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# An alternative flow equation from the regulator-sourced 2PI effective action

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# A few quick thank yous

To the organisers and (in advance) to you for listening in.

To my collaborators:

**Elizabeth Alexander** (former undergraduate at University of Nottingham)

**Björn Garbrecht** (Technische Universität München)

**Jordan Nursey** (former undergraduate at University of Nottingham)

**Paul M. Saffin** (University of Nottingham)

To those who have patiently borne with this before:

participants of UK QFT V, participants of the International Seminars on Asymptotic Safety, the TPPC Group at King’s College London, the Theoretical Physics of the Early Universe Group at TUM, members of the Physics Department at the Università di Genova.

# Outline

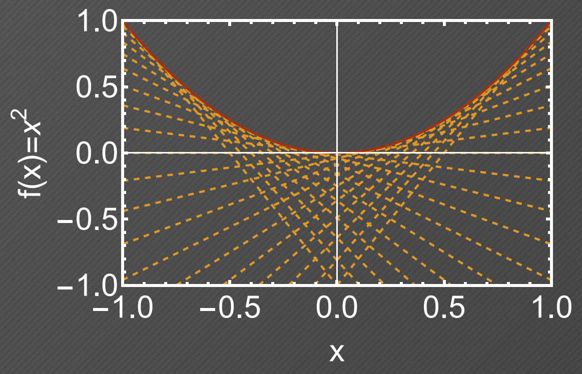
- an **introductory** remark
- a bit of a **recap**
- an alternative take on the **(2PI) effective action**
- an alternative **flow equation** for the **exact RG**
- a few **concluding** remarks

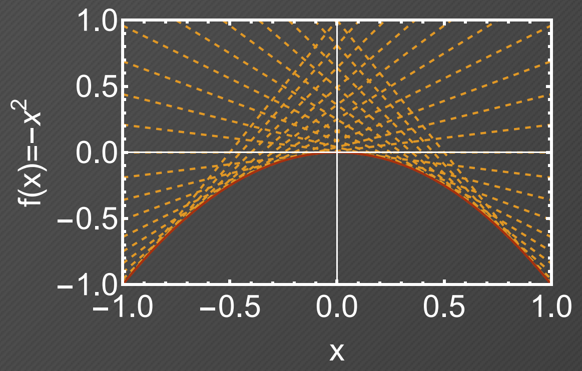
# An introductory remark

**quantum effective actions:**

systematic approaches to non-perturbative effects in quantum field theory

# A bit of a recap




- $f$ is **strictly convex** or **concave** on an interval $I$.
- $f''$ has a fixed sign on $I$.
- $f'$ is monotonic, single-valued and invertible on $I$.
- $f$ can be expressed as the **set of ordered pairs** $\{(x, f(x))\}$ or the **tangents** to $f$.
- **Legendre transform:**
  maps $\{(x, f(x))\}$ to $\{(x^*, f^*(x^*)\}$, specifying the **gradients** and **intercepts** of the **tangents** to $f$.

## An alternative take on the (2PI) effective action

# Using a zero-dimensional QFT

- Take a zero-dimensional QFT with action [PM & Saffin '19]

$$S(\Phi) = \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4$$

- The partition function in the presence of external sources $\{J\}$ is

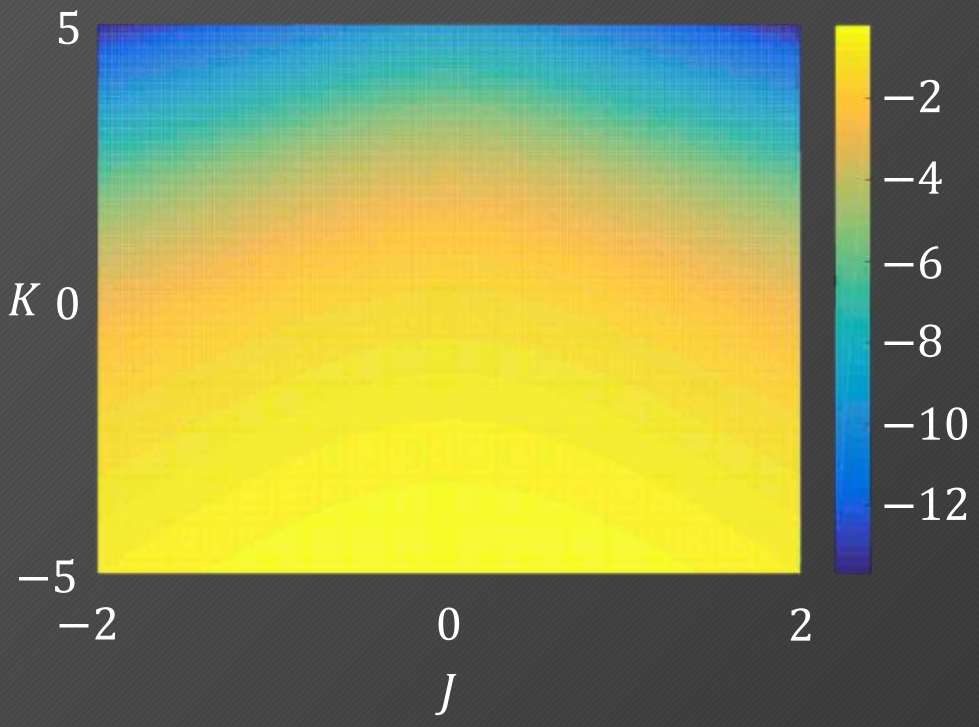
$$Z(\{J\}) = \mathcal{N}\int_{-\infty}^{+\infty} \mathrm{d}\Phi \, \exp\left[-\frac{1}{\hbar}\left(S(\Phi) - \sum_{n=1}^{\infty}\frac{1}{n!}J_n\Phi^n\right)\right]$$

