F-Theorem and ε Expansion

Igor Klebanov

Princeton University

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Talk mostly based on

- IK, S. Pufu, B. Safdi, arXiv:1105.4598
- IK, S. Pufu, B. Safdi, S. Sachdev, arXiv:1112.5342
- S. Giombi, IK, arXiv:1409.1937
- L. Fei, S. Giombi, IK, G. Tarnopolsky, arXiv:1507.01960
- S. Giombi, IK, G. Tarnopolsky, arXiv:1508.06354
- S. Giombi, G. Tarnopolsky, IK, arXiv:1602.01076

The c-theorem

- A deep problem in QFT is how to define a quantity which decreases along RG flows and is stationary at fixed points.
- In two dimensions this problem was beautifully solved by Alexander Zamolodchikov who, using the two-point function of the stress-energy tensor, found the c-function which satisfies these properties.

- At RG fixed points the c-function coincides with the Virasoro central charge, which is also the Weyl anomaly

$$\langle T^a_a \rangle = -\frac{c}{12} R$$

- Determines the thermal free energy.
- Determines the EE of a segment of size r Holzhey, Larsen, Wilczek

$$S(r) = \frac{c}{3} \log(r/\epsilon) + c_0$$

- $c_{IR} < c_{UV}$ follows from boost invariance and SSA Casini, Huerta

$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B)$$

- The central charge can also be found using the 2-d CFT on the sphere of radius R:

F=-log Z = -c/3 log R

The a-theorem

- In d=4 there are two Weyl anomaly coefficients

$$\langle T_a^a \rangle = -\frac{a}{16\pi^2}\left(R_{abcd}^2 - 4R_{ab}^2 + R^2\right) + \frac{c}{16\pi^2} C_{abcd}^2$$

- One of them, called a, is proportional to the 4-d Euler density. It can be extracted from the Euclidean path integral on the 4-d sphere:

F=-log Z = a log R

- Cardy conjectured that the a-coefficient decreases along any RG flow.
- A proof was provided a few years ago. Komargodski, Schwimmer

The F-theorem

- How do we extend these successes to odd dimensions, where there are no anomalies?
- In d=3 there are many CFTs, some of them describing critical points in statistical mechanics and condensed matter physics.
- The free energy on the 3-sphere $F = -\ln |Z_{S^3}|$
- In a CFT, F is a well-defined, regulator independent quantity (there are no Weyl invariant counter terms).
- F-theorem: $F_{IR} < F_{UV}$ Jafferis, IK, Pufu, Safdi

The Entanglement Connection

- -F is the universal entanglement entropy across a circle of radius R in any (2+1)-d CFT. Casini, Huerta, Myers

$$S(R) = \alpha \frac{2\pi R}{\epsilon} - F$$

- Can be tested with Ryu-Takayanagi method.
- Using the language of EE, the F-theorem was formulated and its proof was found. Myers, Sinha; Casini, Huerta
- The interpolating function used in the proof is the Renormalized Entanglement Entropy (REE) Liu, Mezei

$$\mathcal{F}(R) = -S(R) + RS'(R)$$

Calculating F

- The simplest CFT's involve free conformal scalar and fermion fields. Adding mass terms makes such a theory flow to a theory with no massless degrees of freedom in the IR where F=0.
- For consistency with the F-theorem, the F-values for free massless fields should be positive.

Conformal Scalar on S^d

- In any dimension

$$F_S = -\log|Z_S| = \frac{1}{2}\log\det\left[\mu_0^{-2}\mathcal{O}_S\right] \qquad \mathcal{O}_S \equiv -\nabla^2 + \frac{d-2}{4(d-1)}R$$

- The eigenvalues and degeneracies are

$$\lambda_n = \left(n + \frac{d-1}{2}\right)^2 - \frac{1}{4} \qquad n \geq 0 \qquad m_n = \frac{(2n+d-1)(n+d-2)!}{(d-1)!n!}$$

$$F_S = \frac{1}{2}\sum_{n=0}^{\infty} m_n \left[-2\log(\mu_0 a) + \log\left(n + \frac{d}{2}\right) + \log\left(n - 1 + \frac{d}{2}\right)\right]$$

- Using zeta-function regularization in d=3,

$$F_B = -\frac{1}{2}\frac{d}{ds}\left[2\zeta(s-2, 1/2) + \frac{1}{2}\zeta(s, 1/2)\right]\Big|_{s=0} = \frac{1}{16}\left(2\log 2 - \frac{3\zeta(3)}{\pi^2}\right) \approx .0638$$

- An integral representation valid in continuous dimension d:

$$\begin{aligned} F_s &= \frac{1}{2} \log\det\left(-\nabla^2 + \frac{1}{4}d(d-2)\right) \\ &= -\frac{1}{\sin(\frac{\pi d}{2})\Gamma(1+d)} \int_0^1 du\, u \sin\pi u\, \Gamma\left(\frac{d}{2}+u\right)\Gamma\left(\frac{d}{2}-u\right) \end{aligned}$$

- Near even d, it has simple poles whose coefficients are the a-anomalies.
- For example, in

