The Entanglement Hamiltonian and others in one-dimensional critical and gapped systems

Shinsei Ryu
Univ. of Illinois, Urbana-Champaign

Outline:

- Part I:
 Entanglement spectrum in 1+1d a conformal field theories (CFTs) and the sine-square deformation (SSD)
 [with Xueda Wen and Andreas Ludwig]

- Part II (optional):
 Entanglement spectrum in 1+1 d gapped (SPT) phases and boundary conformal field theories (BCFTs)
 [with Gil Cho, Ken Shiozaki, Andreas Ludwig]

Hamiltonians in CFT

- Let's start from a Hamiltonian of (1+1)d CFTs;
  On a lattice (chain), it would look like: $$H = \sum_i h_{i,i+1}$$

- Deformed evolution operator: $$H[f] = \sum_i f\left(\frac{x_i + x_{i+1}}{2}\right) h_{i,i+1}$$

  envelope function

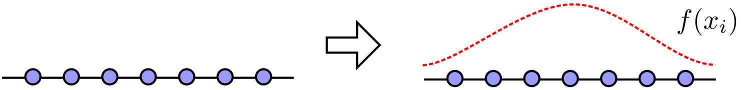
- E.g. Entanglement Hamiltonian: $$f(x) = \frac{R^2 - x^2}{2R}$$

- E.g. Sine-square deformation (SSD): $$f(x) = \sin^2 \frac{\pi x}{L}$$
  [Gendiar-Krcmar-Nishino (2009) ...]

- Other applications: inhomogenous systems,
  quantum energy inequalities, etc.

What is the sine-square deformation (SSD)?

- What it does is to introduce an "optimal" or "infinitely smooth" cutoff.



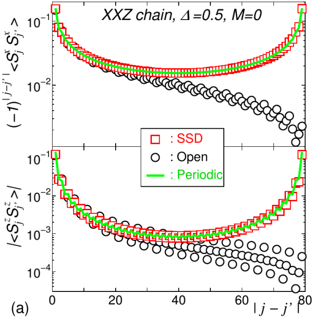
- This may be of interest as a numerical technique. Reducing finite-size error, etc.

- Eearly numerical observations:

  - Correlation functions

  - $\langle \Psi_{SSD} | \Psi_{PBC} \rangle \simeq 1$

  - Entanglement scaling



[Hikihara-Nishino (11)]

Key properties of SSD ?

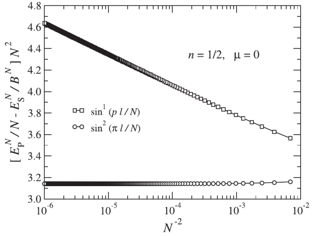
- The ground state of SSD = ground state of periodic chain

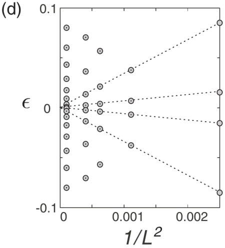
  Numerics: Hikihara-Nishino (11);

  Exactly solvable models: Katsura (11), Maruyama-Katsura-Hikihara (11),
  Okunishi-Katsura (15)

  Proof within CFT: Katsura (12)

- 1/L^2 finite fize scaling of energy levels
  was observed numerically.





[Gendiar-Krcmar-Nishino (09), Hotta-Nishimoto-Shibata (13),
LSM type analysis by Katsura ...]

References:

- Grand canonical numerical analysis -- efficient extraction of physical quantities in the presence of an applied field.

  Shibata and Hotta (11)
  Hotta, Nishimoto and Shibata (13)

- SSD and string theory:

  Tada, arXiv:1404.6346[hep-th]

- Dipolar quantization:

  Ishibashi and Tada, arXiv:1504.00138[hep-th]
  Ishibashi andTada, arXiv: 1602.01190[hep-th]

- Mobius quantization:

  Okunishi arXiv:1603.09543

Strategy

- We will disscuss types of "deformations" $H[f] = \int dx\, f(x)\mathcal{H}$ generated by various conformal maps.

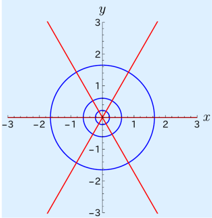
- Put differently, we are interested in deformations which we can "undo" by conformal maps.

