Gravitational Positive Energy Theorems from Infromation Inequalities

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Swampland Question

Given an effective theory of gravity, how can one judge whether it is realized as a low energy appropximation to a consistent quantum theory with **ultra-violet completion**, such as string theory?

Constraints on Symmetry

Conjectures:

- ☆ There are no global symmetry.
- ☆ All continuous gauge symmetries are compact.
- ☆ The spectrum of electric and magnetic charges forms a complete set consistent with the Dirac quantization condition.

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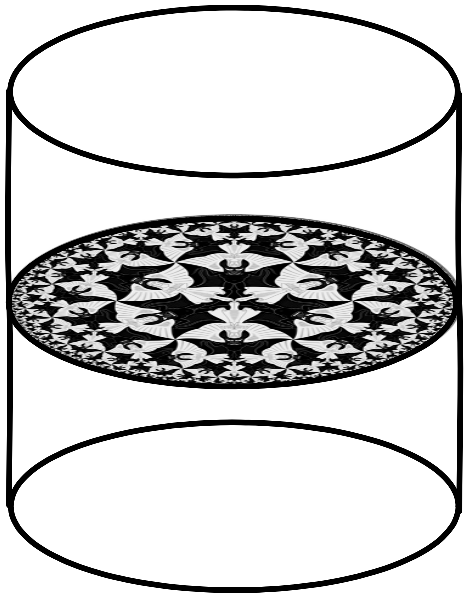

Holographic understanding:

Harlow, arXiv: 1510.07911
Harlow + H.O., to appear

More Conjectures:

Gravity is the weakest force in Nature.

Arkani-Hamed, Motl, Nicolis + Vafa, hep-th/0601001

Every symmetry is gauged.

With a gauge field, there is always a particle
whose **mass is smaller than its charge** in the Planck unit.

☆ Black holes at the Reissner-Nordstrom bound
are unstable unless protected by supersymmetry.

Constraints on Moduli Space

Conjectures:

☆ The moduli space is **non-compact, complete, and has finite volume**.

☆ If we move a large distance *T* from a reference point, **a tower of light particles emerges** with mass of the order exp(- a T) for some a. The number of such light particles becomes infinite at T tends to the infinity.

☆ There is **no non-trivial one-cycle with minimal length** within a given homotopy class in the moduli space.

as formulated by Vafa + H.O., arXiv:0605264

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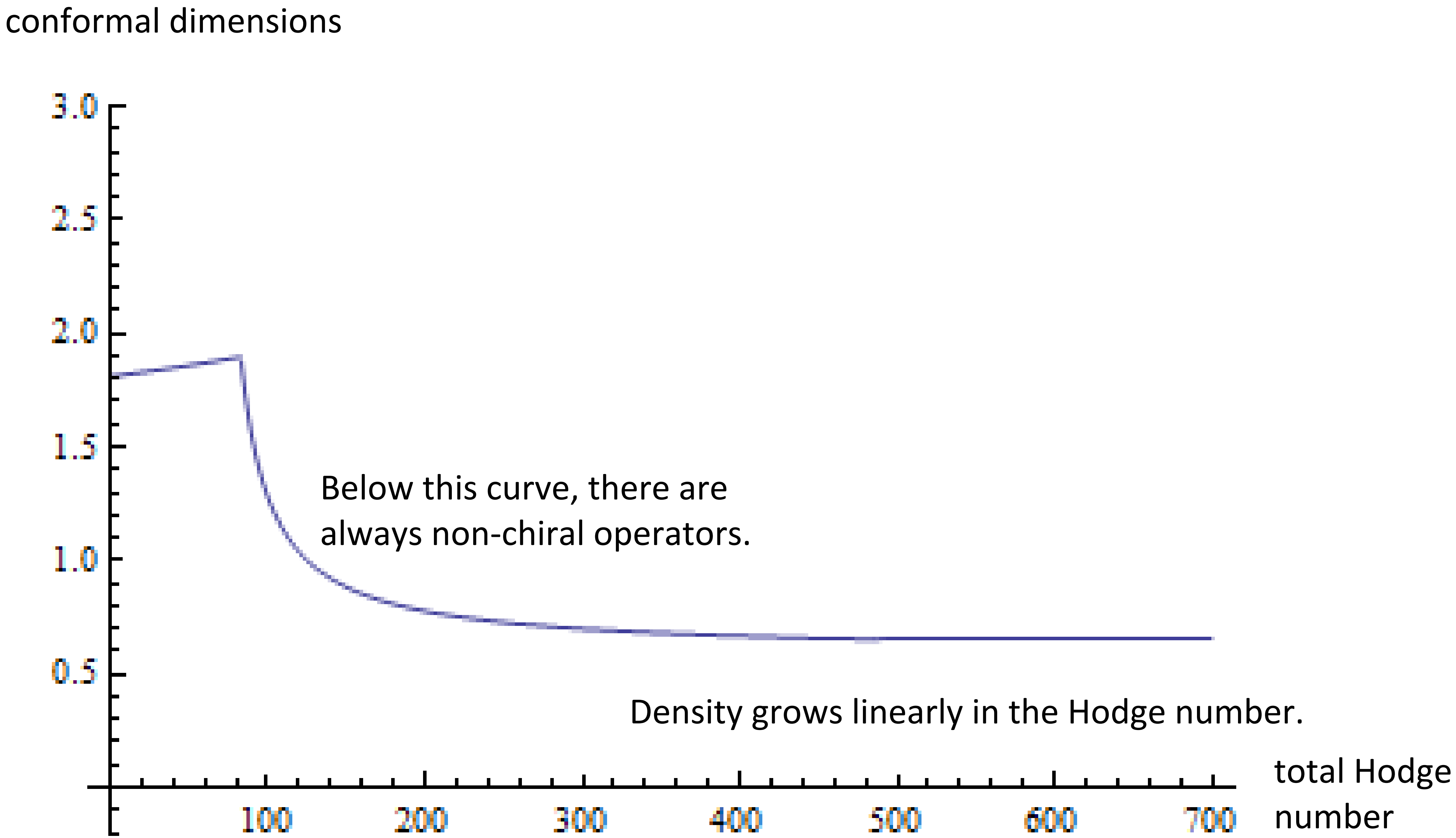
These moduli space constraints have been proven for theories with N=3 or higher supersymmetry.

