Rigorous Renormalization Group

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Motivation and Overview

Rigorous RG is one of many directions in **mathematical physics**.

Only a small selection of results will be discussed here.

Timeline

The main equations

Discrete and continuous RG flows

The use of RG equations for perturbative renormalization

Renormalization group beyond formal perturbation theory

Outlook

Timeline

The subject has a very long history within constructive field theory, starting soon after Wilson and Wegner's works.

An incomplete list is

1980-1987: Gawędzki, Kupiainen and, independently Feldman, Magnen, Rivasseau, Sénéor:
Infrared ϕ^4 in four dimensions, Gross-Neveu model in 2, $2+\varepsilon$ dimensions

1983-1989 Bałaban. Ultraviolet stability of Yang-Mills-theory in $d=4$,
1990-1996 nonlinear sigma models in three dimensions

1984 Polchinski: perturbative renormalization by flow equations
followed up by many people Keller, Kopper, MS, Müller, Hollands

1987/1991 Brydges, Kennedy; Abdesselam Rivasseau.
Tree and forest formulas and their derivation from Wilsonian RG equations.

1990/1991 Benfatto-Gallavotti; Feldman-Trubowitz: perturbative RG for many-fermion systems

1992 Feldman-Magnen-Rivasseau-Trubowitz: nonperturbative fermionic RG; Fermi surface sectorization and intrinsic $1/N$ expansions, followed up by many people (MS, Disertori-Rivasseau, Benfatto-Mastropietro-Giuliani, Pedra)

1994 Benfatto-Gallavotti-Procacci-Scoppola:
1D Fermion systems and Luttinger liquids, followed up by many people (Mastropietro, Giuliani,...)

Generating Functionals

Generating functional for connected Green functions

$$e^{-W(J)} = \int e^{-S(\phi)-(J|\phi)}\,\mathcal{D}\phi$$

Example: scalar theory $S(\phi) = \frac{1}{2}(\phi \mid Q\,\phi) + S_I(\phi)$, $Q \geq 0$,
and $(J \mid \phi) = \int_x J(x)\,\phi(x)$.

$$e^{-W(J)} = Z_0 \int d\mu_C(\phi)\, e^{-(J|\phi)}\, e^{-S_I(\phi)}$$

$d\mu_C$: centered, normalized Gaussian measure with covariance $C = Q^{-1} \geq 0$.

Wilson's effective action generates connected amputated Green functions

$$e^{-G(\phi)} = \int d\mu_C(\phi + \phi')\, e^{-S_I(\phi')}\,.$$

If W is strictly convex, its Legendre transform Γ can be obtained by

$$\Gamma(\varphi) = (j(\varphi) \mid \varphi) - W(j(\varphi))$$

where the functional $j(\varphi)$ is the solution of

$$\frac{\delta W}{\delta J(x)}(j(\varphi)) = \varphi(x)\,.$$

Flow equations

Assume that $s > 0$ is some parameter, and that $C = C(s) > 0$ depends differentiably on s. Source term magic implies the heat equation

$$\tfrac{\partial}{\partial s}\, \mathrm{e}^{-W(J)} = -\frac{1}{2}\, \Delta^{(J)}_{\dot{Q}}\, \mathrm{e}^{-W(J)} \qquad \Delta^{(J)}_{\dot{Q}} = (\tfrac{\delta}{\delta J} \mid \dot{Q}\, \tfrac{\delta}{\delta J}) \qquad \dot{Q} = \frac{\partial Q}{\partial s}$$

W and G obey nonlinear heat equations

$$\dot{W} = -\frac{1}{2}\Delta_{\dot{Q}} W + \frac{1}{2}(\tfrac{\delta W}{\delta J} \mid \dot{Q}\, \tfrac{\delta W}{\delta J})$$

$$\dot{G} = \frac{1}{2}\Delta_{\dot{C}} G - \frac{1}{2}(\tfrac{\delta G}{\delta \phi} \mid \dot{C}\, \tfrac{\delta G}{\delta \phi}) \qquad \text{(Polchinski's equation)}$$

It makes sense to assume that $\dot{Q} < 0$, hence $\dot{C} = -C\dot{Q}C > 0$.

The equation for W implies

$$\dot{\Gamma}(\varphi) = \frac{1}{2}\mathrm{Tr}\left[\dot{Q}\, \Gamma''(\varphi)^{-1}\right] \qquad \text{(Wetterich's equation)}$$

where $\Gamma''(\varphi)$ is the Hessian of Γ : $\qquad \Gamma''(\varphi)_{x,x'} = \frac{\delta^2 \Gamma}{\delta\varphi(x)\delta\varphi(x')}$.