- This talk focuses on the 2PI effective action with

$$\{J\} = \{J_1 \equiv J, J_2 \equiv K\}$$

An alternative take on the (2PI) effective action

# Concavity of the Schwinger function




- The Schwinger function

$$W = -\hbar \ln Z(J, K)$$

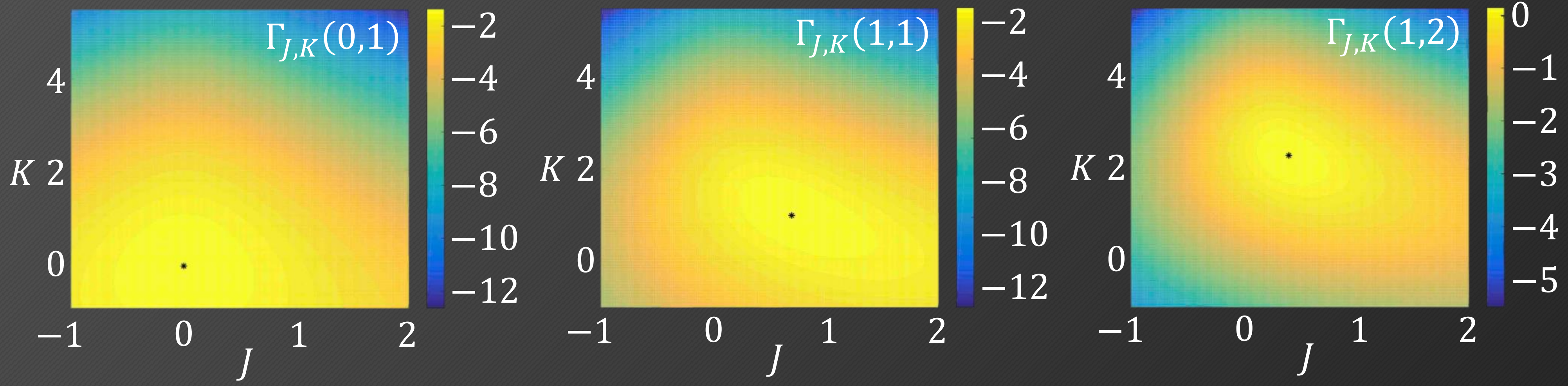
  is concave, even for a non-convex classical potential.
- Left: $m^2 = -1, \lambda = 6$ [PM & Saffin '19].
- Gradient w.r.t. $-J$ is $\langle \Phi \rangle_{J,K}$.
- Gradient w.r.t. $-K/2$ is $\langle \Phi^2 \rangle_{J,K}$.

An alternative take on the (2PI) effective action

# Convex conjugate variables

$$\Gamma_{J,K}(\phi,\Delta) \equiv W(J,K) + J\phi + \frac{1}{2}K \times \left(\phi^2 + \hbar\Delta\right)$$

The maximum in the $(J,K)$ plane is determined by the variables $(\phi,\Delta)$.




[PM & Saffin '19]

An alternative take on the (2PI) effective action

# The 2PI effective action

- The double Legendre transform

$$\Gamma(\phi, \Delta) = \max_{J,K} \Gamma_{J,K}(\phi, \Delta)$$

  corresponds to the value of the maxima as a function of $(\phi, \Delta)$.

