Numerical fluid dynamics for FRG-flow equations: Zero-dimensional QFTs as numerical test cases

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Outline

Motivation:
Calculations in QFT and statistical mechanics

- **How is information stored in QFTs and statistical mechanics?**
- **How can one extract physical quantities?**

All information is contained in the n-point correlation functions (expectation values),

$$\langle \Phi^A \cdots \Phi^\Omega \rangle \,.$$

where

- Φ^A are (microscopic) degrees of freedom,
- $A, \ldots, \Omega =$ generalized indeces (e.g. color, flavor, space-time position, momentum, field, spin, $\ldots$).

- **How can we calculate the n-point correlation functions?**

We need a **path/functional integral** (partition function) that provides the partition of probabilities among the various microstates based on their action $S[\vec{\Phi}]$ (energies)

$$\mathcal{Z} \equiv \sum\!\!\!\!\!\!\int [\mathrm{d}\vec{\Phi}]\, \mathrm{e}^{-S[\vec{\Phi}]}$$

The problem reduces to the calculation of expectation values from a partition function

We calculate the n-correlation functions of the QFT,

$$\langle \Phi^A \cdots \Phi^\Omega \rangle \equiv \frac{1}{\mathcal{Z}} \sum\!\!\!\!\!\!\int [\mathrm{d}\vec{\Phi}]\, \Phi^A \cdots \Phi^\Omega\, \mathrm{e}^{-S[\vec{\Phi}]}\,.$$

Challenge:

- Solve these usually high-dimensional and extremely complicated (functional) integrals.

Problem:

- Analytical solutions are rarely known.
- Direct numerical implementation is often conceptually and computationally extremely challenging.

Solutions are:

- "brute force" numerical integration (e.g. Monte-Carlo-Simulation),
- (if possible) approximations (e.g. perturbation theory, $\frac{1}{N}$-expansion),
- "non-perturbative"/functional methods = "mapping the problem to another problem"
  (e.g. Dyson-Schwinger, FRG, holographic methods),
- combinations of the above, including numerical treatment of PDEs, numerical minimization etc..

Requirements to judge and trust the results (within a given model/theory)**:**

- Check the scope of application of the model/theory.
- Check the range of validity of an approximation or a method.
- Estimate and control the systematic, numerical, statistical errors of a method.

How can we check and guaranty the validity of a method?

- Use analytically known results as test cases.
- Compare different methods, e.g. test approximations and "non-perturbative" methods against "brute force" numerical integration for toy models.
- Check the fulfillment of mathematical theorems, limiting cases, etc..

A testing ground:
Zero-dimensional field theories

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What is zero-dimensional field theory?

- It is QFT "in a point" – zero spacetime dimensions/no momenta.
- The fields are (field-space vectors of) numbers, e.g. C-numbers or Grassmann numbers.
- The functional integral reduces to an ordinary integral.
- It includes all kind of interactions.

J. Keitel and L. Bartosch, J. Phys. A45, 105401 (2012), arXiv:1109.3013.
S. Kemler and J. Braun, J. Phys. G 40, 085105 (2013), arXiv:1304.1161.
S. Moroz, PhD Thesis, 10.11588/heidok.00011602 (2011).
A. S. Wightman, "Mathematical quantum field theory and related topics" Montreal (1977).
F. Strocchi, Int. Ser. Monogr. Phys. 158 (2013).
P. Millington and P. M. Saffin, J. Phys. A 52 40, 405401 (2019), arXiv:1905.09674.
S. Floerchinger, Springer Theses (2010), 10.1007/978-3-642-14113-3 arXiv: 0909.0416.
J. M. Pawlowski, "'Functional' RG-flows for Integrals" (2013), [Online; 2020.10.29]

Zero-dimensional field theory

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Why are we interested in such “toy” models/theories?

- They give us insight into how different methods in QFT and statistics conceptually work.
- They are analytically and/or numerically solvable up to machine precision.
- They serve as test cases for accuracy of approximations, methods and numerical tools.
- They are mathematically well defined without ambiguities.
- Diagrammatic techniques, expansion schemes and methods are independent of dimension and can be studied easier in $d = 0$.
- We can test numerical methods, choices of cutoffs etc. for FRG-flow equations.

The zero-dimensional $O(N)$-model

Example: zero-dimensional $O(N)$-model

- The Field content:
  - N real scalars that transform as a vector $\vec{\phi} = (\phi_1, \ldots, \phi_N)$ under $O(N)$-rotations,

$$\vec{\phi} \longrightarrow \vec{\phi}\,' = O\,\vec{\phi}\,.$$

- The action $S[\vec{\phi}\,]$ of the model:
  - Does not contain spacetime-/momentum-integrals and derivatives: action = Lagrangian = potential,

$$S[\vec{\phi}\,] = \mathcal{L}(\vec{\phi}\,) = U(\vec{\phi}\,)\,.$$

  - $U(\vec{\phi}\,)$ is an arbitrary(!) $O(N)$-invariant function that is bounded from below and decays sufficiently fast for $|\vec{\phi}| \to \infty$,

$$U(\vec{\phi}\,) = U(\rho)\,, \qquad\qquad \rho = \frac{\vec{\phi}^{\,2}}{2}\,.$$

The zero-dimensional $O(N)$-model

- The functional integral = N-dimensional integral,

$$\mathcal{Z} = \int \mathrm{d}\vec{\phi}\, \mathrm{e}^{-U(\vec{\phi})}\,,$$

reduces to a one-dimensional integral (spherical coordinates $\phi = |\vec{\phi}\,|$),

$$\mathcal{Z} = \Omega_N \int_0^\infty \mathrm{d}\phi\, \phi^{N-1}\, \mathrm{e}^{-U(\phi)}\,, \qquad \Omega_N = \frac{2\pi^{\frac{N}{2}}}{\Gamma\big[\frac{N}{2}\big]}\,.$$