Cecotti, "Supersymmetric Field Theories," section 4.9.1

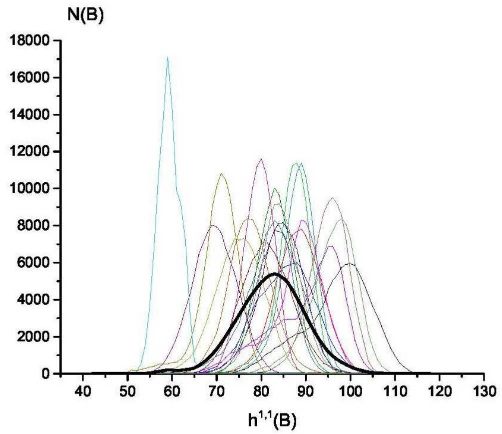
Constraints on Calabi-Yau Topology

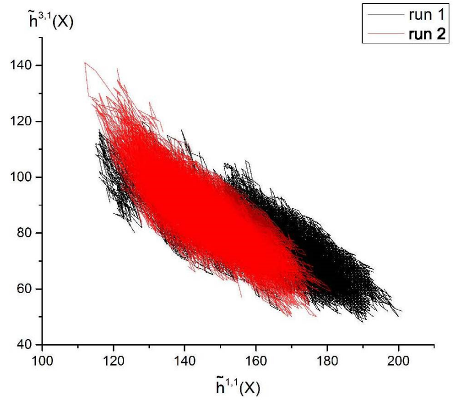
Modular invariance constraints

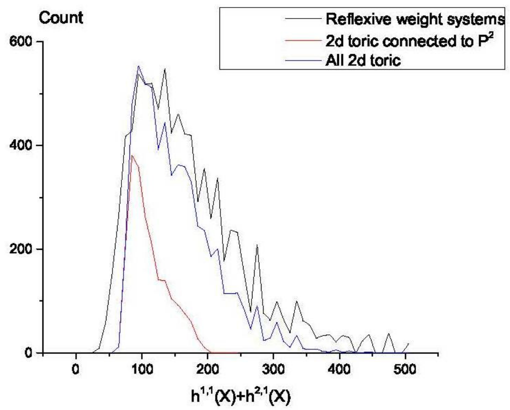
Keller + H.O., arXiv: 1209.4649

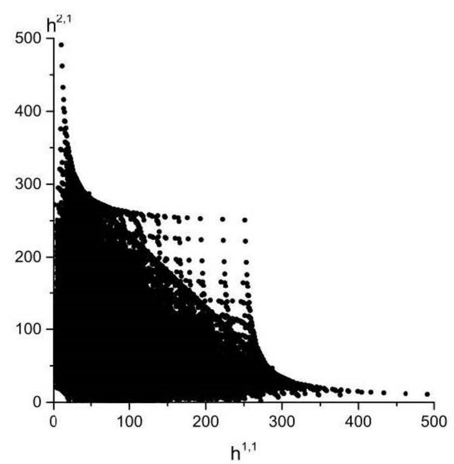


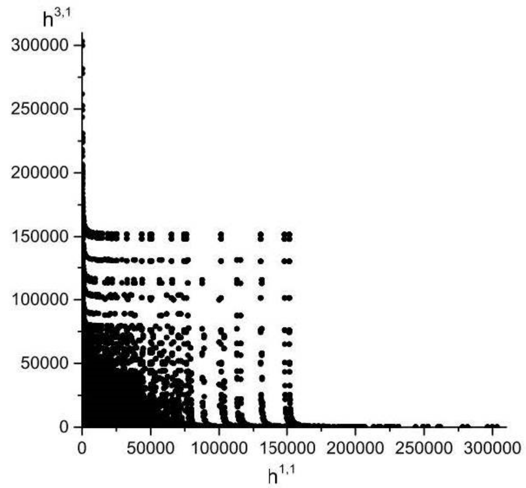
Recent experimental data on Calabi-Yau 3 and 4 folds







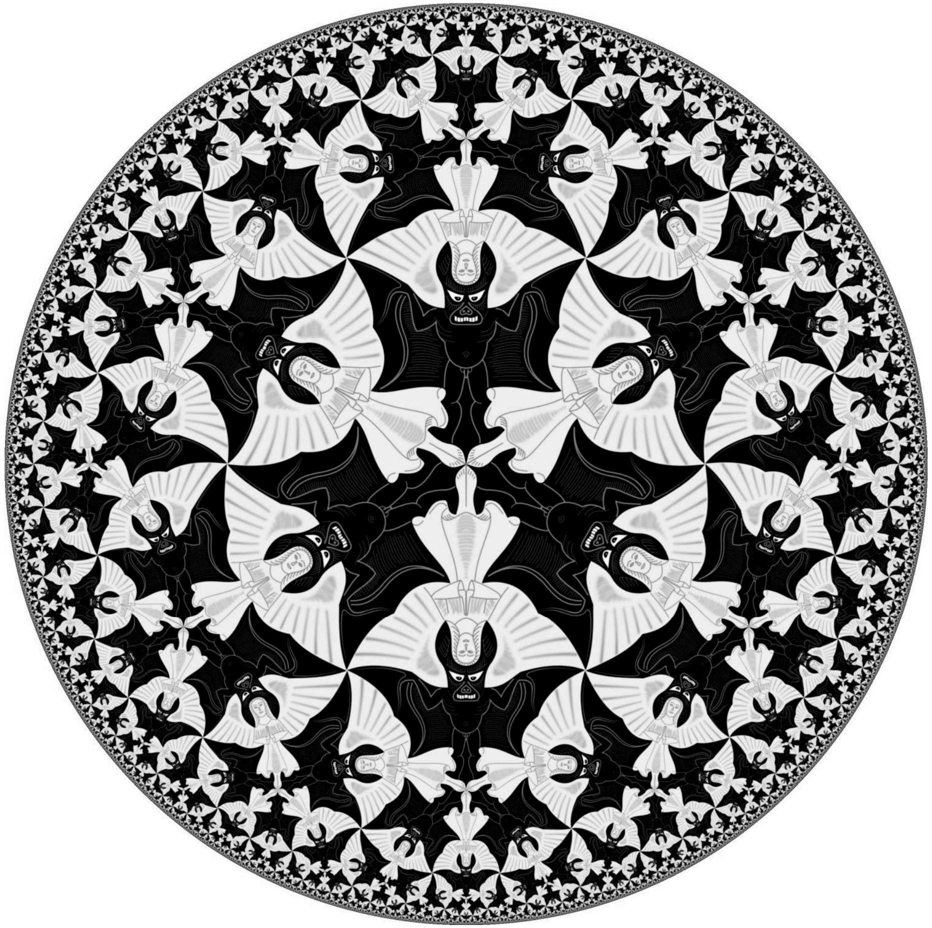




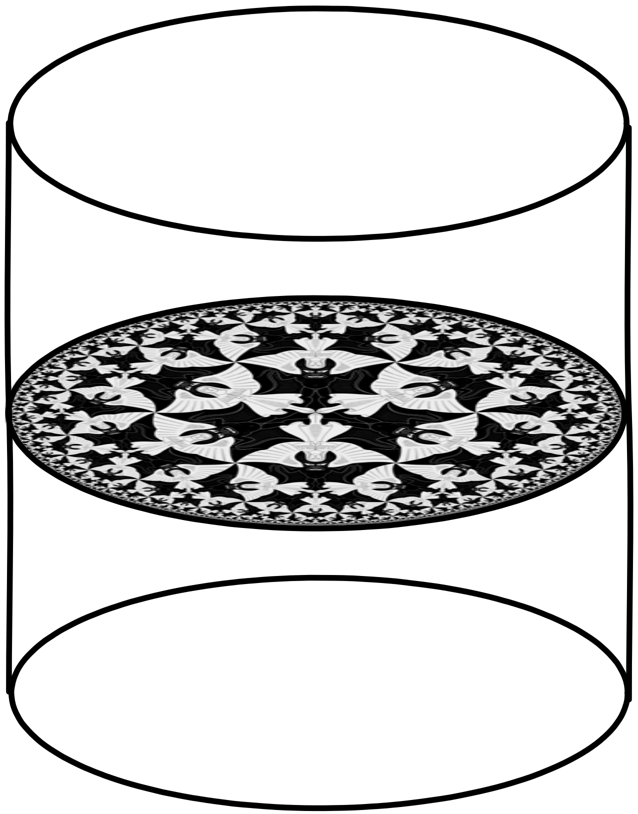
Taylor + Wang,
arXiv: 1510.04978, 1511.03209

Holographic Constraints

Suppose there is a low energy effective field theory whose gravity solutions asymptote to the anti-de Sitter space at the infinity.



Suppose there is a low energy effective field theory whose gravity solutions asymptote to the anti-de Sitter space at the infinity.