$$d = 4 - \epsilon$$

$$F_s = \frac{1}{90\epsilon} + \ldots$$

Sphere free energy in continuous d

- A natural quantity to consider is Giombi, IK

$$\tilde{F} = \sin(\pi d/2) \log Z_{S^d} = -\sin(\pi d/2) F$$

- In odd d, this reduces to IK, Pufu, Safdi

$$\tilde{F} = (-1)^{\frac{d+1}{2}} F = (-1)^{\frac{d-1}{2}} \log Z_{S^d}$$

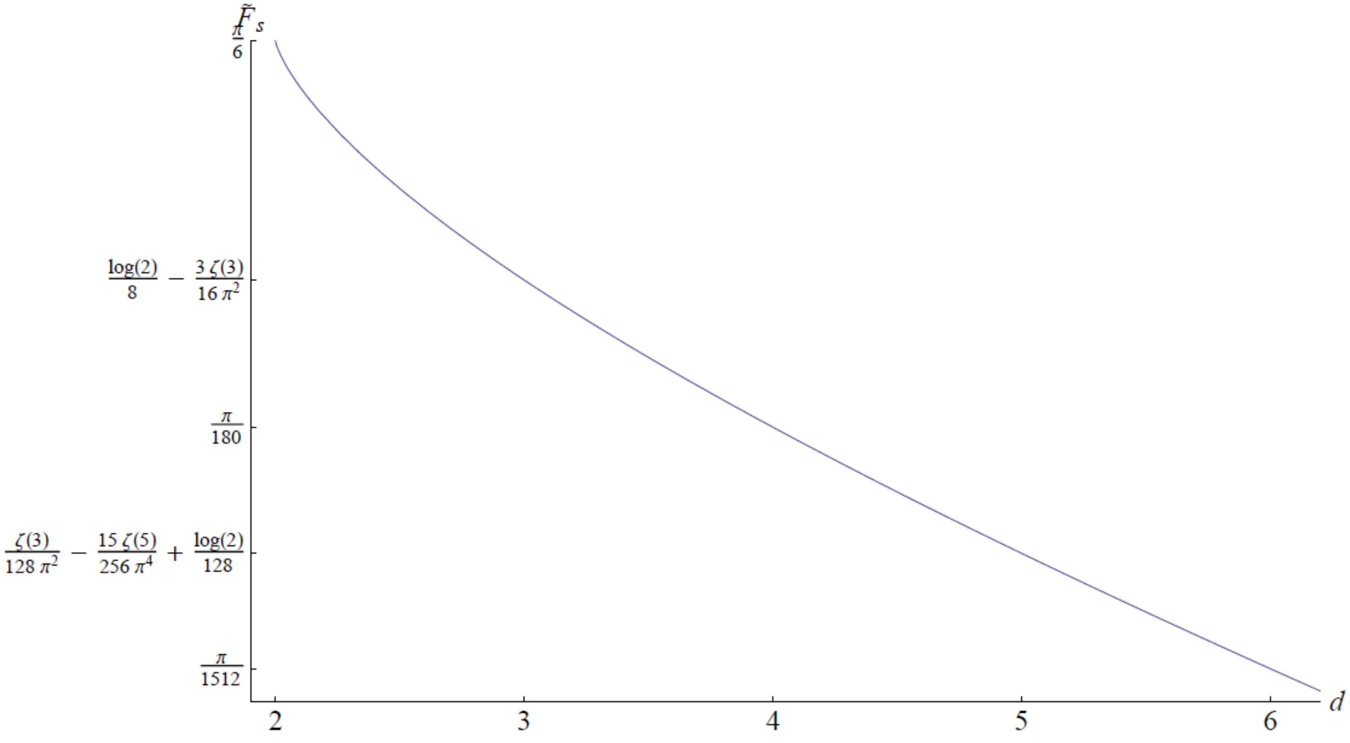
- In even d, -log Z has a pole in dimensional regularization whose coefficient is the Weyl *a*-anomaly. The multiplication by $\sin(\pi d/2)$ removes it.
- $\tilde{F}$ smoothly interpolates between *a*-anomaly coefficients in even and ``F-values" in odd d.

Free conformal scalar in continuous d

- For a free conformal scalar on S^d

$$\tilde{F}_s = \frac{1}{\Gamma(1+d)} \int_0^1 du\, u \sin \pi u\, \Gamma\left(\frac{d}{2}+u\right) \Gamma\left(\frac{d}{2}-u\right)$$

- This is positive for all d and smoothly interpolates between *a* and F



Generalized F-theorem in continuous d?

- Based on the known F- and a-theorems, it is natural to ask whether

$$\tilde{F}_{UV} > \tilde{F}_{IR}$$

holds in continuous dimension *d*.

- We have calculated $\tilde{F}$ in various examples of CFTs that can be defined in continuous dimension, including double-trace flows in large N CFTs and perturbative Wilson-Fisher fixed points in the epsilon-expansion.
- In all unitary examples that we considered, we find that $\tilde{F}$ indeed decreases under RG flow. For non-unitary fixed points, the inequality $\tilde{F}_{UV} > \tilde{F}_{IR}$ does not have to hold.

Weakly Relevant Flows

- A special class of RG flows is obtained by perturbing a CFT by a slightly relevant operator O(x) with dimension $\Delta = d - \varepsilon$ ($\varepsilon << 1$)

$$S_g = S_{\mathrm{CFT}_0} + g_b \int d^d x \, O(x)$$

- Working in conformal perturbation theory, one finds the β–function

$$\beta(g) = -\epsilon g + \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} C g^2 + \mathcal{O}(g^3)$$

- Here $C = \mathcal{C}_3/\mathcal{C}_2$, where

$$\langle O(x)O(y)\rangle_0 = \frac{\mathcal{C}_2}{|x-y|^{2\Delta}} \qquad \langle O(x)O(y)O(z)\rangle_0 = \frac{\mathcal{C}_3}{|x-y|^{\Delta}|y-z|^{\Delta}|z-x|^{\Delta}}$$

- There is a perturbative IR fixed point, $\beta(g_*) = 0$, at

$$g_* = \frac{\Gamma\left(\frac{d}{2}\right)\epsilon}{\pi^{\frac{d}{2}} C} + \mathcal{O}(\epsilon^2)$$

- To compute the change in F from UV to IR, we conformally map to the sphere S^d and obtain

$$\delta F = F - F_0 = -\frac{g_b^2}{2}\mathcal{C}_2 I_2(d-\epsilon) + \frac{g_b^3}{6}\mathcal{C}_3 I_3(d-\epsilon) + \mathcal{O}(g_b^4)$$

- I_2 and I_3 are the 2-point and 3-point integrals on S^d (*Cardy*)

$$I_2(\Delta) = \int \frac{d^d x d^d y \sqrt{g_x}\sqrt{g_y}}{s(x,y)^{2\Delta}} = (2R)^{2(d-\Delta)} \frac{2^{1-d}\pi^{d+\frac{1}{2}}\Gamma\left(\frac{d}{2}-\Delta\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma(d-\Delta)},$$

$$I_3(\Delta) = \int \frac{d^d x d^d y d^d z \sqrt{g_x}\sqrt{g_y}\sqrt{g_z}}{[s(x,y)s(y,z)s(z,x)]^{\Delta}} = R^{3(d-\Delta)} \frac{8\pi^{\frac{3(1+d)}{2}}\Gamma(d-\frac{3\Delta}{2})}{\Gamma(d)\Gamma(\frac{1+d-\Delta}{2})^3}$$

- The final result for the change of $\tilde{F} = -\sin(\pi d/2)F$ is:

$$\delta\tilde{F} = \frac{2\pi^{1+d}\mathcal{C}_2}{\Gamma(1+d)}\left[-\frac{1}{2}\epsilon g^2 + \frac{1}{3}\frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}Cg^3\right] = \frac{2\pi^{1+d}\mathcal{C}_2}{\Gamma(1+d)}\int_0^g \beta(g)dg$$

- Setting $g=g_*$,

$$\delta\tilde{F} = \tilde{F}_{\rm IR} - \tilde{F}_{\rm UV} = -\frac{\pi\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(1+d)}\frac{\mathcal{C}_2}{3C^2}\epsilon^3$$

- In a unitary CFT, $\mathcal{C}_2$ is positive and *C* is real, so we find agreement with the generalized F-theorem in all *d:*

$$\tilde{F}_{UV} > \tilde{F}_{IR}$$

- This generalizes to all *d* previous computations in odd *d* (*Klebanov, Pufu, Safdi*) and even *d* (*Komargodski*)

εE of EE

- The fact that $\tilde{F}$ is a smooth function of dimension suggests that, in the spirit of the Wilson-Fisher ε expansion, it is a useful tool to estimate the value of F for interacting CFTs.
- Consider the 3d Ising model, and more generally the O(N) Wilson-Fisher CFTs in d=3.
- They are strongly coupled CFTs in d=3, but they have a perturbative description in d=4-ε.
- We will compute the sphere free energy perturbatively and extrapolate the result to $\varepsilon=1$ to estimate the value of F.
- This gives an ε expansion of the universal Entanglement Entropy across a circle.