- The locations of the maxima correspond to extremal sources $(\mathcal{J}, \mathcal{K})$:

$$\partial_J \Gamma_{J,K}(\phi, \Delta)\Big|_{J=\mathcal{J},K=\mathcal{K}} = 0 = \partial_K \Gamma_{J,K}(\phi, \Delta)\Big|_{J=\mathcal{J},K=\mathcal{K}}$$

- The extremization yields the 2PI effective action with

$$\phi = -\hbar \partial_J W(J, K)\Big|_{J=\mathcal{J},K=\mathcal{K}} \quad \text{and} \quad \hbar\Delta = 2\hbar \partial_K W(J, K)\Big|_{J=\mathcal{J},K=\mathcal{K}} - \phi^2$$

# The 2PI effective action(s)

- $\mathcal{J} \equiv \mathcal{J}(\phi, \Delta)$ and $\mathcal{K} \equiv \mathcal{K}(\phi, \Delta)$
- Fixing one of $\mathcal{J}$ or $\mathcal{K}$ determines $\Delta \equiv \Delta(\phi)$ or similarly $\phi \equiv \phi(\Delta)$.

In the full field-theoretic case: [Garbrecht & PM '16]

- $\mathcal{K}_{x,y} = 0$ gives the 1PI effective action (EA) [Jackiw '74].
- $\mathcal{J}_x = \mathcal{K}_{x,y} = 0$ gives the CJT EA [Cornwall, Jackiw & Tomboulis '74].
- $\mathcal{J}_x = 0$ and $\mathcal{K}_{x,y} \propto \delta^d(x-y)$ gives the 2PPI EA [Verschelde & Coppens '92].
- Fixing $\mathcal{K}_{x,y}$ by Ward identities $\sim$ the symmetry-improved EA [Pilaftsis & Teresi '13].

An alternative take on the (2PI) effective action

# A caveat on recovering the CJT EA

- $\mathcal{J}$ and $\mathcal{K}$ can be chosen so that the saddle point in the path integral of the partition function coincides with the quantum trajectory:

$$\left.\frac{\delta S[\phi]}{\delta\phi}\right|_{\phi=\varphi} - \mathcal{J}[\phi,\Delta] - \mathcal{K}[\phi,\Delta]\varphi = \left.\frac{\delta\Gamma[\phi,\Delta]}{\delta\phi}\right|_{\phi=\varphi,\Delta=\mathcal{G}} = 0$$

- Self-consistency then requires [Garbrecht & PM ’16; PM & Saffin ‘19]

$$\mathcal{J}[\varphi,\mathcal{G}] + \mathcal{K}[\varphi,\mathcal{G}]\varphi = 0$$

- Key if the quantum trajectory is non-perturbatively far from the classical one, e.g., radiatively induced instabilities. [Garbrecht & PM ’15; Plascencia & Tamarit ‘16]

An alternative flow equation for the exact RG

# The regulator-sourced 2PI effective action

What if $\mathcal{K}_{x,y}[\phi, \Delta] = -\mathcal{R}_{k;x,y}$ is (minus) the inverse Fourier transform of the regulator and $\mathcal{J}_x[\phi, \Delta]$ is such that $\phi$ is independent of the scale $k$? [Alexander, PM, Nursey & Saffin '19]

$$\partial_k \Gamma[\phi, \Delta] = \frac{\delta \Gamma[\phi, \Delta]}{\delta \phi_x} \partial_k \phi_x + \frac{\delta \Gamma[\phi, \Delta]}{\delta \Delta_{x,y}} \partial_k \Delta_{x,y}$$

$$\partial_k \phi_x = -\partial_k \frac{\delta W[\mathcal{J}, \mathcal{K}]}{\delta \mathcal{J}_x} = 0$$

$$\partial_k \Gamma\big[\phi, \Delta_k[\phi]\big] = -\frac{\hbar}{2} \mathcal{R}_{k;x,y} \partial_k \Delta_{k;x,y}[\phi]$$

An alternative flow equation for the exact RG

# The regulator-sourced 2PI effective action

What if $\mathcal{K}_{x,y}[\phi, \Delta] = -\mathcal{R}_{k;x,y}$ is (minus) the inverse Fourier transform of the regulator and $\mathcal{J}_x[\phi, \Delta]$ is such that $\phi$ is independent of the scale $k$? [Alexander, PM, Nursey & Saffin ‘19]

$$\partial_k \Gamma\big[\phi, \Delta_k[\phi]\big] = -\frac{\hbar}{2}\mathcal{R}_{k;x,y}\partial_k \Delta_{k;x,y}[\phi]$$

versus

$$\partial_k \Gamma_{\text{av}}^{1\text{PI}}[\phi] = +\frac{\hbar}{2}\Delta_{k;x,y}[\phi]\partial_k \mathcal{R}_{k;x,y}$$

$$\partial_k \Gamma\big[\phi, \Delta_k[\phi]\big] = \partial_k \Gamma_{\text{av}}^{1\text{PI}}[\phi] - \frac{\hbar}{2}\partial_k\big(\mathcal{R}_{k;x,y}\Delta_{k;x,y}[\phi]\big)$$