Using the invariant $\rho = \frac{\vec{\phi}^{\,2}}{2}$ it reads,

$$\mathcal{Z} = \Omega_N\, 2^{\frac{N-2}{2}} \int_0^\infty \mathrm{d}\rho\, \rho^{\frac{N-2}{2}}\, \mathrm{e}^{-U(\rho)}\,.$$

The zero-dimensional $O(N)$-model

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- n-point correlation functions:

$$G^{(n)}_{i_1 \cdots i_n} \equiv \langle \phi_{i_1} \cdots \phi_{i_n} \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{d}\vec{\phi}\, \phi_{i_1} \cdots \phi_{i_n}\, \mathrm{e}^{-U(\vec{\phi})} \,,$$

- Constraints on correlation functions by the symmetry of the model
  - $\langle \phi_{i_1} \cdots \phi_{i_n} \rangle$ with $n =$ odd are zero,
  - $G^{(2)}_{ij} = \langle \phi_i\, \phi_j \rangle = \frac{1}{N}\, \delta_{ij} \langle \vec{\phi}^{\,2} \rangle$,
  - $G^{(4)}_{ijkl} = \langle \phi_i\, \phi_j\, \phi_k\, \phi_l \rangle = \frac{1}{N(N+2)}\, (\delta_{ij}\, \delta_{kl} + \delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}) \langle (\vec{\phi}^{\,2})^2 \rangle$,
  - $\ldots$.

The zero-dimensional $O(N)$-model

Define generating functionals for

- n-point correlation functions,

$$\mathcal{Z}[\vec{J}\,] \equiv \mathcal{N} \int \mathrm{d}\vec{\phi}\, \mathrm{e}^{-U(\vec{\phi})+\vec{J}\cdot\vec{\phi}}\,, \quad G^{(n)}_{i_1\cdots\, j_n} = \frac{1}{\mathcal{Z}[\vec{J}\,]} \frac{\delta^{(n)}\mathcal{Z}[\vec{J}\,]}{\delta J_{i_1}\cdots\,\delta J_{i_n}}\bigg|_{\vec{J}=0}\,,$$

- connected n-point correlation functions,

$$\mathcal{W}[\vec{J}\,] = \ln \mathcal{Z}[\vec{J}\,]\,, \qquad \mathcal{W}^{(n)}_{i_1\cdots\, j_n} = \frac{\delta^{(n)}\mathcal{W}[\vec{J}\,]}{\delta J_{i_1}\cdots\,\delta J_{i_n}}\bigg|_{\vec{J}=0}\,,$$

- 1-PI-irreducible n-point correlation functions,

$$\Gamma[\vec{\varphi}\,] = \sup\{\vec{J}(\vec{\varphi}\,)\cdot\vec{\varphi} - \mathcal{W}[\vec{J}(\vec{\varphi})]\}\,, \quad \Gamma^{(n)}_{i_1\cdots\, j_n} = \frac{\delta^{(n)}\Gamma[\vec{\varphi}\,]}{\delta \varphi_{i_1}\cdots\,\delta \varphi_{i_n}}\bigg|_{\vec{\varphi}=0}\,.$$

- We choose the normalization condition $\mathcal{Z}[0] \stackrel{!}{=} 1$, that sets

$$\mathcal{N}^{-1} = \int \mathrm{d}\vec{\phi}\, \mathrm{e}^{-U(\vec{\phi})}$$

and fixes zero-point functions

$$\mathcal{W}^{(0)} = \mathcal{W}[0] = \ln \mathcal{Z}[0] = 0\,, \qquad \Gamma^{(0)} = 0\,.$$

The zero-dimensional $O(N)$-model

- There are all kind of relations between the different higher n-point correlation functions

- In this talk, we need:

$$G^{(2)} = \mathcal{W}^{(2)} = \left(\Gamma^{(2)}\right)^{-1} = \frac{1}{N} \langle \vec{\phi}^{\,2} \rangle$$

and

$$G^{(4)} - 3\left(G^{(2)}\right)^2 = \mathcal{W}^{(4)} = -\Gamma^{(4)} \left(\Gamma^{(2)}\right)^{-4},$$

where

$$G^{(4)} = \frac{3}{N\,(N+2)} \langle (\vec{\phi}^{\,2})^2 \rangle \,.$$

J. Keitel and L. Bartosch, J. Phys. A45, 105401 (2012), arXiv:1109.3013.

The zero-dimensional $O(N)$-model

- Relation between two-point functions of $\mathcal{Z}$ and $\mathcal{W}$:

$$G^{(2)}_{ij} = \frac{1}{N}\,\delta_{ij}\,\langle \vec{\phi}^{\,2} \rangle\,, \qquad \longrightarrow \qquad \mathcal{W}^{(2)}_{ij} = G^{(2)}_{ij} + \cancel{G^{(1)}_i\,G^{(1)}_j}\,,$$

- We define "measured values",

$$G^{(2)} \equiv G^{(2)}_{11} = \frac{1}{N}\,\delta_{11}\,\langle \vec{\phi}^{\,2} \rangle = \frac{1}{N}\langle \vec{\phi}^{\,2} \rangle = \mathcal{W}^{(2)}_{11} \equiv \mathcal{W}^{(2)}\,.$$

- The connection to the 1-PI-two-point function is

$$\Gamma^{(2)}_{ij} = \big(\mathcal{W}^{(2)}_{ij}\big)^{-1} = \delta_{ij}(\mathcal{W}^{(2)})^{-1}\,.$$

- For $\Gamma^{(2)} \equiv \Gamma^{(2)}_{11}$ we find

$$G^{(2)} = \mathcal{W}^{(2)} = \big(\Gamma^{(2)}\big)^{-1} = \frac{1}{N}\,\langle \vec{\phi}^{\,2} \rangle\,.$$

First example:
ϕ^4-theory and the failure of perturbation theory

Example: zero-dimensional $O(N)$-model

- Consider quartic potential $\lambda > 0$,

$$U(\vec{\phi}\,) = +\frac{1}{2}\,\vec{\phi}^{\,2} + \frac{\lambda}{4!}\,(\vec{\phi}^{\,2})^2\,.$$

- The "mass parameter" is absorbed by redefinition of fields and quartic coupling.

