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Covariant Holographic Entanglement: A Derivation

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Xi Dong, Aitor Lewkowycz, MR
to appear….

Introduction

- The holographic entanglement entropy prescription of Ryu-Takayanagi relates geometric data in the bulk with quantum features on the boundary

Ryu, Takayanagi ‘06

- Motivated by analogy between entanglement entropy and Bekenstein-Hawking formula.
- Derived by invoking the basic entry in the AdS/CFT dictionary: mapping of boundary and bulk path integrals and evaluate the latter in the semiclassical saddle point approximation. Lewkowycz, Maldacena ‘13
- Derivation makes clear how the bulk gravity dynamics picks out the minimal surface prescription.
- Gives us insight into stringy/quantum corrections.
- Relates nicely bulk and boundary relative entropies.

Faulkner, Lewkowycz, Maldacena ‘ 13; Jafferis, Lewkowycz, Maldacena, Suh ‘16

Covariant Holographic Entanglement Entropy

- Given the boundary region $\mathcal{A}$ the prescription to compute entanglement holographically involves finding a bulk extremal surface $\mathcal{E}_\mathcal{A}$ which is anchored on $\partial\mathcal{A}$ and is homologous to $\mathcal{A}$.

$$S_\mathcal{A} = \frac{\text{Area}(\mathcal{E}_\mathcal{A})}{4G_N}$$

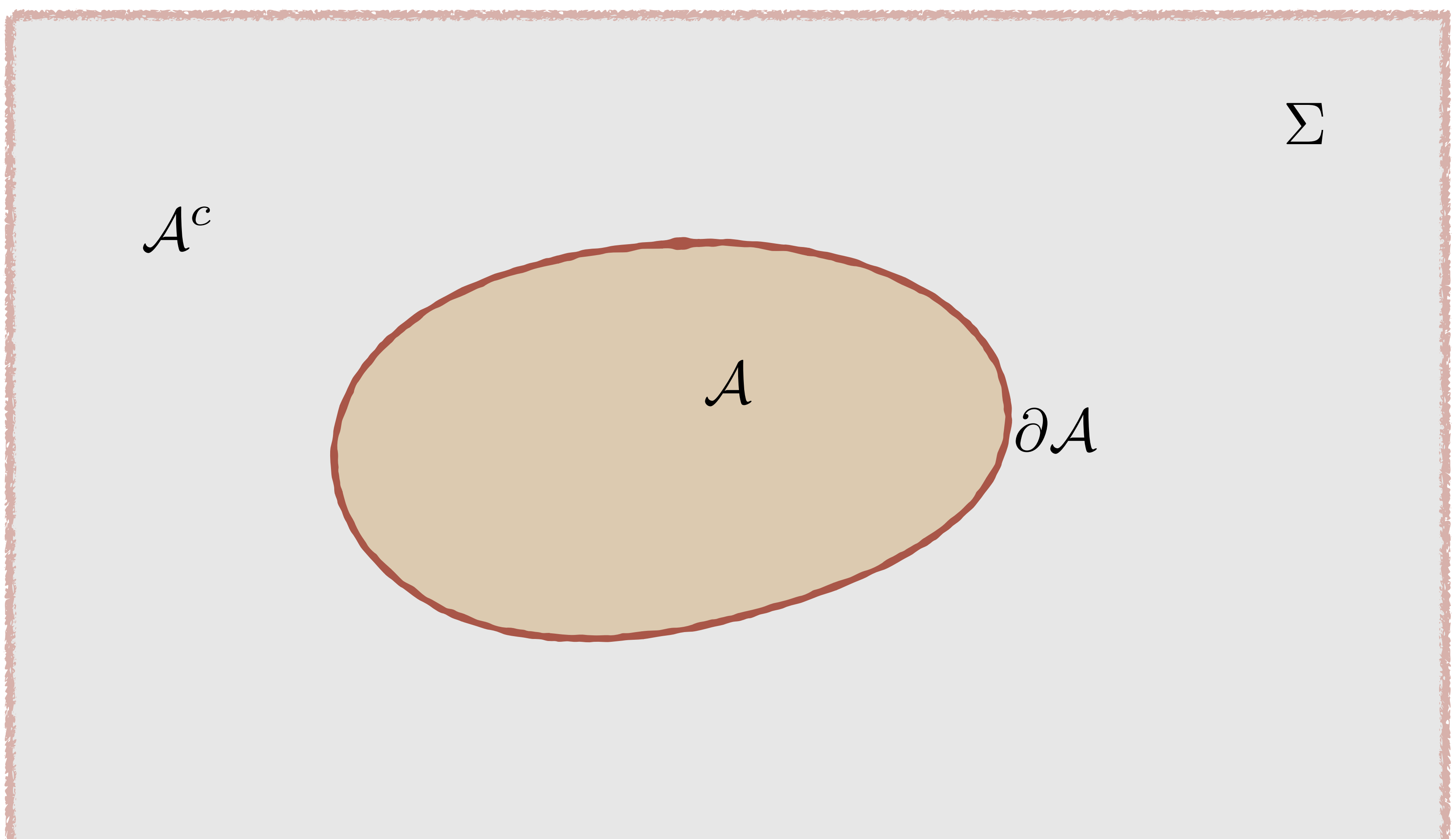
- The extremal surface $\mathcal{E}_\mathcal{A}$ is a codimension-2 surface in the bulk asymptotically AdS spacetime $\mathcal{M}$ (nb: $\partial\mathcal{M} = \mathcal{B}$)

Hubeny, MR, Takayanagi '07

- Motivated by using bulk covariance as a guiding principle.
- Covariance does not pick out a unique prescription, but supplemented with intuition arising from covariant entropy bounds we land on the extremal surface prescription.
- Various consistency checks; an attempt at a derivation using Lorentzian AdS/CFT.

Entanglement in QFT

- Consider a QFT in a density matrix, living on a background $\mathcal{B}$ which is globally hyperbolic spacetime with a nice time foliation (Cauchy slices Σ).
- $\mathcal{A}$ is a subregion of the Cauchy slice, with an *entangling surface* $\partial\mathcal{A}$.