[for average 1PI, see Wetterich ’91 & ’93; Morris ’94; Ellwanger ‘94]

An alternative flow equation for the exact RG

# Boundary conditions and closure

**Boundary conditions:**

- $k \to 0$, $\mathcal{R}_k \to 0$, and the regulator-sourced 2PI and average 1PI effective actions coincide with the 1PI effective action $\Gamma^{1\mathrm{PI}}[\phi] = W[\mathcal{J}] + \mathcal{J}_x\phi_x$.
- $k \to \infty$, both coincide with the classical action $S[\phi]$.

**Closure:**

- It follows from convexity of the 2PI effective action when $\delta\phi/\delta\mathcal{K} = 0$ that

$$\Delta^{-1}_{k;x,y}[\phi] = \frac{\delta^2\Gamma[\phi,\Delta_k]}{\delta\phi_x\delta\phi_y} = \frac{\delta^2 S[\phi]}{\delta\phi_x\delta\phi_y} + \mathcal{R}_{k;x,y} + \mathcal{O}(\hbar)$$

# An alternative flow equation for the exact RG

## Imports

**Observation:**

We have two closed systems with the same boundary conditions, but with different flow equations.

**Take home question:**

To which should we apply the standard zoology of Ansaetze: the average 1PI effective action or the regulator-sourced 2PI effective action?

An alternative flow equation for the exact RG

# An example

- Take the derivative expansion, making the Ansatz [see, e.g., Berges, Tetradis & Wetterich '02]

$$\Gamma[\phi, \Delta_k] = \int \mathrm{d}^d x \left[ U_k(\rho) + \frac{1}{2} Z_k\big(\rho, (\partial\phi)^2\big) \partial\phi \cdot \partial\phi + \mathcal{O}(\partial^4) \right], \qquad \rho \equiv \phi^2/2$$

with $U_k(\rho) = \frac{1}{2} g_k \left(\rho - \bar{\rho}_k\right)^2 + \Lambda_k$ and $\Delta_k(\rho; q^2) = \dfrac{1}{Z_k(\rho; q^2) q^2 + \mathcal{R}_k(q^2) + U_k'(\rho) + 2\rho U_k''(\rho)}$

- Define

$$\kappa_k \equiv Z_k(\bar{\rho}_k, k^2) k^{2-d} \bar{\rho}_k \quad \text{and} \quad \lambda_k \equiv Z_k^{-2}(\bar{\rho}_k, k^2) k^{d-4} g_k$$

- Casually neglect the anomalous dimension and take the Litim regulator [Litim '02]

$$\mathcal{R}_k(q^2) = Z_k(\bar{\rho}_k, k^2)(k^2 - q^2)\theta(k^2 - q^2)$$

An alternative flow equation for the exact RG

# An example (threshold functions)

**Regulator-sourced 2PI:**

$$\partial_t U_k(\rho) = -\frac{1}{2}\int \frac{\mathrm{d}^d q}{(2\pi)^d}\mathcal{R}_k(q^2)\partial_t\Delta_k(\rho;q^2)$$

$$\partial_t\Lambda_k = \frac{8v_d k^d}{d(d+2)}\frac{1}{(1+2\kappa_k\lambda_k)^2}$$

$$\partial_t\kappa_k = (2-d)\kappa_k + \frac{48v_d}{d(d+2)}\frac{1}{(1+2\kappa_k\lambda_k)^3}$$

$$\partial_t\lambda_k = (d-4)\lambda_k + \frac{432v_d}{d(d+2)}\frac{\lambda_k^2}{(1+2\kappa_k\lambda_k)^4}$$

**Average 1PI:**

$$\partial_t U_k(\rho) = +\frac{1}{2}\int \frac{\mathrm{d}^d q}{(2\pi)^d}\Delta_k(\rho;q^2)\partial_t\mathcal{R}_k(q^2)$$