- Will discuss spectral properties (finite size scaling) of $H[f]$

Warm-up

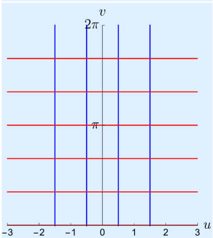
-

$$z = x + iy$$



$$w(z) = \log z$$

$$w = u + iv$$



- CFT on the plane <--> CFT on a cylinder (quantum lattice model on a circle)

- Radial evolution <--> Hamiltonian (1/L scaling)
  (Dilatation)

- Angular evolution <--> Hamiltonian with boundary
  ("Rindler" or "Modular"
  or "Entanglement" Hamiltonian)

Finite size scaling of CFT [Cardy]

- CFT on a cylinder of circumference L:

$$\tilde{H} = \frac{1}{2\pi} \int_0^L dv\, \tilde{T}_{uu}(u_0, v)$$

$$\tilde{H} = \frac{1}{2\pi} \oint_{C_w} dw\, \tilde{T}(w) + (\mathrm{anti-hol})$$

$$\tilde{T}_{uu}(w) = \tilde{T}(w) + \bar{\tilde{T}}(\bar{w})$$

- Conformal map: cylinder --> plane

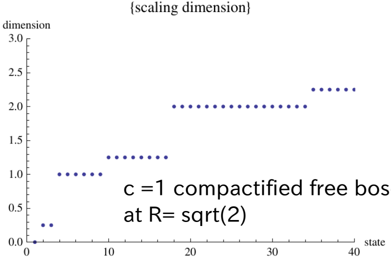
$$w = \frac{L}{2\pi} \log z.$$

$$\tilde{T}(w) = \left(\frac{2\pi}{L}\right)^2 \left[z^2 T(z) - \frac{c}{24}\right]$$

$$\begin{aligned}
&\oint_{C_w} dw\, \tilde{T}(w) \\
&= \oint_{C_z} dz\, \frac{dw}{dz} \left(\frac{2\pi}{L}\right)^2 \left[z^2 T(z) - \frac{c}{24}\right] \\
&= \oint_{C_z} dz\, \left(\frac{L}{2\pi}\right) \left[z\, T(z) - \frac{c}{24}\frac{1}{z}\right].
\end{aligned}$$

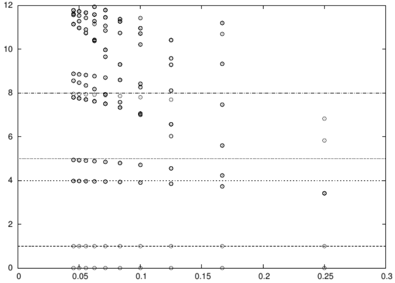
$$\tilde{H} = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{24}\right)$$

-- For a given tower of states,
all levels are equally spaced
(with degenearcy, which depends
on details of the theory)



c =1 compactified free boson
at R= sqrt(2)

$(E - E_{\mathrm{GS}})/(E_1 - E_{\mathrm{GS}})$

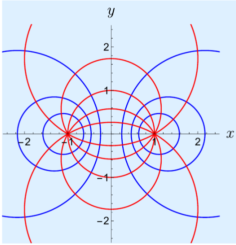


$1/L$

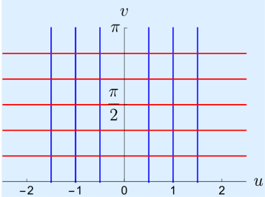
-- Level spacing scales as 1/L

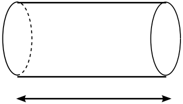
$$H = \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y)$$

-



$$w(z) = \ln \frac{(z+R)}{(z-R)}$$





$\sim \log R$

- Entangement Hamiltonian on finite interval [-R, R]
  <--> Hamiltonian with boundaries

- Transforming from strip to plane:

$$H = \int du\, T_{vv}(u, v_0 = \pi) = \int_{-R}^{+R} dx\, \frac{(x-R)(x+R)}{2R} T_{yy}(x, y=0)$$

- Entanglement spec: 1/Log(R) scaling.

[See, e.g: Casini-Huerta-Myers (11), Cardy @ 2015 KITP conference]

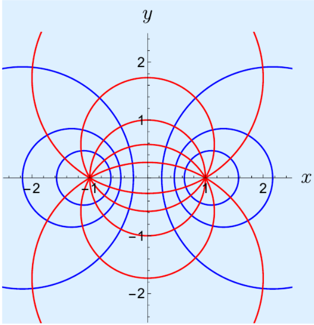
What is the evolution orthogonal to the evolution by Entanglement H ?

