Space-time random tensor networks and holographic duality

Xiao-Liang Qi
Stanford University

YITP Kyoto University, 6/16/2016

Outline

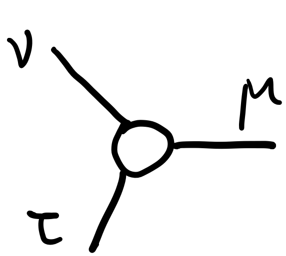
- Background: tensor networks and holographic duality
- Definition of space-time random tensor networks (RTN)
- Entanglement properties of space-time RTN
  --HRT formula with quantum corrections
  --Quantum error correction properties of the
    holographic mapping
- Gauge fixing and finite D correction

Reference

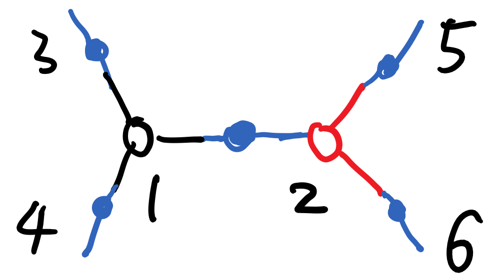
- Zhao Yang, XLQ, in preparation
- Closely related previous work on spatial RTN
  Patrick Hayden, Sepehr Nezami, XLQ, Nathaniel Thomas, Michael Walter, Zhao Yang, arxiv: 1601.01694

Tensor networks

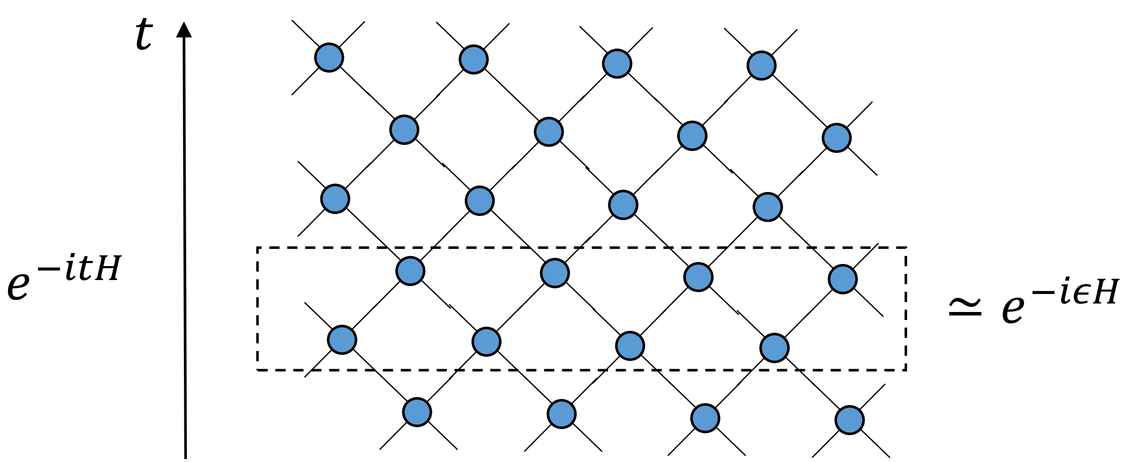
- Spatial tensor networks: many-body entangled states made from few-qubit states. (PEPS, F. Verstraete, J.I. Cirac, 04')



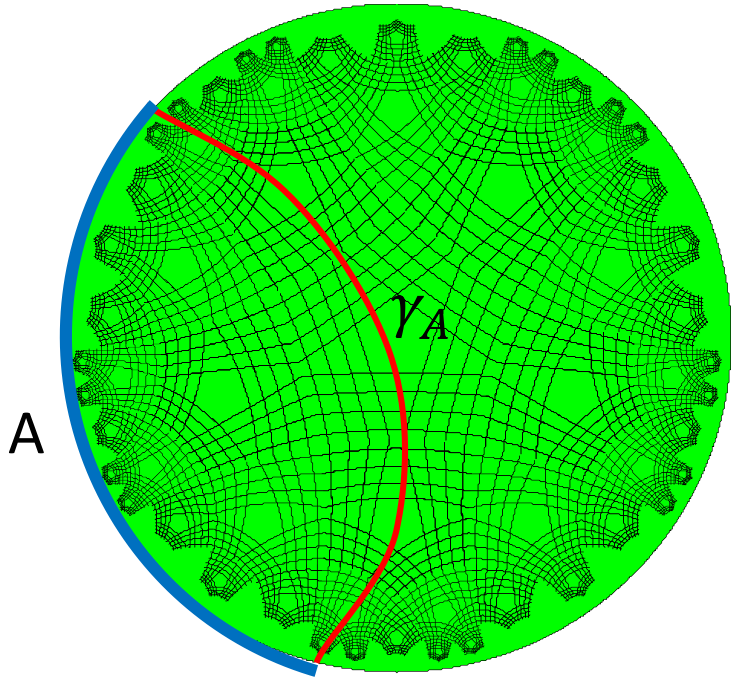
$$T_{1\mu\nu\tau}|\mu\nu\tau\rangle = |V_1\rangle$$

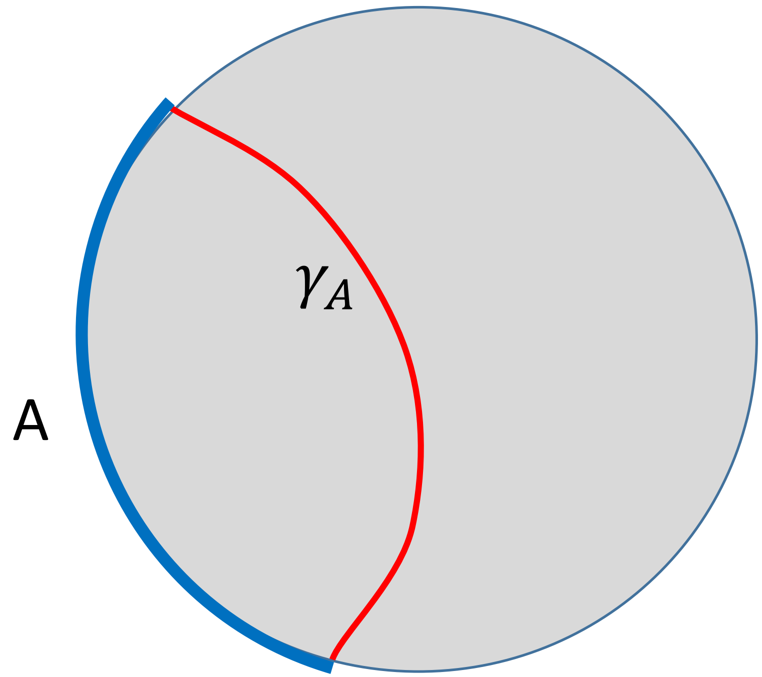


- Space-time tensor network: a discretization of path integral.