(if one takes $Q(s) = Q(s_0) + R(s)$, then $\dot{\Gamma}(\varphi) = \frac{1}{2}\mathrm{Tr}\left[\dot{R}\, (R + \Gamma''(\varphi))^{-1}\right]$).

Scalar field propagators

$$C_{x,x'} = \int_p \mathrm{e}^{\mathrm{i}(x-x')p}\, \hat{C}(p) \quad \text{with} \quad \hat{C}(p) = \frac{1}{p^2+m^2} = \int_0^\infty \mathrm{e}^{-(p^2+m^2)\tau}\, \mathrm{d}\tau$$

For $\Lambda = \Lambda_0 \mathrm{e}^{-s}$, $s = 0$ corresponds to $\Lambda = \Lambda_0$, and $\Lambda \to 0$ means $s \to \infty$. The integral

$$\hat{C}^{\Lambda\Lambda_0}(p) = \int_{\Lambda_0^{-2}}^{\Lambda^{-2}} \mathrm{e}^{-(p^2+m^2)\tau}\, \mathrm{d}\tau = \frac{1}{p^2+m^2} \left(\mathrm{e}^{-(p^2+m^2)/\Lambda_0^2} - \mathrm{e}^{-(p^2+m^2)/\Lambda^2} \right)$$

gives a propagator with ultraviolet cutoff Λ_0 and infrared cutoff Λ.

By decomposing the integration interval into finitely many, e.g. corresponding to $s \in \mathbb{N}$, one gets a discrete decomposition of C in terms of a geometric progression of scales: If $s_k = k$, then $\Lambda_k = \Lambda(s_k) = \Lambda_0 \mathrm{e}^{-k}$, and

$$\hat{C}^{\Lambda_k\Lambda_0}(p) = \sum_{j=1}^{k} \hat{C}_j(p) \qquad \text{with} \quad \hat{C}_j = \hat{C}^{\Lambda_j\Lambda_{j-1}}$$

The semigroup property for the Wilsonian effective action then leads to a discrete flow

$$G_0 = S_I, G_1, \ldots, G_k \quad \text{with} \quad \mathrm{e}^{-G_j(\phi)} = \int \mathrm{d}\mu_{C_j}(\phi')\, \mathrm{e}^{-G_{j-1}(\phi+\phi')}$$

Discrete decompositions are standard in mathematical RG studies.

Perturbative renormalizability

Polchinski used his equation to prove perturbative renormalizability of scalar field theory. Idea: expand in the loop order ℓ and orders n of the fields,

$$G^{\Lambda\Lambda_0}(\phi) = \sum_{\ell=0}^{\infty} \hbar^{\ell} \sum_{n=0}^{\bar{n}(\ell)} G^{\Lambda\Lambda_0}_{\ell,n}(\phi)$$

and

$$G^{\Lambda\Lambda_0}_{\ell,n}(\phi) = \int_{x_1,\ldots,x_n} \mathcal{G}^{\Lambda\Lambda_0}_{\ell,n}(x_1,\ldots,x_n)\ \phi(x_1)\ldots\phi(x_n)$$

Translation invariance

$$\int_{x_1,\ldots,x_n} \mathcal{G}^{\Lambda\Lambda_0}_{\ell,n}(x_1,\ldots,x_n)\mathrm{e}^{-\mathrm{i}\sum p_j\cdot x_j} = \delta\Big(\sum_{j=1}^{n} p_j\Big)\ \tilde{\mathcal{G}}^{\Lambda\Lambda_0}_{\ell,n}(p_2,\ldots,p_n)$$

$$G^{\Lambda\Lambda_0}_{\ell,n}(\phi) = \int_{p_2,\ldots,p_n} \tilde{\mathcal{G}}^{\Lambda\Lambda_0}_{\ell,n}(p_2,\ldots,p_n)\ \hat{\phi}(-p_2-\ldots-p_n)\hat{\phi}(p_2)\ldots\hat{\phi}(p_n)$$