- Let's take a circle as a Cauchy surf

$$\left(x+\frac{\cosh u_0}{\sinh u_0}R\right)^2+y^2=\frac{R^2}{(\sinh u_0)^2}$$

We have chosen: $r_0 := \dfrac{R}{\sinh u_0}$

Circumference: $L = 2\pi r_0$



- Evolution operator

$$H=\int_0^{\pi} dv\, T_{uu}(u_0,v)=r_0^2\int_0^{2\pi} d\theta\,\frac{\cos\theta+\cosh u_0}{\sinh u_0}\,T_{rr}(r,\theta)$$

- "Regularized" version of the SSD:

$$H=\frac{L}{2\pi}\frac{1}{\sinh u_0}\int_0^{L} ds\,\left(\cos\frac{2\pi s}{L}+\cosh u_0\right)T_{rr}\left(r=\frac{L}{2\pi},\theta=\frac{2\pi s}{L}\right)$$

"Regularized" SSD

$$H = \frac{L}{2\pi}\frac{1}{\sinh u_0}\int_0^L ds \left(\cos\frac{2\pi s}{L} + \cosh u_0\right) T_{rr}\left(r = \frac{L}{2\pi}, \theta = \frac{2\pi s}{L}\right)$$

- By construction, this operator has the spectrum of CFT on a circle with level spacing of order one.

- Define: $$H_{\mathrm{rSSD}} = \int_0^L ds \left(\cos\frac{2\pi s}{L} + \cosh u_0\right) T_{rr}\left(\frac{L}{2\pi}, \frac{2\pi s}{L}\right)$$

- The envelope function:

$$\begin{aligned} f(s) &= \cos\left(\frac{2\pi s}{L}\right) + \cosh u_0 \\ &= \cos\left(\frac{2\pi s}{L}\right) + \sqrt{1 + \left(\frac{2\pi R}{L}\right)^2}. \end{aligned}$$

$$\begin{aligned} f(s) &\stackrel{R\to 0}{\to} \cos\left(\frac{2\pi s}{L}\right) + 1 \\ &= \cos^2(\frac{\pi s}{L}) = \sin^2\left[\frac{\pi}{L}\left(s - \frac{L}{2}\right)\right]. \end{aligned}$$

- "Regularized" version of the SSD:
  R, the distance between vortices, is the regularziation parameter.

- Scalilng:
  (i) fix u0, change R --> 1/L scaling

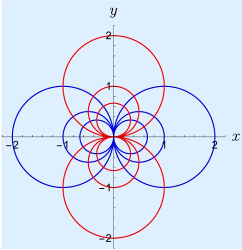
  $$\sim \frac{\sinh u_0}{L} = \frac{1}{2\pi}\frac{(\sinh u_0)^2}{R} \sim \frac{1}{R}.$$

  (ii) fix R, change u_0 --> 1/L^2 scaling

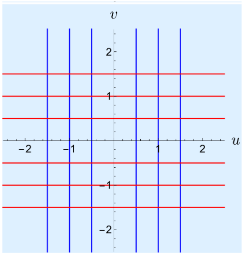
  $$\sim \frac{\sinh u_0}{L} = \frac{1}{2\pi}\frac{(\sinh u_0)^2}{R} \sim \frac{1}{L^2}.$$

The dipolar limit

- Can take the dipolar limit $R \to 0$ rSSD --> SSD:



$$w(z) = 1/z$$



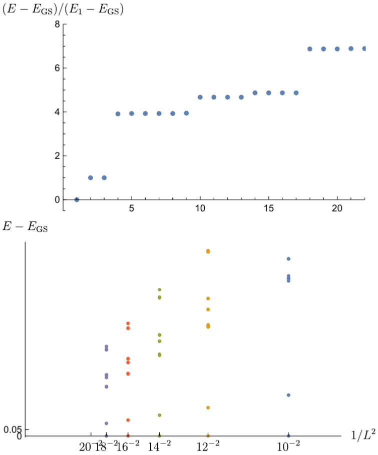
- In the dipolar limit, the w-plane (u-v plane) is an infinit plane
  --> Infinite system length limit, continuum spectrum [Ishibashi-Tada (15,16)]

$$\begin{aligned} H &= \int_{-\infty}^{+\infty} dv\, T_{uu}(u_0, v) = 4r_0^3 \int_0^{2\pi} d\phi\, \sin^2(\phi/2) T_{rr}(r_0, \theta) \\ &= \frac{L^2}{\pi^2} \int_0^L ds\, \sin^2\left(\frac{\pi s}{L}\right) T_{rr}\left(\frac{L}{2\pi}, \frac{2\pi s}{L}\right) \end{aligned}$$