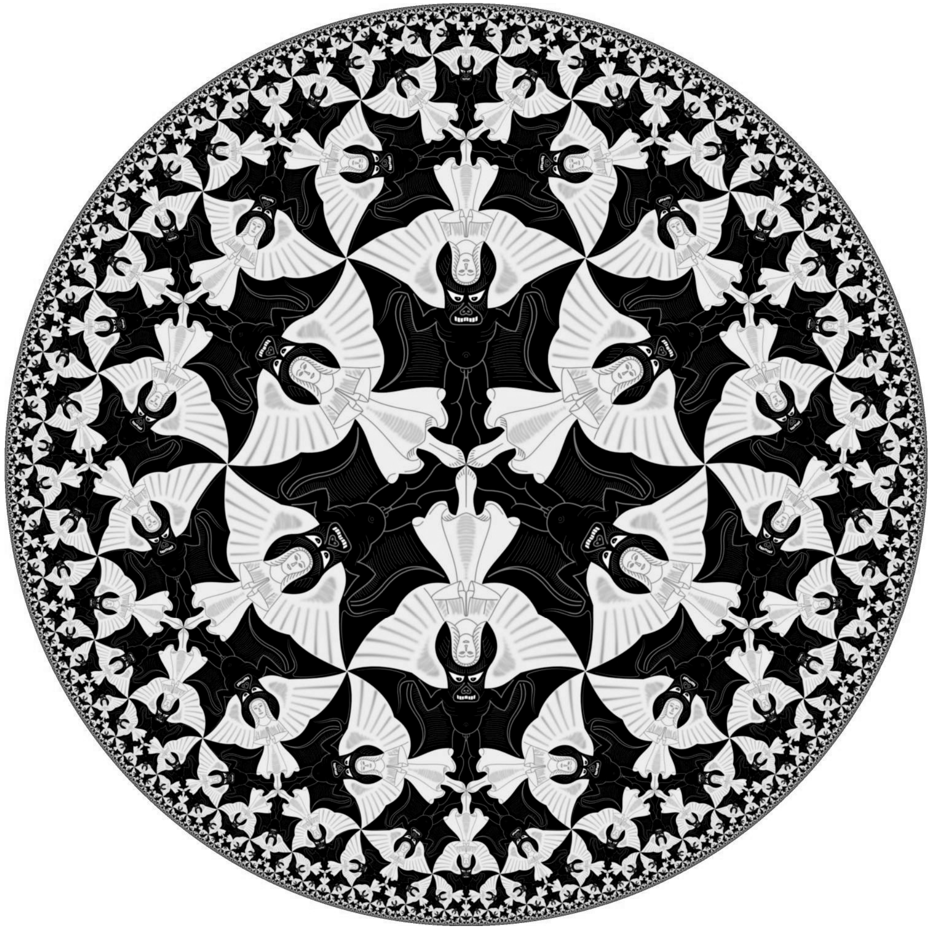


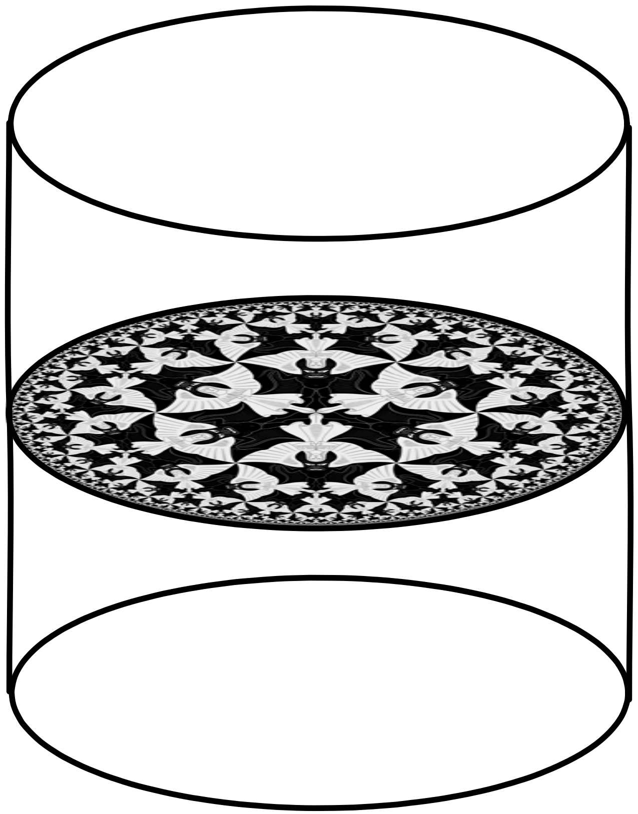
Holography of Quantum Gravity:

Consistent quantum gravity in AdS is equivalent to a conformal field theory on the boundary.

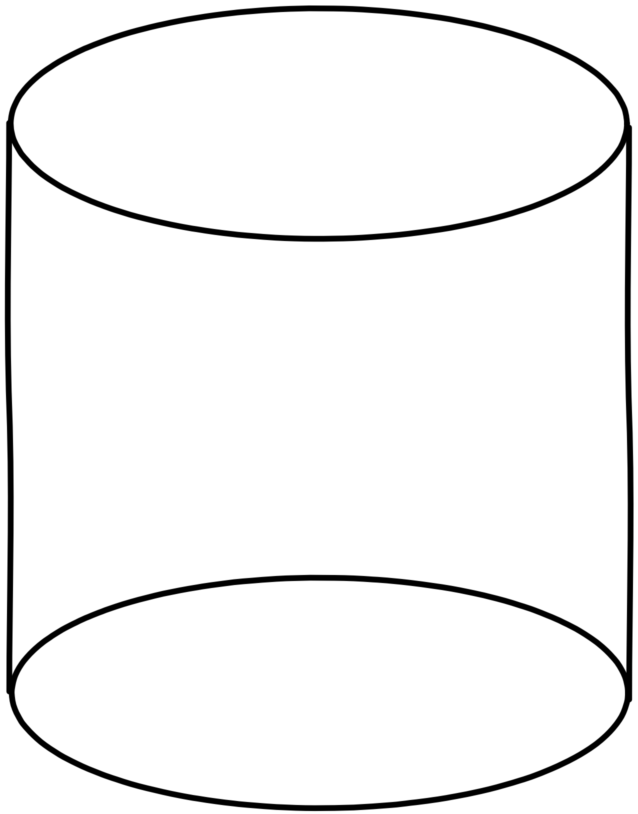
AdS/CFT Correspondence

Question: What does consistency of the conformal field theory mean for the gravity theory?

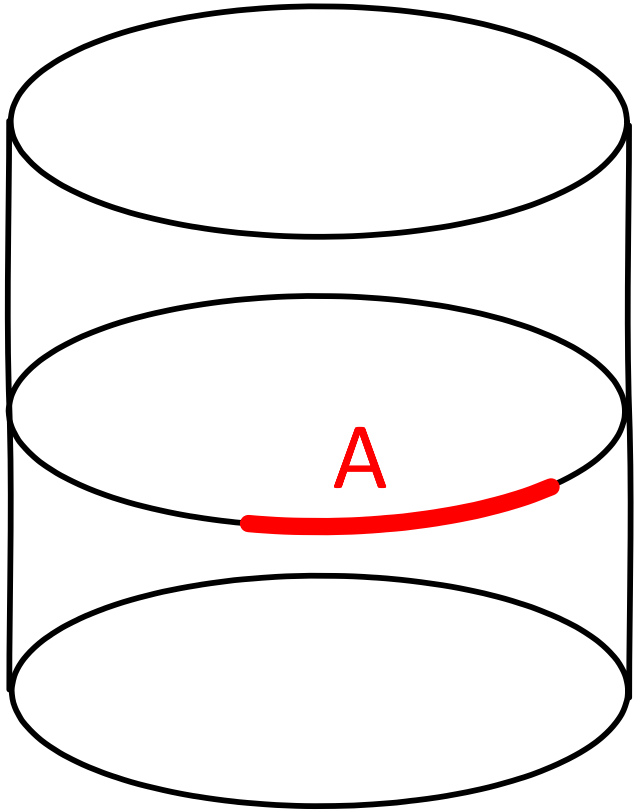




Gravity theory in (d+1)-dim AdS



Gravity theory in (d+1)-dim AdS is equivalent to d-dim CFT.



