Entanglement for gauge theories: An operational point of view

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PRL 91, 010404 (2003); PRL 92, 087904 (2004)

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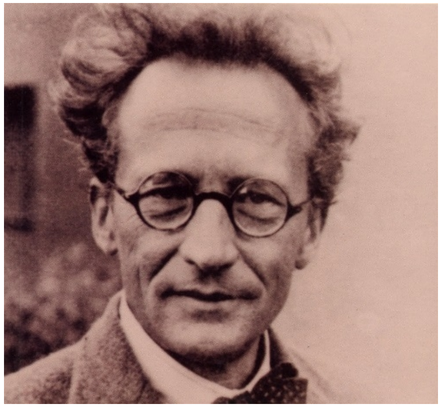
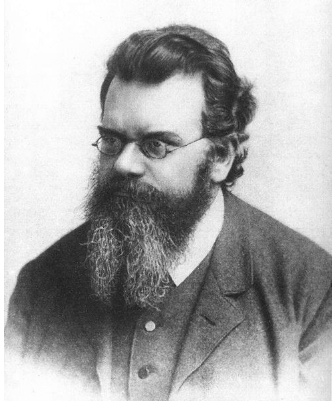
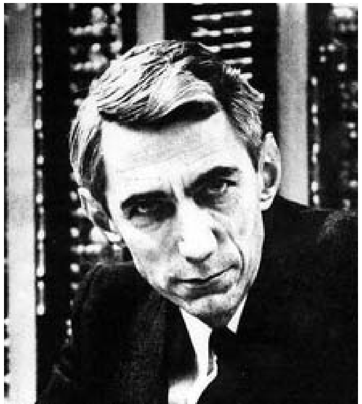
ArXiv 1511.04369

Quantum Many body physics: the 21st century point of view



Entanglement





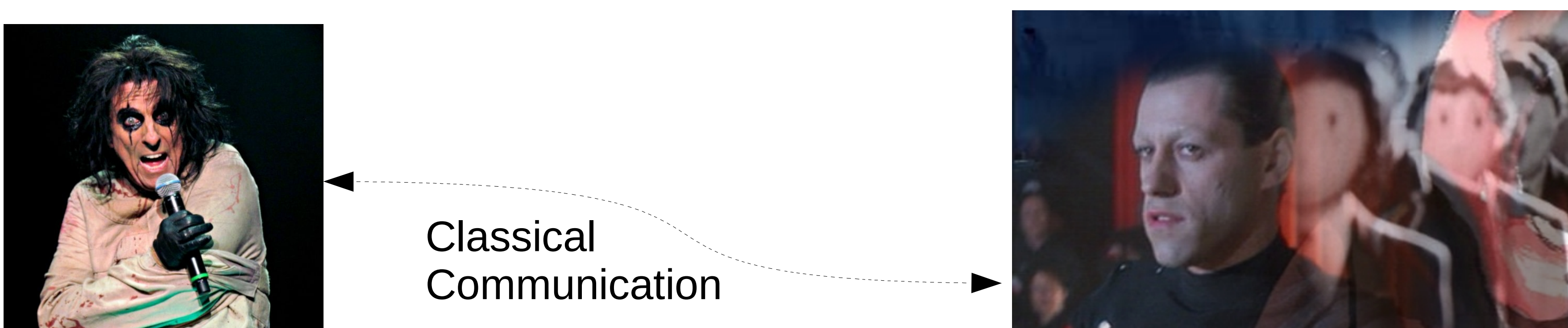
Outline

- Entanglement as a resource theory
- Entanglement in the presence of superselection rules
- Entanglement in gauge theories

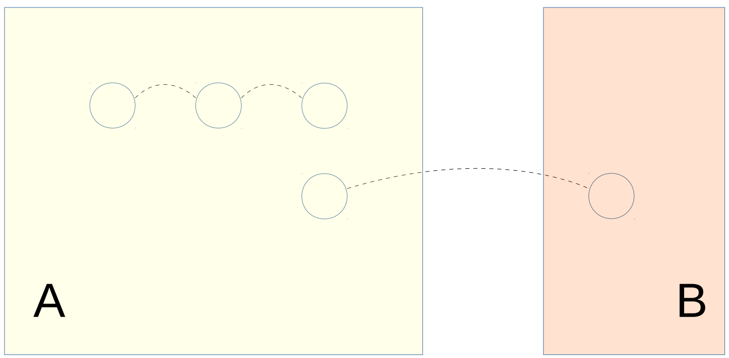
Entanglement as a resource theory

- The beauty of quantum information theory stems from the subtle interplay between
  - the Massive Hilbert space (“a convenient illusion”)
  - The limitations of the allowed quantum operations
  - Resources that allow to overcome those limitations