- There is an exact analytic closed solution this choice of $U(\vec{\phi}\,)$.

J. Keitel and L. Bartosch, J. Phys. A45, 105401 (2012), arXiv:1109.3013.
A. S. Wightman, "Mathematical quantum field theory and related topics" Montreal (1977).
F. Strocchi, Int. Ser. Monogr. Phys. 158 (2013).

ϕ^4-theory and the failure of perturbation theory

Perturbation theory:

- Expand the interaction term in orders of λ,

$$\langle(\vec{\phi}^{\,2})^n\rangle = \frac{\int_0^\infty \mathrm{d}\phi\, \phi^{N-1}\, \phi^{2n}\left(1 - \frac{\lambda}{4!}\,\phi^4 + \ldots\right)\mathrm{e}^{-\frac{1}{2}\,\phi^2}}{\int_0^\infty \mathrm{d}\phi\, \phi^{N-1}\left(1 - \frac{\lambda}{4!}\,\phi^4 + \ldots\right)\mathrm{e}^{-\frac{1}{2}\,\phi^2}}\,.$$

- There is an analytic solution for

$$\Gamma^{(2)}(\lambda) = 1 + \frac{2}{3}\,\lambda - \frac{2}{3}\,\lambda^2 + \frac{14}{9}\,\lambda^3 - \frac{46}{9}\,\lambda^4 + \frac{562}{27}\,\lambda^5 - \frac{8054}{81}\,\lambda^6 + \ldots\,,$$
$$\Gamma^{(4)}(\lambda) = \lambda - \frac{5}{3}\,\lambda^2 + \frac{61}{9}\,\lambda^3 - \frac{299}{9}\,\lambda^4 + \frac{4987}{27}\,\lambda^5 - \frac{30685}{27}\,\lambda^6 + \ldots\,.$$

J. Keitel and L. Bartosch, J. Phys. A45, 105401 (2012), arXiv:1109.3013.

ϕ^4-theory and the failure of perturbation theory

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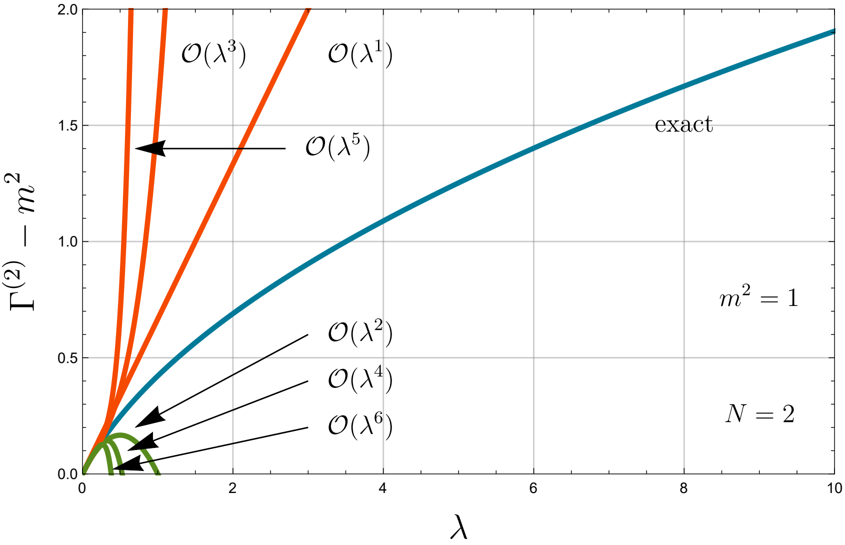
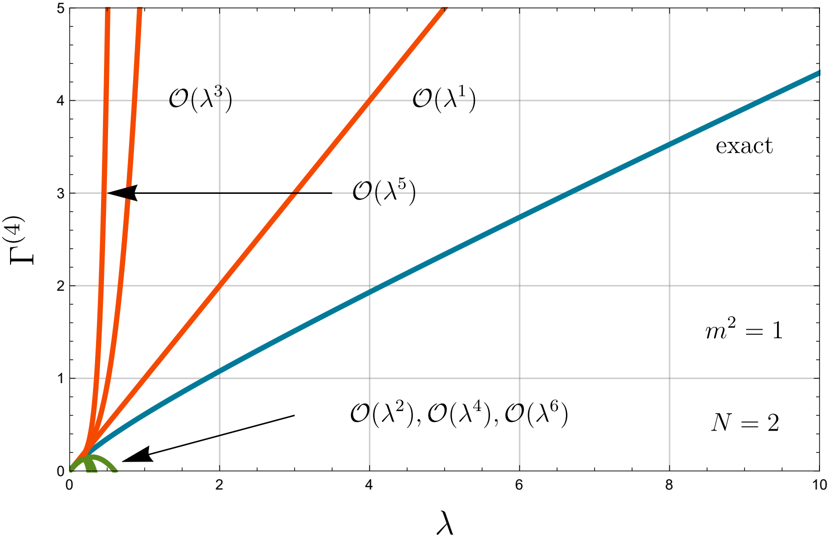


J. Keitel and L. Bartosch, J. Phys. A45, 105401 (2012), arXiv:1109.3013.

ϕ^4-theory and the failure of perturbation theory



J. Keitel and L. Bartosch, J. Phys. A45, 105401 (2012), arXiv:1109.3013.

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Problems of perturbation series:

- It does not converge and is only asymptotic to the correct solution around $\lambda = 0$.