Gravity theory in (d+1)-dim AdS is equivalent to d-dim CFT.

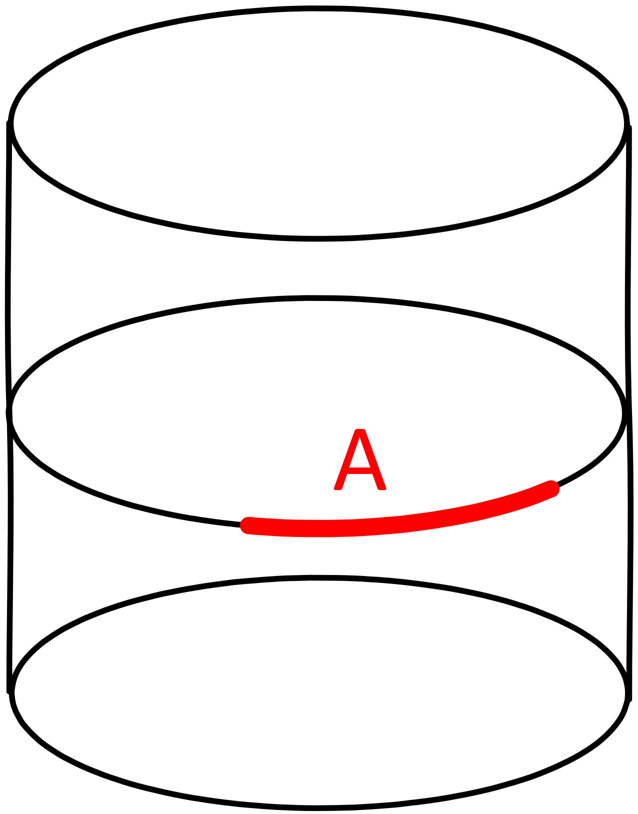
Entanglement Density Matrix ρ

For any state $|\psi\rangle$ in CFT,
choose a spacelike region A.

$$\rho = tr_{\bar{A}}\, |\psi\rangle\langle\psi|$$

☆ The trace is on the Hilbert space over the complement of A.

☆ It is an operator acting on the Hilbert space over A.



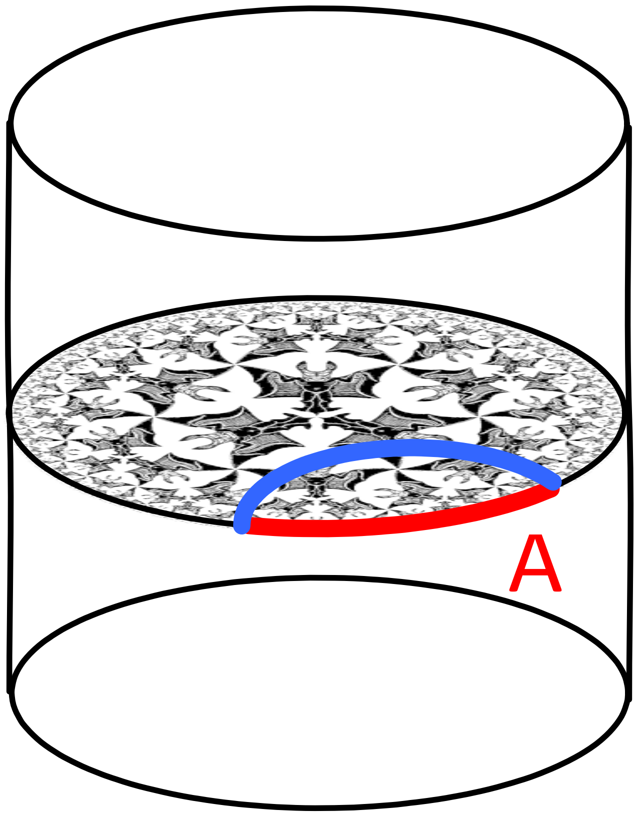
Entanglement Density Matrix ρ

$$\rho = tr_{\bar{A}} \, |\psi\rangle\langle\psi|$$

Entanglement Entropy *S*

$$S = -\mathrm{tr}\, \rho \log \rho$$

S measures the amount of entanglement between the region A and its complement.



Entanglement Entropy S

$$S = -\,\mathrm{tr}\,\rho \log \rho$$

When the bulk gravity theory is described with smooth geometry, the entanglement entropy ***S*** is proportional to the area of the minimum surface ending of the boundary of A.

$$S = \frac{1}{4G_N} Area(\Sigma)$$

Ryu-Takayanagi (2006)

Entanglement Entropy
satisfies inequalities:

$$S = -\operatorname{tr}\rho\log\rho$$

☆ Some inequalities are satisfied both by any CFT and by AdS gravity.

☆ Some inequalities are satisfied by any CFT but not always by AdS gravity.

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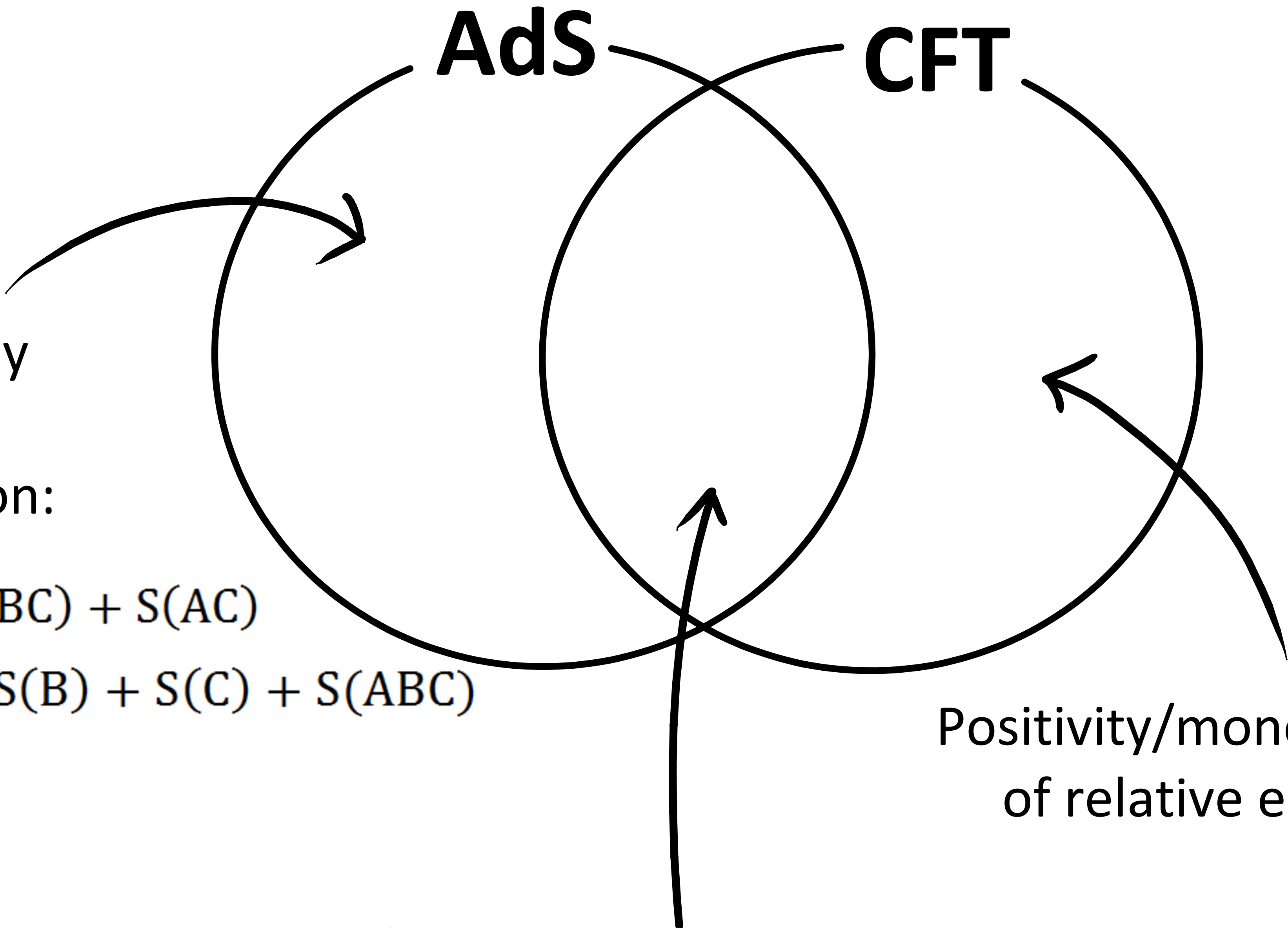