The O(N) models in d=4-ε

$$S = \int d^d x \left(\frac{1}{2} \left(\partial_\mu \phi_0^i\right)^2 + \frac{\lambda_0}{4} (\phi_0^i \phi_0^i)^2 \right)$$

$$\beta = -\epsilon\lambda + \frac{N+8}{8\pi^2}\lambda^2 - \frac{3(3N+14)}{64\pi^4}\lambda^3 + \dots$$

$$\lambda_* = \frac{8\pi^2}{N+8}\epsilon + \frac{24(3N+14)\pi^2}{(N+8)^3}\epsilon^2 + \dots$$

- The ε-expansion works well for operator dimensions:

$$\Delta_\phi = \frac{d}{2} - 1 + \gamma_\phi = 1 - \frac{\epsilon}{2} + \frac{N+2}{4(N+8)^2}\epsilon^2 + \mathcal{O}(\epsilon^3) ,$$

$$\Delta_{\phi^2} = d - 2 + \gamma_{\phi^2} = 2 - \frac{6}{N+8}\epsilon + \mathcal{O}(\epsilon^2) .$$

- The 1/N expansion is worse for the small N models, such as Ising. Wilson, Kogut

The Wilson-Fisher fixed points in curved space

- To renormalize the theory in curved space in d=4-ε, one starts with the bare action (*Brown-Collins '80; Hathrell '82*)

$$S = \int d^d x \sqrt{g} \left(\frac{1}{2}((\partial_\mu \phi_0^i)^2 + \frac{d-2}{4(d-1)}\mathcal{R}(\phi_0^i)^2) + \frac{\lambda_0}{4}(\phi_0^i \phi_0^i)^2 + \frac{1}{2}\eta_0 H(\phi_0^i)^2 + a_0 W^2 + b_0 E + c_0 H^2 \right)$$

$$W^2 = \mathcal{R}_{\mu\nu\lambda\rho}\mathcal{R}^{\mu\nu\lambda\rho} - \frac{4}{d-2}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \frac{2}{(d-2)(d-1)}\mathcal{R}^2 \qquad H = \frac{\mathcal{R}}{d-1}$$

$$E = \mathcal{R}_{\mu\nu\lambda\rho}\mathcal{R}^{\mu\nu\lambda\rho} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2$$

- Divergences in the free energy are removed by expressing all bare couplings in terms of renormalized ones

$$\lambda_0 = \mu^\epsilon \left(\lambda + \frac{(N+8)}{8\pi^2 \epsilon}\lambda^2 + \ldots \right),$$

$$a_0 = \mu^{-\epsilon} \left(a + \sum_{i=0}^{\infty} \frac{L_a^{(i)}(\lambda)}{\epsilon^i} \right), \qquad b_0 = \mu^{-\epsilon} \left(b + \sum_{i=0}^{\infty} \frac{L_b^{(i)}(\lambda)}{\epsilon^i} \right), \quad \text{etc.}$$

- Each renormalized coupling λ, *a*, *b*,… then acquires a non-trivial beta function β_λ, β_a, β_b,…
- The renormalized free energy is a finite function of the renormalized couplings and renormalization scale μ that satisfies the Callan-Symanzik equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta_\lambda\frac{\partial}{\partial\lambda}+\beta_\eta\frac{\partial}{\partial\eta}+\beta_a\frac{\partial}{\partial a}+\beta_b\frac{\partial}{\partial b}+\beta_c\frac{\partial}{\partial c}\right)F=0$$

- The conformally invariant IR fixed point is obtained by setting to zero *all* beta functions in *d*=4-ε

$$\beta_\lambda=\beta_a=\beta_b=\beta_c=\beta_\eta=0$$

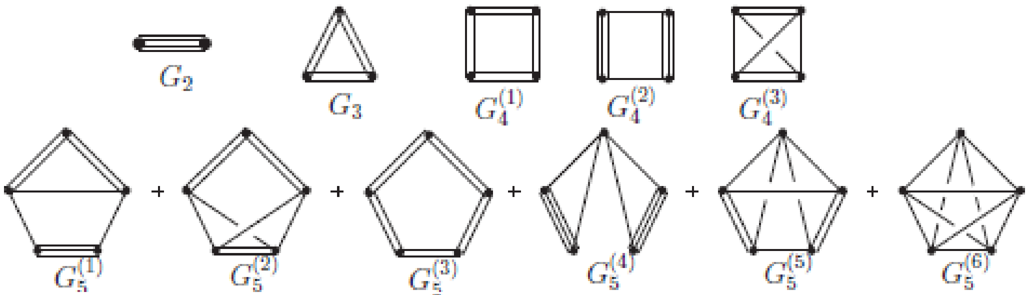
- The sphere free-energy at the IR fixed point in *d*=4-ε

$$F_{\rm IR}(\epsilon)=F(\lambda_*,a_*,b_*,c_*,\eta_*,\mu R)$$

is then a R-independent quantity which is a function of ε only

F for the O(N) scalar theory in *d*=4-ε

- We performed a perturbative calculation of F to order λ^5 (*Fei, Giombi, IK, Tarnopolsky*)



- The poles in the above diagrams fix the curvature beta functions to the needed order. At the IR fixed point, we get the final result for $\tilde{F} = -\sin(\pi d/2) F$:

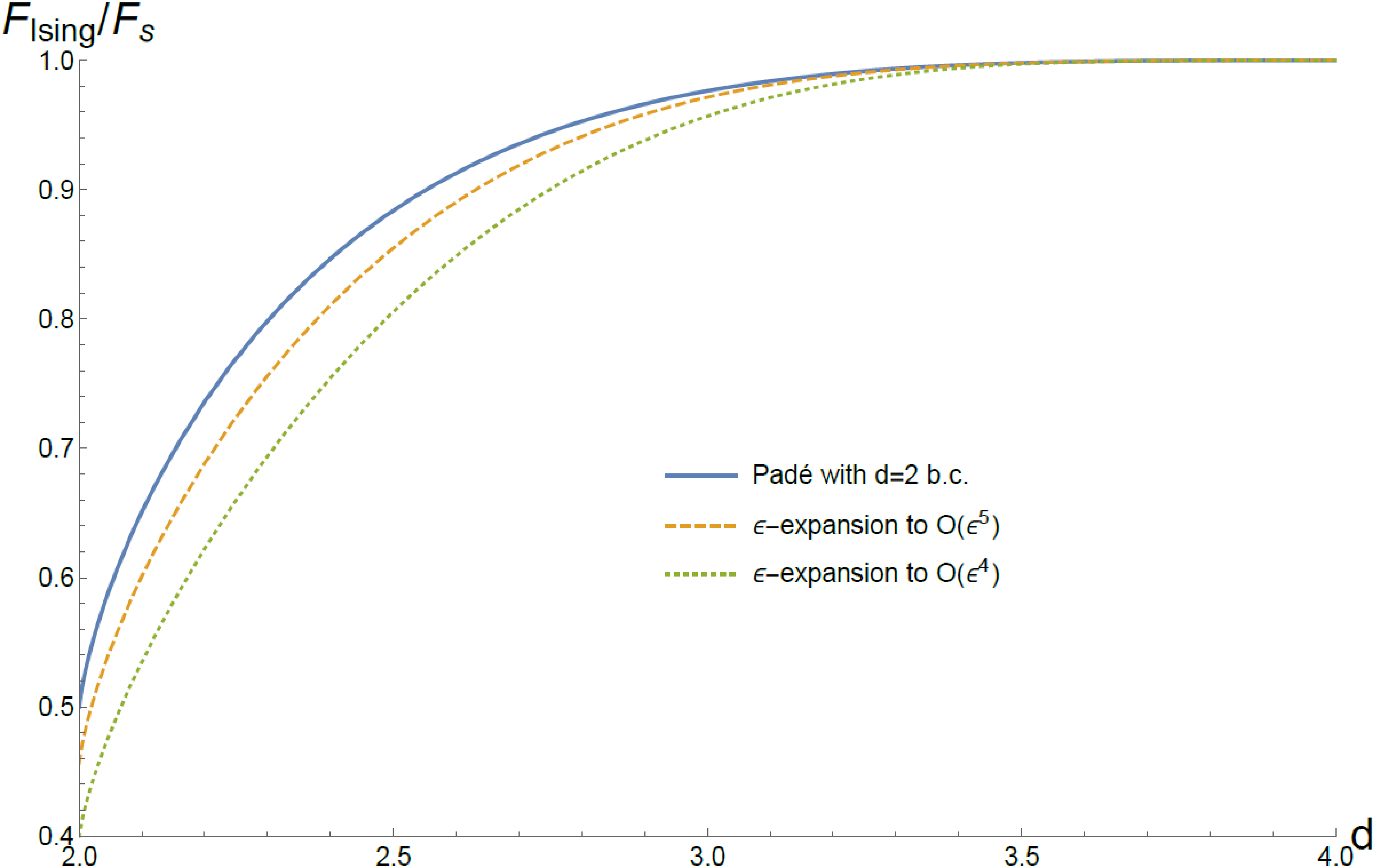
$$\begin{aligned}\tilde{F}_{\rm IR} =& N\tilde{F}_s(\epsilon) - \frac{\pi N(N+2)\epsilon^3}{576(N+8)^2} - \frac{\pi N(N+2)(13N^2+370N+1588)\epsilon^4}{6912(N+8)^4} \\ &+ \frac{\pi N(N+2)}{414720(N+8)^6}\Big(10368(N+8)(5N+22)\zeta(3) - 647N^4 - 32152N^3 \\ &\quad -606576N^2 - 3939520N + 30\pi^2(N+8)^4 - 8451008\Big)\epsilon^5 + \mathcal{O}(\epsilon^6)\end{aligned}$$

F for the 3d Ising model

- Extracting precise estimates from the ε–expansion requires some resummation technique. A simple approach is to use Pade approximants

$$\text{Padé}_{[m,n]}(\epsilon) = \frac{A_0 + A_1\epsilon + A_2\epsilon^2 + \ldots + A_m\epsilon^m}{1 + B_1\epsilon + B_2\epsilon^2 + \ldots + B_n\epsilon^n}$$

- For the Ising model (N=1), we expect $\tilde{F}$ to be a smooth function of *d*, such that near *d*=4 it reproduces the perturbative ε–expansion we computed, and in *d*=2 it reproduces the exact central charge of the 2d Ising CFT: c=1/2.
- The accuracy of the Pade approximants can be improved if we impose the exact value c=1/2 (which in terms of $\tilde{F}$ corresponds to $\tilde{F} = \pi/12$) as a boundary condition at *d*=2



$$\tilde{F} = \tilde{F}_s + \tilde{F}_{\text{int}} = \frac{\pi}{180} + 0.0205991\epsilon + 0.0136429\epsilon^2 + 0.00670643\epsilon^3 + 0.00264883\epsilon^4 + 0.000927589\epsilon^5 + O(\epsilon^6)$$

$$\tilde{F}_s = \frac{\pi}{180} + 0.0205991\epsilon + 0.0136429\epsilon^2 + 0.00690843\epsilon^3 + 0.00305846\epsilon^4 + 0.0012722\epsilon^5 + O(\epsilon^6)$$

- Using the constrained Pade approximant method, we get

$$\frac{F_{\text{3d Ising}}}{F_{\text{free sc.}}} \approx 0.976$$

- Consistent with the F-theorem.
- The value of F for 3d Ising is very close to the free field value!
- A similar result was found for c_T in the conformal bootstrap approach *El-Showk et al.*

$$c_T^{\text{3d Ising}} / c_T^{\text{3d free scalar}} \approx 0.9466$$

- The dimension of ϕ is 0.518… which is only 3.6% above the free field value.

- The large N expansion for the d=3 O(N) model is Pufu, Safdi, IK

$$F_{\rm crit} = N\left(\frac{1}{8}\log 2 - \frac{3\zeta(3)}{16\pi^2}\right) - \frac{\zeta(3)}{8\pi^2} + O(1/N)$$

The first correction to F_{crit}/F_{free} is 24% for N=1.

- In the ε expansion the first correction is only 1% for ε=1:

$$\frac{\tilde{F}_{\rm Ising}}{\tilde{F}_s} = 1 - 0.0115737\epsilon^3 - 0.00981025\epsilon^4 + 0.000880686\epsilon^5 + \mathcal{O}(\epsilon^6)$$

- The ε expansion gives a much more accurate approximation method for small N than the large N expansion.
- We have also used our results and Pade approximants to find F for the critical O(N) models in d=3 for N>1.