Tensor networks and holographic duality





Tensor networks

$$S \leq |\gamma_A| \log D$$

⟺

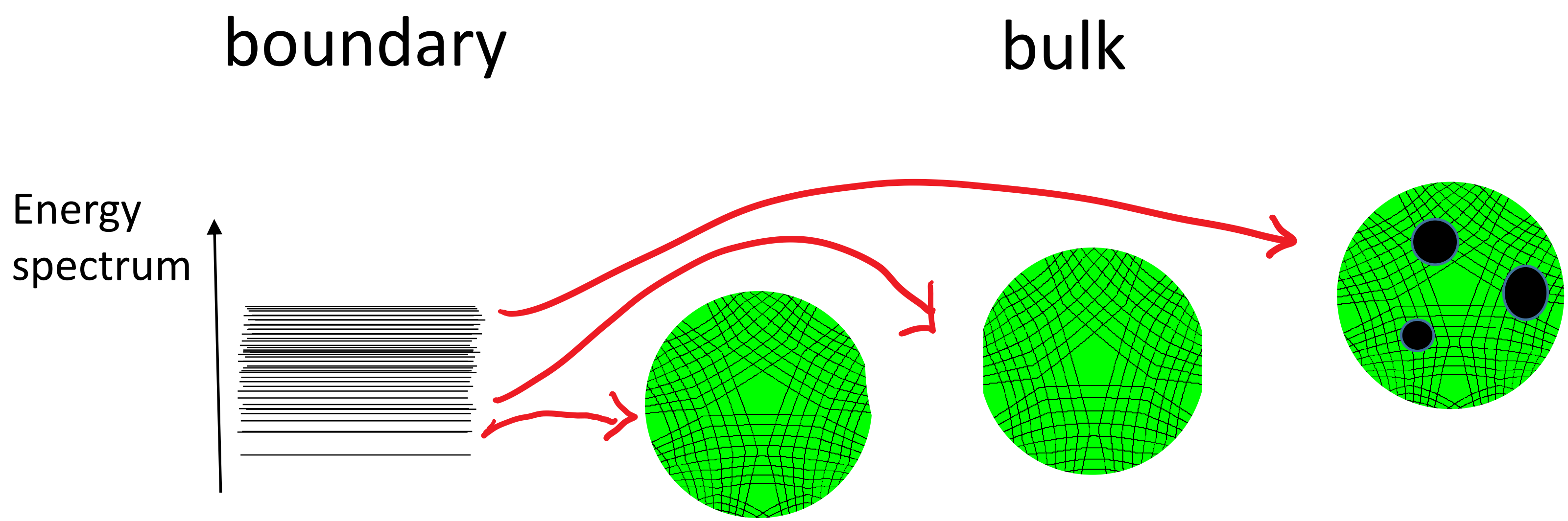
Holographic duality

Ryu-Takayanagi formula

$$S = \frac{1}{4\pi G} |\gamma_A|$$

Ryu&Takayanagi '06, Swingle '09

Tensor networks and holographic duality

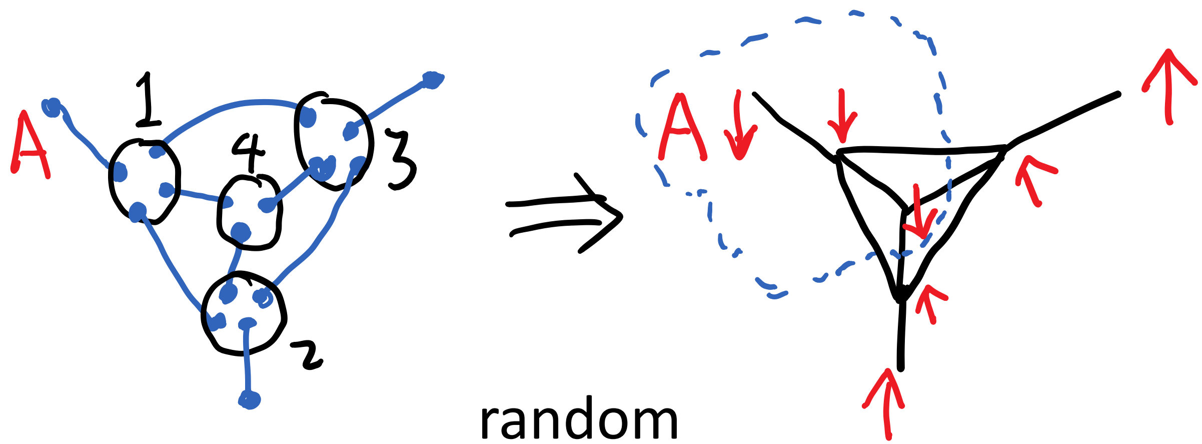


- The tensor network proposal: geometry emerges from the entanglement structure of quantum states
- Various tensor network proposals (Nozaki et al ’12, XLQ ’13, Pastawski et al ’15, Yang et al ’15, Hayden et al ‘16)
- Entanglement properties can be studied for tensors with good properties (Pastawski et al ‘15, Yang et al ‘15)

Random tensor networks

Hayden et al arxiv: 1601.01694

- Tensor networks with a random tensor at each vertex



n-th Renyi entropy $\xrightarrow{\text{random average}}$ Free energy of S^n spin model $T^{-1} = \log D$

- In $D \to \infty$ limit, many interesting properties emerge
  ---RT formula and its quantum correction
  ---Error correction properties of bulk-boundary operator correspondence
  ---Scaling dimension gap
- Problem: Need to pick a time slice. Hard to talk about dynamics. Solution: going to space-time networks.

Space-time tensor networks

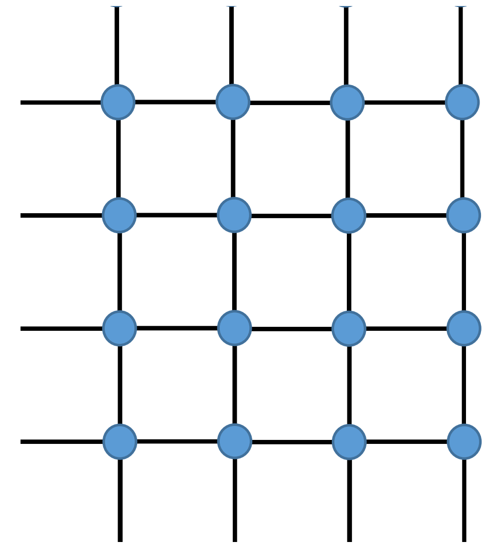
- Space-time tensor networks are discretized version of path integral.

- Or $Z = \int D\phi e^{-S(\phi)}$

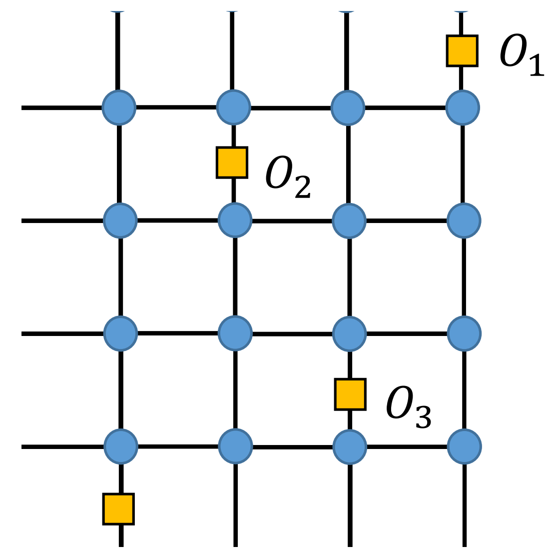
  $Z = \int D\phi e^{iS(\phi)}$

discretize

$Z =$

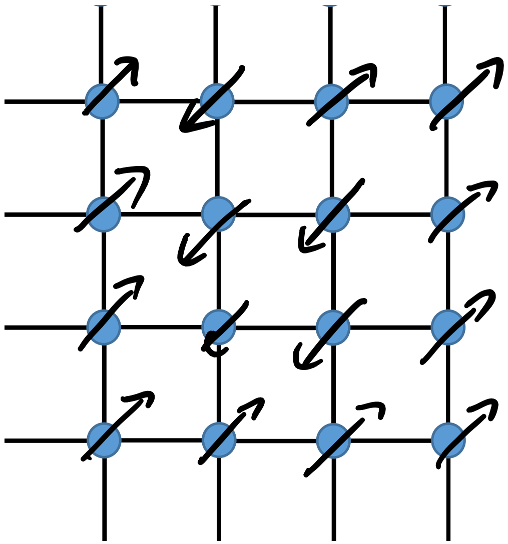


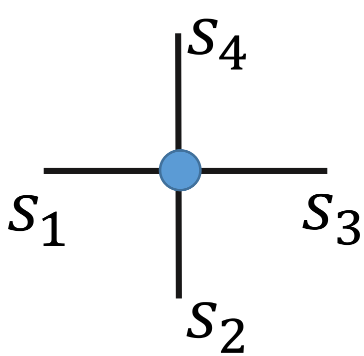
$\langle T O_1 O_2 O_3 \dots O_n \rangle = \frac{1}{Z} \times$