- The theory of entanglement is the resource theory when faced with the limitation of local operations and classical communication (LOCC)



- Basic premise: entanglement is a resource (just as e.g. energy in thermodynamics) which allows to perform certain information tasks much more efficient than classically
  - Mother of all tasks is quantum teleporation: entanglement + LOCC allows to do any global operation locally



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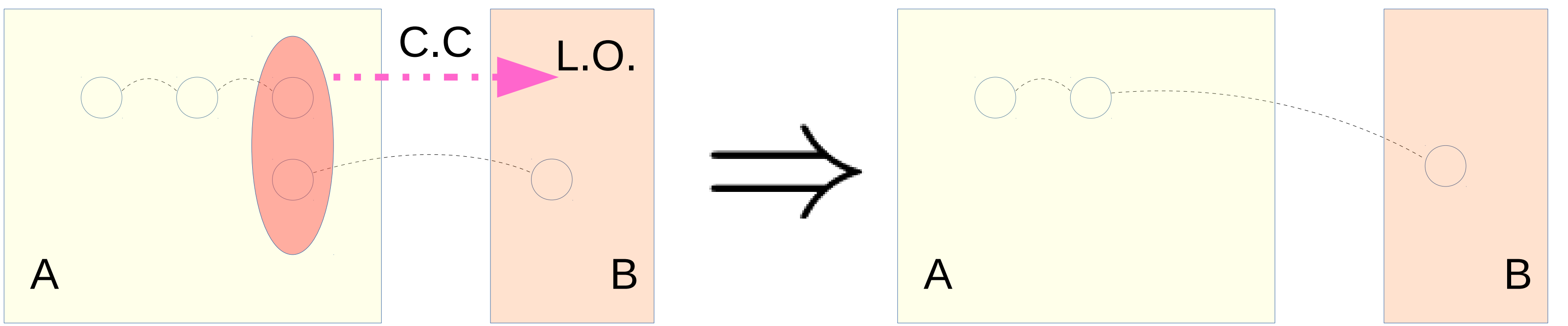
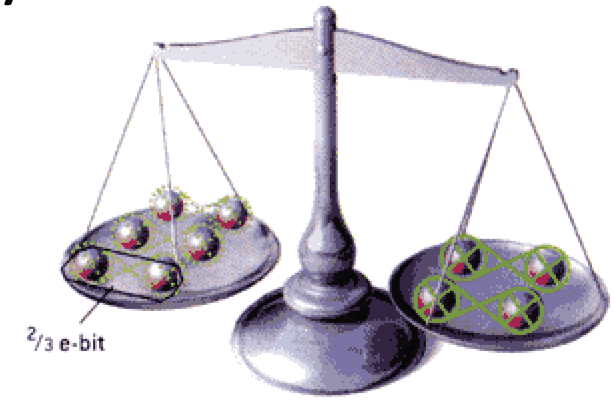


  - Similarly: quantum cryptography using Bell states, dense coding, measurement based quantum computation

- Basis unit of currency: Bell state $|I\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$

Entanglement Entropy



- Theory of quantum information is to a large extent concerned with constructing a resource theory of entanglement: how (efficient) can I convert states to Bell states? What about mixed states? What about the entangling power of a Hamiltonian?

- Fundamental theorem in the theory of entanglement:
  - Given a pure bipartite quantum state $|\psi_{AB}\rangle$ with marginal ρ_A , then I can convert N copies of $|\psi_{AB}\rangle$ into S.N copies of Bell states and vice versa using only local operations and classical communication, with S the entanglement entropy of the state $|\psi_{AB}\rangle$

  - Hence the amount of bipartite quantum correlations can be completely quantified by the entanglement entropy

$$|\psi\rangle = \sum_{ij} X_{ij}|i\rangle|j\rangle \qquad \rho_A = X.X^\dagger \qquad S(A) = -\mathrm{Tr}\,(\rho.\log\rho)$$