What about

- potentials of more than one coupling constant? E.g.,

$$U(\vec{\phi}) = +\frac{\vec{\phi}^{\,2}}{2} + \frac{\lambda}{4!}\,(\vec{\phi}^{\,2})^2 + \frac{\kappa}{6!}\,(\vec{\phi}^{\,2})^3\,,$$

- potentials including a negative "mass term"?

$$\langle(\vec{\phi}^{\,2})^n\rangle = \frac{\int_0^\infty \mathrm{d}\phi\,\phi^{N-1}\,\phi^{2n}\,\big(1 - \frac{\lambda}{4!}\,\phi^4 + \ldots\big)\,\mathrm{e}^{+\frac{1}{2}\,\phi^2}}{\int_0^\infty \mathrm{d}\phi\,\phi^{N-1}\,\big(1 - \frac{\lambda}{4!}\,\phi^4 + \ldots\big)\,\mathrm{e}^{+\frac{1}{2}\,\phi^2}}\,.$$

A. S. Wightman, "Mathematical quantum field theory and related topics" Montreal (1977).
F. Strocchi, Int. Ser. Monogr. Phys. 158 (2013).

Second example:
The limits of the large-N-expansion

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Large-N-expansion:

- Expand in the intrinsically "dimensionless" parameter N.
- It is not limited to positive "mass terms" and small couplings.
- Introducing $\tilde{y} \equiv \phi^2/N$ and $\tilde{\lambda} \equiv N\lambda$ the partition function of ϕ^4-theory reads

$$\mathcal{Z} = \Omega_N \, N^{N/2} \int_0^\infty \frac{\mathrm{d}\tilde{y}}{2\tilde{y}} \, \mathrm{e}^{-N f(\tilde{y})} \,, \quad f(\tilde{y}) \equiv \frac{m^2}{2}\tilde{y} + \frac{\tilde{\lambda}}{24}\tilde{y}^2 - \frac{1}{2}\ln(\tilde{y}) \,.$$

- At large N this integral can be calculated in the saddle-point approximation around the classical minimum of $f(\tilde{y})$,

$$\tilde{y}_0 \equiv \frac{3m^2}{\tilde{\lambda}} \, \mathrm{sgn}(m^2) \left[\sqrt{1 + 2\tilde{\lambda}/(3m^4)} - 1 \right] .$$

J. Keitel and L. Bartosch, J. Phys. A45, 105401 (2012), arXiv:1109.3013.

Saddle-point approximation: Integrals of the type

$$\mathcal{I}[g,f](N) \equiv \Omega_N \, N^{N/2} \int_0^\infty \mathrm{d}\tilde{y}\, g(\tilde{y})\, \mathrm{e}^{-N f(\tilde{y})}$$

can be computed for large N by expanding $g(\tilde{y})$ and $f(\tilde{y})$ around the minimum $\tilde{y}_0$ of $f(\tilde{y})$:

1. Introduce $\tilde{y} \equiv \tilde{y}_0 + v/\sqrt{N}$.
2. Expand $g(\tilde{y}_0 + v/\sqrt{N})$ and $\exp(-N f(\tilde{y}_0 + v/\sqrt{N}))$ for large N.
3. Perform the remaining Gaussian integrals.

$$\mathcal{I}[g,f](N) = \Omega_N \, N^{N/2} \, g(\tilde{y}_0)\, \mathrm{e}^{-N f(\tilde{y}_0)} \sqrt{\frac{2\pi}{N f''(\tilde{y}_0)}} \left(1 + \sum_{i=1}^{\infty} \frac{C_i[g,f]}{n^i}\right).$$

G.B. Arfken and H.J. Weber, 2005, Mathematical Methods for Physicists 6th ed.

Large-N-expansion for ϕ^4-theory

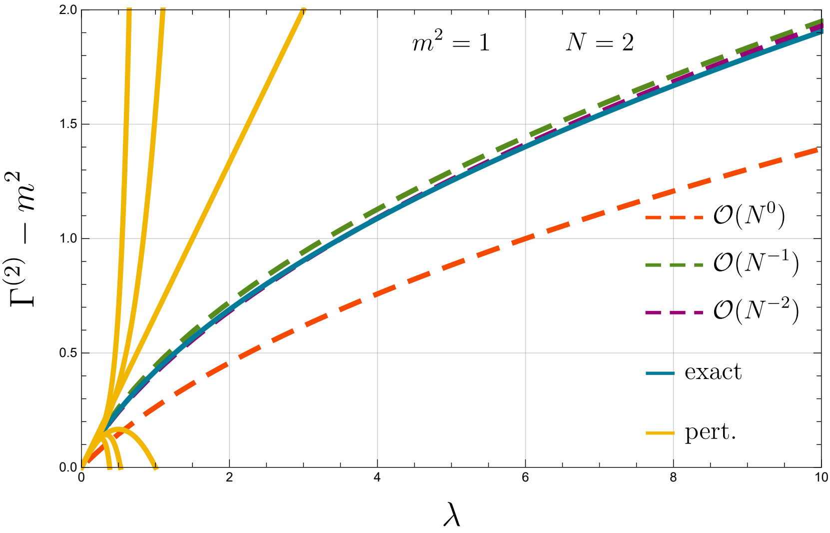
Using $\mathcal{I}[g,f](N)$, we can compute $\mathcal{Z}$ and $\langle(\vec{\phi}^{\,2})^n\rangle$ for large N and ultimately:

$$\Gamma^{(2)} = \tilde{y}_0 + \frac{2\tilde{y}_0\,(m^2\,\tilde{y}_0 - 1)}{N(m^2\,\tilde{y}_0 - 2)^2} - \frac{4\tilde{y}_0\,(2m^2\,\tilde{y}_0 + 1)\,(m^2\,\tilde{y}_0 - 1)^2}{N^2(m^2\,\tilde{y}_0 - 2)^5} + \mathcal{O}(N^{-3})\,,$$

$$\begin{aligned}\Gamma^{(4)} = {} & \frac{6(m^2\,\tilde{y}_0 - 1)}{N\tilde{y}_0^2\,(m^2\,\tilde{y}_0 - 2)} - \frac{12(m^4\,\tilde{y}_0^2 - 3m^2\,\tilde{y}_0 + 6)\,(m^2\,\tilde{y}_0 - 1)^2}{N^2\,\tilde{y}_0^2\,(m^2\,\tilde{y}_0 - 2)^4} + \\ & + \frac{24(m^8\,\tilde{y}_0^4 - 8m^6\,\tilde{y}_0^3 + 35m^4\,\tilde{y}_0^2 - 49m^2\,\tilde{y}_0 + 56)\,(m^2\,\tilde{y}_0 - 1)^3}{N^3\,\tilde{y}_0^2\,(m^2\,\tilde{y}_0 - 2)^7} + \\ & + \mathcal{O}(N^{-4})\,.\end{aligned}$$

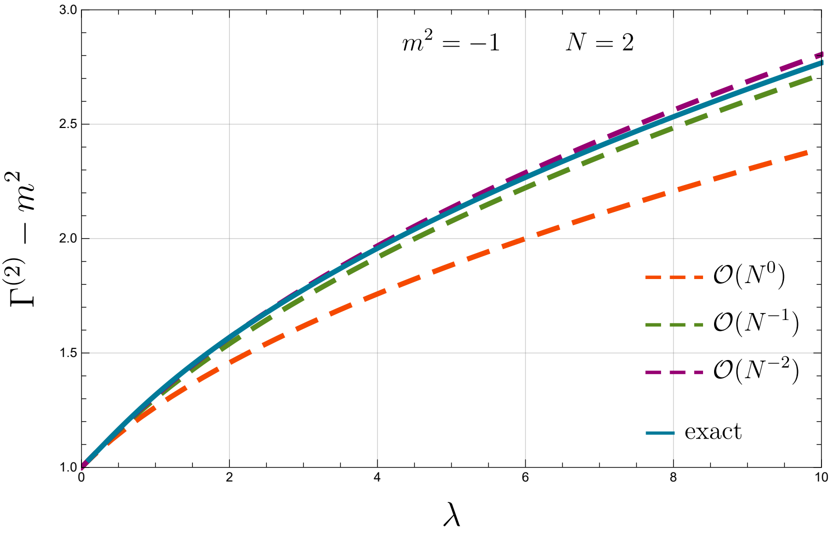
J. Keitel and L. Bartosch, J. Phys. A45, 105401 (2012), arXiv:1109.3013.