  They are also slightly below the corresponding free field values.

QED_3

- Consider QED in d=3 with massless fermions

$$S = \int d^3x \left(\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} - \sum_{i=1}^{N_f} \bar{\psi}_i \gamma^\mu (\partial_\mu + i A_\mu) \psi^i \right)$$

- Here ψ^i are N_f *4-component* spinors, and γ^μ are (three of) the usual 4x4 Dirac matrices.
- Writing $\psi^i = (\chi_1^i, \chi_2^i)$, the action can be stated equivalently in terms of $2N_f$ two-component spinors χ_1^i, χ_2^i.
- The model has $SU(2N_f)$ global symmetry.
- This is often called a "chiral" symmetry. A parity invariant mass term $m\bar{\psi}_i\psi^i$ would explicitly break $SU(2N_f)$ down to $SU(N_f)xSU(N_f)xU(1)$.

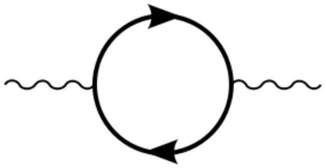
- In d=4 the electric charge is dimensionless and has positive beta function, i.e. QED is free in the IR
- In d=3, e^2 has dimension of mass, so the theory is free in the UV, but it can have non-trivial behavior in the IR.
- In the UV we have the free Maxwell theory and a collection of free fermions. This free UV theory is not conformally invariant. Its sphere free energy has an explicit radius dependence due to the Maxwell term IK, Pufu, Safdi, Sachdev

$$F_{\text{Maxwell}} = -\frac{1}{2}\log(e^2 R) + \dots$$

- In the IR, it is believed that theory flows to an interacting conformal phase for $N_f > N_{crit}$

QED_3 at large N_f

- Integrating out the fermions, the one-loop vacuum polarization diagram



yields an induced kinetic term Appelquist, Pisarski

$$N_f F_{\mu\nu}(-\nabla^2)^{\frac{d}{2}-2}F^{\mu\nu}$$

- At low momenta, this induced term dominates over the Maxwell term, and one gets an interacting CFT where $F_{\mu\nu}$ has dimension 2 instead of the UV dimension 3/2.
- Scaling dimensions and correlation functions of other operators can be computed in $1/N_f$ expansion.

F at large N_f

- At large N_f, the free energy on S^3 in the IR CFT can also be computed in the $1/N_f$ expansion. The first non-trivial correction comes from the determinant of the induced kinetic operator for the gauge field IK, Pufu, Sachdev, Safdi

$$F_{\rm conf} = N_f \left(\frac{\log(2)}{2} + \frac{3\zeta(3)}{4\pi^2} \right) + \frac{1}{2} \log\left(\frac{\pi N_f}{4} \right) + \mathcal{O}(\frac{1}{N_f})$$

- In the UV limit, the free energy F_{UV} diverges due to the log(R) dependence of the Maxwell term
- Thus, there is no contradiction with F-theorem $F_{UV}>F_{IR}$, despite the fact that the $\log(N_f)$ term in F_{conf} can grow without bound for large N_f

QED$_3$ at finite N$_f$ and symmetry breaking

- For sufficiently large N_f, this interacting fixed point is expected to be stable, because there are no relevant operators preserving $SU(2N_f)$ and parity.
- As we lower N_f, a widely discussed scenario is that for N_f less or equal than some critical value N_{crit}, the model displays spontaneous symmetry breaking according to the pattern

$$SU(2N_f) \to SU(N_f) \times SU(N_f) \times U(1)$$

- This symmetry breaking is due to the condensation of the fermion bilinear $\bar{\psi}\psi = (\bar{\chi}_1\chi_1 - \bar{\chi}_2\chi_2)$

  which breaks $SU(2N_f)$ but preserves parity Pisarski; Appelquist et al
- If SSB occurs, then the IR theory consists of the $2N_f^2$ Nambu-Goldstone bosons and an extra massless scalar (dual photon)

QED_3

- At $N_f = N_{crit}$ a quartic fermion operator (invariant under $SU(2N_f)$ and parity) can become relevant in the IR, and render the IR fixed point unstable towards the symmetry breaking Di Pietro et al; Kaveh, Herbut; Braun et al
- We would like to use the F-theorem to provide a constraint on the value of N_{crit} (a similar F-theorem approach to N_{crit} was considered earlier by Grover)
- Since we are interested in finite N_f physics, we use the epsilon expansion and the dimensional continuation of

$$\tilde{F} = -\sin(\pi d/2) F$$

to estimate the value of F in QED_3

QED_d in the epsilon expansion

$$S = \int d^d x \left(\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} - \sum_{i=1}^{N_f} \bar{\psi}_i \gamma^\mu (\partial_\mu + i A_\mu) \psi^i \right)$$

- The dimensional continuation of the model is defined in such a way that ψ^i are 4-component spinors in all *d*, i.e. the gamma matrices γ^μ are a formal set of 4x4 matrices where the vector index is continued to d-dimensions
- In this way, the usual QED in 4d with N_f massless Dirac fermions is connected to QED_3 with $2N_f$ two-component fermions.
- The even number of 3-d Dirac flavors avoids the "parity anomaly." A. N. Redlich

QED_d in the epsilon expansion

- In d=4-ε , the coupling *e* has dimension (4-d)/2=ε/2. So one finds the beta function

$$\beta = -\frac{\epsilon}{2}e + \frac{4N_f}{3}\frac{e^3}{(4\pi)^2} + \mathcal{O}(e^5)$$

- Thus, similarly to the Wilson-Fisher fixed point of the O(N) scalar field theory, we get a perturbative IR stable fixed point at

$$e_*^2 = 6\epsilon\pi^2/N_f + \mathcal{O}(\epsilon^2)$$

- This fixed point can be studied by usual perturbative methods for any N_f , for instance scaling dimensions of some local operators can be determined as series expansions in ε

F of QED_d in the ε-expansion

- As in the O(N) model discussed earlier, the calculation requires careful renormalization of the model in curved space in $d=4-\varepsilon$

$$S = \int d^d x \sqrt{g_x} \Big(\frac{1}{4e_0^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\nabla_\mu A^\mu)^2 - \sum_{i=1}^{N_f} \bar{\psi}_i \gamma^\mu (\nabla_\mu + i A_\mu) \psi^i + a_0 W^2 + b_0 E + c_0 \mathcal{R}^2/(d-1)^2 \Big)$$

- Building on previous results for the renormalization of QED on S^d (Adler; Drummond, Shore; Hathrell; Jack, Osborn;…) we have performed the calculation of F at the IR fixed point working to order $e^4 \sim \varepsilon^2$ in perturbation theory