    – Open question: neccessary and sufficient conditions for interconvertibility of pure multipartite entangled states
      - e.g. interconvertibility of W-states into GHZ-states is related to border rank problem for multiplication of matrices)

Entanglement for mixed states

- Consider a mixed bipartite state $\rho = \sum_{\alpha} p_\alpha |\psi_\alpha\rangle_{AB} \langle\psi_\alpha|$
- There are 2 natural entanglement measures with an operational meaning :
  - The entanglement cost $E_C(\rho)$

$$\begin{aligned} E_C(\rho) &= \lim_{\epsilon \to 0} \lim_{N \to \infty} S(\epsilon, N) \\ S(\epsilon, N) &= \mathrm{argmin}_{\mathrm{S}, \$ \in \mathrm{LOCC}} \left\| \rho^{\otimes \mathrm{N}} - \$ \left((|\mathrm{I}\rangle\langle\mathrm{I}|)^{\otimes \mathrm{S.N}} \right) \right\| \leq \epsilon \end{aligned}$$

  - The enanglement of distillation $E_D(\rho)$

$$\begin{aligned} E_D(\rho) &= \lim_{\epsilon \to 0} \lim_{N \to \infty} S(\epsilon, N) \\ S(\epsilon, N) &= \mathrm{argmax}_{\mathrm{S}, \$ \in \mathrm{LOCC}} \left\| \$ \left(\rho^{\otimes \mathrm{N}} \right) - (|\mathrm{I}\rangle\langle\mathrm{I}|)^{\otimes \mathrm{S.N}} \right\| \leq \epsilon \end{aligned}$$

- A state is seperable (unentangled) iff it can be written as a mixture of product states:

$$E_D(\rho) = E_C(\rho) = 0 \Rightarrow \exists\, p_\alpha, \rho_\alpha, \sigma_\alpha \, : \rho = \sum_{\alpha} p_\alpha \rho_\alpha \otimes \sigma_\alpha$$

- In general, we have $E_D(\rho) < E_C(\rho)$
  - Otherwise, we could build an "entanglement perpetuum mobile"
  - There exist cases where $E_D(\rho) = 0 < E_C(\rho)$; this phenomenon is called bound entanglement

- For pure states, we obviously have $E_D(\rho) = E_C(\rho) = S\left(|\psi\rangle\right)$
  - But equality can also happen for mixed states iff there exists a local basis in which it holds that

$$(U_A \otimes U_B)\,\rho\,(U_A \otimes U_B)^\dagger = \sum_\alpha p_\alpha |\alpha,\alpha\rangle_{AB}\langle\alpha,\alpha| \otimes |\psi_\alpha\rangle_{AB}\langle\psi_\alpha|$$

  - In that case, $E_D(\rho) = E_C(\rho) = \sum_\alpha p_\alpha S\left(|\psi_\alpha\rangle\right)$

Entanglement and superselection rules

- What happens with the operational definitions of entanglement if there are extra restrictions on the operations that one can implement?
  - e.g. in atomic physics: all physical operations commute with particle number operator
  - In spirit of QIT: what is the resource which allows me to overcome the new limitations?

Entanglement and superselection rules

FV, Cirac PRL '03

- Let us give a consider the simplest example: a system of 2 qubits, but with a SSR of particle number
  - All physical states are of the form

$$\rho = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & x_3 & 0 \\ 0 & \bar{x}_3 & x_4 & 0 \\ 0 & 0 & 0 & x_5 \end{pmatrix}$$

  - Let us consider the state

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

    - This state is seperable (non-entangled) in the usual sense

$$\rho = \frac{1}{16}\sum_{k=0}^{3}\left(|0\rangle + e^{i\pi k/2}|1\rangle\right)\left(\langle 0| + e^{-i\pi k/2}\langle 1|\right) \otimes \left(|0\rangle + e^{i\pi k/2}|1\rangle\right)\left(\langle 0| + e^{-i\pi k/2}\langle 1|\right)$$

but cannot be created using LOCC and local SSR, as its seperable decomposition involves terms of the form $|\pm\rangle = |0\rangle \pm |1\rangle$

Entanglement and superselection rules

- General framework: given a SSR rule which decomposes the Hilbert space into a direct sum of projectors

$$\mathcal{H} = \oplus_{r=1}^{\infty} P_r$$

then under LOCC +local SSR operations, any bipartite state ρ is indistinguishable from

$$\tilde{\rho} = \sum_{r_a, r_b} (P_{r_a} \otimes P_{r_b})\, \rho\, (P_{r_a} \otimes P_{r_b})$$