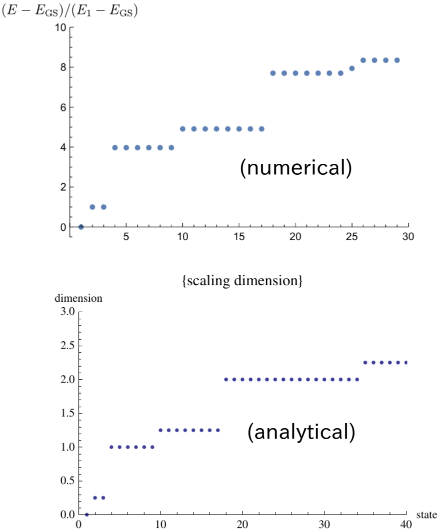
- The prefactor L^2 is indicative of the 1/L^2 scaling seen in numerics.

Numerics (rSSD)

$$H = \sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y)$$

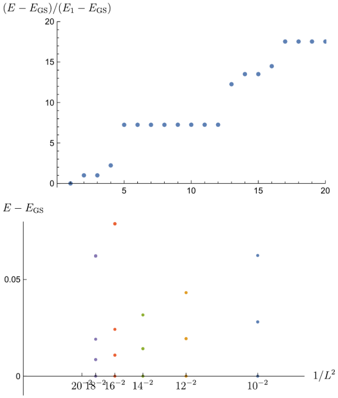


Regularized SSD spectrum

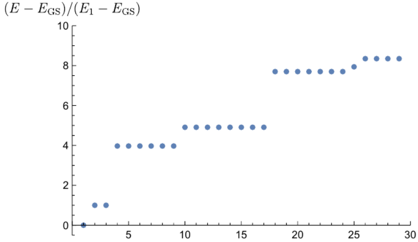


Physical spectrum (PBC)

Numerics (SSD)



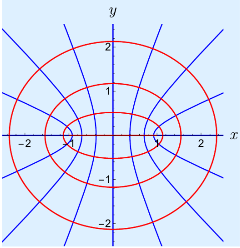
SSD



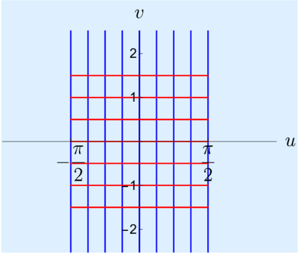
Physical spectrum (PBC)

SSD spectrum does not much physical spectrum,
1/L^2 scaling

Other examples

- 

$$z = \sin(w)$$



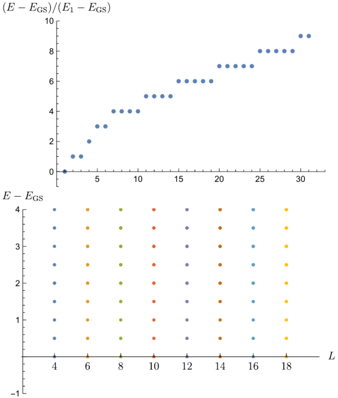
infinite stripe

$$-\pi/2 < u < \pi/2$$
$$-\infty < v < +\infty$$

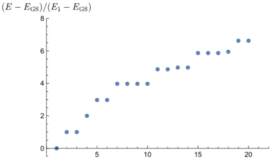
- Engineering conformal map and evolution operator

- "square root deformation" $\tilde{H} = \int_{-R+\epsilon}^{+R-\epsilon} dx \, \sqrt{R^2 - x^2} \, T_{yy}$

- Known in the context of "perfect state transfer"
  (Thanks: Hosho Katsura)



"Square root" deformation



Physical spectrum (OBC)

Summary (Part I)

- Setup a general discusion of "deformed" Hamiltonians in CFTs

- Proposed a "regularized" version of SSD (rSSD).

- Original SSD can be viewed as a "singular" limit of rSSD

- Spectrum of rSSD is easy to understand.
  Shed light on 1/L^2 scaling of SSD.