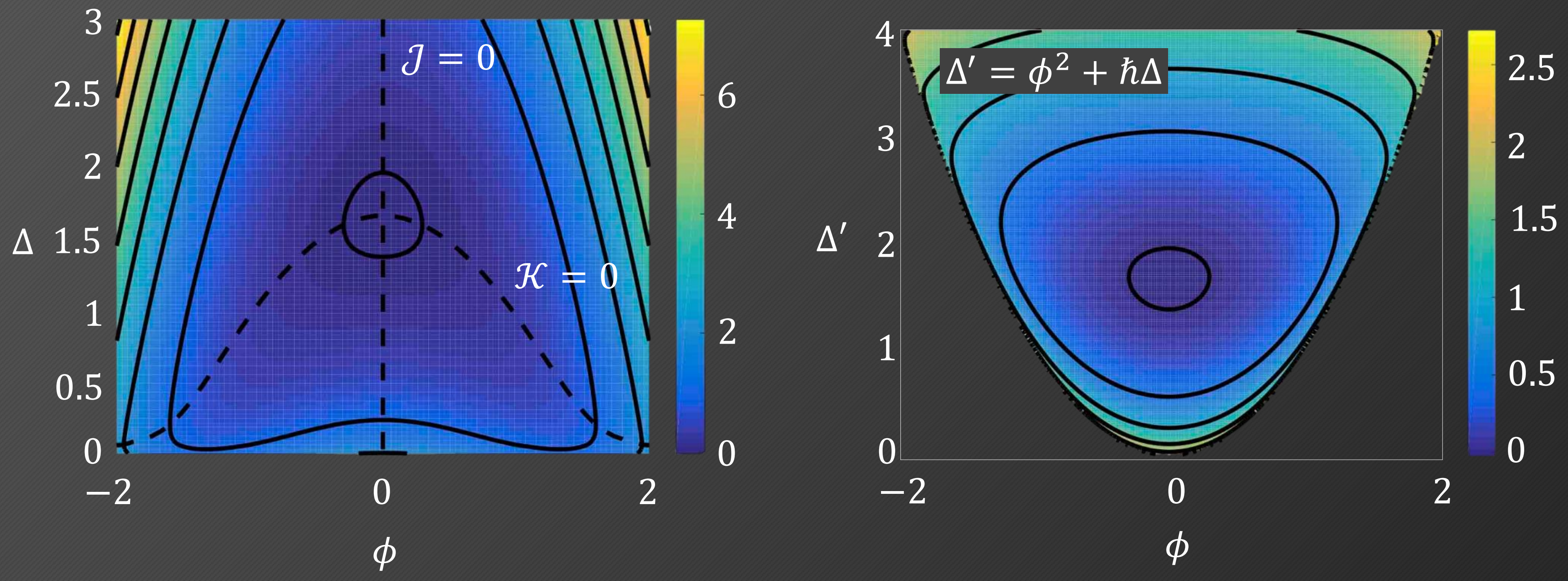
$$\partial_t\Lambda_k = \frac{4v_d k^d}{d}\frac{1}{1+2\kappa_k\lambda_k}$$

$$\partial_t\kappa_k = (2-d)\kappa_k + \frac{12v_d}{d}\frac{1}{(1+2\kappa_k\lambda_k)^2}$$

$$\partial_t\lambda_k = (d-4)\lambda_k + \frac{72v_d}{d}\frac{\lambda_k^2}{(1+2\kappa_k\lambda_k)^3}$$

[Alexander, PM, Nursey & Saffin ’19; cf. Berges, Tetradis & Wetterich ‘02]