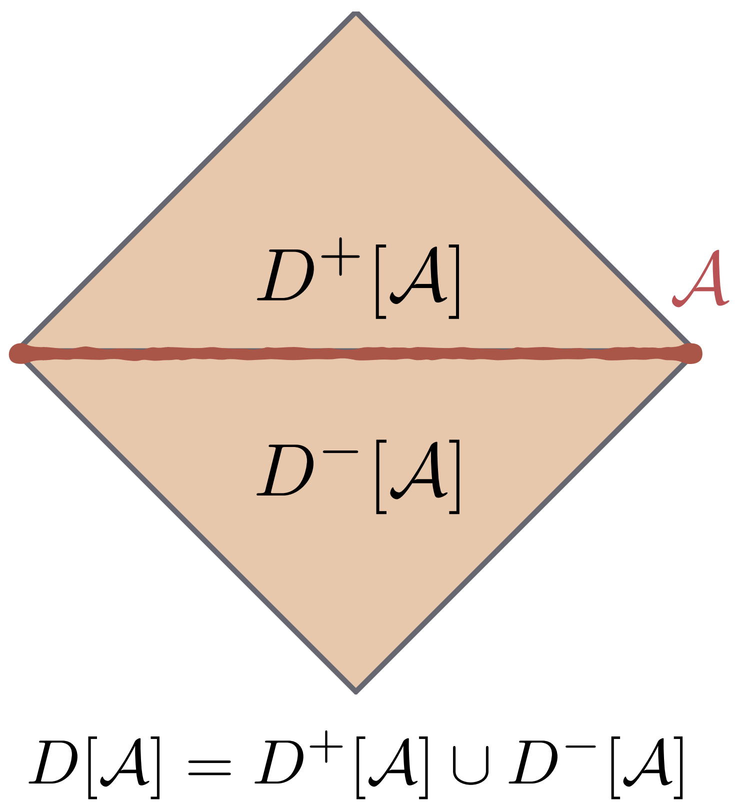


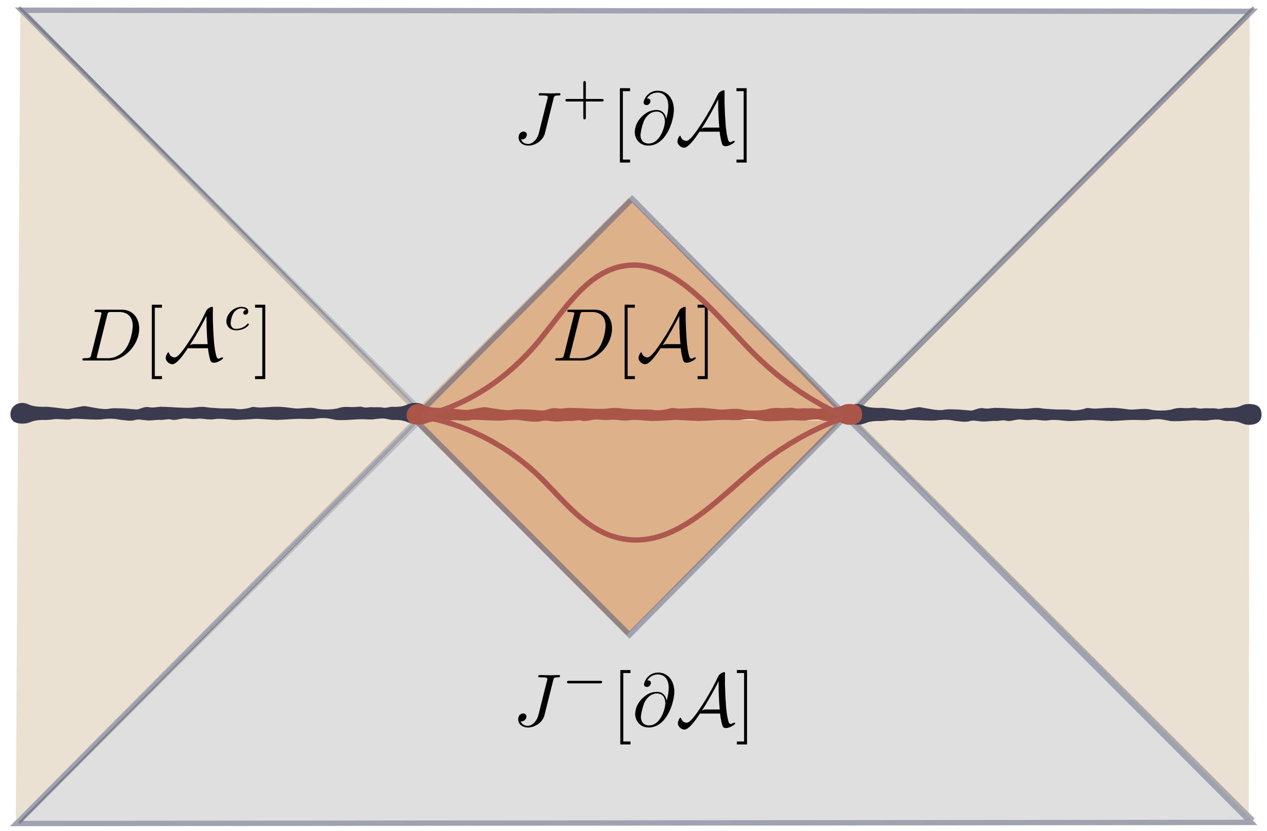
$\rho_{\mathcal{A}}$ reduced density matrix

$$S_{\mathcal{A}} = -\mathrm{Tr}\left(\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}\right)$$

Causality and Entanglement

✦ Entanglement entropy in QFT is a *wedge observable*.





✦ The entanglement entropy can only be influenced by changing conditions in the past domain $J^-[\partial\mathcal{A}]$.

Implementing causality in the path integral

- The real time path integral which is cognizant of the causality constraints invokes the Schwinger-Keldysh formalism.

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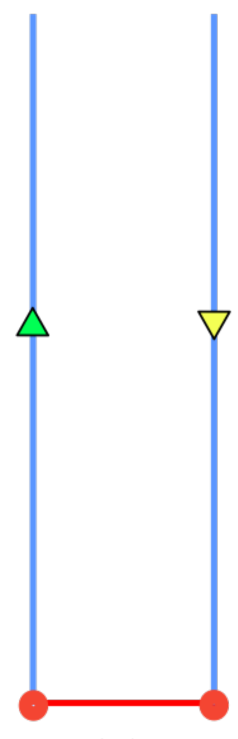

- We compute the matrix elements of the total density matrix at a given time by evolving forward from the initial state, to the instant of interest and thence retrace our footsteps to the far past.

- This ensures that sources inserted in the future do not affect the operator of interest.

- The forward and backward evolutions are glued together on some Cauchy slice Σ_t .

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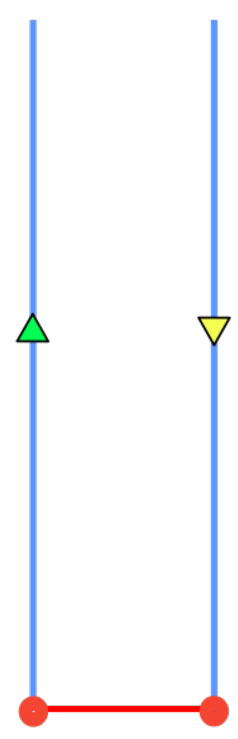

$$\rho = | \psi \rangle \langle \psi | \to e^{-iHt} \, | \psi \rangle \langle \psi | \, e^{iHt}$$

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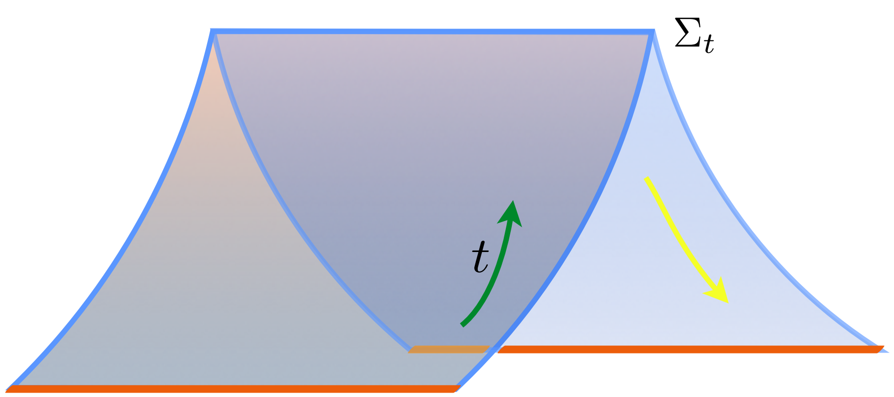

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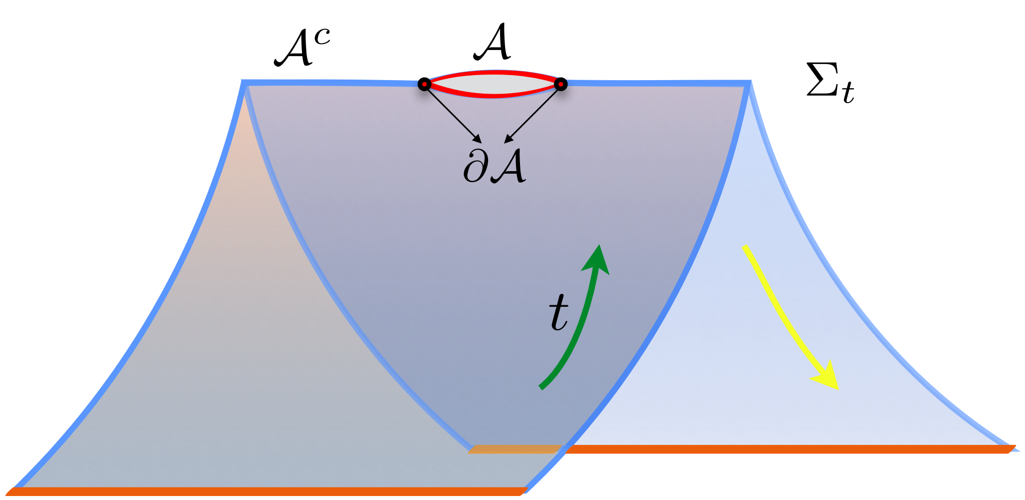
The reduced density matrix elements

- This general prescription can be minimally modified to obtain the reduced density matrix elements.