- Issues:

Exciations?
Relation to the classification of conformal vacua [Candelas-Dowker (1979)]

(Part II) Going away from CFTs

- Add a relevant deformation --> go into a massive phase

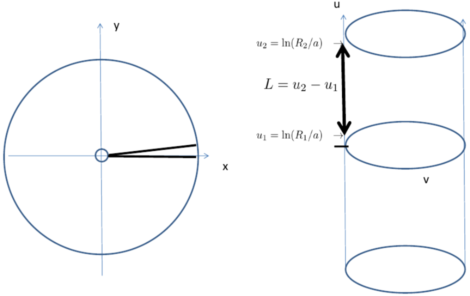
$$S_{z,\bar{z}} = S_* + g \int d^2z \; \phi(z, \bar{z})$$

- Consider the entanglement Hamiltonian for the half space; The entanglement spectrum?

$$\boxed{z = (x + iy) = \exp(w) = \exp(u + iv)}$$

- Repeat the conformal map analysis

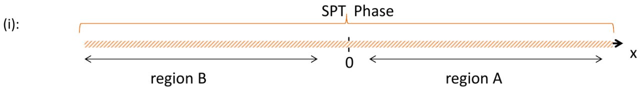
$$S_* + \; g \int_{u_1}^{\infty} du \int_0^{2\pi} dv \;\; e^{yu} \;\; \Phi(w, \bar{w})$$

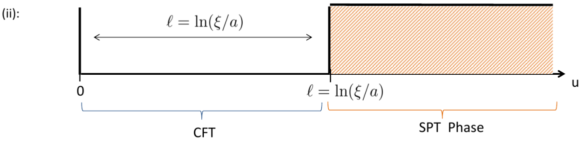


- Massive perturbation creates an exponentially growing potential

- Massive perturbation creates an artificial boundary in the entanglement Hamiltonian

- To a good approximation, the entanglement Hamiltonian is the Hamiltonian of a CFT with boundaries; Boundary CFT.





$$\rho_A \propto \exp(-H_e)$$

$$H_e = const. \frac{L_0}{\log(\xi/a_0)}$$

- Comment: this argument shows the low-energy part of the entanglement spec. of a massive theory is given by BCFT.
  There are integrable massive models, whose corner transfer matrices are given exactly by Virasoro characters (BCFT).

Question and result

-ES for gapped phases is given by nearby boundary conformal field thoery

$$\rho_A \propto \exp(-H_e) \qquad H_e = const. \frac{L_0}{\log(\xi/a_0)}$$

- Q: Which boundary condition ? <----> Which gapped phase?

- Let's focus on the case when the massive phase is a SPT phase
  I.e.: (i) unique ground state
  (ii) topologically distinct in the presence of some symmetry

- Result:
  For a given symmetry G, and a given boundary state |B>,
  found a method to compute the topoogical invariant
  of the corresponding SPT phase.

$$\hat{g}|B\rangle_h = \varepsilon(g|h)|B\rangle_h$$

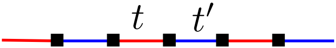
- Related to symmetry-protected degeneracy of ES

- Relation to physics of fractional branes

Symmetry-protected degeneracy

- E.g. 1d lattice fermion model ("SSH" model)

$$H = t\sum_i (a_i^\dagger b_i + h.c.) + t'\sum_i (b_i^\dagger a_{i+1} + h.c.)$$

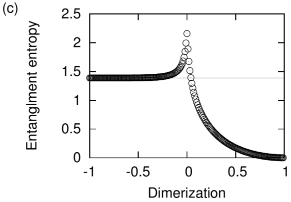


Symmetry: $a_i \to a_i^\dagger \quad b_i \to -b_i^\dagger$

Phase diagram:

- Symmetry-protected contribution to EE [SR-Hatsugai (06)]

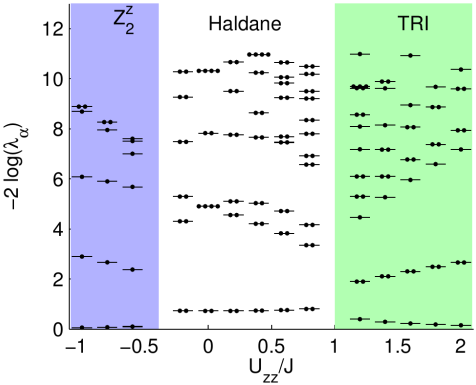
$$S_A \sim (1/6)\log(\xi/a_0) + \ln 2$$



Symmetry-protected degenerac

- (1+1)d SPT phase:
E.g. the Haldane phase, the Kitaev chain

- Symmetry-protected degenearcy in ES: [Pollmann-Berg-Turner-Oshikawa (10)]



$$H = \sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + U_{zz} \sum_i (S_j^z)^2$$