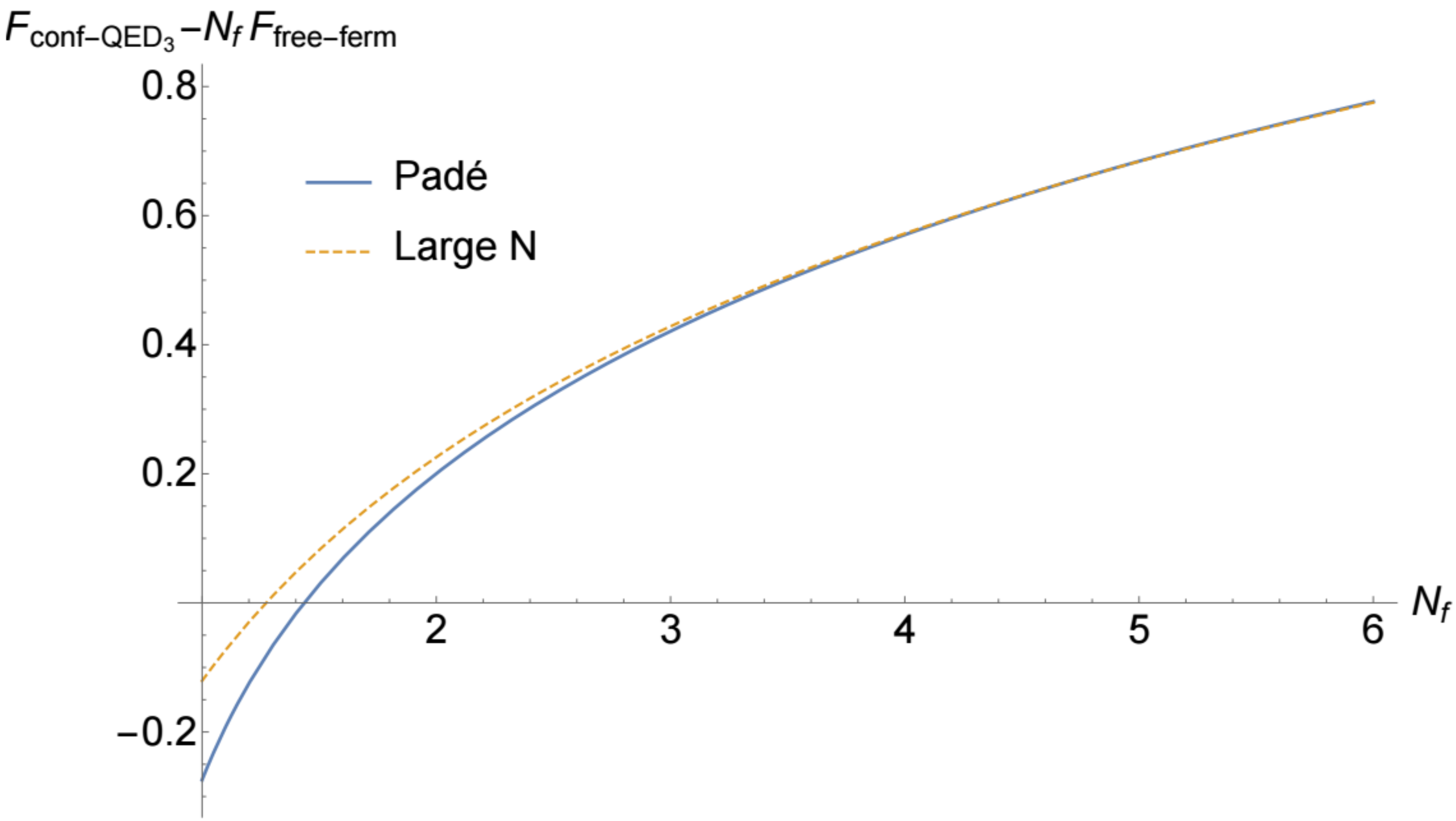
- At the IR fixed point Giombi, IK, Tarnopolsky

$$\tilde{F}_{\rm conf} = N_f \tilde{F}_{\rm free\text{-}ferm} - \frac{1}{2}\sin(\frac{\pi d}{2})\log(\frac{N_f}{\epsilon}) + \frac{31\pi}{90} - 1.2597\epsilon - 0.6493\epsilon^2 + 0.8429\epsilon^3 + \frac{0.4418\epsilon^2}{N_f} - \frac{0.6203\epsilon^3}{N_f} - \frac{0.5522\epsilon^3}{N_f^2} + \mathcal{O}(\epsilon^4)$$

- The term 31π/90 is from the a-anomaly coefficient of the d=4 Maxwell field.
- A new feature is the non-analytic term ~ $\log(N_f/\varepsilon)$. This originates from the free Maxwell contribution, which contains the term $\log(e^2R^{4-d})$
- The R dependence drops out at $e=e_*$ as a result of delicate cancellations between the free Maxwell term and terms due to interactions. Consistent with the expected conformal invariance in the IR.

Schwinger Model

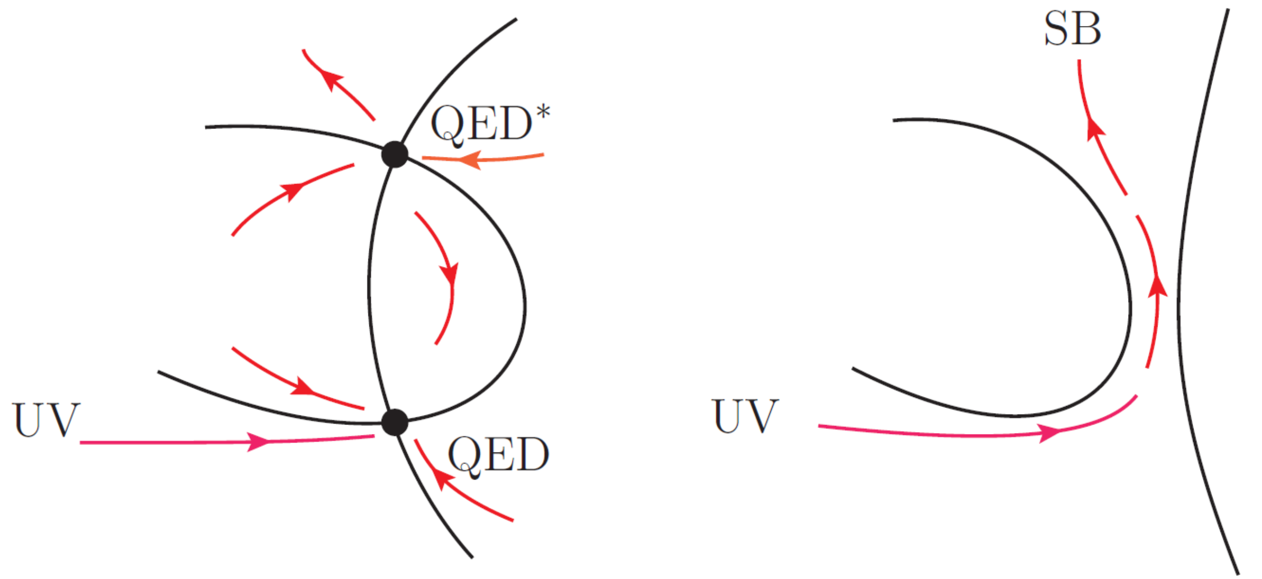
- In d=2, the IR behavior of QED with $2N_f$ two-component fermions is that of the multi-flavor Schwinger model, which has a description in terms of the level 1 SU($2N_f$) WZW model. This has c=$2N_f$-1, or $\tilde{F}$ =π($2N_f$-1)/6. This expectation is also supported by a large N_f calculation of F in $2 \leq d \leq 4$
- Therefore we use a "two-sided" Pade approximant with this d=2 boundary condition in order to better estimate the value in d=3.