Space-time tensor networks

- Example: Ising model $S = -J\sum_{\langle xy\rangle} s_x s_y$

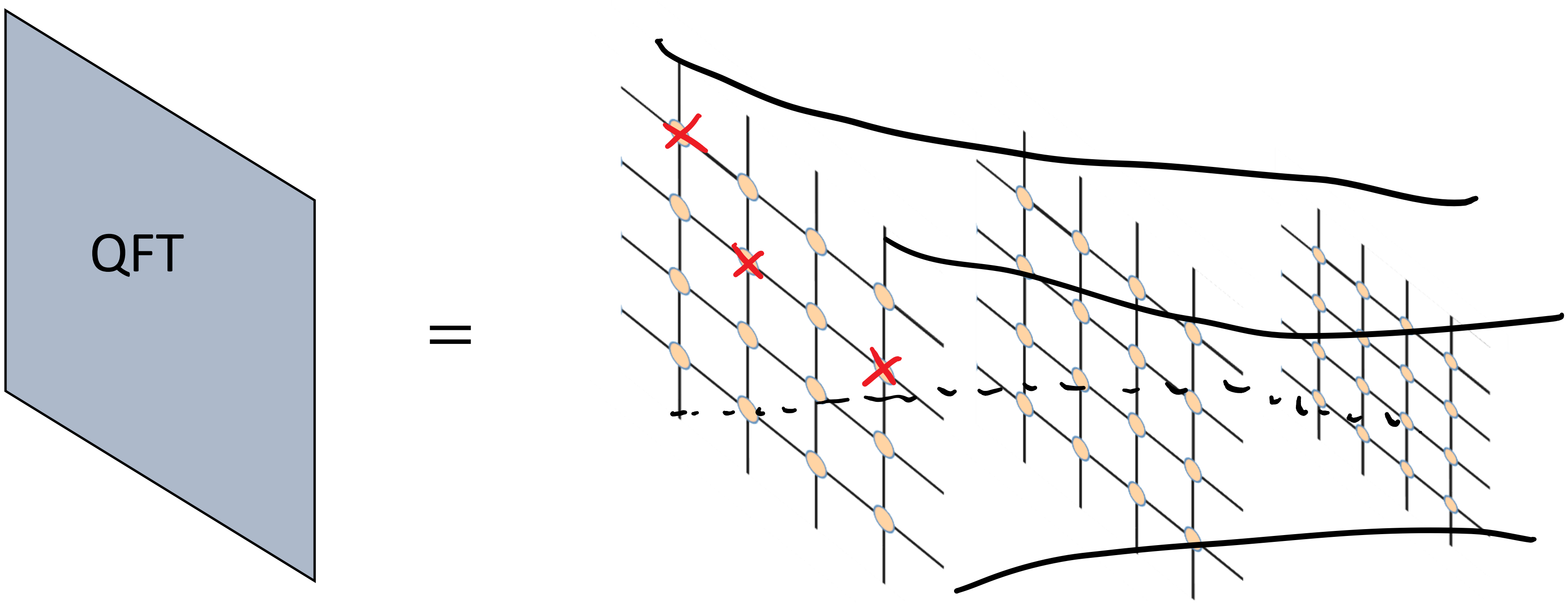
$Z =$ 



$$= \delta_{s_1 s_2} e^{J s_1 s_4 + J s_2 s_3}$$

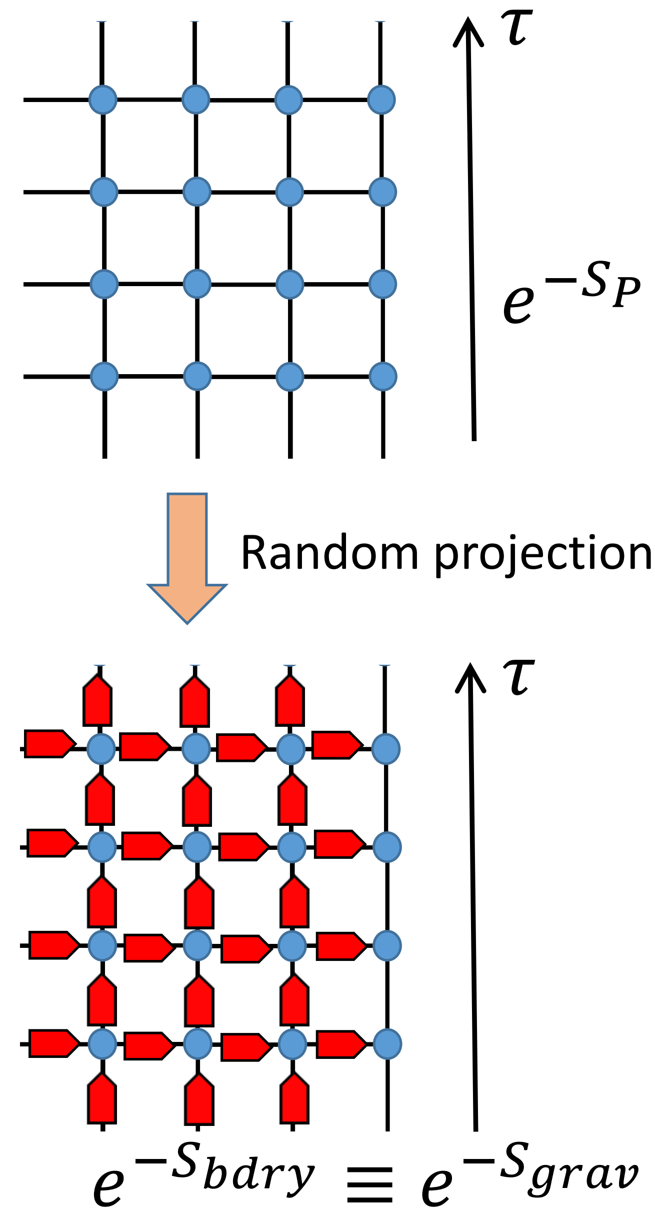
Space-time tensor network for holography

- **Wanted:**
  ---a tensor network representation of the $(d+1)$-dimensional bulk geometry
  ---The tensor network defines $Z_{bulk} = Z_{boundary}$, a d-dimensional QFT.

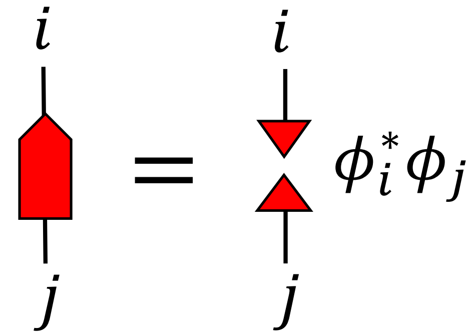


Space-time tensor network for holography

- **Our proposal**:
- 1. Introduce a space-time tenor network in the bulk
- This defines a bulk QFT, which we called the "*parent theory*". (Can be defined for either Euclidean or Lorentzian time)
- 2. Introduce a random projection at each bulk link
- **Key result: n-th Renyi entropy⇒ S^n discrete gauge theory**
- Subtlety: gauge fixing (will discuss later)