- Conversely, a state can be prepared using LOCC and local SSR iff $\rho = \tilde{\rho}$

FV, Cirac PRL '03

Entanglement and superselection rules

- Central mantra in QIT: constraints lead to new resources (e.g. LOCC, SSR) and hence must have quantum information theoretic uses
  - Indeed, let us consider the 2 bipartite states

$$|\psi_\pm\rangle = \frac{1}{2}\left(|0\rangle|1\rangle \pm |1\rangle|0\rangle\right)$$

  - If the operations at A and B have to commute with $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then it is impossible to detect the phase $\pm$ with LOCC
    - Off-diagonal elements are invisible for observables commuting with σ_z
    - QIT task: data hiding.
      - The bit can only revealed when the two parties come together
    - Which resource do we need to overcome this limitation?
      - $|\psi_+\rangle$
      - Indeed: a local entangled measurement can then reveal info about the bit:

$$P_1 \otimes P_1 |\psi_\pm\rangle|\psi_+\rangle = \left(|0\rangle|1\rangle\right)_A \left(|1\rangle|0\rangle\right)_B \pm \left(|1\rangle|0\rangle\right)_A \left(|0\rangle|1\rangle\right)_B$$

Entanglement and superselection rules

- In the case of 1 global superselection rule (e.g. U(1)):
  - Quantification of resource is terms of superselection induced variance:

$$V(\psi) = 4\left(\langle\psi|\hat{N}_A^2 \otimes \mathbb{1}_B|\psi\rangle - \langle\psi|\hat{N}_A \otimes \mathbb{1}_B|\psi\rangle^2\right)$$

  - Using standard QIT tools, we can then prove:

Theorem *Consider N copies of a state $|\phi\rangle$ with entropy of entanglement $E(\phi)$ and SiV $V(\phi)$, then there exists an asymptotically reversible conversion*

$$|\phi\rangle^N \leftrightarrow \Big[|01\rangle_A|10\rangle_B+|10\rangle_A|01\rangle_B\Big]^{\otimes NE(\phi)} \otimes \sum_n c_n|n\rangle_A|N-n\rangle_B,$$

where the coefficients c_n are Gaussian distributed with SiV $NV(\phi)$, and $|n\rangle$ denotes the state $|\underbrace{1\cdots1}_{n}0\cdots0\rangle$.