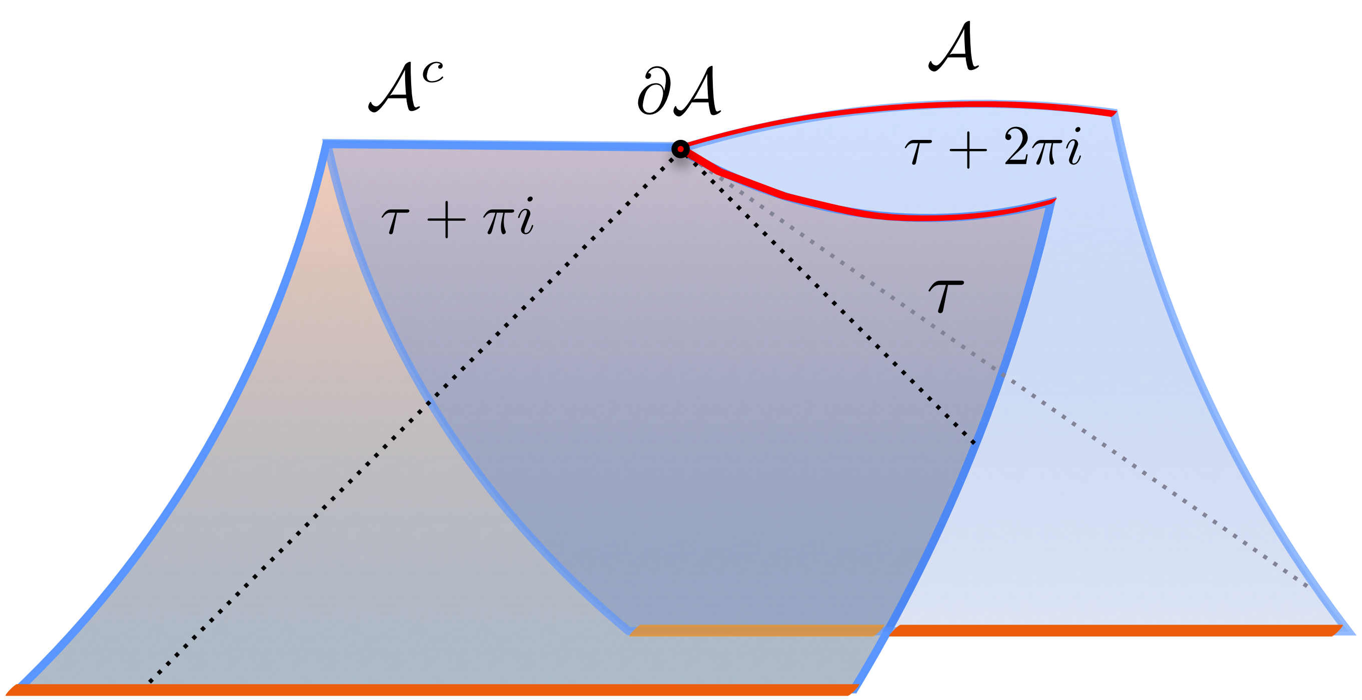


- We cut open the path integral along the region $\mathcal{A}$ on the Cauchy slice Σ_t .
- Imposing suitable boundary conditions in the future/past segments leads to the matrix elements of the reduced density matrix $(\rho_{\mathcal{A}})_\pm$.

- Knowing the reduced density matrix elements we can then attempt to compute the von Neumann entropy by the replica trick.

Local Rindler structure

- In what follows, it will be useful to pay attention to the local geometry near the entangling surface.



- We can adapt local Rindler like coordinates in the neighbourhood, which also allows for a suitable complexification.

Out-of-time-ordered Schwinger-Keldysh

- To compute powers of the reduced density matrix we need a slight generalization of the Schwinger-Keldysh contour.
- Consider computing not time-ordered, but out-of-time ordered correlators.
- These can be generated by suitably stringing together Schwinger-Keldysh contours, with multiple switchbacks or timefolds.

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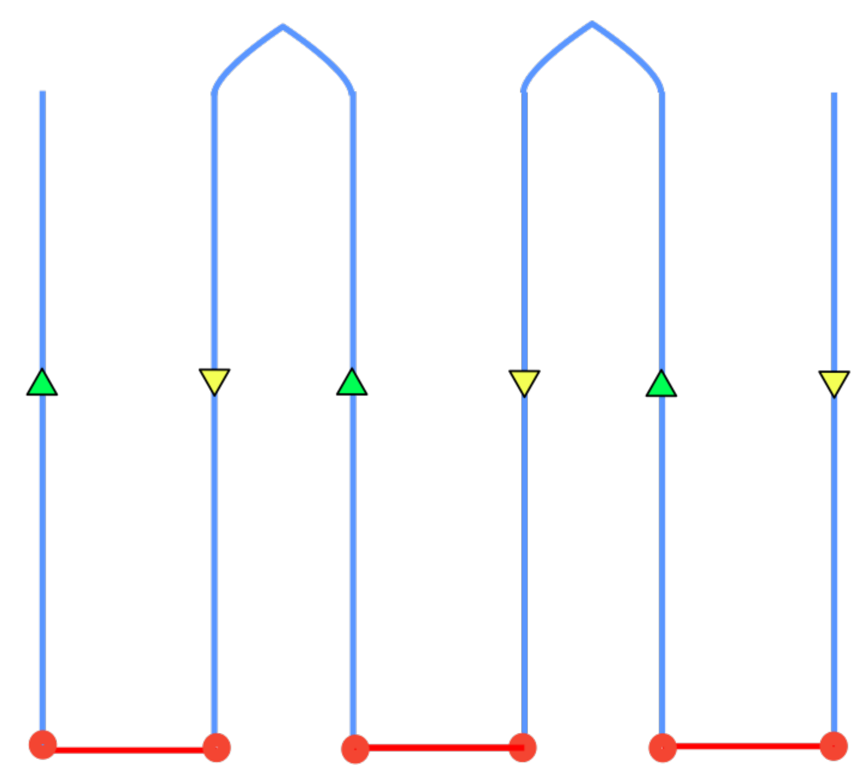
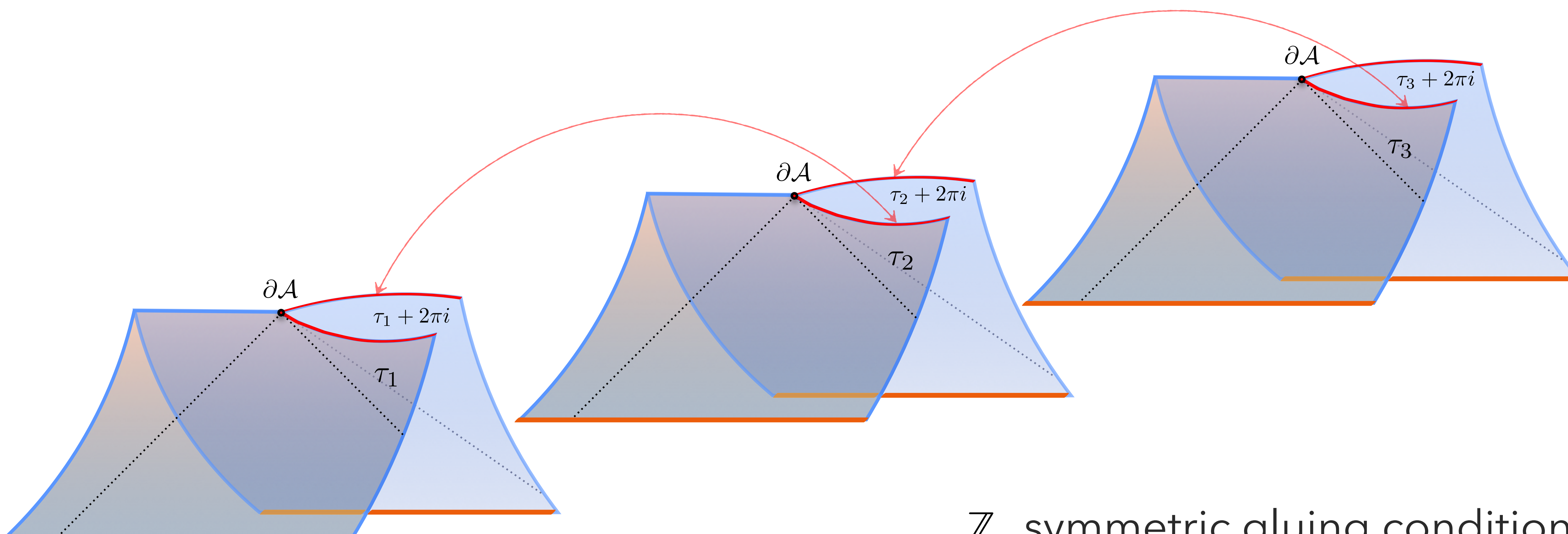
- Consider computing not time-ordered, but out-of-time ordered correlators.
- These can be generated by suitably stringing together Schwinger-Keldysh contours, with multiple switchbacks or timefolds.
- For instance computing a three-fold out-of-time ordered correlation function will involve some contour of the form:



Replicated reduced density matrix

- In this timefold picture we can straightforwardly compute powers of the reduced density matrix.
- We string together copies of the path integral with the pieces identified cyclically as required for multiplying out the matrix elements



$\mathbb{Z}_q$ symmetric gluing conditions

From replicas to entanglement

- Our aim is to compute the Renyi entropies and then analytically continue to obtain the von Neumann entropy

$$S_{\mathcal{A}} = -\mathrm{Tr}_{\mathcal{A}} \left(\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}\right) = \lim_{q\to 1} S^{(q)}_{\mathcal{A}}$$

$$S^{(q)}_{\mathcal{A}} = \frac{1}{1-q} \log \mathrm{Tr}_{\mathcal{A}}(\rho_{\mathcal{A}})^q \,.$$

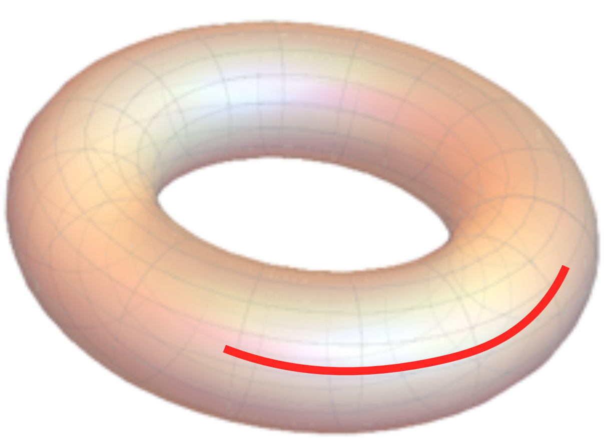
- It is useful to consider a particular auxiliary quantity related to the Renyi entropies, which will turn out to have a clean geometric avatar.

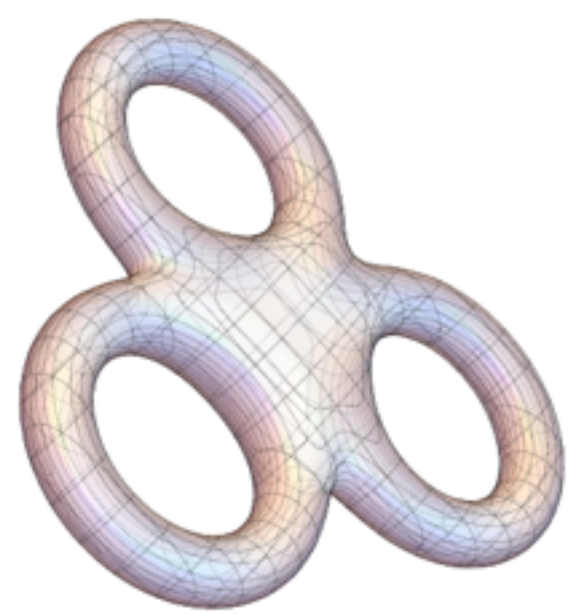
$$\tilde{S}^{(q)}_{\mathcal{A}} = -q^2\, \partial_q \left[\frac{1}{q}\, \log \mathrm{Tr}_{\mathcal{A}}(\rho_{\mathcal{A}})^q\right]$$

Dong '16

Time reflection symmetric case

- If the quantum state is at a moment of time symmetry, then we can eschew various complications of the Schwinger-Keldysh construction.
- Essentially we unwrap the switchbacks and work with regular path integral contours; cutting them open to obtain $(\rho_{\mathcal{A}})_{\pm}$ which may then be glued together cyclically.
- Useful to view this Euclidean path integral in terms of QFT on a new background: q-fold branched cover of the original background, branched at the entangling surface.





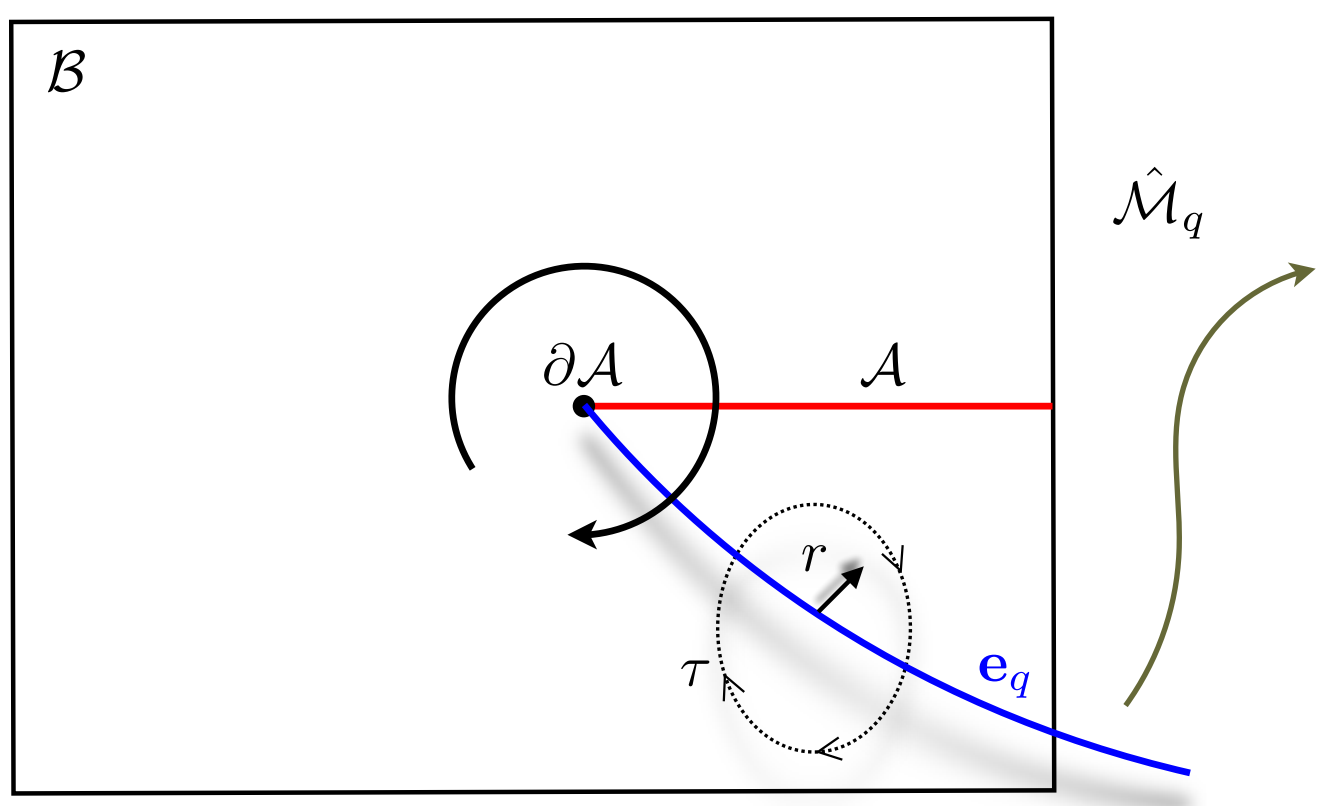
Review: Lewkowycz-Maldacena

- AdS/CFT relates holographic field theories on some background $\mathcal{B}$ to a gravitational theory on a bulk spacetime $\mathcal{M}$ subject to boundary conditions which demand that $\partial\mathcal{M} = \mathcal{B}$.
- The bulk spacetime can be obtained in the semiclassical limit using a saddle point solution of the quantum gravity path integral.
- The branched cover construction gives us a boundary manifold $\mathcal{B}_q$ which we use as boundary conditions to determine $\mathcal{M}_q$.
- This sets up the gravitational problem. Our job is to find the appropriate solution, determine the boundary partition function, which is related to the on-shell action, and thence analytically continue.
- LM argue that the analytic continuation is simpler in gravity and employ it to directly derive the RT prescription.