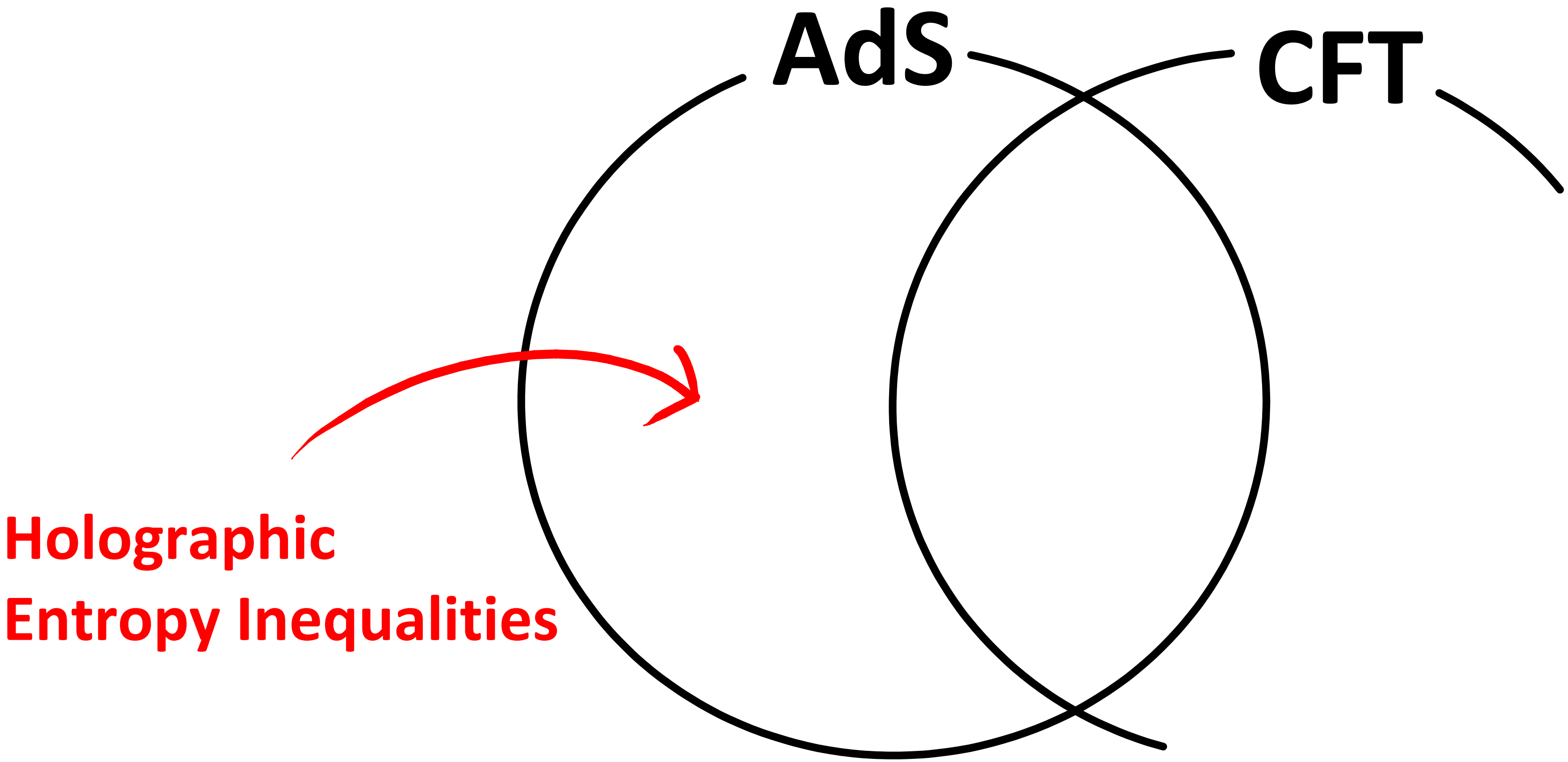
Monogamy
of mutual
information:

$$S(AB) + S(BC) + S(AC) \geq S(A) + S(B) + S(C) + S(ABC)$$

Positivity/monotonicity
of relative entropy

Strong subadditivity:

$$S(AB) + S(BC) \geq S(B) + S(ABC)$$



CFT states with gravitational duals have interesting entanglement properties.

Entropy Inequalities

(Classical) Shannon Entropy:

There are ***infinite number*** of independent entropy inequalities for more than 3 regions.

$\Rightarrow$ Asymptotic performance for information processing tasks

Matus (2007)

(Quantum) von Neumann Entropy:

For more than 3 regions, the complete set of independent inequalities is ***not known***.

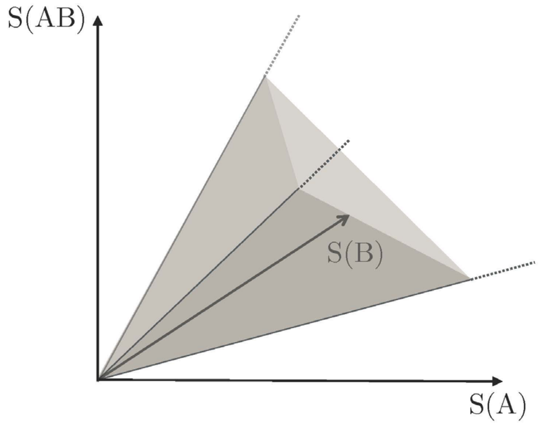
$\Rightarrow$ Numerical evidences that the number is infinite.

For holographic states:

- ☆ **Finite algorithm** to classify all inequalities.
- ☆ There are **finitely many independent inequalities** for a fixed number of regions.
- ☆ Complete classification for 2, 3, 4 regions.
- ☆ A new family of inequalities for 5 and more regions.