- The plot of the Pade resummed ε-expansion evaluated at $\varepsilon=1$, compared to the d=3 large N_f expansion result shows that they are very close already at $N_f \sim 3$

F-theorem and N_{crit}

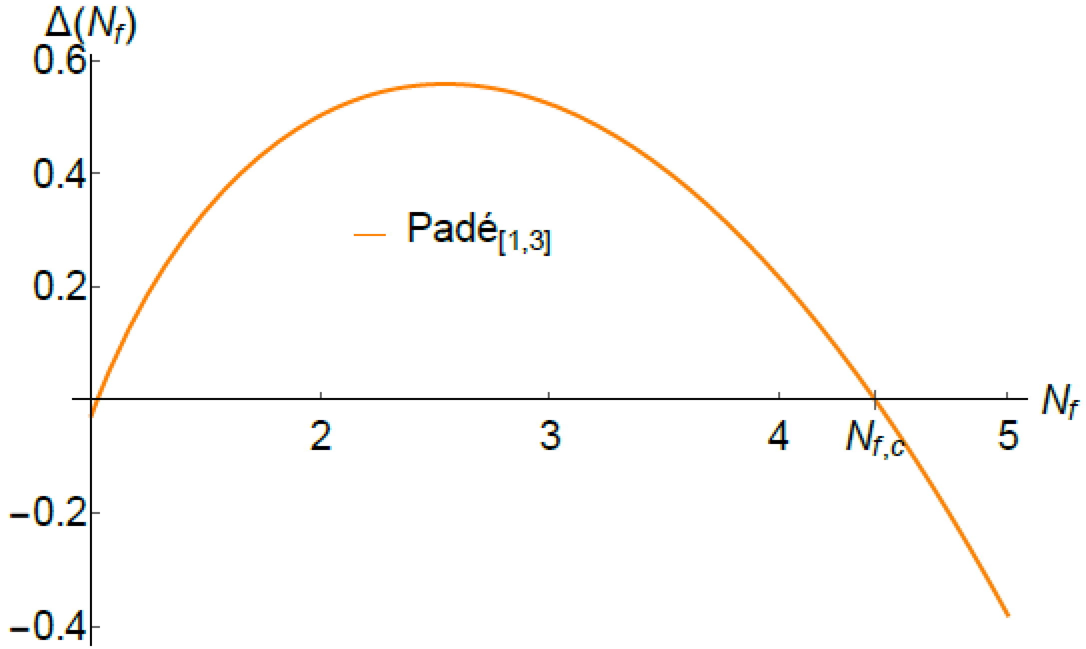
- Assume that the mechanism for SSB is that of a quartic fermion operator becoming relevant at some $N_f=N_{crit}$
- For N_f slightly above N_{crit}, we have a slightly irrelevant operator, and by conformal perturbation theory we expect a nearby UV fixed point, which we may call QED_3*
- A commonly discussed scenario is that as N_f approaches N_{crit}, the two fixed points QED3 and QED_3* merge and annihilate at $N_f=N_{crit}$ Herbut, Kaveh; Braun et al; Kaplan et al.



$(a).\ N_f \gtrsim N_{\rm crit}$ $(b).\ N_f \lesssim N_{\rm crit}$

- For $N_f \lesssim N_{\rm crit}$, the conformal phase no longer exists, but the RG flow originating in the UV can "hover" near the complex fixed points before running away to large quartic coupling and presumably towards SSB. During this "hovering" F can be made parametrically close to $F_{conf}(N_{crit})$
- The F-theorem then requires $F_{conf}(N_{crit}) > F_{SB}$, where F_{SB} is that of $2N_f^2$ +1 massless scalars

F-theorem and N_{crit}



$$\text{Plot of } \Delta(N_f) = F_{\rm conf}(N_f) - F_{\rm SB}(N_f), \text{ using Padé}_{[1,3]}$$

- QED_3 is in the $SU(2N_f)$ invariant conformal phase for $N_f>4.4$
- For $N_f=1$ the SB seems to be disallowed again (just barely!). QED_3 is conformal? What about $N_f=1/2$?

C_J and C_T

$$\langle J^a_\mu(x_1) J^b_\nu(x_2)\rangle = C_J \frac{I_{\mu\nu}(x_{12})}{(x_{12}^2)^{d-1}}\delta^{ab} \,,$$

$$\langle T_{\mu\nu}(x_1) T_{\lambda\rho}(x_2)\rangle = C_T \frac{I_{\mu\nu,\lambda\rho}(x_{12})}{(x_{12}^2)^{d}}$$

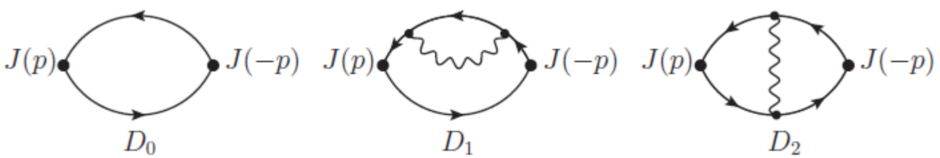
$$I_{\mu\nu}(x) \equiv \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2} \,,$$

$$I_{\mu\nu,\lambda\rho}(x) \equiv \frac{1}{2}\left(I_{\mu\lambda}(x) I_{\nu\rho}(x) + I_{\mu\rho}(x) I_{\nu\lambda}(x)\right) - \frac{1}{d}\delta_{\mu\nu}\delta_{\lambda\rho}$$

- C_J determines the universal charge or spin conductivity.
- C_T enters in many contexts including the entanglement entropy. It is one of the natural measures of degrees of freedom in a CFT.
- In d=2 satisfies the Zamolodchikov C-theorem, but there are counterexamples in d>2.