LM: Kinematics

- ✦ Assume bulk saddles are replica symmetric and construct the orbifold

$$\hat{\mathcal{M}}_q = \mathcal{M}_q/\mathbb{Z}_q$$
$$\partial\hat{\mathcal{M}}_q = \mathcal{B}_q/\mathbb{Z}_q = \mathcal{B}$$



locus of orbifold singularities

opening angle $\frac{2\pi}{q}$

cosmic brane of tension

$$T_q = \frac{1}{4G_N^{(d+1)}}\,\frac{q-1}{q}$$

- ✦ On-shell action we want is simply related $\quad I[\mathcal{M}_q] = q\, I[\hat{\mathcal{M}}_q]$

- ✦ Subtleties arising from global topology…

Hartman, Haehl, Marolf, Maxfield, MR '14

LM: Dynamics

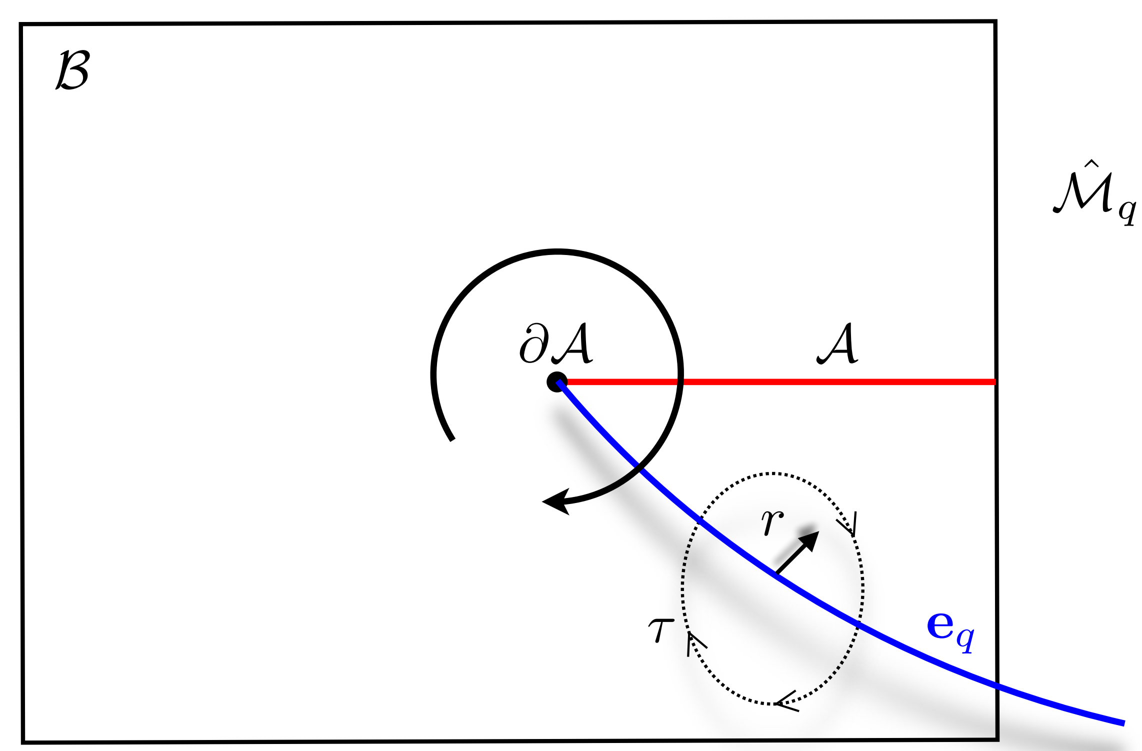
- Gravitational analytic continuation: dial the tension of cosmic brane!
- Local analysis of eom in the vicinity of singular locus gives RT prescription.

$$ds^2 = \left(q^2\, dr^2 + r^2\, d\tau^2\right) + \left(\gamma_{ij} + 2\, K^x_{ij}\, r^q \cos\tau + 2\, K^t_{ij}\, r^q \sin\tau\right) dy^i\, dy^j + \cdots$$

$$\lim_{q\to 1} \mathbf{e}_q \to \mathcal{E}_\mathcal{A}\,, \qquad \mathcal{E}_\mathcal{A} \in \mathcal{M} \text{ with } \; t = 0,\; K^x = 0 \qquad K^t \;=\; 0$$ by symmetry

$$\partial_q I[\hat{\mathcal{M}}_q] = \int_{\mathbf{e}_q(\epsilon)} \Theta((g_q)_{AB}, \partial_q (g_q)_{AB})$$

$$\partial_q I[\hat{\mathcal{M}}_q] = \frac{\text{Area}(\mathbf{e}_q)}{4\, q^2 G_N}$$



Lorentzian AdS/CFT

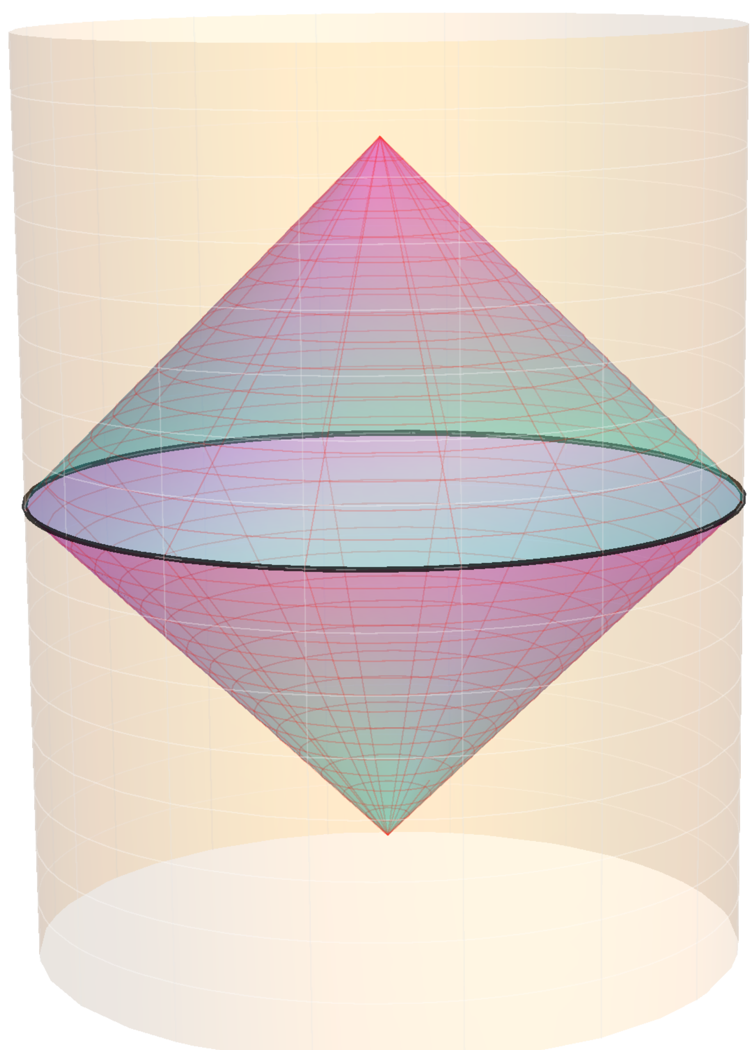
- ✦ To compute entanglement entropy from the bulk we need to figure out how to set up the bulk quantum gravity path integral.
- ✦ Gravity dual of the Schwinger-Keldysh path integral contour?