Bao, Nezami, Stoica, Sully, Walter + H.O., arXiv:1505.07839

Holographic Entropy Cone

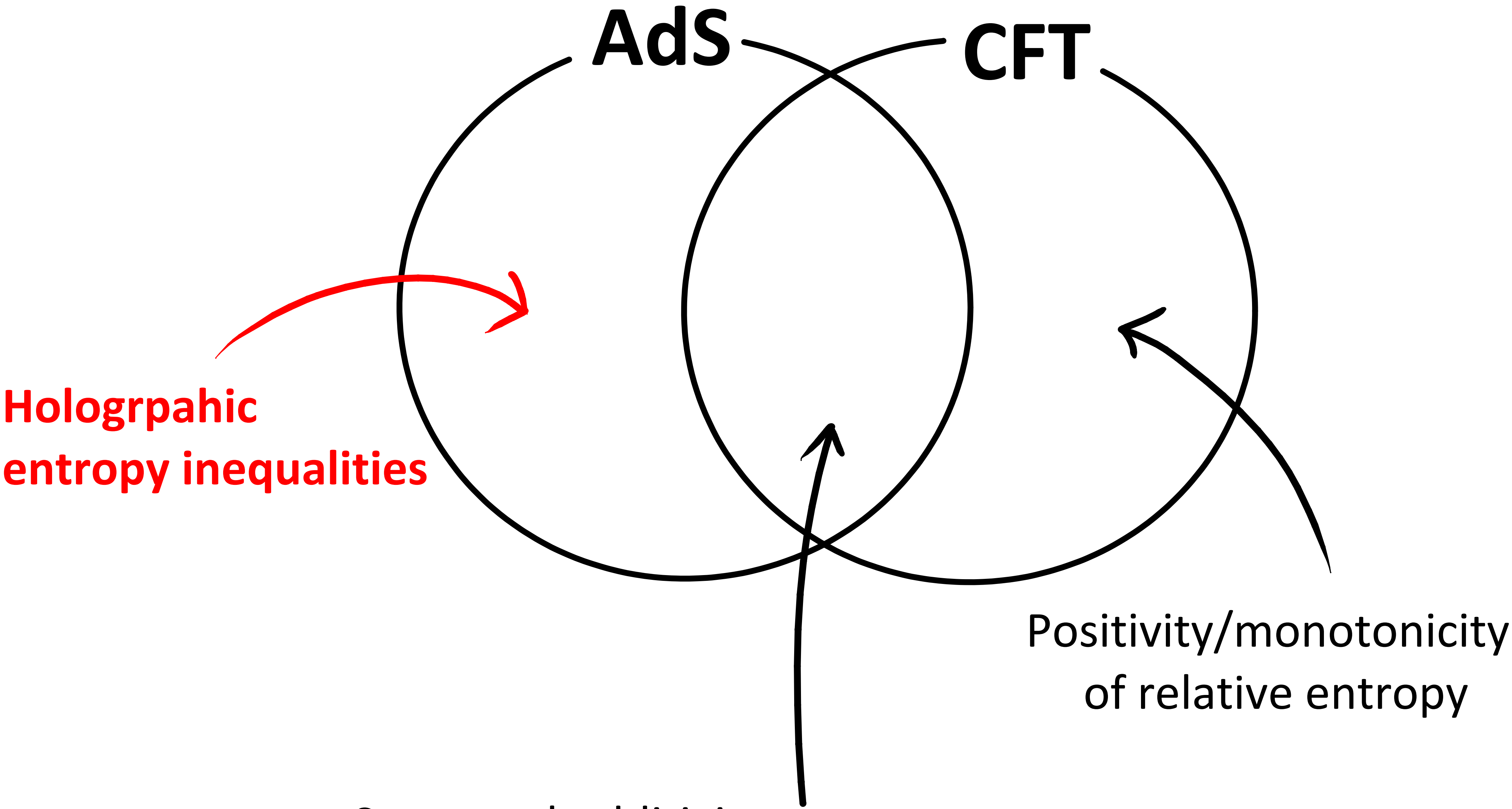


Entanglement entropies for n regions make a vector in $(2^n - 1)$ dimensions.

Entropy vectors of holographic states populate inside of a **convex rational polyhedral cone**.

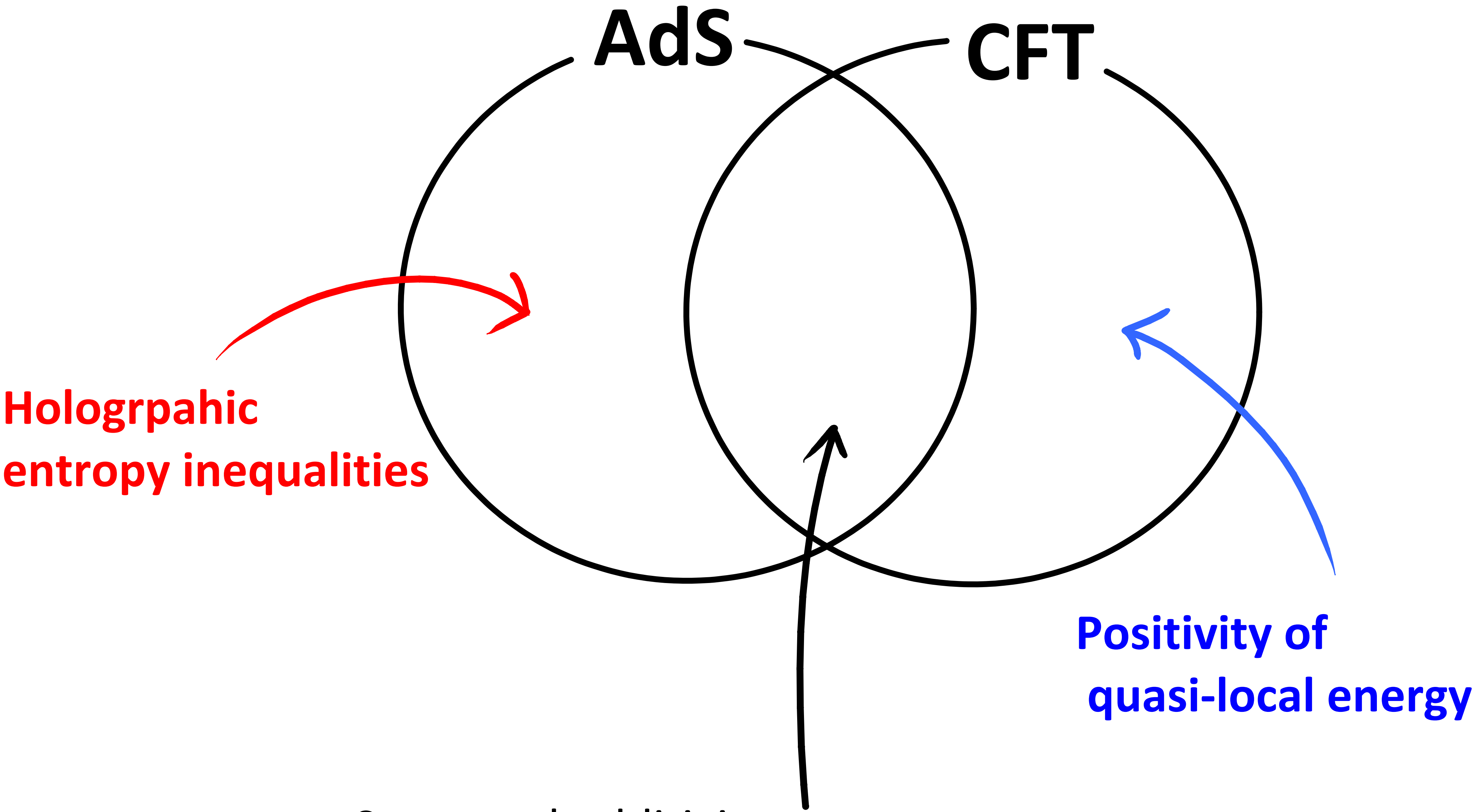
The number of independent inequalities is finite for each n.

Are these implied by the gap conditions on CFT?