- Symmetry-protected degeneracy --> Vanishing of part. function:

$$Z_{AB}^{h} = \mathrm{Tr}_{\mathcal{H}_{AB}} \left[\hat{h}\ e^{-\beta \hat{H}_{AB}^{open}} \right] = 0,$$

- Exchange time and space:

$$Z_{AB}^{h} = {}_{h}\langle A | e^{-\frac{\ell}{2}\hat{H}^{closed}} | B \rangle_{h} = {}_{h}\langle A | \tilde{q}^{\frac{1}{2}(\hat{H}_L + \hat{H}_R)} | B \rangle_{h},$$

$${}_{h}\langle A | e^{-\frac{\ell}{2}\hat{H}^{closed}} | B \rangle_{h} = 0.$$

- Act with a symmetry on |B>

$$\hat{g}|B\rangle_h = \varepsilon_B(g|h)|B\rangle_h, \quad \text{when} \quad g \in N_h.$$

- Symmetry-enforced vanishing of partition function

$${}_{h}\langle A | \hat{g}\tilde{q}^{\frac{1}{2}(\hat{H}_L + \hat{H}_R)} | B \rangle_{h} = {}_{h}\langle A | \tilde{q}^{\frac{1}{2}(\hat{H}_L + \hat{H}_R)} \hat{g} | B \rangle_{h},$$

$$\begin{aligned} &\varepsilon_A(g|h)^*\, {}_{h}\langle A | \tilde{q}^{\frac{1}{2}(\hat{H}_L + \hat{H}_R)} | B \rangle_{h} \\ &\quad = \varepsilon_B(g|h)\, {}_{h}\langle A | \tilde{q}^{\frac{\ell}{2}(\hat{H}_L + \hat{H}_R)} | B \rangle_{h}. \end{aligned}$$

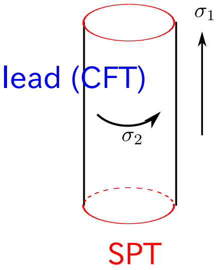
$$\varepsilon_A(g|h)^* \ = \ \varepsilon_B(g|h)$$

Anomalous boundary states

- Ideal lead obeys B.C. set by SPT

$$\Phi(\sigma_2) - U \cdot \Phi(\sigma_2) = 0$$

$$[\Phi(\sigma_2) - U \cdot \Phi(\sigma_2)]|B\rangle = 0$$



- Symmetry G acts on fundamental fields

$$\mathcal{G} \cdot \Phi(\sigma_2) \cdot \mathcal{G}^{-1} = U_G \cdot \Phi(\sigma_2)$$

- B.C. is invariant under G:

$$\mathcal{G}\left[\Phi - U \cdot \Phi\right]\mathcal{G}^{-1} = U_G \cdot \Phi - U_G \cdot U \cdot \Phi$$

- But boundary state may not be:

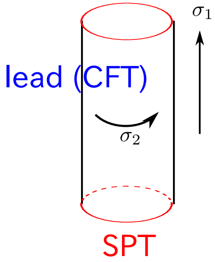
$$\mathcal{G} \cdot |B\rangle \neq |B\rangle$$

- Z_8 classification of TRS Kitaev chain, Haldane phase

Analysis and result: Fidkowski-Kitaev problem

- Ideal lead

$$H = \sum_{a=1}^{N_f} \int_0^{\ell} dx \, [\psi_L^a(-vi\partial_x)\psi_L^a + \psi_R^a(+vi\partial_x)\psi_R^a]$$



- Symmetry group: $\{T, G_f, T \times G_f\}$

- Boundary states

$$[\psi_L(\sigma_2) - i\eta_1\psi_R(\sigma_2)]\,|B(\eta_1,\eta_2)\rangle = 0$$
$$[\psi_R(\sigma_2) + i\eta_2\psi_L(\sigma_2)]\,|B(\eta_1,\eta_2)\rangle = 0$$

- Symmetry action on fermion number parity:

$$G_f|B(\eta_1 = -\eta_2)\rangle = |B(\eta_1 = -\eta_2)\rangle$$
$$G_f|B(\eta_1 = \eta_2)\rangle = (-1)^{N_f}|B(\eta_1 = \eta_2)\rangle$$

Anomalous relative sign goes away for 2N copies --> Z2

- Time reversal:

$$T|B(\eta_1 = \eta_2)\rangle = e^{i\pi N_f/4}|B(\eta_1 = \eta_2)\rangle$$