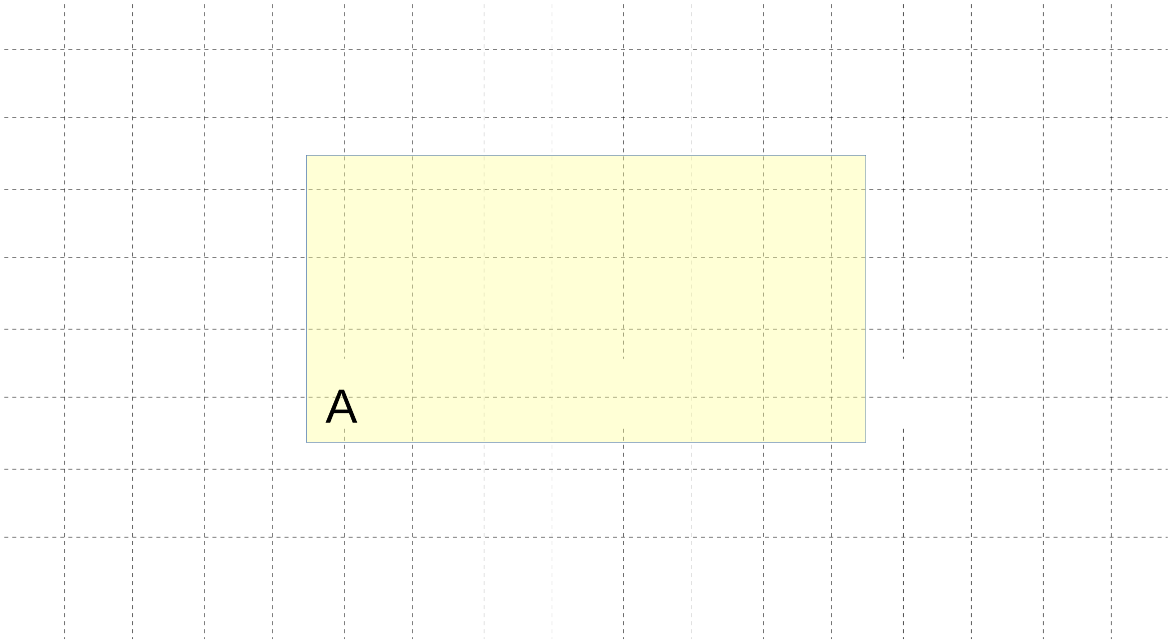
Schuch, FV, Cirac '04

Entanglement in symmetry protected phases of matter

- The same discussion can of course be done for other global symmetries (e.g. SU(2), CPT, ...)
- When dividing the system into 2 regions, we assume that all local observables and Hamiltonian terms commute with the global symmetry in the respective regions

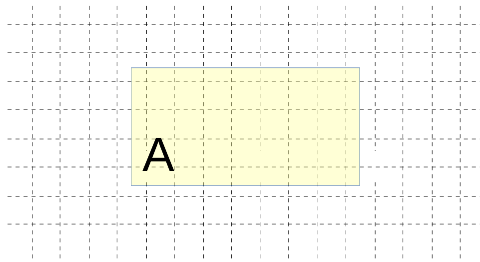
$$U(g)^{\otimes N}|\psi\rangle = |\psi\rangle$$

$$\left[\otimes_{k\in A} U_k(g), \hat{O}_A\right] = 0 = \left[\otimes_{k\in B} U_k(g), \hat{O}_B\right] =$$



Entanglement in symmetry protected phases of matter

- Let r label the irreps of $\otimes_{k\in A} U_k(g)$ and hence denote the charge in region A, and r* the dual irrep denoting the charge in region B, then the group action on the Hilbert space in A can be represented as



$$|r,\alpha\rangle \to \Gamma^r(g)|r,\alpha\rangle$$
$$|r^*,\beta\rangle \to \bar{\Gamma}^r(g)|r^*,\beta\rangle$$

- The physical Hilbert space is given by

$$\Pi_{symm} = \sum_r \Pi_r^A \otimes \Pi_{r^*}^B \qquad |\psi\rangle = \sum_{r,\alpha,\beta} c_{r\alpha\beta}|r\alpha\rangle|r^*\beta\rangle$$

and therfore the reduced density matrix on A is given by

$$\begin{aligned}\rho_A &= \sum_r p_r|r\rangle\langle r| \otimes \left(\frac{1}{p_r}\sum_{\alpha\alpha'}\left(\sum_\beta c_{r\alpha\beta}\bar{c}_{r\alpha'\beta}\right)|\alpha\rangle\langle\alpha'|\right) \\ &= \sum_r p_r|r\rangle\langle r| \otimes \rho_r^A\end{aligned}$$

Entanglement in symmetry protected phases of matter

$$\rho_A = \sum_r p_r |r\rangle\langle r| \otimes \rho_r^A$$

- The entanglement entropy of this state consists of 2 parts:

$$S(\rho_A) = H(p_r) + \sum_r p_r S\left(\rho_r^A\right)$$

- Any local observable/operation has to leave the irrep label invariant, and hence the entropy associated to r is purely "classical"(can only be used for creating classical correlations); the useful entanglement (entanglement of distillation) is given by the second part:

$$E(|\psi\rangle) = \sum_r p_r S\left(\rho_A^r\right)$$

  – Note however that this is not longer true when we have several copies available: cfr. Superselection induced variance

Gauging SPT phases: zero coupling gauge theories

- Given an SPT state on a lattice, it is possible to lift the global symmetry to a local one by introducing gauge degrees of freedom on the links (Levin & Gu '12, Haegeman et al. '15)

$$G|\psi\rangle = \prod_{v\in\Lambda} \int dg_v U_v(g_v)|\psi\rangle \bigotimes_e |g_{v_{e-}} g_{v_{e+}}^{-1}\rangle_e$$