Strong subadditivity:

$$S(AB) + S(BC) \geq S(B) + S(ABC)$$



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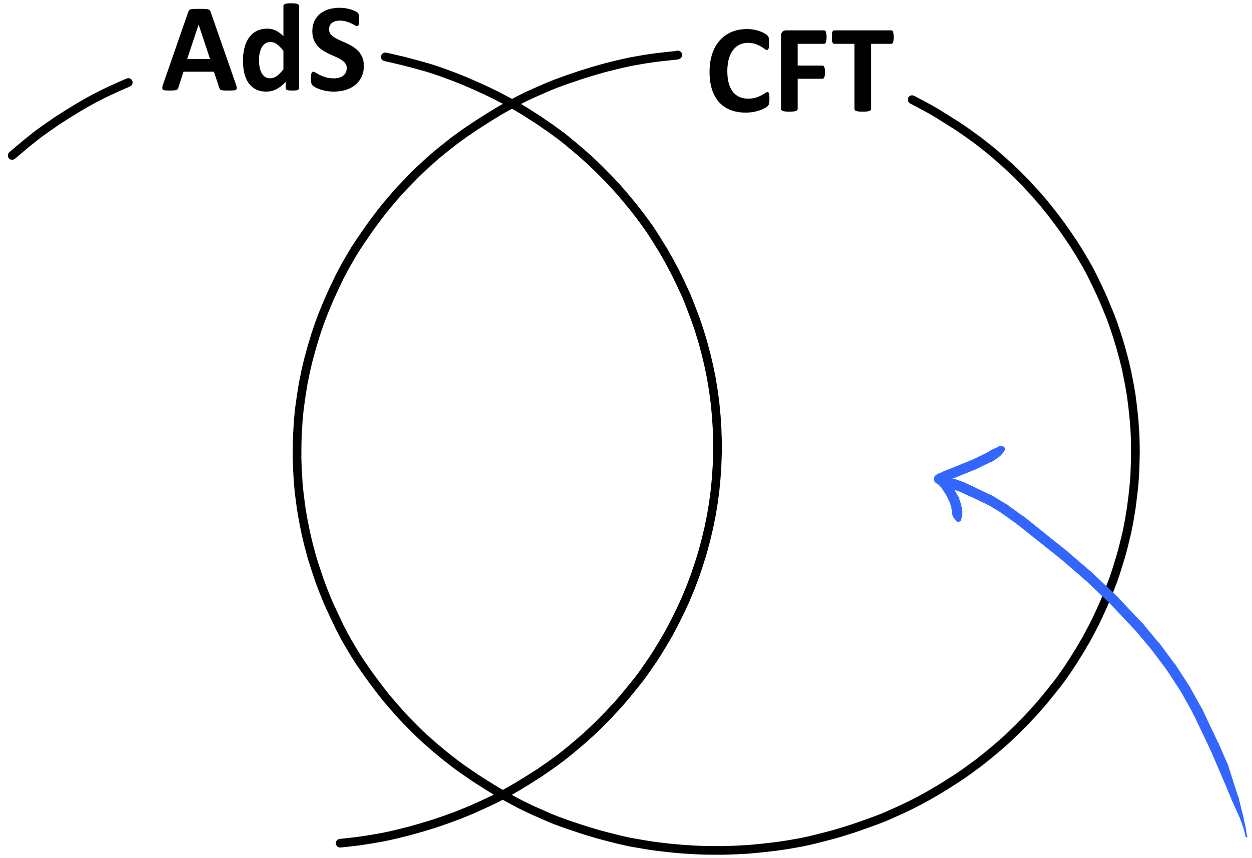

Positivity of
quasi-local energy

Delineate the Swampland by positive energy conditions.

Positivity of Quasi-Local Energy

Energy and Entropy

based on formalism
developed by Wald & collaborators

Σ : subspace of a Cauchy surface

We will choose Σ to be part of an asymptotically AdS geometry bounded by the Ryu-Takayanagi surface and the AdS boundary.

$\Sigma \subset$ Cauchy surface, g : metric + matter on Σ.

$L(g)$: Lagrangian density

$$\delta L(g) = d\,\theta(\delta g) + \text{e.o.m.}$$

$$\Downarrow$$

Symplectic form

$$\Omega(\delta_1 g, \delta_2 g)$$

$$= \int_\Sigma \omega(\delta_1 g, \delta_2 g)$$

$$\equiv \int_\Sigma \delta_1 \theta(\delta_2 g) - \delta_2 \theta(\delta_1 g)$$

Analogy :

$$L(Q) = \frac{1}{2}\left(\frac{dQ}{dt}\right)^2 - V(Q)$$

$$\delta L(Q) = \frac{d}{dt}\left(\frac{dQ}{dt}\delta Q\right) + \text{e.o.m.}$$

$$= \frac{d}{dt}\theta(\delta Q) + \text{e.o.m.}$$

$$\theta(\delta Q) = P\,\delta Q$$

$$\delta\theta = \delta P \wedge \delta Q$$

Hamiltonian H_ξ for a vector field ξ on Σ to generate $\mathcal{L}_\xi g$

$$\delta H_\xi = \Omega(\delta g, \mathcal{L}_\xi g)$$

$$= \int_\Sigma \delta\theta(\mathcal{L}_\xi g) - \mathcal{L}_\xi \theta(\delta g)$$

$$\left(\mathcal{L}_\xi \theta = \xi \cdot \underbrace{d\theta}_{\delta L + \text{e.o.m.}} + d(\xi \cdot \theta) \right)$$

$$= \int_\Sigma \delta\left(\theta(\mathcal{L}_\xi g) - \xi \cdot L\right)$$

$$- \oint_{\partial\Sigma} \xi \cdot \theta(\delta g)$$

Analogy :

$$\delta H = \delta p \frac{dQ}{dt} - \delta Q \frac{dP}{dt}$$

$$= \delta\left(p \frac{dQ}{dt}\right) \underbrace{- \frac{d}{dt}(p\,\delta Q)}_{\delta L + \text{e.o.m.}}$$

$$= \delta\left(p \frac{dQ}{dt} - L\right)$$

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← boundary terms are important in gravity.

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$$= \delta\left(p \frac{dQ}{dt} - L\right)$$

For a vector field ξ on Σ,

$$\delta H_\xi = \int_\Sigma \delta\left(\theta(\mathcal{L}_\xi g) - \xi \cdot L \right) - \oint_{\partial\Sigma} \xi \cdot \theta(\delta g).$$

If $\exists\, B$ on $\partial\Sigma$ such that $\xi \cdot \theta(\delta g) = \delta(\xi \cdot B)$,

$$H_\xi = \int_\Sigma J_\xi - \oint_{\partial\Sigma} \xi \cdot B$$

where $J_\xi = \theta(\mathcal{L}_\xi g) - \xi \cdot L$.

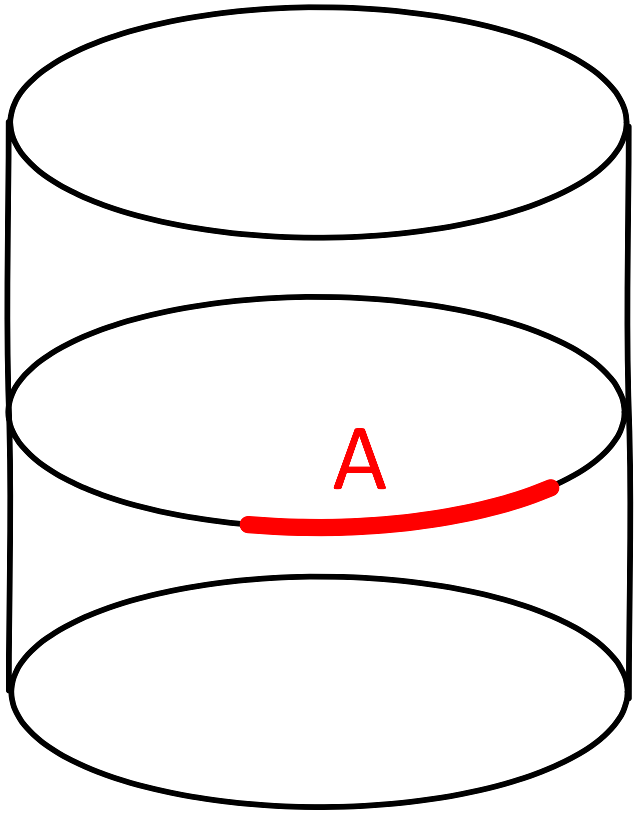
e.g. pure Einstein gravity,

$L = \frac{1}{2}(R - \Lambda)\, e$, $\quad e$: spacetime volume form

$\theta(\delta g) = \frac{1}{2}\left(g^{\mu\nu} D^\rho - g^{\nu\rho} D^\mu \right) \delta g_{\nu\rho}\, e_\mu$, $\quad e_\mu$: volume form on Σ