Lorentzian AdS/CFT

- To compute entanglement entropy from the bulk we need to figure out how to set up the bulk quantum gravity path integral.
- Gravity dual of the Schwinger-Keldysh path integral contour?
  - Assumption: The boundary Schwinger-Keldysh contour is piecewise extended into the bulk gravity theory.
  - If the global state admits a semiclassical dual, then each segment of the bulk path integral is dominated by the the corresponding geometry.
  - We have to respect the time ordering constraint in the bulk, which now has some extra redundancy…

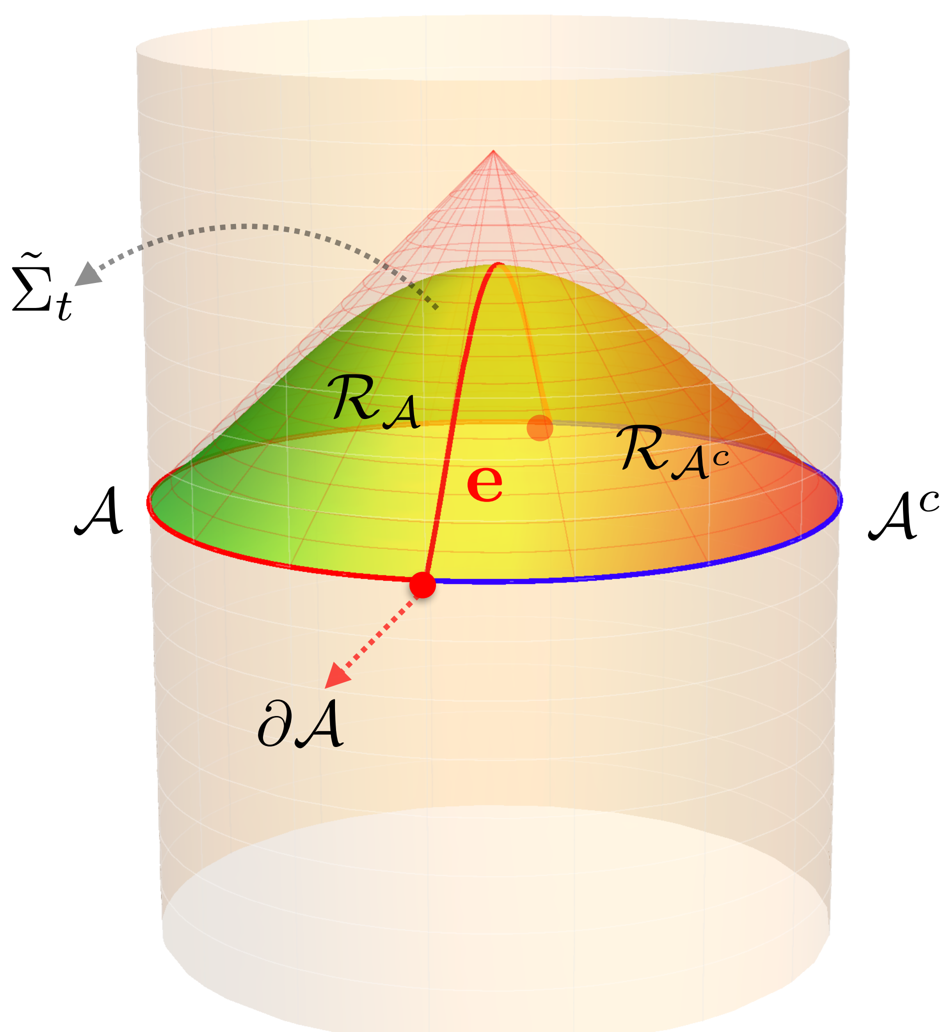
Skenderis, van Rees '08

A bulk redundancy



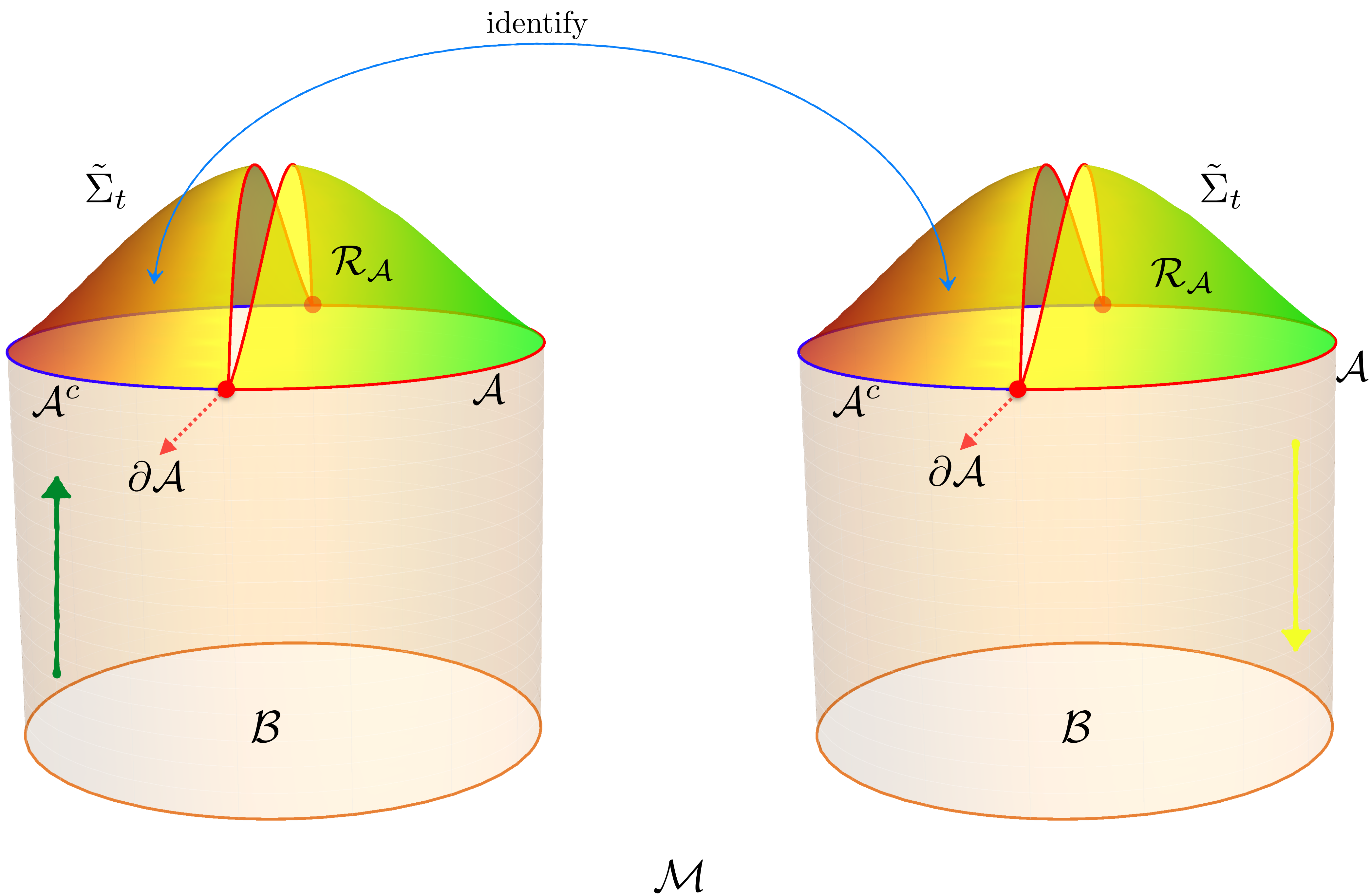
- A boundary time slice does not uniquely extend into the bulk.
- The redundancy is captured by bulk Cauchy surfaces that are anchored on and spacelike to our boundary slice.
- This ambiguity maps out a causal domain in the bulk, the *FRW wedge*.

