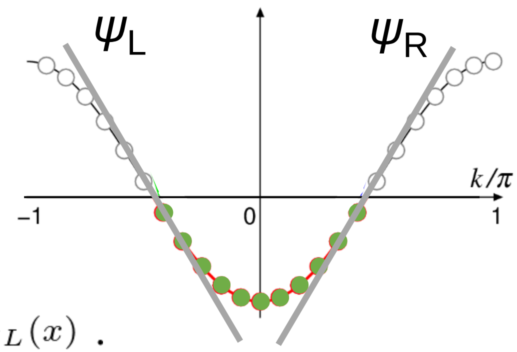
Similar to Calabrese-Cardy!





***K* is almost independent of g and is close to 1 (free-Dirac).**

Bosonization and RG



- Lattice fermion → Dirac (*a*: lattice spacing)

$$c_j \sim \sqrt{a}\left(e^{ik_F x}\psi_R(x) + e^{-ik_F x}\psi_L(x)\right)$$

- Dirac fermion → boson

$$\psi_R(x) = \frac{1}{\sqrt{2\pi a}} : e^{i\varphi_R(x)} : \qquad \psi_L(x) = \frac{1}{\sqrt{2\pi a}} : e^{-i\varphi_L(x)} :$$

- Sin-Gordon Hamiltonian

$$H \sim \frac{v}{2}\int dx \left\{\frac{1}{K}\,\partial_x\varphi(x)^2 + K\,\Pi(x)^2\right\}$$

← ***Free boson!***

$$[\varphi(x),\, \Pi(y)] = \mathbf{i}\,\delta(x-y)$$

$$+\gamma\int dx\, \cos(\sqrt{16\pi}\varphi(x))$$

← Cos term

Velocity

$$v = 2\sqrt{\left(g - \frac{15}{8\pi}\right)\left(g - \frac{27}{8\pi}\right)}$$

TL parameter

$$K = \sqrt{\frac{1-15/(8\pi g)}{1-27/(8\pi g)}}$$

- Scaling dim. of cos

$$4K \sim 4 + \frac{3}{\pi g}$$

Cos term is irrelevant. Gapless!
~ Massless Thirring model

Summary

- Introduced one-parameter extension of Nicolai's model
  Lattice model with exact supersymmetry
  1. Original Nicolai (g=0), 2. Free fermions (g=∞)

- Spontaneous SUSY breaking
  1. Finite chain: broken for any g >0.
  2. Infinite chain: broken when g > $4/\pi = 1.2732...$

- Nambu-Goldstone fermions
  1. Rigorous result: $E(p) \leqq (\text{const.})\,|p|$
  2. Numerical result: $E(p) = v\,|p|$, gapless, linear
  3. Field theory: gapless c=1 CFT with K close to 1