Large-N-expansion for ϕ^4-theory

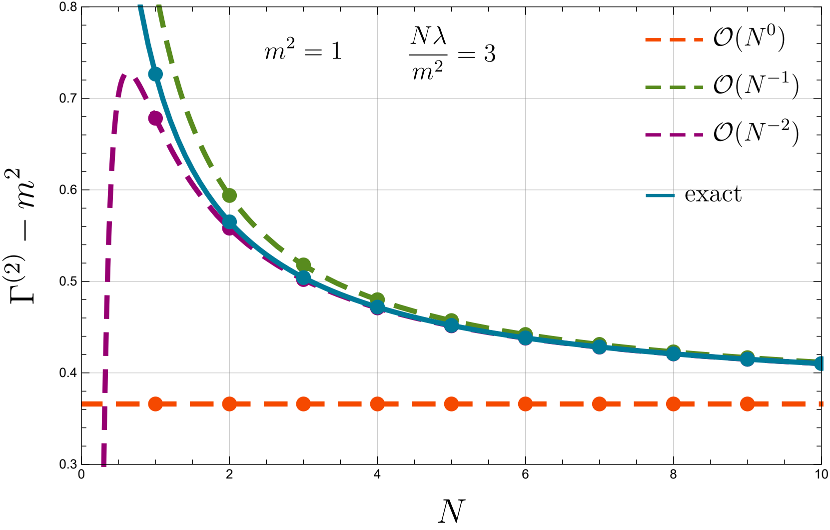


J. Keitel and L. Bartosch, J. Phys. A45, 105401 (2012), arXiv:1109.3013.

Large-N-expansion for ϕ^4-theory

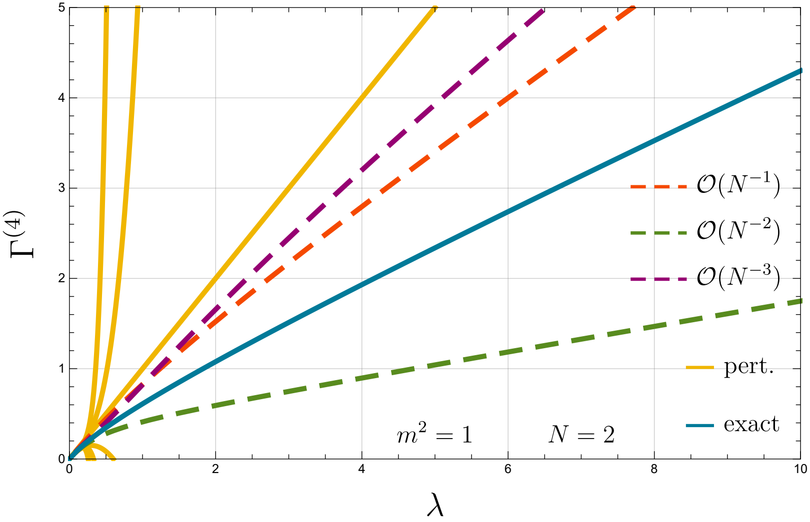


J. Keitel and L. Bartosch, J. Phys. A45, 105401 (2012), arXiv:1109.3013.



J. Keitel and L. Bartosch, J. Phys. A45, 105401 (2012), arXiv:1109.3013.

Large-N-expansion for ϕ^4-theory



J. Keitel and L. Bartosch, J. Phys. A45, 105401 (2012), arXiv:1109.3013.

The limits of the large-N-expansion

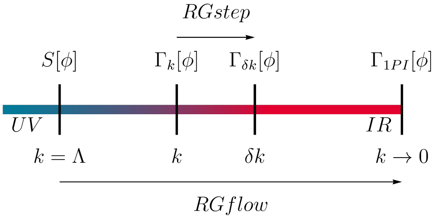
Problems of the large-N-expansion:

- It does not work for "small" N.
- It is getting less acurate for higher n-point-correlation functions.
- It gets problematic for more than one coupling constant.
- It is highly complicated (especially in non-zero dimensions).

A short recap:
The Functional Renormalization Group

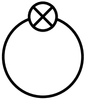
The Functional Renormalization Group

- FRG is an implementation of Wilson's RG approach.



- The Functional Renormalization Group equation is exact!

$$\partial_t \bar{\Gamma}_t[\Phi] = \mathrm{STr}\left[\left(\tfrac{1}{2}\,\partial_t R_t\right)\left(\bar{\Gamma}_t^{(2)}[\Phi] + R_t\right)^{-1}\right] =$$



C. Wetterich, Phys. Lett. B 301 (1993) 90-94.
K. G. Wilson, Phys. Rev. B 4, (1971) 3174, Phys. Rev. B 4, (1971) 3184.
J. Berges, N. Tetradis, C. Wetterich, Phys.Rept. 363 (2002) 223-386.

The ERG-equation and numerical fluid dynamics

- RG-flow/Callan-Symanzik equations can be understood as flow-equations in the true sense of the word.
- Maybe most explicit (F)RG-flow equations can be recast into conservative form in a fluid-dynamical picture?!
- One particular example is the LPA flow equation for the effective potential.

S. Coleman, *Aspects of Symmetry* (1985).

M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory* (1995).

E. Grossi and N. Wink (2019), arXiv:1903.09503.

LPA flow equation of the quark-meson-model

- Consider the LPA-flow eq. of the N_f flavour, N_c color quark-meson-model (QMM) in $d+1$ spacetime dimensions at temperature T and quark chemical potential μ:

$$
\begin{aligned}
-k\,\partial_k U_k(\rho) = & -\frac{S_{d-1}}{(2\pi)^d}\frac{k^{d+1}}{d}\frac{N_\mathrm{f}^2-1}{2E_k^\pi}\left\{1+2\,n_\mathrm{b}[\beta E_k^\pi]\right\} - \\
& -\frac{S_{d-1}}{(2\pi)^d}\frac{k^{d+1}}{d}\frac{1}{2E_k^\sigma}\left\{1+2\,n_\mathrm{b}[\beta E_k^\sigma]\right\} + \\
& +\frac{S_{d-1}}{(2\pi)^d}\frac{k^{d+1}}{d}\frac{d_\gamma N_\mathrm{f} N_\mathrm{c}}{2E_k^\psi}\left\{1-n_\mathrm{f}[\beta(E_k^\psi+\mu)]-n_\mathrm{f}[\beta(E_k^\psi-\mu)]\right\},
\end{aligned}
$$

$$
(E_k^\pi)^2 \equiv k^2+\partial_\rho U_k(\rho)\,, \quad (E_k^\sigma)^2 \equiv k^2+\partial_\rho U_k(\rho)+2\rho\,\partial_\rho^2 U_k(\rho)\,, \quad (E_k^\psi)^2 \equiv k^2+\tfrac{\rho g^2}{N_\mathrm{f}}.
$$

D. U. Jungnickel and C. Wetterich, Phys.Rev.D 53 (1996) 5142-5175.

D. D. Scherer, J. Braun and H. Gies, J. Phys. A 46 (2013) 285002.

LPA flow equation in dimensionless units

- Introducing flow time $t \equiv -\ln(k/\Lambda)$ and dimensionless variables $x = \sqrt{2\rho}/\Lambda$, $\tilde{\beta} \equiv \Lambda\beta$, $\tilde{\mu} \equiv \mu/\Lambda$, the LPA flow equation of the QMM can be written as:

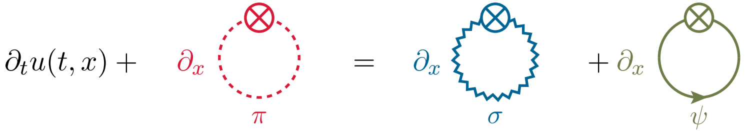
$$\partial_t U_t(x) = -\frac{S_{d-1}}{(2\pi)^d}\frac{\mathrm{e}^{-(d+1)\,t}}{d}\frac{N_\mathrm{f}^2-1}{2E_t^\pi}\left\{1+2\,n_\mathrm{b}[\tilde{\beta}E_t^\pi]\right\} - \frac{S_{d-1}}{(2\pi)^d}\frac{\mathrm{e}^{-(d+1)\,t}}{d}\frac{1}{2E_t^\sigma}\left\{1+2\,n_\mathrm{b}[\tilde{\beta}E_t^\sigma]\right\} + \frac{S_{d-1}}{(2\pi)^d}\frac{\mathrm{e}^{-(d+1)\,t}}{d}\frac{d_\gamma N_\mathrm{f} N_\mathrm{c}}{2E_t^\psi}\left\{1-n_\mathrm{f}[\tilde{\beta}(E_t^\psi+\mu)]-n_\mathrm{f}[\tilde{\beta}(E_t^\psi-\mu)]\right\},$$

$$(E_t^\pi)^2 \equiv \mathrm{e}^{-2t} + \tfrac{1}{x}\partial_x U_t(x)\,, \qquad (E_t^\sigma)^2 \equiv \mathrm{e}^{-2t} + \partial_x^2 U_t(x)\,, \qquad (E_t^\psi)^2 \equiv \mathrm{e}^{-2t} + \tfrac{x^2 g^2}{2N_\mathrm{f}}\,.$$