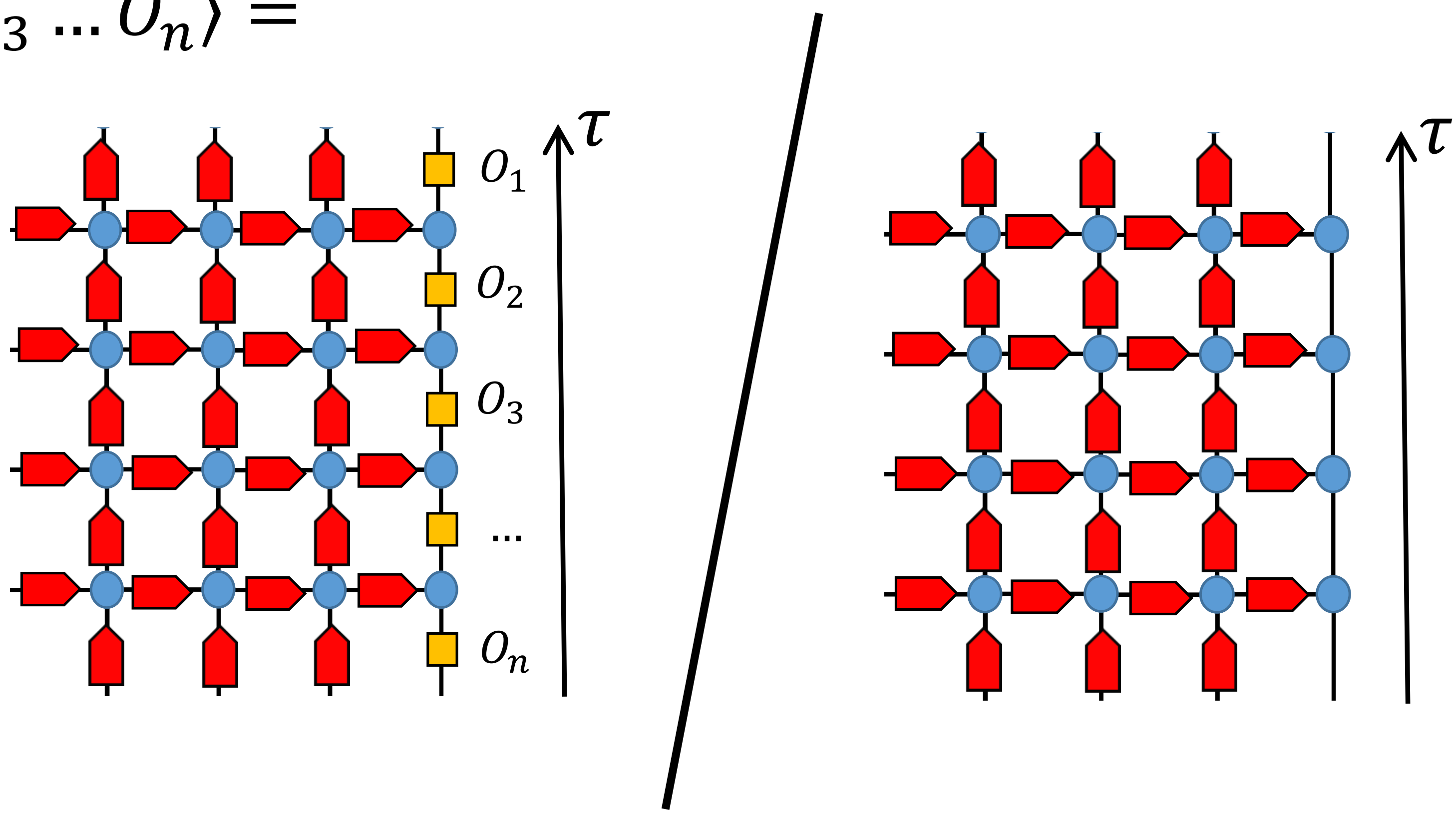


Holographic duality from space-time tensor networks



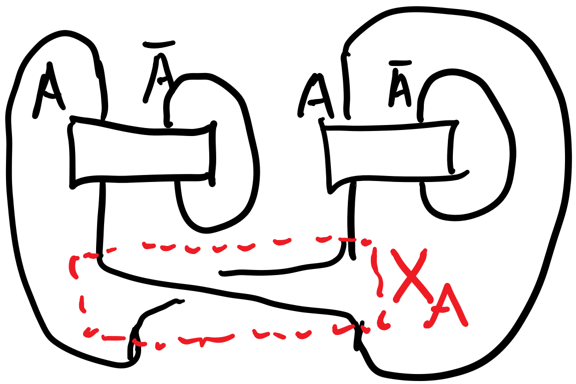
- Random projection at each link
- Correlation function

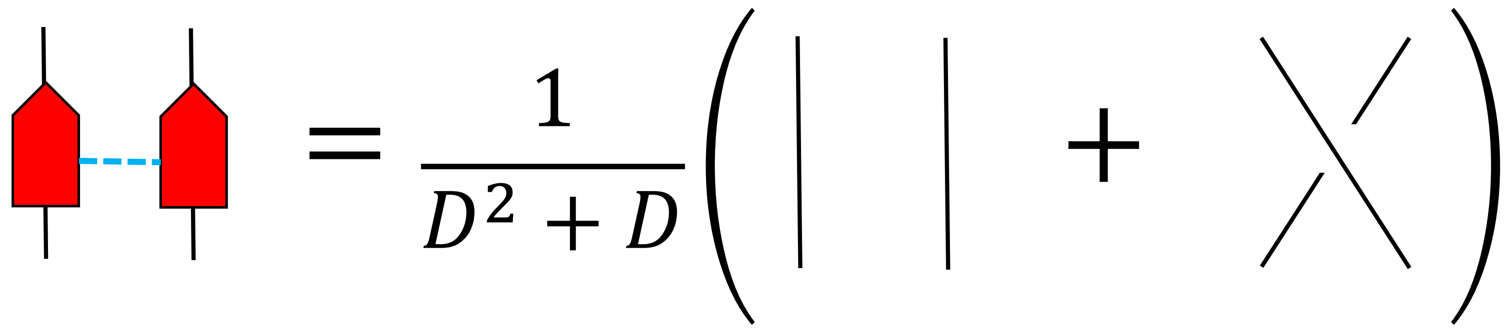
$\langle T O_1 O_2 O_3 \ldots O_n \rangle =$



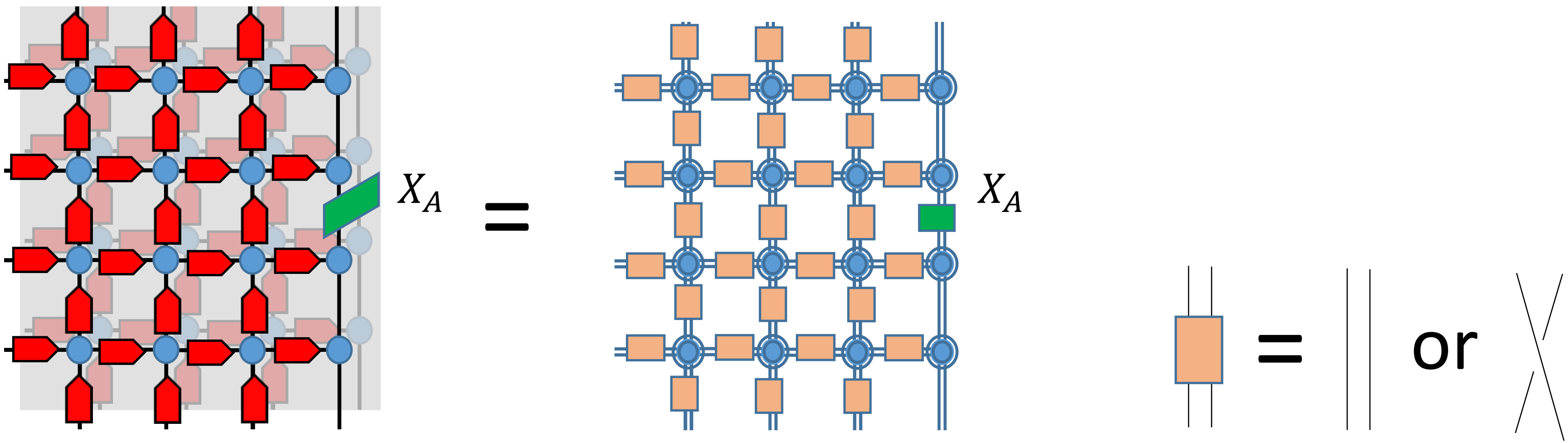
Second Renyi entropy of random tensor networks

- Second Renyi entropy definition. $tr\left(\rho_A^2\right) = tr(\rho \otimes \rho \cdot$