Convergence of the derivative expansion

Theorem [Hollands-Kopper, Comm. Math. Phys. 313, 257]. In Euclidian scalar field theory in $d = 4$ with $\phi \to -\phi$ symmetry, for all ℓ and n, $G^{\Lambda\Lambda_0}_{\ell,n}$ has a derivative expansion

$$G^{\Lambda\Lambda_0}_{\ell,n}(\phi) = \sum_w g_{\ell,n,w} \int_x \phi(x)\, \partial^{w_2}\phi(x)\ \ldots \partial^{w_2}\phi(x)$$

which converges if ϕ is a Schwartz function and its Fourier transform $\hat{\phi}$ is supported in a sufficiently small ball around $p = 0$. The sum runs over $(n-1)$-tuples of 4-multiindices $w = (w_2, \ldots, w_n)$, and the coefficients are

$$g_{\ell,n,w} = \frac{-\mathrm{i}^{|w|}}{w!}\ \left(\partial^w \tilde{\mathcal{G}}^{\Lambda\Lambda_0}_{\ell,n}\right)(0, \ldots, 0)$$

where $|w| = |w_2| + \ldots + |w_n|$ and $w! = w_2! \ldots w_n!$ (where $w_j! = w_{j,1}! \ldots w_{j,4}!$).

The expansion is uniform in Λ and Λ_0 and the same derivative expansion holds in the limit $\Lambda_0 \to \infty$ and $\Lambda \to 0$.

They also prove the convergence of the operator-product expansion.

This was generalized to massless ϕ^4_4 in a further paper by Holland, Hollands and Kopper.

Convergence of the sums over n and ℓ is not shown.

Taylor expansion

$$\tilde{\mathcal{G}}^{\Lambda\Lambda_0}_{\ell,n}(p_2,\ldots,p_n) = \sum_w \frac{1}{w!} \partial^w \tilde{\mathcal{G}}^{\Lambda\Lambda_0}_{\ell,n}(0,\ldots,0)\; p_2^{w_2}\ldots p_n^{w_n}$$

gives

$$G^{\Lambda\Lambda_0}_{\ell,n}(\phi) = \sum_w \frac{1}{w!} \partial^w \tilde{\mathcal{G}}^{\Lambda\Lambda_0}_{\ell,n}(\mathbf{0}) \int_{p_2,\ldots,p_n} \hat{\phi}(-p_2-\ldots-p_n)\; p_2^{w_2}\hat{\phi}(p_2)\;\ldots\; p_n^{w_n}\hat{\phi}(p_n)$$

The Fourier identities

$$p_j^{w_j}\hat{\phi}(p_j) = (-\mathrm{i})^{|w_j|}\; \left(\partial^{w_j}\phi\right)^{\wedge}(p_j)$$

and

$$\int_{p_2,\ldots,p_n} \hat{\psi}_1(-p_2-\ldots-p_n)\; \hat{\psi}_2(p_2)\ldots\hat{\psi}_n(p_n) = \int_x \psi_1(x)\ldots\psi_n(x)$$

imply that

$$G^{\Lambda\Lambda_0}_{\ell,n}(\phi) = \sum_w g_{\ell,n,w} \int_x \phi(x)\; \partial^{w_2}\phi(x)\; \ldots \partial^{w_2}\phi(x)\;.$$

The hard part is to prove that $\tilde{\mathcal{G}}^{\Lambda\Lambda_0}_{\ell,n}$ remains finite and infinitely often differentiable as $\Lambda_0 \to \infty$ and $\Lambda \to 0$, and that the sum over w converges.

Bounds for the Green functions

Set $\mathbf{p} = (p_2, \ldots, p_n)$ and $\Lambda_+ = \max\{\Lambda, m\}$. For $4 - n - |w| < 0$,

$$\left| \partial^w \tilde{\mathcal{G}}^{\Lambda\Lambda_0}_{\ell,n}(\mathbf{p}) \right| \le \gamma_{\ell,n,w} \, \Lambda_+^{4-n-|w|} \, \mathcal{P}\left[\log(\sup\{ \frac{|\mathbf{p}|}{\Lambda_+} \, , \, \frac{\Lambda_+}{m} \}) \right] \tag{1}$$

$\gamma_{\ell,n,w}$ and all the coefficients of the polynomial $\mathcal{P}$ are independent of Λ_0.

The limits $\Lambda_0 \to \infty$ and $\Lambda \to 0$ of $\partial^w \tilde{\mathcal{G}}^{\Lambda\Lambda_0}_{\ell,n}$ exist for all ℓ, n, w.

This is proven using Polchinski's induction scheme.

$$\dot{\mathcal{G}}^{\Lambda\Lambda_0}_{\ell,n} = \mathcal{L}_\Lambda \left(\mathcal{G}^{\Lambda\Lambda_0}_{\ell-1,n+2} \right) + \mathcal{B}_\Lambda \left(\mathcal{G}^{\Lambda\Lambda_0}_{\ell_1,n_1} \, , \, \mathcal{G}^{\Lambda\Lambda_0}_{\ell_2,n_2} \right)$$

Double induction on $v = \ell + \frac{n}{2} - 1 \ge 1$ and at fixed v, on $\ell \ge 0$.