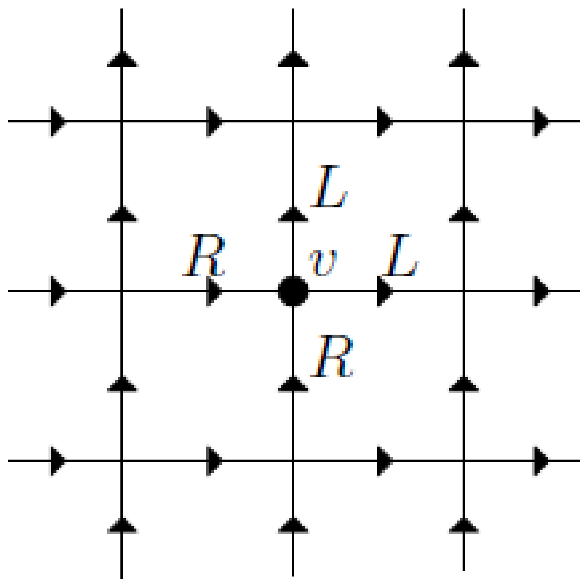
  - Wavefunction analogue of minimal coupling procedure
  - Lifts SPT phases to so-called quantum doubles
    - Example: paramagnet $|+\rangle^{\otimes N} \rightarrow$ toric code (Z2 gauge theory)

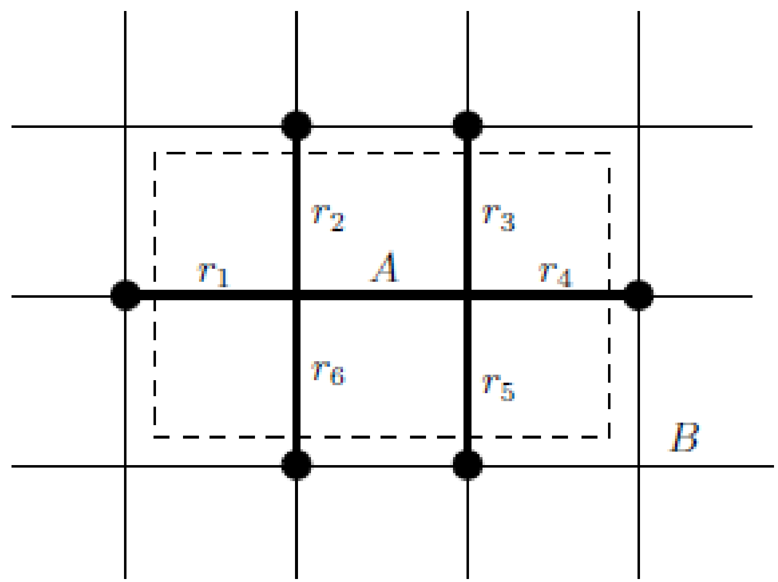
- This gauging procedure obviously leads to an extensive number of superselection rules; how should we quantify “useful” entanglement in those gauge theories?
  - Spoiler: gauging leaves entanglement of distillation invariant (e.g. zero for the case of toric code)

Entanglement in lattice gauge theories

- Setting: system with extensive number of superselection rules

$$U_v(g) = V_v(g) \otimes_{e\in E_v^+} R_e(g) \otimes_{e\in E_v^-} L_e(g) \qquad\qquad U_v(g)|\psi\rangle = |\psi\rangle$$





  - Operational definition of entanglement can be obtained along the lines explained before

Entanglement in lattice gauge theories

- Question has been adressed (and solved) many times, although a precise operational meaning of entanglement seems to have only been obtained very recently:
  - Buividovich and Polikarpov '08
  - Donnely '12
  - Casini, Huerta, Rosabal '14
  - Radividic '14
  - Gromov and Santos '14
  - Aoki, Iritani, Nozaki, Numassawa, Shiba, Tasaki '15
  - Ghosh, Soni, Trivedi '15
  - ...

Entanglement in lattice gauge theories

- We proceed as before (and as done in all references given before):
  - Let us consider the gauge constraints crossing the border between regions A and B
  - For region A, let's define the basis $|\vec{r}, \vec{i}, \alpha\rangle$ with $\vec{r}$ labeling the equivalence classes of irreps of G on the different links, $\vec{i}$ enumerating a basis in the associated representation space, and α the multiplicity space. The local Hilbert space therefore has the following direct sum structure:

    $$\mathcal{H}_A = \oplus_{\vec{r}} \mathcal{H}^{\vec{r}}_{A_g} \otimes \mathcal{H}^{\vec{r}}_{A_m}$$