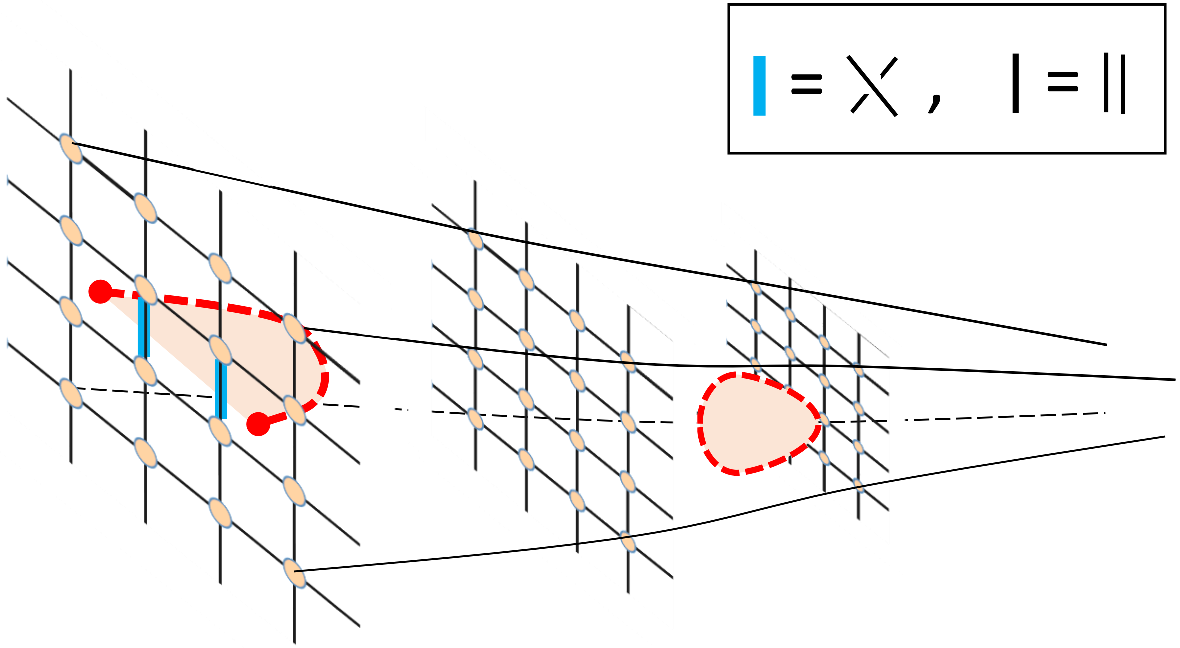
Second Renyi entropy



- The random average leads to the partition function of a Z_2 gauge theory.
- $\overline{tr\left(\rho_A^2\right)} = \sum_{g_{xy}=I\ or\ X} e^{-\mathcal{A}[g_{xy}]}$
- The choice of boundary region A determines the boundary condition of gauge field g_{xy}.
- Gauge invariance = permutation symmertry between two replica

Second Renyi entropy

- The Z_2 gauge field is minimally coupled to two copies of the parent theory

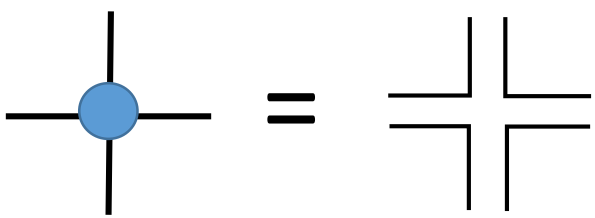


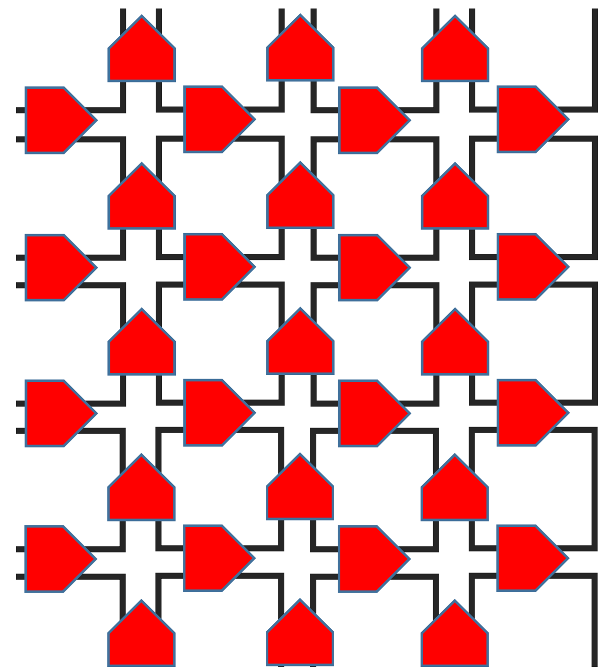
Second Renyi entropy

- For example, if the parent theory is an Ising model
- $S_P = -\sum_{\langle xy \rangle} s_x s_y$
- The random average gives ($\alpha = 1,2$)
- $S_{bulk}\left[s_x^\alpha, g_{xy}\right] = -\sum_{\langle xy \rangle} s_x^\alpha g_{xy}^{\alpha\beta} s_y^\beta$
- $[g_{xy}^{\alpha\beta}] = I$ or σ_x is the Z_2 connection
- Schematically, if we could take continuum limit,
- $e^{-\mathcal{A}_{eff}[g]} \equiv \int D\phi e^{-\mathcal{A}_P\left[\phi^1,\phi^2,g\right]}$
  (or corresponding form in real time)

HRT formula

- The dynamics of the gauge field is induced by integrating out the bulk "parent theory".
- Consider a particular example: the valence bond solid (VBS) state

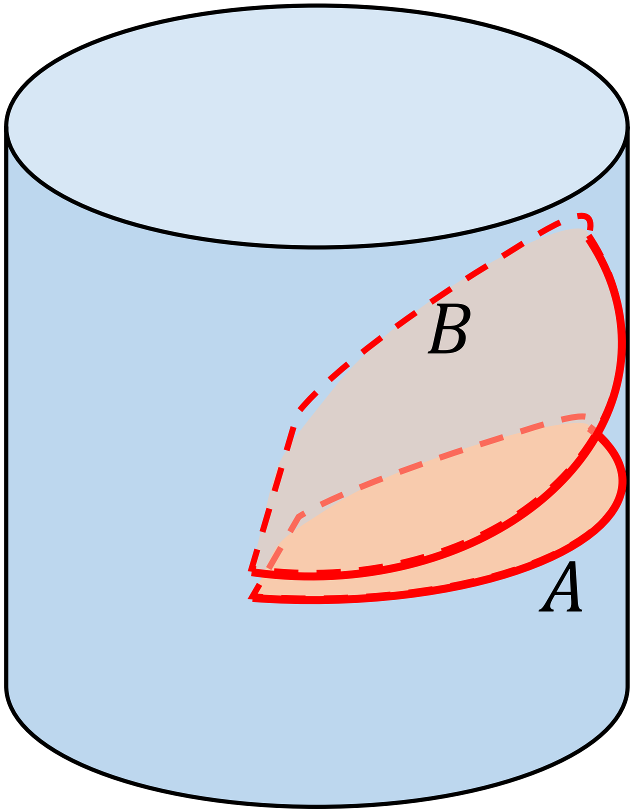




- Gauge field action: $S = -\frac{1}{2}\log D \sum_{p\in\square} F_p$
- $F_p = \prod_{\langle xy\rangle\in p} g_{xy}$ is the flux in plaquette p.

HRT formula

- For the VBS state, $Z = \sum_{\gamma=flux\ config.} e^{-\log D|\gamma|}$
- In large D limit, $Z \simeq e^{-\log D|\gamma_A|}$, and $S_2 \simeq \log D\,|\gamma_A|$
- $|\gamma_A| = \min|\gamma|$ is the area of minimal co-dimension-2 surface bounding A.
- In general, when the bulk parent theory is a massive theory, the effective action of gauge theory has a similar Maxwell form, leading to HRT formula (Hubeny Rangamani Takayanagi '07).