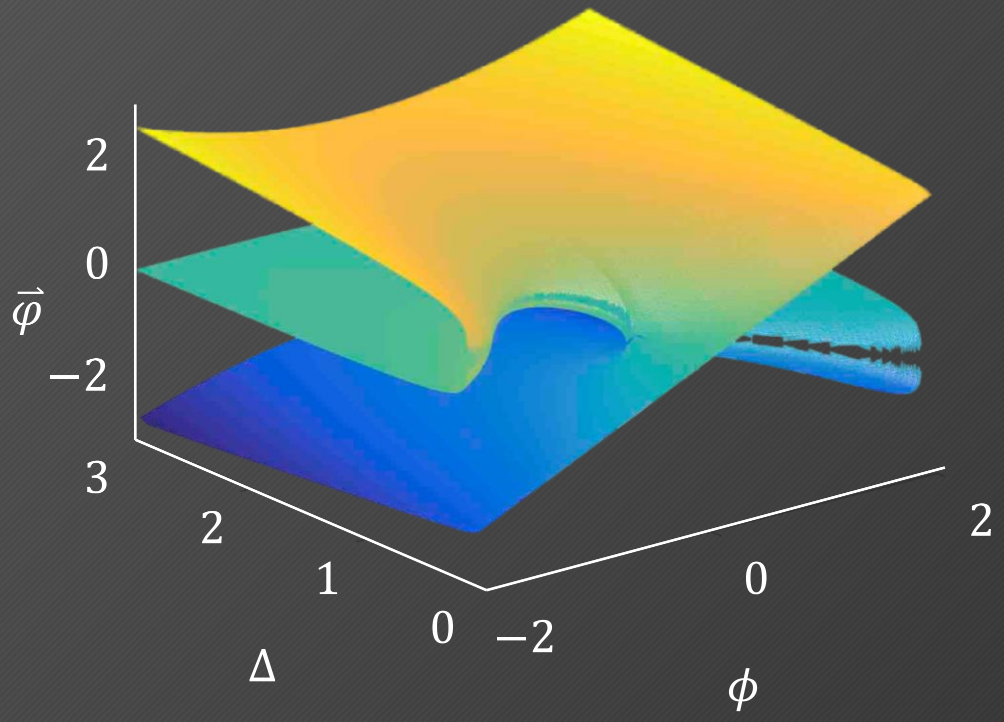
# A remark on convexity




[PM & Saffin '19]

An alternative flow equation for the exact RG

# A remark on multiple saddle points




- The variables $(\phi, \Delta)$ determine the type and number of saddle points, i.e.,

$$\{\varphi_i\} \equiv \{\varphi_i\}(\phi, \Delta)$$

- In our zero-dimensional example, with $m^2 = -1$ and $\lambda = 6$, we have 1 to 3 saddles (left).

[PM & Saffin '19]

An alternative flow equation for the exact RG

# A remark on symmetry preservation

- Consider a globally $O(2)$-invariant model with SSB ($m^2 < 0$):

$$\mathcal{L} = \frac{1}{2}\partial\Phi_i \cdot \partial\Phi_i + \frac{1}{2}m^2\Phi_i^2 + \frac{\lambda}{4}\Phi_i^2\Phi_j^2, \qquad i,j = 1,2$$

- In the Hartree-Fock approximation, the Goldstone boson is spuriously massive in the SSB phase.
- Using the Ward identities to constrain $\mathcal{K}_{ij}[\phi,\Delta]$ resolves this problem [Garbrecht & PM '16]:

$$\mathcal{K}^{GG}_{x,y}[\phi,\Delta] = \mathcal{K}^{HH}_{x,y}[\phi,\Delta] = -2\hbar \frac{\delta\Gamma_2^{(\mathrm{HF})}[\phi,\Delta]}{\delta\Delta^{HH}_{x,y}}\bigg|_{\phi=\varphi,\Delta=\mathcal{G}} = -\hbar\lambda(3\mathcal{G}^{HH}_{x,x} + \mathcal{G}^{GG}_{x,x})\delta^4(x-y)$$

  i.e., replace the HF Goldstone self-energy by the Higgs self-energy. [cf. Pilaftsis & Teresi '13]

- The pathological mass cancels algebraically, and we get the correct second-order phase transition.

# A few concluding remarks

- A new perspective on the quantum effective action:
  - Exploiting the full role of the external sources.
  - Allowing us to map between different realisations of the quantum effective action.
  - And derive an alternative flow equation for the exact RG.

- Outstanding questions:
  - Is there a unique realisation of the effective action to which we should apply the usual Ansaetze for solving the flow equations?
  - How significant are the differences between the two flow equations?
  - Can we improve issues of gauge dependence in the exact RG by constraining sources appropriately? [cf. Garbrect & PM '16; Lavrov, '20]

# Thank you

- **Questions** or **comments**?

- Chat with me on the **slack channel** **11-2_millington-peter**.
- Chat with me on **Remo** later: **https://live.remo.co/e/erg2020-nov2**.
- Message me on **twitter** **@pwmillington** or on **Skype** at **peterwmillington**, or **email** me at **p.millington@nottingham.ac.uk**.

- **Slides** and **key references** on the **slack channel** **11-2_millington-peter**.

# References

## Please see references therein too.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# Back up slides

# A bit of a recap

## The average 1PI effective action

- The average 1PI effective action is

$$\Gamma_{\text{av}}^{\text{1PI}}[\phi, \mathcal{R}_k] = W[\mathcal{J}, \mathcal{R}_k] + \mathcal{J}_x \phi_x - \frac{1}{2} \phi_x \mathcal{R}_{k;x,y} \phi_y$$

- The regulator-sourced 2PI effective action is (schematically)

$$\Gamma[\phi, \Delta_k] = \text{"} \max_{-\mathcal{R}} \Gamma_{\text{av}}^{\text{1PI}}[\phi, \mathcal{R}_k] \text{"}$$

  i.e., the two are related by a Legendre transform.