    Note that the representation space is trivial in the Abelian case.
  - Following the Peter-Weyl theorem, any physical state (exhibiting the gauge symmetry) is now defined in the physical space

    $$\mathcal{H}_{\text{phys}} = \bigoplus_{\vec{r} \in \Lambda^n} \left(|\phi^{\vec{r}}\rangle_{gAB} \otimes \mathcal{H}^{\vec{r}}_{Am} \otimes \mathcal{H}^{\vec{r}}_{Bm} \right) \qquad |\phi^{\vec{r}}\rangle_{gAB} = \sum_{\vec{i}} |\vec{i}\rangle_A |\vec{i}\rangle_B / \sqrt{d_{\vec{r}}}$$

    where $d_{\vec{r}} = \prod_v d_{r_v}$ denotes the dimension of the direct product irrep $\vec{r}$

Entanglement in lattice gauge theories

- All local observables have to commute with the projection

$$P_{\vec{r}} = |\phi^{\vec{r}}\rangle\langle\phi^{\vec{r}}|_{gAB} \otimes \mathbb{I}_{\mathcal{H}^{\vec{r}}_{Am}} \otimes \mathbb{I}_{\mathcal{H}^{\vec{r}}_{Bm}} = W_{\vec{r}} W_{\vec{r}}^{\dagger} \qquad W_{\vec{r}} = |\phi^{\vec{r}}\rangle_{gAB} \otimes \mathbb{I}_{\mathcal{H}^{\vec{r}}_{Am}} \otimes \mathbb{I}_{\mathcal{H}^{\vec{r}}_{Bm}}$$

and hence the algebra of local observable has a nontrivial center spanned by $P_{\vec{r}}$ (see e.g. papers of Casini et al.)

- Due to the direct sum structure and the structure in representation space, the entropy of the reduced density matrix of any quantum state in $\mathcal{H}_{\text{phys}}$ can be decomposed in three terms: (see also Donnely and other references)

$$S_A = -\mathrm{Tr}\rho_{\mathrm{A}} \log \rho_{\mathrm{A}} = -\sum_{\tilde{\mathrm{r}}} \mathrm{p}_{\tilde{\mathrm{r}}} \log \mathrm{p}_{\tilde{\mathrm{r}}} + \sum_{\tilde{\mathrm{r}}} \mathrm{p}_{\tilde{\mathrm{r}}} \log \mathrm{d}_{\tilde{\mathrm{r}}} + \sum_{\tilde{\mathrm{r}}} \mathrm{p}_{\tilde{\mathrm{r}}} \mathrm{S}^{\tilde{\mathrm{r}}}_{\mathrm{A}}$$

$$\rho^{\vec{r}}_A = \frac{1}{p_{\vec{r}}} \mathrm{Tr}_{\mathcal{H}^{\vec{r}}_{Bm}} [W^{\dagger}_{\vec{r}} |\psi\rangle\langle\psi| W_{\vec{r}}]$$

- As in the case of SSR, the first two terms are useless from the entanglement point of view, as we cannot touch them. By depolarizing, we get a mixed state

$$\sigma^{\psi} = \sum_{\vec{r} \in \Lambda^n} P_{\vec{r}} |\psi\rangle\langle\psi| P_{\vec{r}}$$ and hen $$E^{\text{gauge}}_D(|\psi\rangle) = E_D(\sigma^{\psi}) = \sum_{\vec{r}} p_{\vec{r}} S^{\vec{r}}_A$$

- Examples:
  - Z2 lattice gauge theory (toric code): $E_D = 0$
  - Double semion string net: $E_D = 0$
  - Pure Abelian lattice gauge theory at zero coupling: $E_D = 0$
  - Pure non-Abelian gauge theory at zero coupling:

    $$E_D^{\text{gauge}}(|\psi_0\rangle) = \frac{1}{|G|^{n-1}} \sum_{\vec{r}\in\Lambda^n} d_{\vec{r}} N_{\vec{r}}^1 \log N_{\vec{r}}^1$$