LPA flow equation as advection-diffusion equation

- Taking one ∂_x-derivative and introducing $u(t,x) \equiv \partial_x U_t(x)$ the flow eq. can be understood as an **advection-diffusion equation**:

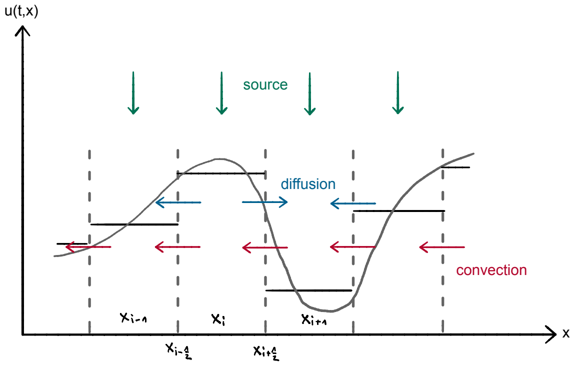
$$\partial_t u(t,x) + \partial_x F[t,x,u(t,x)] = \partial_x Q[t,\partial_x u(t,x)] + \partial_x S[t,x]$$



  - $u(t,x)$ conserved quantity
  - $\partial_x F[t,x,u(t,x)]$ nonlinear advection flux (generated by π-modes)
  - $\partial_x Q[t,\partial_x u(t,x)]$ dissipation flux (generated by σ-mode)
  - $\partial_x S(t,x)$ (internal) source term (generated by fermions)

E. Grossi and N. Wink (2019), arXiv:1903.09503.

A. Kurganov and E. Tadmor, J. Comput. Phys. 160, 241 – 282 (2000).



- **Idea:** Time evolution of a set of cell averages $\{\bar{u}_j\}$ maintaining detailed balance at the cell interfaces $\{x_{j+1/2}\}$

$$\bar{u}_j \equiv \bar{u}(t, x_j) \equiv \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathrm{d}\xi \, u(\xi, t)$$

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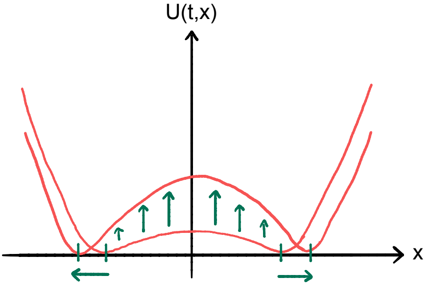
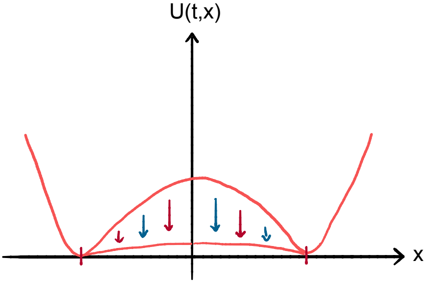
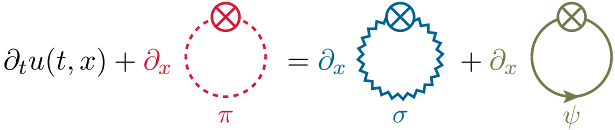


fermionic contributions



bosonic contributions

The structure of the flow equation determines **conceptual behavior**.



Kurganov and Tadmor (KT) $O(\Delta x^2)$ scheme

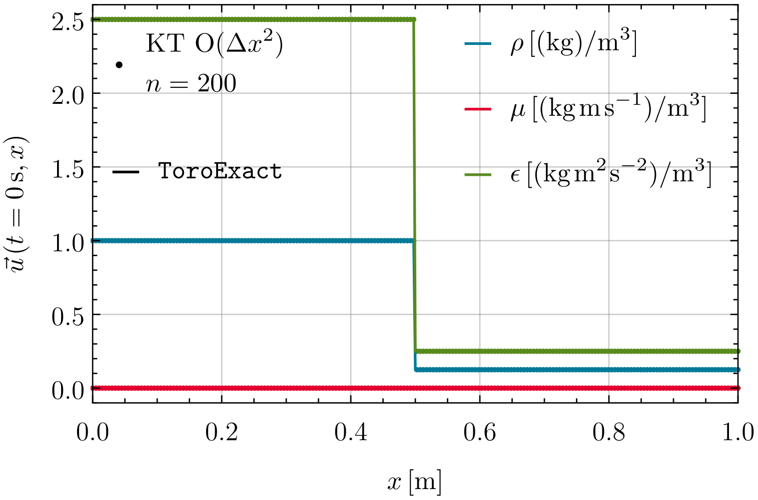
- **Semi-discrete central scheme**: Method of Lines (MoL) finite volume PDE discretization
- Total RHS flux for $\partial_t \bar{u}_j$ is based on the 5-point stencil $\{\bar{u}_{j-2}, \bar{u}_{j-1}, \bar{u}_j, \bar{u}_{j+1}, \bar{u}_{j+2}\}$
- Approximate derivatives are reconstructed from the cell averages using ϕ-Limiter
- MUSCL reconstruction for function values $u^{\pm}_{j+1/2}$ at cell interfaces
- Only additional information (apart from PDE and grid): spectral radius of jacobian $\rho = \max|\lambda_i(\partial F/\partial u)|$ to approximate local speeds
- Second order accurate in Δx and total variation diminishing (TVD)
- Generalizations to more then one spatial dimension are available

A. Kurganov and E. Tadmor, J. Comput. Phys. 160, 241 – 282 (2000).

B. van Leer, J. Comput. Phys. 32, 101 – 136 (1979).

A. Koenigstein and M. J. Steil, Talk in Heidelberg (2019), [Online; 2020.10.30].

Classical test case: Riemann problem for Euler eqs.