C_J and C_T in Conformal QED

- Here the 1/N corrections are calculated using the induced photon propagator.
- To find C_J we calculated Giombi, Tarnopolsky, IK



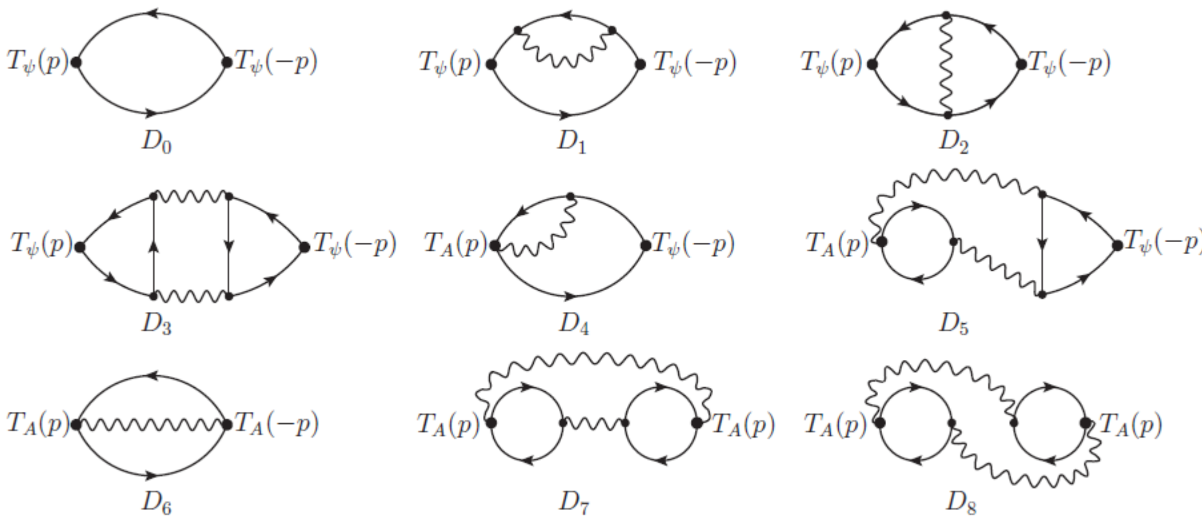
$$C_{J0} = \mathrm{Tr}\mathbf{1}\frac{1}{S_d^2}$$

$$C_{J1}(d) = \eta_{m1}\left(\frac{3d(d-2)}{8(d-1)}\Theta(d) + \frac{d-2}{d}\right) \qquad \Theta(d) \equiv \psi'(d/2) - \psi'(1)$$

- The electron mass anomalous dimension is

$$\eta_{m1}(d) = -\frac{2(d-1)\Gamma(d)}{\Gamma(\frac{d}{2})^2\Gamma(\frac{d}{2}+1)\Gamma(2-\frac{d}{2})}$$

- The calculation of C_T requires more diagrams because $T = T_\psi + T_A$ Huh and Strack



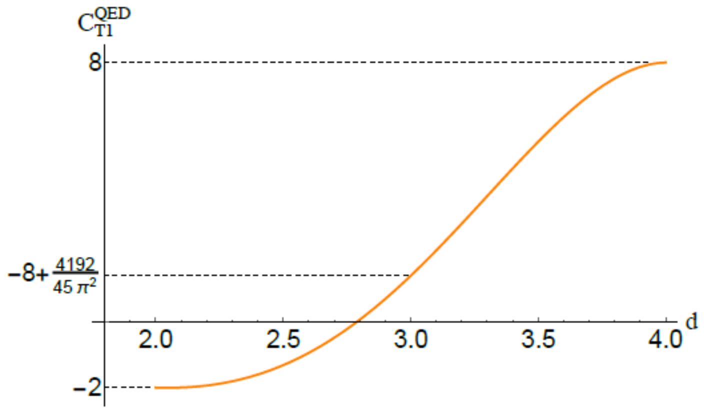
- With an analytic regulator we find

$$C_{T1}(d) = \eta_{m1}\left(\frac{3d(d-2)}{8(d-1)}\Theta(d) + \frac{d(d-2)}{(d-1)(d+2)}\Psi(d) - \frac{(d-2)(3d^2+3d-8)}{2(d-1)^2 d(d+2)}\right)$$

$$\Psi(d) \equiv \psi(d-1) + \psi(2-d/2) - \psi(1) - \psi(d/2-1)$$

- Agrees with the 4-ε expansion for QED.
- In d=2 agrees with the exact result for multi-flavor Schwinger model

$$C_T|_{d=2} = \frac{N}{S_2^2}\left(1 - \frac{2}{N}\right)$$



- In d=3 we find

$$C_{J1}(3) = \frac{736}{9\pi^2} - 8 \approx 0.285821$$
$$C_{T1}(3) = \frac{4192}{45\pi^2} - 8 \approx 1.43863$$

Another Estimate for SB?

- The far UV theory of free fermions and decoupled Maxwell field is not conformal. Define C_T via the 2-point function of

$$T \equiv z^\mu z^\nu T_{\mu\nu} \qquad\qquad z^\mu z^\nu \delta_{\mu\nu} = 0$$

- Then $C_T^{\rm UV} = \dfrac{12N_f + 9}{32\pi^2}$
- For the SB phase with Nambu-Goldstone bosons $C_T^{\rm IR} = \dfrac{3(2N_f^2 + 1)}{32\pi^2}$
- IF we assume $C_T^{\rm UV} > C_T^{\rm IR}$ then

$$N_{f,\rm crit} = 1 + \sqrt{2} \approx 2.414$$

Conclusion and Discussion

- We studied the dimensional continuation of the sphere free energy and provided some evidence for a “generalized F-theorem” in continuous *d,* interpolating between F-theorems in odd *d* and *a*-theorems in even *d*.
- The quantity that appears to decrease under RG flow is

$$\tilde{F} = \sin(\pi d/2) \log Z_{S^d} = -\sin(\pi d/2) F$$

- $\tilde{F}$ a smooth function of d, and its ε–expansion is a useful tool for finding F of CFTs in the physical integer dimension.

Conclusion and Discussion

- For the critical 3d Ising model, using the ε–expansion we found that F is only 2-3% below that of the free conformal scalar.
- Can this be compared with a direct numerical calculation of F, or of the disk Entanglement Entropy, for the 3d Ising CFT?
- For the QED_3 with N_f massless fermions, we have computed F using the ε–expansion.
- Used F-theorem to argue that the theory is conformal for $N_f>4.4$.
- Is QED_3 conformal again for $N_f=1$?
- What about $N_f=1/2$ (a single charged 2-component fermion)?