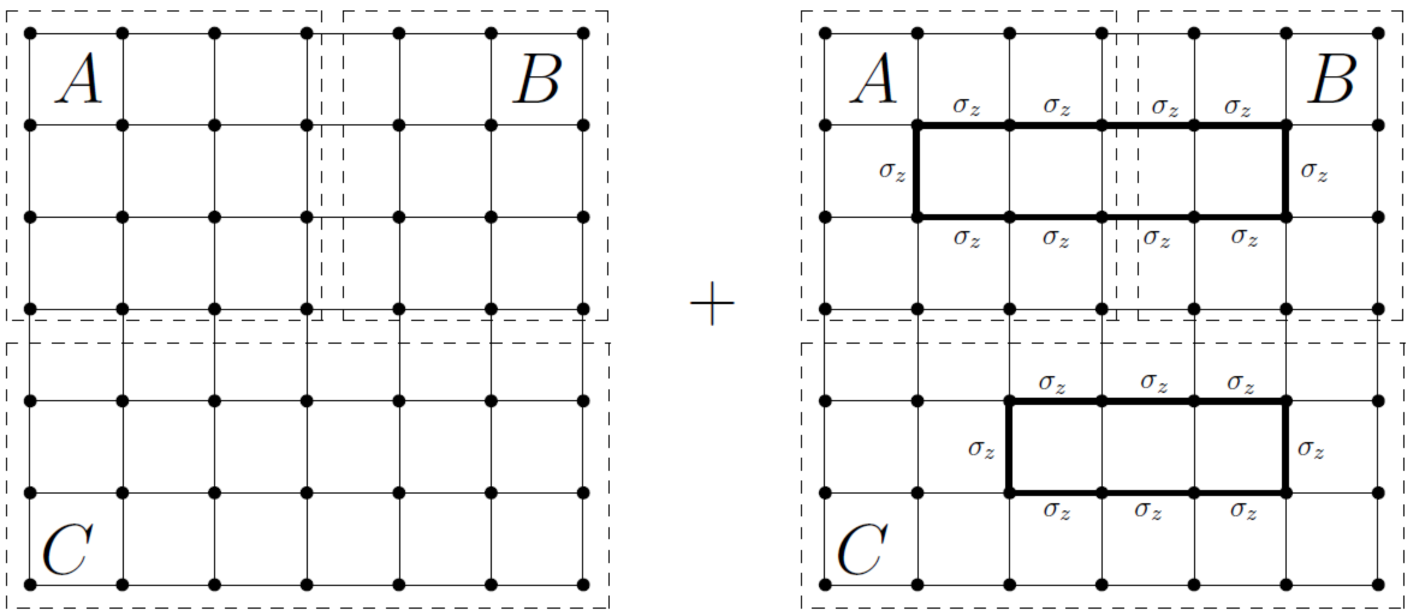
    with $N_{\vec{r}}^1$ the number of inequivalent ways the representations $\vec{r}$ can fuse to the scalar one
  - Perturbed toric code (Z2 gauge) with a magnetic field:
    - Careful: entropy is not an observable (unstable under perturbations: Fannes inequality) and therefore we have to use formalism of quasi-adiabatic evolution of Hastings

$$E_D(|\psi(\epsilon)\rangle) = nH(\cos^2\epsilon, \sin^2\epsilon) - \log 2 \qquad (\epsilon > 0, n \to \infty)$$

Notice the topological quantum entropy correction!

Entanglement of distillation violates subadditivity

- Let's consider a Z2 gauge theory and a quantum state which is a superposition of the product state $|\psi\rangle = |+\rangle|+\rangle|+\rangle\cdots$ and the state obtained by applying two Wilson loops to it:



- We have $\mathrm{E}_D(A/BC) = 0 = E_D(B/AC)$, $E_D(AB/C) = 1$
and hence $E_D(AB) > E_D(A) + E_D(B)$

Conclusion

- Theory of entanglement is a theory of resources
- In the presence of global superselection rules, the rules of the game change, and we get new restrictions (but also new resources to overcome them)
  - Distillable entanglement quantifies the “useful” entanglement (i.e. number of Bell states which can be extracted), and is generically strictly smaller than entanglement entropy in the presence of superselection rules
- Lattice gauge theories exhibit an extensive number of superselection rules, and the distillable entanglement can be obtained by looking at the average entropy in the different superselection blocks
  - For Abelian lattice gauge theories at zero coupling, $E_D = 0$, while for non-Abelian ones, we get non-zero distillable entanglement
  - Distillable entanglement exhibits topological quantum entropy correction
  - Distillable entanglement violates (strong) subadditivity
  - To get universal quantity: look at regions A,B seperated by a region C