$B \propto$ extrinsic curvature (Gibbons-Hawking term)

Relative Entropy



$|\psi_0\rangle$: vacuum in CFT

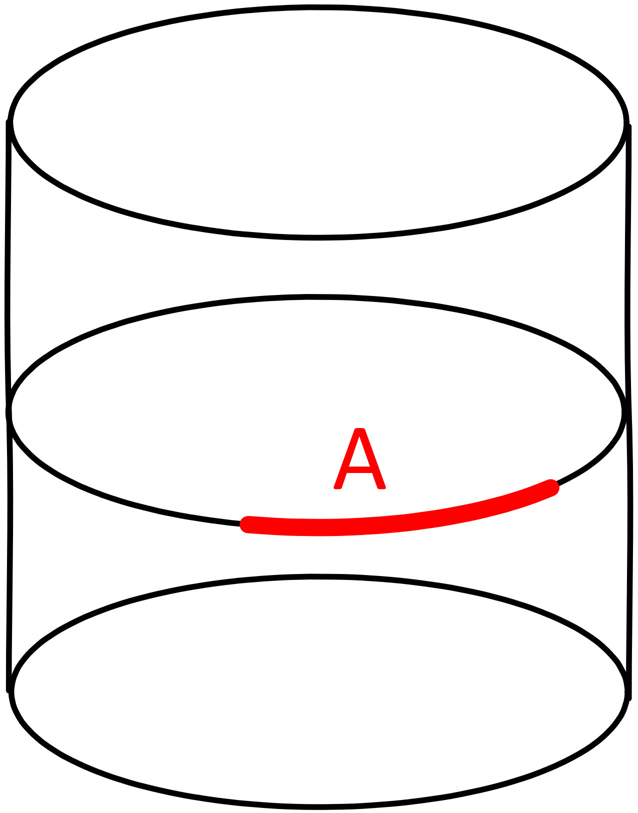
$\Leftrightarrow$ pure AdS geometry

$|\psi\rangle$: any CFT state

$\Leftrightarrow$ gravity solution

$$\rho_0 = \mathrm{tr}_{\bar{A}} |\psi_0\rangle\langle\psi_0|$$

$$\rho = \mathrm{tr}_{\bar{A}} |\psi\rangle\langle\psi|$$



Relative entropy :

$$S(\rho | \rho_0) = - \mathrm{tr}\,[\rho \log \rho_0] + \mathrm{tr}\,[\rho \log \rho]$$

measures the distance between

$$\rho_0 = \mathrm{tr}_{\bar{A}} |\psi_0\rangle\langle\psi_0|$$

$$\rho = \mathrm{tr}_{\bar{A}} |\psi\rangle\langle\psi|$$

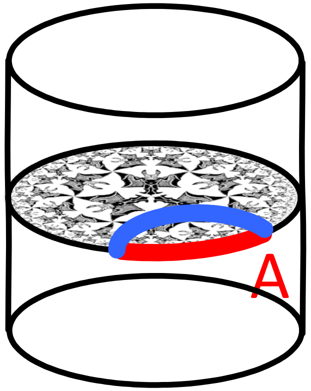
When A is a ball,

the modular Hamiltonian $= -\log \rho_0$ is simplified,

and $S(\rho | \rho_0)$ has a holographic expression.

Relative Entropy

$$S(\rho | \rho_0) = \underbrace{-\,\mathrm{tr}[\rho \log \rho_0]}_{=\ \langle \text{modular Hamiltonian} \rangle_\rho} + \underbrace{\mathrm{tr}[\rho \log \rho]}_{=\ -(\text{Entanglement Entropy})}$$



Metric asymptotics on A

Minimum surface area

Hamiltonian $\quad H_\xi = \int_\Sigma J_\xi - \oint_{\partial\Sigma} \xi \cdot B$

$dJ_\xi = 0$ by e.o.m. $\Rightarrow \exists\, Q_\xi, \; J_\xi = dQ_\xi$.

$$H_\xi = \oint_{\partial\Sigma} (Q_\xi - \xi \cdot B)$$

$\exists\, \xi$, such that

$$S(\rho | \rho_0) = H_\xi(\rho) - H_\xi(\rho_0).$$

Relative Entropy = Quasi-local Energy

$$S(\rho \,|\, \rho_0) = H_\xi(\rho) - H_\xi(\rho_0)$$

Since $S(\rho \,|\, \rho_0) \geq 0$,

$H_\xi(\rho)$ is bounded below by the vacuum energy.

For linear variation, $\rho = \rho_0 + \delta\rho$

$$S(\rho_0 + \delta\rho, \rho_0) = 0$$

implies the linearized Einstein equation in the bulk.

Faulkner, Guica, Hartman, Myers + Van Raamsdonk,
arXiv: 1312.7856

In the next leading order with backreaction from matters,

$$S(\rho|\rho_0) \geqq 0, \quad \frac{d}{dR} S(\rho|\rho_0) \geqq 0$$

↖ Radius of A

imply integrated positivity conditions on $T_{\mu\nu}$ of matters,

such as $$\int_\Sigma T_{00} \sqrt{g_\Sigma} \geqq 0$$

Lin, Marcolli, Stoica + H. O. arXiv: 1412.1879

Lashkari, Rabideau, Sabella-Garnier, Van Raamsdonk arXiv: 1412.3514

Using $S(\rho | \rho_0) = H_\xi(\rho) - H_\xi(\rho_0)$,

we can discuss holographic interpretions of

positivity : $S(\rho | \rho_0) \geqq 0$

monotonicity : $\frac{d}{dR} S(\rho | \rho_0) \geqq 0$

when ρ and ρ_0 are not necessarily close to each other.

$\Rightarrow$ next talk by Mark.

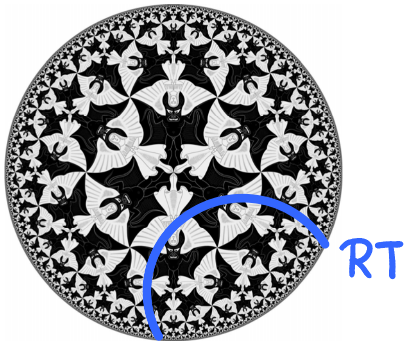
Lashkari, Lin, Stoica, van Raamsdonk + H.O.

arXiv:1605.01075

Radon transform

For a perturbation near AdS,

$$\frac{d}{dR}\left(\frac{d}{dR}+\frac{1}{R}\right) S(\rho\,|\,\rho_0) = 16\pi^2 \int_{RT} T_{00}$$



For AdS, this is invertible.

$\Rightarrow$ T_{00} is reconstructible from $S(\rho\,|\,\rho_0)$.

For a general solution, for some vectors v and τ,

$$\frac{d}{dR}\left(\frac{d}{dR}+\frac{1}{R}\right) S(\rho\,|\,\rho_0) = 2\pi \int_{RT} v\cdot\left(J_\tau - d(\tau\cdot B)\right)$$

Can we invert this?

A low energy effective theory of a consistent **ultraviolet complete quantum theory** of gravity must satisfy the **positive energy conditions** implied by the positivity and monotonicity of the relative entropy.

How strong are these conditions?

Which low energy theories are ruled out by them?

Do they follow from the bulk reconstruction argument?

Swampland Question:

How to characterize an effective gravity theory that can emerge in a low energy approximation to a consistent quantum theory, such as string theory.

Constraints on Symmetry

Constraints on Moduli Space

Constraints on Calabi-Yau Topology

Positive Energy Theorems