The bulk ansatz

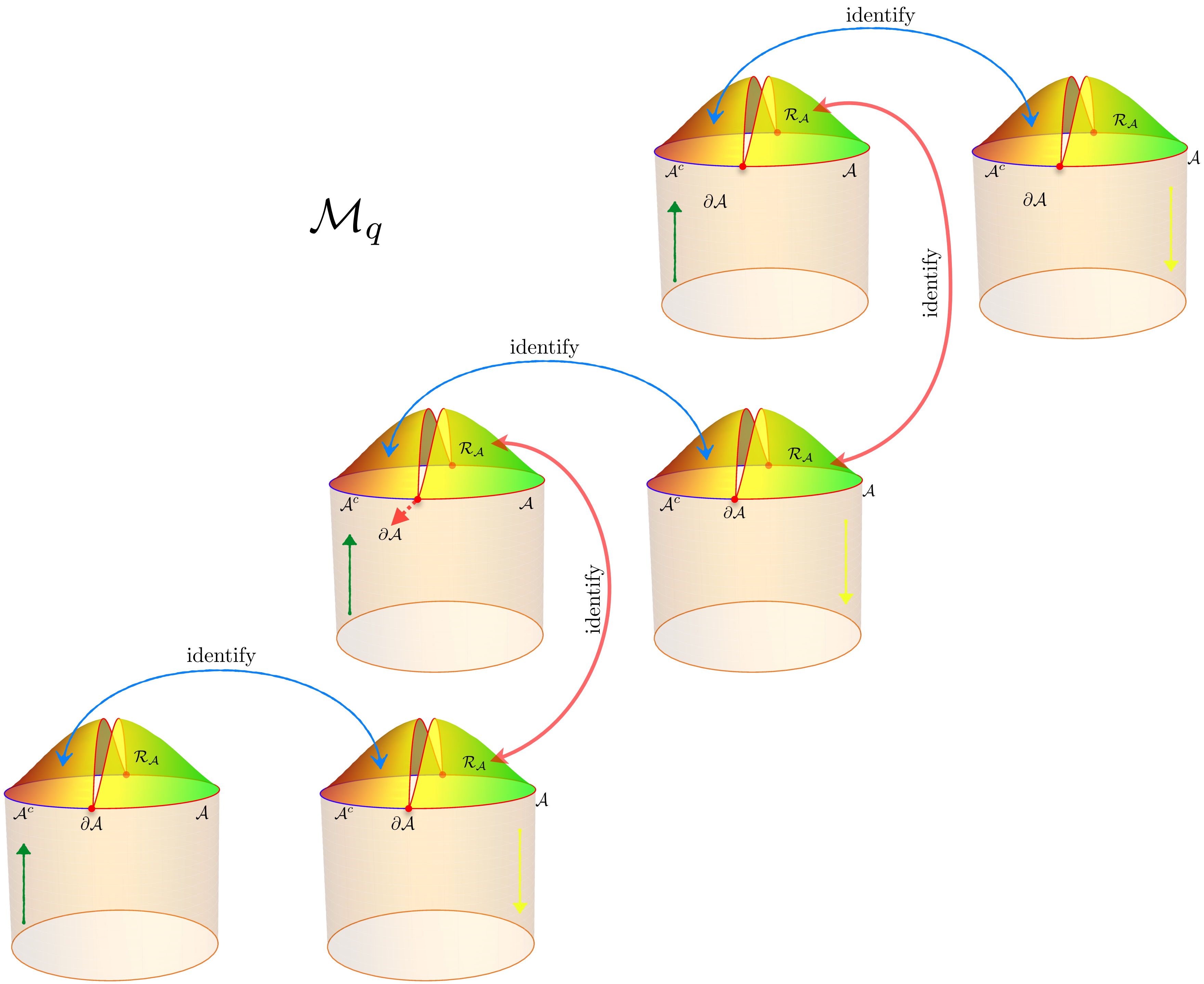


- ✦ Prescription: Pick some bulk Cauchy slice $\tilde{\Sigma}_t$ within the FRW wedge.
- ✦ We will glue copies of the geometry past of $\tilde{\Sigma}_t$ to obtain the dual of the SK contour.
- ✦ The choice of $\tilde{\Sigma}_t$ is irrelevant for computing time-ordered correlation functions.
- ✦ For entanglement entropy we will find that $\tilde{\Sigma}_t$ is forced to contain the extremal surface.