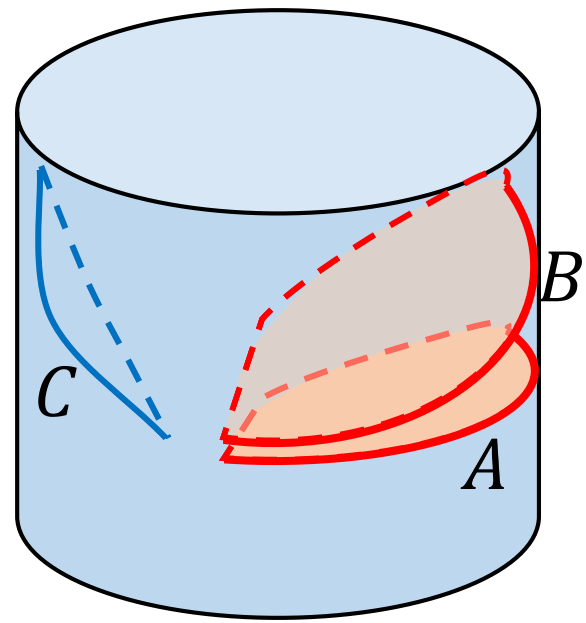


HRT formula for Lorentzian time

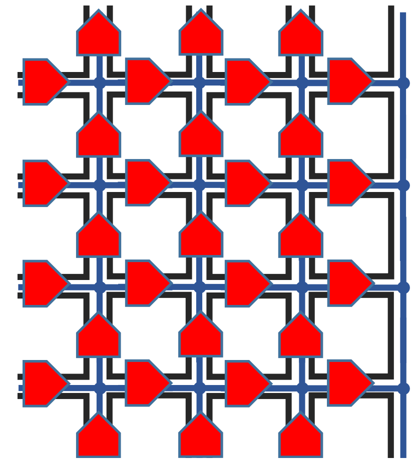
- The bulk parent theory can be defined with either Euclidean or Lorentzian signature.
- For Lorentzian bulk theory, the effective action of gauge field is also Lorentzian.
- Lorentzian Maxwell action of a Z_2 gauge theory (continuum limit)
- $S = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \rho_s \left(\partial_\mu \theta - 2A_\mu\right)(\partial^\mu \theta - 2A^\mu)$
- In the classical limit $(\frac{1}{g^2}, \rho_s \to \infty)$, $Z \simeq e^{i\alpha|\gamma_A|}$
- The area of saddle point surface $|\gamma_A|$ is imaginary (real) for surface with Euclidean (Lorentzian) signature. $\alpha \propto \log D$

HRT formula for Lorentzian time

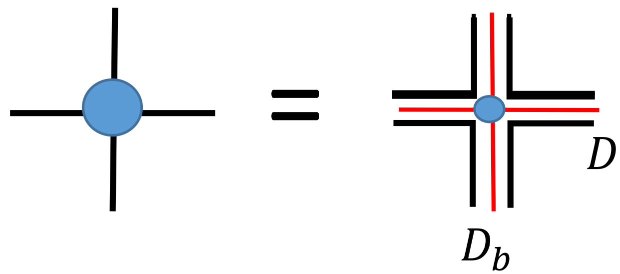
- In our approach, HRT surface can be defined even for time-like regions.
- Renyi entropies are generalized to multipoint functions of twist operators.
- $e^{-S_n(n-1)} = \langle T X_n(x_2, t_2) X_n(x_1, t_1) \rangle = \sum_\gamma e^{i\mathcal{A}(\gamma)}$
- HRT formula can be generalized *if* $\mathcal{A}(\gamma) \propto (n-1)|\gamma|$ *and if the saddle point approximation applies.*



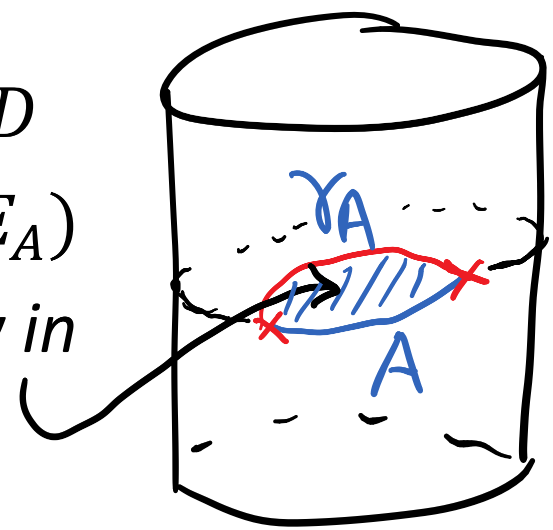
Quantum corrections to HRT



- Consider a bulk parent theory with VBS$\otimes$low energy field theory



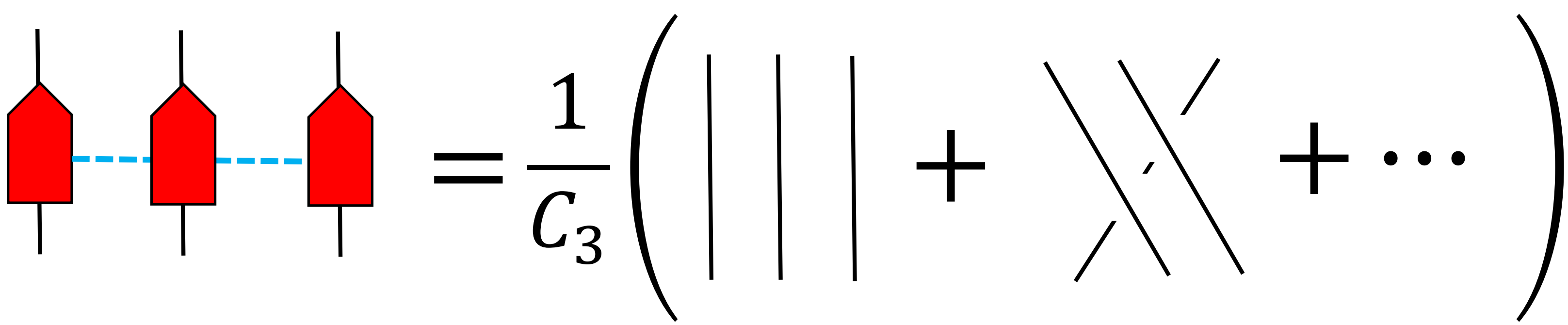
- Int. out VBS leads to $S = -\frac{1}{2}\log D \sum_p F_p$ for gauge field.
- Gauge field is coupled to the low energy theory with dimension $D_b \ll D$
- $S_2(A) \simeq \mathcal{A}_{cl} = \log D\,|\gamma_A| + S_{2bulk}(E_A)$
- *Both terms are entanglement entropy in the bulk*



Consistent with Faulkner et al '13

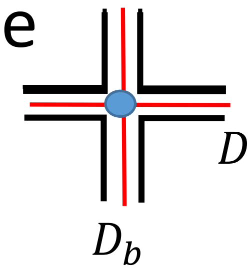
Generalization to higher Renyi entropies

- The random average can be generalized to higher Renyi entropies $tr(\rho_A^n) = tr\left(\rho^{\otimes n} X_{An}\right)$
- $\overline{\phi_{i_1}^* \phi_{j_1} \phi_{i_2}^* \phi_{j_2} \dots \phi_{i_n}^* \phi_{j_n}} = \frac{1}{C_n} \sum_{g \in S^n} g_{i_1 i_2 \dots i_n}^{j_1 j_2 \dots j_n}$
- Summation over all permutation group elements



Generalization to higher Renyi entropies

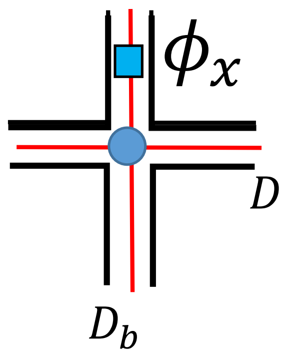
- Random average of n copies of the network ➔ S^n gauge theory minimal coupled to n replica of the parent theory