The induction turns into a recursion for the coefficients $\gamma_{\ell,n,w}$.

More detailed bounds

For an action invariant under $\phi \to -\phi$, $\tilde{\mathcal{G}}^{\Lambda\Lambda_0}_{\ell,n} = 0$ for odd n.

For $n = 2k$ and $4 - 2k - |w| < 0$, one can choose [Hollands-Kopper]

$$\gamma_{\ell,2k,w} = \sqrt{|w|!(|w| + 2k + 4)!}\; K^{(2k+4\ell-4)(|w|+1)}\; \frac{(k+\ell-1)!}{k!}$$

$$\mathcal{P}[x] = \sum_{\mu=0}^{\ell} \frac{x^\mu}{2^\mu \mu!}$$

Proof of convergence

w-dependence of $\gamma_{\ell,2k,w}$: use $(|w|+2k+4)! = \binom{|w|+2k+4}{2k+4}|w|!\,(2k+4)!$ and $\binom{a}{b} \le 2^a$

$$\gamma_{\ell,2k,w} \le |w|!\sqrt{(2k+4)!}\;2^{\frac{|w|}{2}+k+2}\;K^{(2k+4\ell-4)(|w|+1)}\;\frac{(k+\ell-1)!}{k!}$$

Thus

$$\gamma_{\ell,2k,w} \le \tilde{\gamma}_{\ell,2k}\;|w|!\;\tilde{K}^{|w|}$$

The polynomial $\mathcal{P}$ is independent of w. Set $\mathbf{p}=0$ in (1).

$$\left|\partial^w \tilde{\mathcal{G}}^{\Lambda\Lambda_0}_{\ell,n}(\mathbf{0})\right| \le \tilde{\gamma}_{\ell,2k}\;|w|!\;\tilde{K}^{|w|}\Lambda_+^{4-n-|w|}\;\mathcal{P}\left[\log(\frac{\Lambda_+}{m})\right]$$

Finally, use

$$\frac{|w|!}{w!} \le (4n)^{|w|}$$

to get

$$|g_{\ell,n,w}| \le (4n\tilde{K})^{|w|}\;\Lambda_+^{4-n-|w|}\;\tilde{\gamma}_{\ell,n}\;\mathcal{P}\left[\log(\frac{\Lambda_+}{m})\right]$$

At fixed ℓ and n, this is bounded by $\text{const}^{\,|w|}$.

Further remarks

The series in $\hbar$ is formal, and is not expected to converge.

The restriction on the support of $\hat{\phi}$ is plausible (small gradients).

Similar proofs of perturbative renormalizability can be done using the one-particle-irreducible functional.

So far, there has not been success in doing nonperturbative proofs using Polchinski's induction scheme. For bosonic fields, the combinatorics grows too fast when expanding in the fields. For fermionic fields, the sign cancellations help, but a proof has not yet been found.

Another justification of gradient expansions is given in the work of Bałaban, Feldman, Knörrer, and Trubowitz, in their work on the proof of Bose condensation for weakly interacting boson systems. The hypothesis there is the so-called small-field condition, where they prove that the action is analytic in the fields and its derivatives.

Brydges-Kennedy formula

Brydges and Kennedy first showed that integrating Polchinski's equation over a finite scale interval leads to a convenient reorganization of perturbation theory in terms of trees. For

$$\mathrm{e}^{-G_j(\phi)} = \int \mathrm{d}\mu_{C_j}(\phi')\, \mathrm{e}^{-G_{j-1}(\phi+\phi')}$$

it reads

$$G_j(\phi) = \sum_{p=1}^{\infty} G_j^{(p)}(\phi)$$

$G_j^{(p)}$ is of order p in G_{j-1}, and given by

$$G_j^{(p)}(\phi) = \frac{1}{p!} \sum_{T \in \mathcal{T}_p} \int_{[0,1)^{p-1}} \mathrm{d}^T s \int \mathrm{d}\mu_{C_{j,p,s,T}}(\phi'_1, \ldots, \phi'_p) \prod_{\theta \in T} \Delta_{C_j}^{(\theta)} \prod_{q=1}^{p} G_{j-1}(\phi'_q + \phi)$$

$(C_{j,p,s,T})_{(q,x),(q',x')} = C_j(x,x') \int_0^1 \mathrm{d}t 1\,(s_\ell < t\ \forall \ell \in P_{q,q'}(T))$

$P_{q,q'}(T)$ the unique path on the tree T from q to q'.