Bulk density matrix elements



Replicating the bulk



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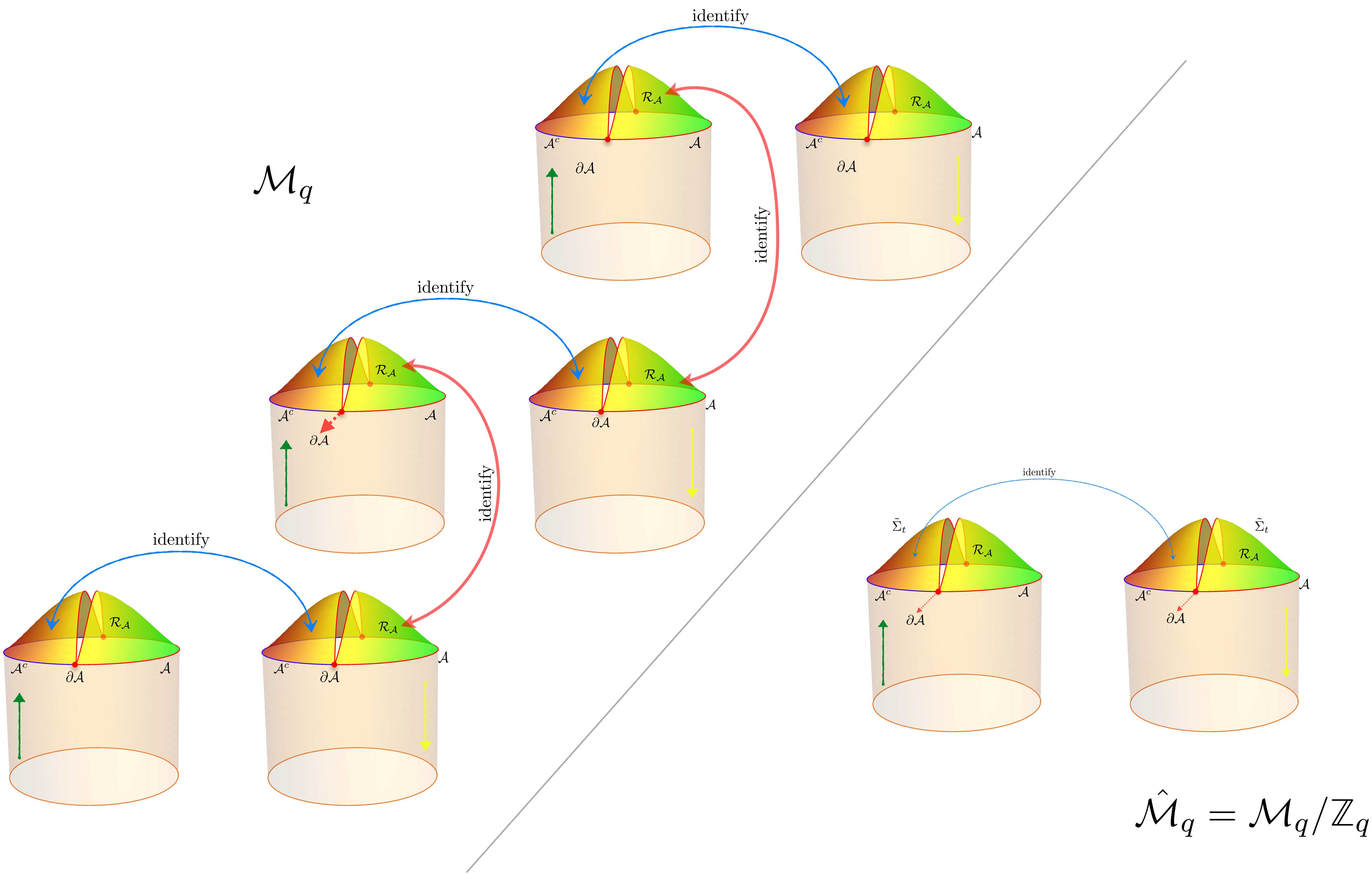
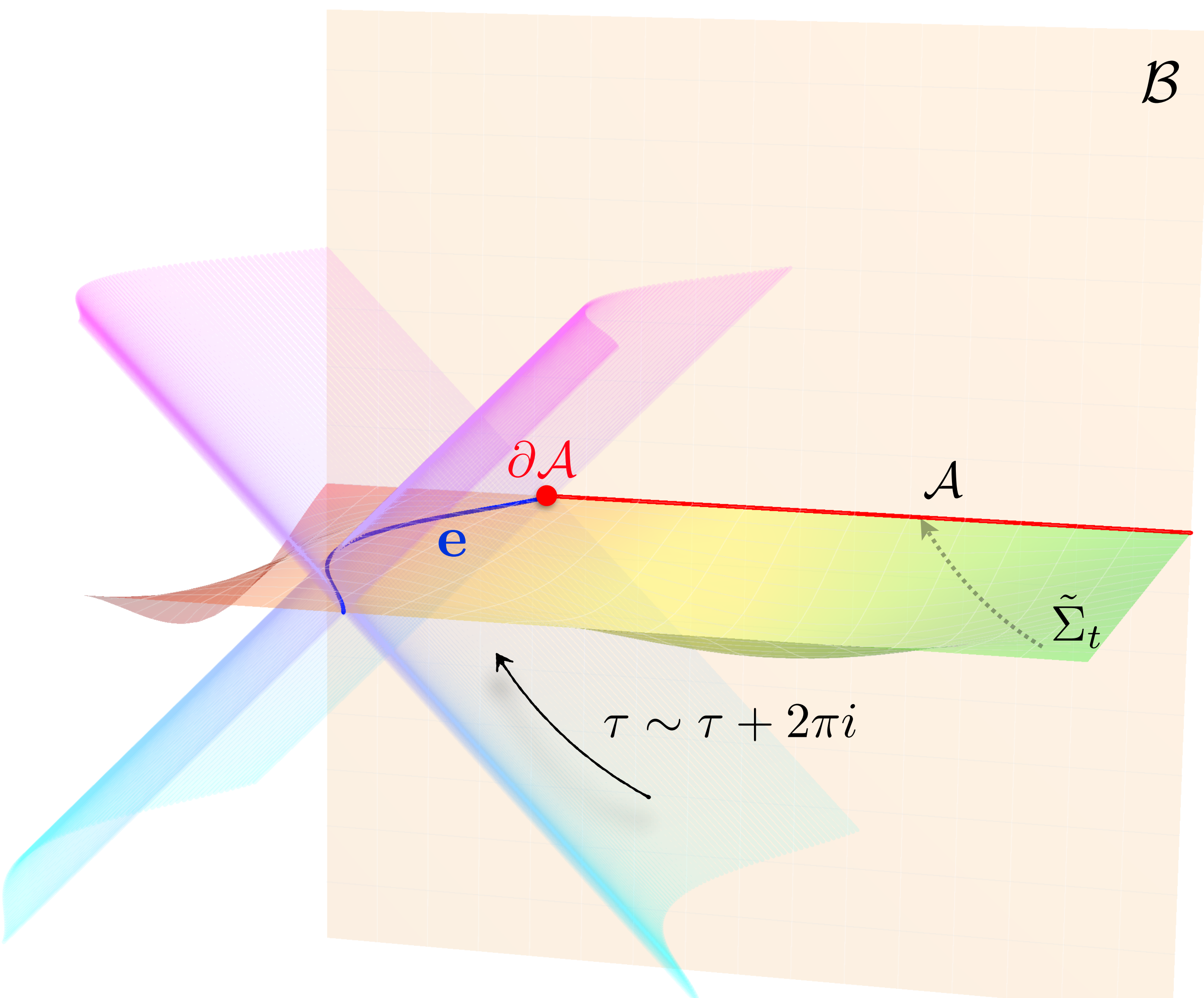


Kinematics for covariant construction

- Construct the boundary Schwinger-Keldysh contour which gives a suitably doubled spacetime $\mathcal{B}$.
- Filling in this contour with appropriate bulk pieces we have the bulk geometry which serves as the seed ansatz $\mathcal{M}$ for computing the reduced density matrix elements.
- We string together copies of $\mathcal{M}$ sewing together successive segments cyclically to obtain the covering space $\mathcal{M}_q$.
- Gravitational dynamics is supposed to pin down $\mathcal{M}_q$.
- We again exploit the relative simplicity of the gravitational analytic continuation, by working in the quotient spacetime $\hat{\mathcal{M}}_q = \mathcal{M}_q/\mathbb{Z}_q$ which now features a spacelike cosmic brane $\mathbf{e}_q$.

Local boundary conditions



- The q=1 geometry locally looks like Rindler spacetime with the entangling surface extended out into the bulk as a Rindler horizon.
- The replica boundary conditions for $\hat{\mathcal{M}}_q$ can equivalently be stated in terms of modulating this local structure; the Rindler temperature is lowered to $\dfrac{2\pi}{q}$.

Gravitational dynamics

- Once we have the ansatz and the replica boundary conditions, all that remains is to solve the bulk equations of motion. Work in local coordinates adapted to the normal bundle of the singular locus:

$$ds^2 = \left(q^2 dr^2 - r^2\, d\tau^2\right) + \left(\gamma_{ij} + 2\, K^x_{ij}\, r^q \cosh\tau + 2\, K^t_{ij}\, r^q \sinh\tau\right) dy^i\, dy^j \\ + \left[r^{f_q\,(q-1)} - 1\right] \delta g_{\mu\nu}\, dx^\mu\, dx^\nu + \cdots$$

- Bulk equations of motion then fix the geometry of the singular locus. To leading order in q-1 we fix the geometry. In Einstein-Hilbert theory this gives the extremal surface condition

$$\mathrm{EOM}^a \propto \frac{q-1}{r}\, K^a + \mathrm{regular}^a$$

$$\begin{aligned} K^a = 0 \quad &\Longrightarrow \quad \theta^{\pm} = \frac{1}{\sqrt{2}}\left(K^0 \pm K^1\right) = 0\,, \\ &\Longrightarrow \quad \lim_{q\to 1} \mathbf{e}_q = \mathcal{E}_{\mathcal{A}}\,, \qquad \mathcal{E}_{\mathcal{A}} \in \mathcal{M} \text{ is extremal.} \end{aligned}$$

The on-shell gravitational action

- Since the singular locus is a spacelike source, it influences the geometry along the past light cones. This complicates the analysis of the on-shell action, which requires a suitable regulating procedure.

- The non-trivial part is to argue that the Renyi entropies work out correctly; the limit $q \mapsto 1$ turns out to be simpler. In any event we can show that

$$\tilde{S}_q = -i\,\partial_q I[\hat{\mathcal{M}}_q] = -\frac{i}{8\pi G_N}\,\partial_q \int_{\mathbf{e}_q(\epsilon)} \mathcal{K}_\epsilon = \frac{\text{Area}(\mathbf{e}_q)}{4q^2 G_N}$$

Farhi, Guth, Guven '90, Neiman '12, '13

Comments…

- The construction builds in the homology constraint.
- The bulk Cauchy surface we pick is forced to admit the extremal surface.
- The construction explicitly ensures that the proposal satisfies boundary causality; the extremal surface lies in the causal shadow of the boundary region's domain of dependence .

Wall' 12; Headrick, Hubeny, Lawrence, MR '14

- In spirit the construction has elements of the maximin reformulation of the HRT prescription.

Wall' 12

  - Pick a bulk slice and find a minimal surface on the slice.
  - Maximize the area of the surface across all bulk slices in the FRW wedge.

Open Questions

- Can we rule out the occurrence of complex saddle points?

Fischetti, Marolf + Wall '14; Maxfield '14

- Should they be relevant, how does one reconcile their presence with causality restrictions?

- A cleaner derivation of the bulk dual to the Schwinger-Keldysh prescription?

- Can we put the topological symmetries inherent in Schwinger-Keldysh to use efficiently for this purpose?

Haehl, Loganayagam, MR '15