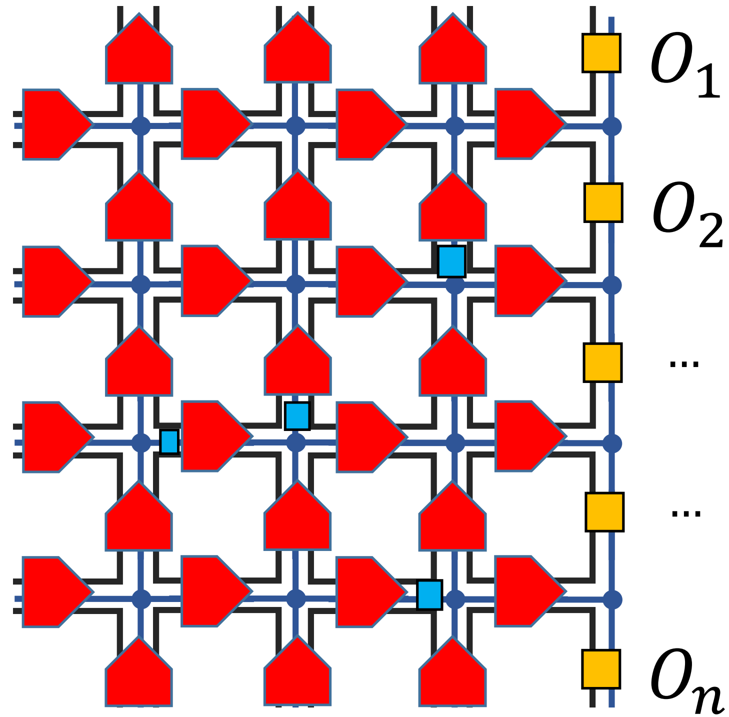


- $e^{-\mathcal{A}[g^{(n)}]} \equiv \int D\phi^{(i)} e^{-\mathcal{A}_P[\phi^{(1)},\phi^{(2)},\ldots,\phi^{(n)},g^{(n)}]}$
- For short-range entangled states such as VBS, in large D limit one obtains
$$S_n \simeq \frac{1}{n-1}\mathcal{A}_{cl} = \log D\,|\gamma_A| + S_{n\,bulk}(E_A)$$
with $S_{n\,bulk}(E_A)$ the bulk low energy field contribution
- Leading order Renyi entropies are n independent, different from CFT results. Reason: absence of back reaction. (c.f. Xi Dong 1601.06788)

Operator correspondence

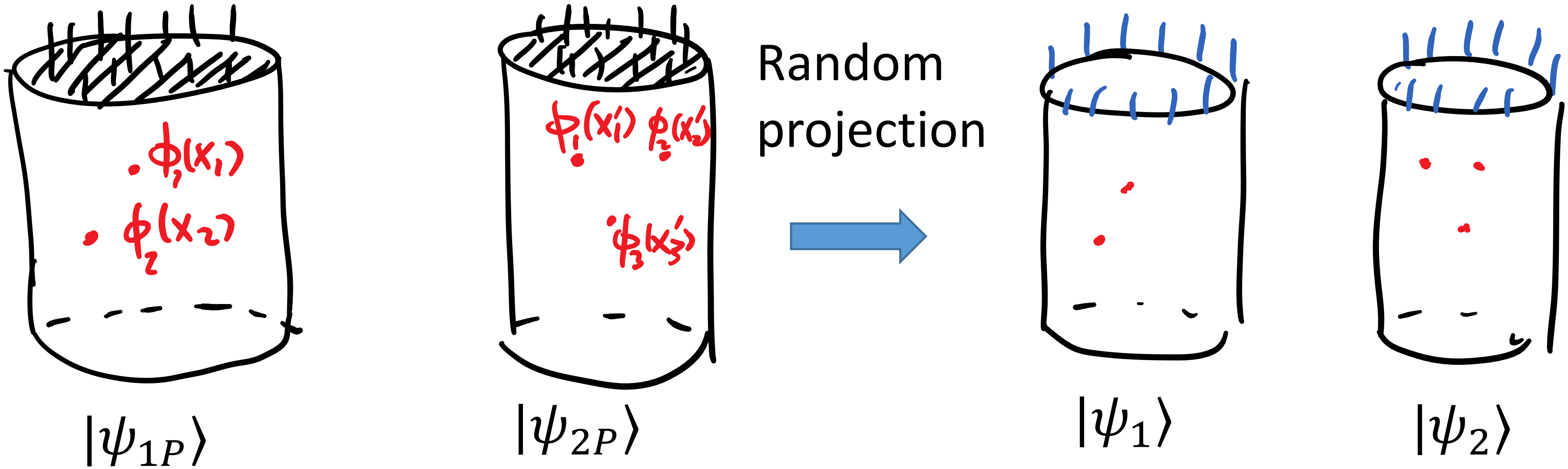
- Definition of Bulk local operators:
- Bulk operators ϕ_x *acting on low energy subspace* have nontrivial effect to the boundary after random projection.





- Does each bulk *low energy* operator correspond to a boundary operator? If yes, can the bulk operator be represented in a region of the boundary?

Operator correspondence

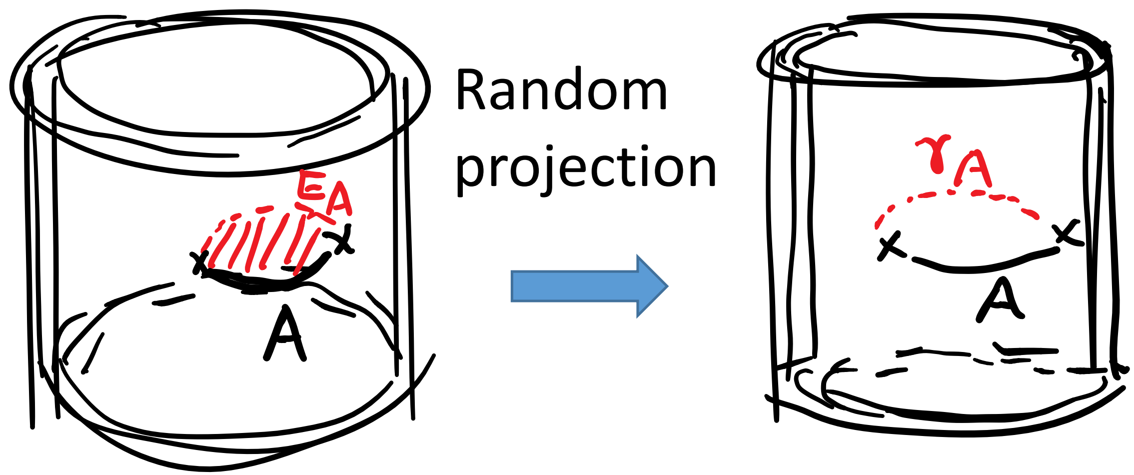


- Different operators create different states in the parent theory $|\psi_{1P}\rangle = \prod \phi_i(x_i, t_i)|G\rangle$
- Correspondingly there are different boundary states after the random projection.
- Generically each state gives different gauge field action.

Error correction properties

- A "code subspace" can be defined as the subspace of bulk states with the same gauge field action (i.e. quantum corrections to HRT can be ignored)
- For such states in $D \to \infty$, $tr_P(\rho_{1P}(E_A)\rho_{2P}(E_A)^n) = tr(\rho_1(A)\rho_2(A)^n)\ \forall n. \Rightarrow S(\rho_{1P}(E_A)|\rho_{2P}(E_A)) = S(\rho_1(E_A)|\rho_2(E_A))$

$\Rightarrow$ (Jaffer et al '15, Harlow et al '16)