Brydges-Kennedy formula

$$G_j^{(p)}(\phi) = \frac{1}{p!} \sum_{T \in \mathcal{T}_p} \int_{[0,1)^{p-1}} \mathrm{d}^T s \int \mathrm{d}\mu_{C_{j,p,s,T}}(\phi'_1, \ldots, \phi'_p) \prod_{\theta \in T} \Delta_{C_j}^{(\theta)} \prod_{q=1}^{p} G_{j-1}(\phi'_q + \phi)$$

Fermions: basic convergence theorem

$$\|G_j\|_h = \sum_{m\geq 2} h^{2m} \int_{x_2,\ldots,x_m} |\mathcal{G}_m^{(j)}(0,x_2,\ldots,x_m)|$$

Assume $\|\hat{C}_j\|_1 = \int_p |\hat{C}_j(p)| = \delta_j^2 < \infty$ and $\alpha_j = \|C_j\|_1 = \int_x |C_j(0,x)| < \infty$.

Set $\omega = \alpha_j\ \delta_j{}^{-2}$.

If $\omega\|G_{j-1}\|_{3\delta} < 1$, then for all $P \geq 0$

$$\left\| G_j - \sum_{p=1}^{P} \frac{1}{p!} G_j^{(p)} \right\|_\delta \leq \frac{(\omega\|G_{j-1}\|_{3\delta})^P}{1-\omega\|G_{j-1}\|_{3\delta}} \|G_{j-1}\|_{3\delta}$$

With more general weighted norms, one can get much more detailed, pointwise estimates.

Thus under these hypotheses the discrete RG iteration can be done using convergent expansions, and the effective action is analytic in the fields.

Gross-Neveu model, many-fermion models in $d = 1$ and $d = 2$ with pointlike singularities (graphene) and extended Fermi surfaces have been treated this way. The 'sector' method used in these proofs is similar to the N-patch technique of the fermionic fRG. A classic in the field is

J. Feldman, H. Knörrer, E. Trubowitz, A Two-Dimensional Fermi Liquid Comm. Math. Phys. 247 (2004), Rev. Math. Phys. 15 (2003) (12 papers)

Some recent works using fermionic expansions

W. de Roeck, M.S., Persistence of Exponential Decay and Spectral Gaps for Interacting Fermions

Comm. Math. Phys. 365 (2019) 773-796

Alessandro Giuliani, Vieri Mastropietro, Marcello Porta, Quantization of the interacting Hall conductivity in the critical regime

Journal of Statistical Physics 180 (2020) 332-365

J. Magnen, J. Unterberger, A mathematical derivation of zero-temperature 2D superconductivity from microscopic Bardeen-Cooper-Schrieffer model

https://arxiv.org/abs/1902.02337

Nonperturbative aspects: bosons

A straightforward expansion in the fields does not work for bosons: e.g. in the Brydges-Kennedy formula, the remaining Gaussian integral, which creates the loops, grows too fast in p.

Decompositions in 'large and small fields' allow to bypass this problem, but create technical overhead. One alternative to this in scalar field theories is the 'loop-vertex expansion' of J. Magnen, V. Rivasseau, *Constructive field theory without tears*, Ann. Henri Poincaré **9** (2008) 403. Combining this method with RG also creates complicated proofs. But a similar method works also for quantum many-boson systems MS, arXiv:2006:12281.

Proving Bose-Einstein condensation as spontaneous symmetry breaking in a weakly interacting gas at low temperature is currently under study. For an overview, see

T. Bałaban, J. Feldman, H. Knörrer, E. Trubowitz, Complex Bosonic Many-body Models: Overview of the Small Field Parabolic Flow. Annales Henri Poincaré, 18, 2873-2903 (2017).

Perspective

Mathematical results on a variety of physically interesting models have been achieved by RG methods (many-body models, prototypical models of high-energy physics).

Almost all of these proofs use discrete RG iterations, not continuous flows. This is a technical limitation (controlling combinatorics).

Many rigorous results exist about the Wilson-Polchinski effective action. It will be interesting to derive similar results for the 1PI generating functional.

Three major open problems for mathematical physicists working with RG methods are:

proving Bose-Einstein condensation in $d \geq 3$ dimensions,

proving that a three-dimensional many-electron-system with round Fermi surface and short-range interaction is a Fermi liquid above a critical temperature, and a superconductor below that temperature,

proving decay of correlations for Yang-Mills theory in 4 dimensions.

Thank you for listening!