# Closure
## Extra details

- Convexity implies that

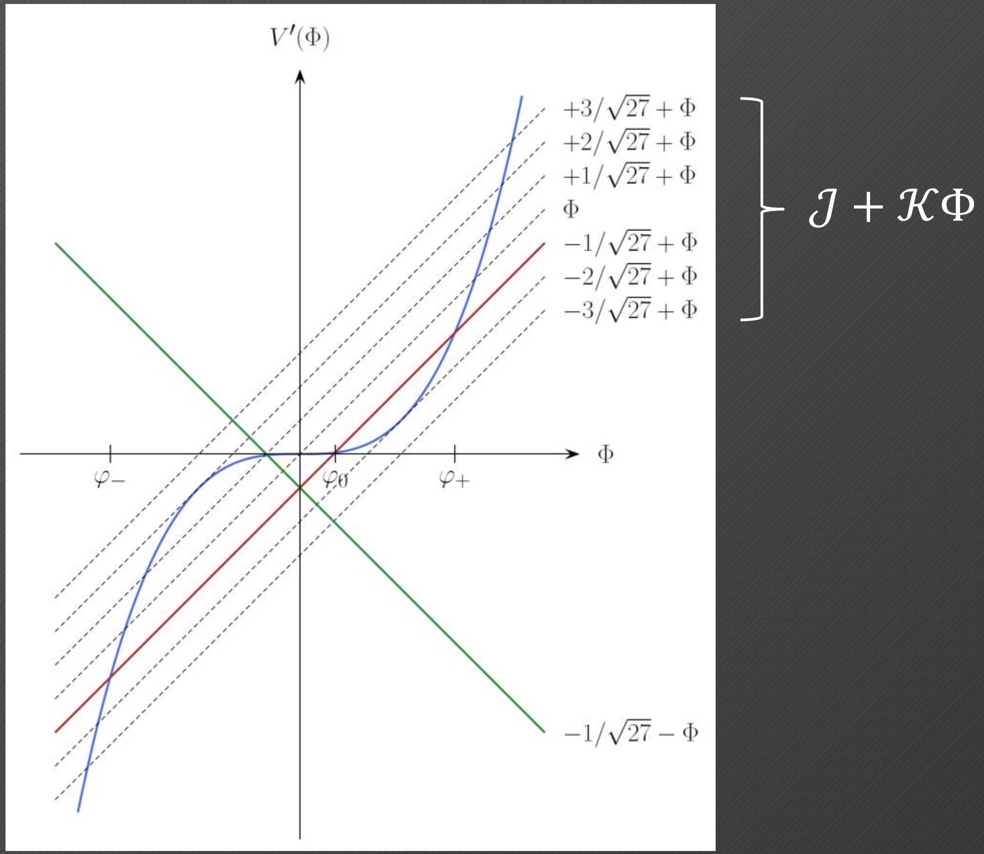
$$-\frac{\delta^2\Gamma[\phi,\Delta_k]}{\delta\phi_x\delta\phi_y}\frac{\delta^2 W[\mathcal{J}_k,\mathcal{K}_k]}{\delta\mathcal{J}_{k;x}\delta\mathcal{J}_{k;y}}-\frac{\delta^2\Gamma[\phi,\Delta_k]}{\delta\phi_x\delta\Delta'_{k;y,z}}\frac{\delta^2 W[\mathcal{J}_k,\mathcal{K}_k]}{\delta\mathcal{J}_{k;x}\delta\mathcal{K}'_{k;y,z}}=1, \quad \mathcal{K}'_{k;x,y}\equiv\mathcal{K}_{k;x,y}/2$$