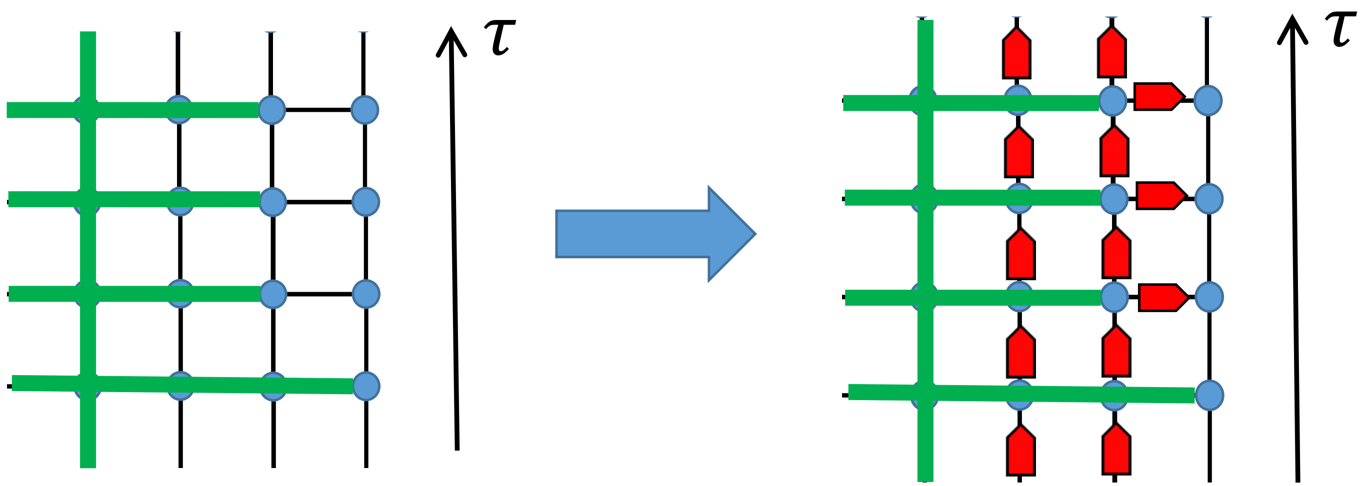
- For each bulk operator ϕ acting in E_A, $\exists$ boundary operator O_A, s.t. $\langle\psi_1|O_A|\psi_2\rangle = \langle\psi_{1bulk}|\phi|\psi_{2bulk}\rangle$ for any two states in the code subspace.

Gauge redundancy and gauge fixing

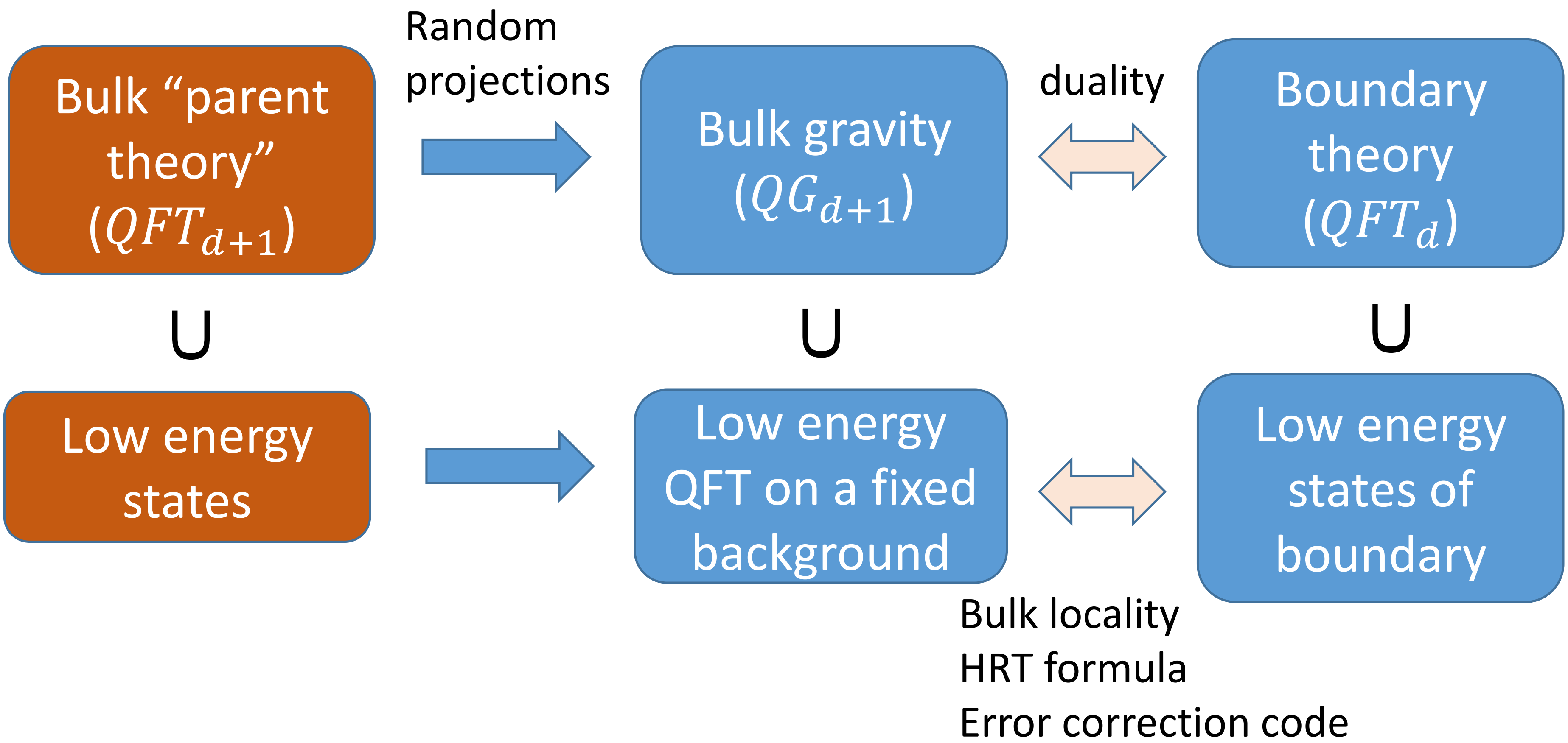
- The random average leads to a gauge field partition function with gauge redundancy $\overline{\mathrm{Z_n(A)}} = \sum_{\{g_{xy}\in S^n\}} e^{-\mathcal{A}[g_{xy}]} = Z_{S^n}(A)\cdot\Omega$
- Ω is the volume of gauge orbit.
- Not a problem if we calculate ratios such as $\frac{\overline{Z_n(A)}}{\overline{Z_n(\emptyset)}}$
- However, problem happens when we consider fluctuations:
- $\overline{\delta Z_2^2} = \overline{tr(\rho^2)^2} - \overline{tr(\rho^2)}^2 = Z_4(\emptyset) - Z_2(\emptyset)^2 = \Omega_4 Z_{S^4}(\emptyset) - \Omega_2^2 Z_{S^2}(\emptyset)^2$
- The fluctuation is large because $\Omega_4 \gg \Omega_2^2$. We need to remove the gauge redundancy.

Gauge redundancy and gauge fixing

- For discrete gauge theories, gauge fixing can be done by directly fixing g_{xy} on some links, without constraining any gauge flux.
- This corresponds to picking a spanning tree in the bulk, and only impose random projection on links off the tree.
- Fluctuations are suppressed in large D limit after gauge fixing. (Similar to spatial RTN Hayden et al '16)



Summary



- "Low energy states" actually means states that share a common entanglement structure in large D limit.

Open questions

- How to start from the boundary and construct the bulk theory?
- How to take into account of the back reaction and describe correct entanglement properties of CFTs?
- How to obtain the bulk geometry equation (Einstein equation)?
- A formalism in the continuum limit?
- Does the space-time RTN helps us to understand of the Black hole information paradox?
- Does this approach allow us to define holography in flat and positively curved space?

More details about finite D fluctuations

- If we find an upper bound for

$$\frac{\overline{Z_{nA}^2}}{e^{-\mathcal{A}_A^{(2n)}}} - 1 \leq f(D)$$

and $f(D) \ll 1$ in the large D limit, then in the limit $f(D) \ll 1$,

$$\text{Prob}\left(\left|S_n(A) - S_n^{RT}(A)\right| \leq \delta\right) \geq 1 - \frac{32}{\delta^2} f(D)$$

- $f(D) = \frac{e^{\lambda\Omega}}{D}$ gives a very loose bound that can be prooved.
- $f(D) = \frac{\Omega}{D^{\frac{\Delta_{2n}}{2}}}$ gives a better bound that’s less rigorous.
- Ω is spacetime volume.