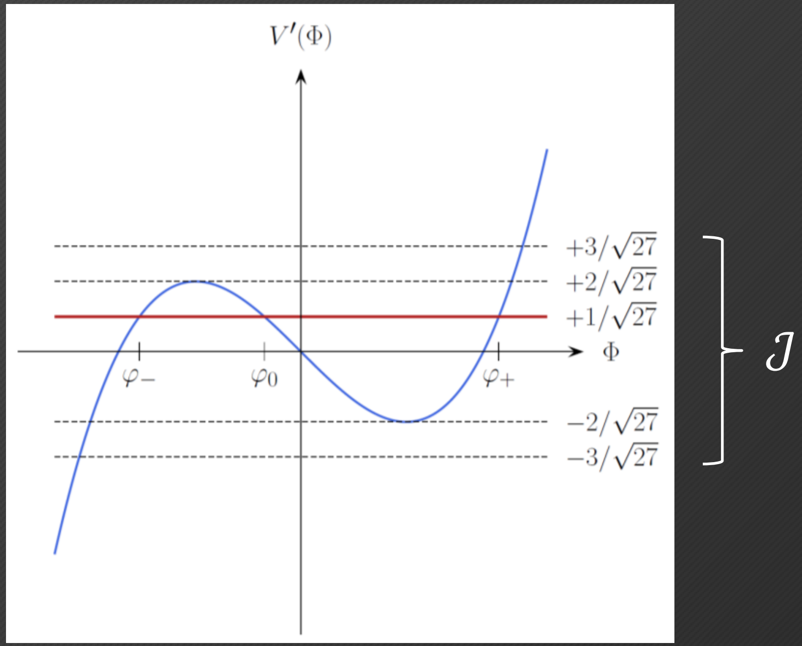
- $\frac{\delta^2\Gamma[\phi,\Delta_k]}{\delta\phi_x\delta\phi_y}\Delta_{k;x,y}+\frac{\delta^2\Gamma[\phi,\Delta_k]}{\delta\phi_x\delta\Delta'_{k;y,z}}\frac{\delta\phi_x}{\delta\mathcal{K}'_{k;y,z}}=1$
- By construction,

$$\frac{\delta\phi_x}{\delta\mathcal{K}'_{k;y,z}}=0, \qquad \text{so} \qquad \frac{\delta^2\Gamma[\phi,\Delta_k]}{\delta\phi_x\delta\phi_y}\Delta_{k;x,y}=1$$

# Multiple saddle points

## Source dependence

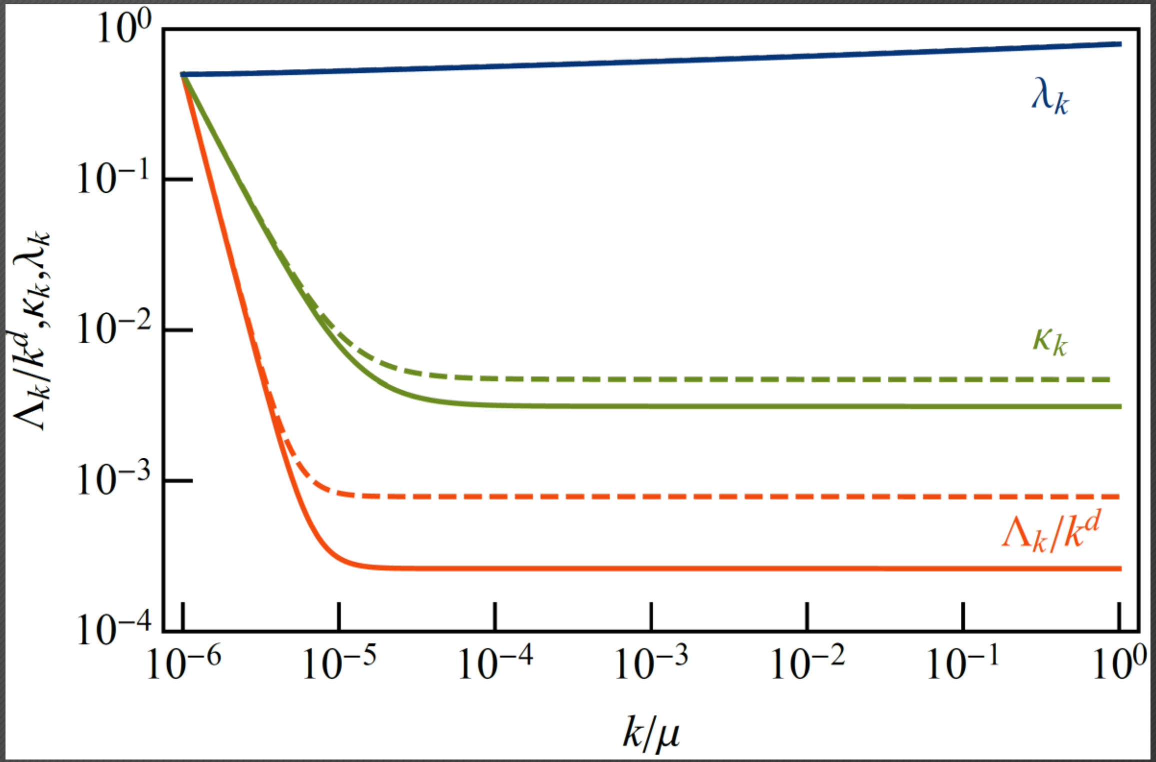



[PM & Saffin ‘19]

# Example

## Evolution in four dimensions




[Alexander, PM, Nursey & Saffin ’19]