Sod shock tube problem – G. A. Sod, J. Comput. Phys. 27.1 (1978).
A. Koenigstein and M. J. Steil, Talk in Heidelberg (2019), [Online; 2020.10.30].
T. Handy/M. J. Steil, `ToroExact` Python program (2017/2020), [Github.com, 2020.10.02].

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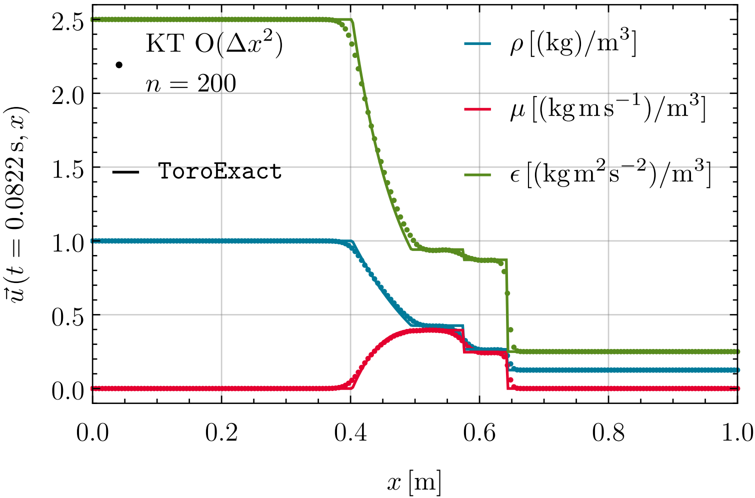

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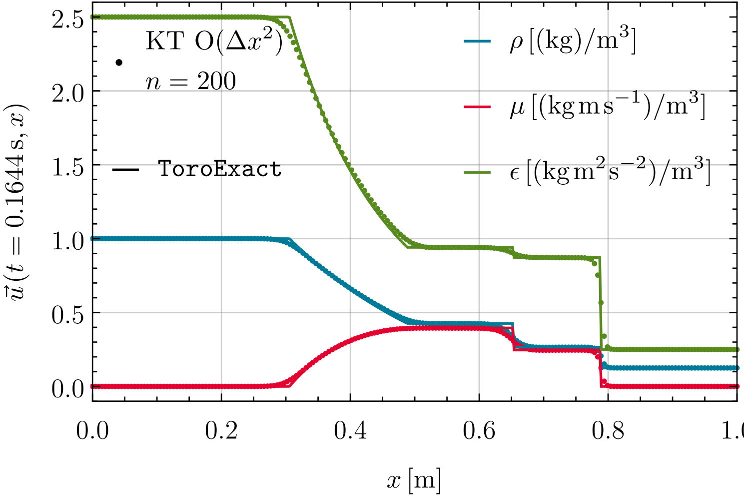

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T. Handy/M. J. Steil, `ToroExact` Python program (2017/2020), [Github.com, 2020.10.02].

FRG-precision test with the zero-dimensional $O(N)$-model

The FRG-formulation of the $O(N)$-model

- The ansatz for the effective average action is exact/complete

$$\bar{\Gamma}_t[\vec{\varphi}] = U(t, \vec{\varphi}\,)\,,$$

  where $U(t, \vec{\varphi}\,)$ is the $O(N)$-invariant effective potential and the only possible(!) contribution to $\bar{\Gamma}_t[\vec{\varphi}]$.

- The flow for $U(t, \vec{\varphi}\,)$ is an exact(!) PDE without(!) truncation.

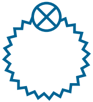
- The 1PI-n-point correlation functions calculated from $U(t_{\mathrm{IR}}, \vec{\varphi}\,)$ have to coincide with the exact results from the functional integral.

- The zero-dimensional $O(N)$-model can thus be used to test numerical schemes, cutoff scales etc..

A. Koenigstein, M. J. Steil, et al. in preparation.

The FRG-formulation of the $O(N)$-model

- The flow-equation for $u(t,\sigma) = \partial_\sigma U(t,\sigma)$ can be reformulated in conservative form as a convection-diffusion equation with $x = \sigma$,

$\partial_t u(t,x) =$ ∂_x  π $+$ ∂_x  σ

$$= -\,\partial_x F[t,x,u(t,x)] + \partial_x Q[t,\partial_x u(t,x)]\,.$$

- Convection and diffusion terms:

$$F[t,x,u(t,x)] = -\big[\tfrac{1}{2}\,\partial_t r(t)\big]\,\frac{N-1}{r(t)+\frac{u(t,x)}{x}}\,,$$

$$Q[t,\partial_x u(t,x)] = \big[\tfrac{1}{2}\,\partial_t r(t)\big]\,\frac{1}{r(t)+\partial_x u(t,x)}\,.$$

A. Koenigstein, M. J. Steil, et al. in preparation.

Challenging testing scenario

- Choose an exponential regulator shape function with UV-cutoff Λ,

$$r(t) = \Lambda \, \mathrm{e}^{-t} \, ,$$

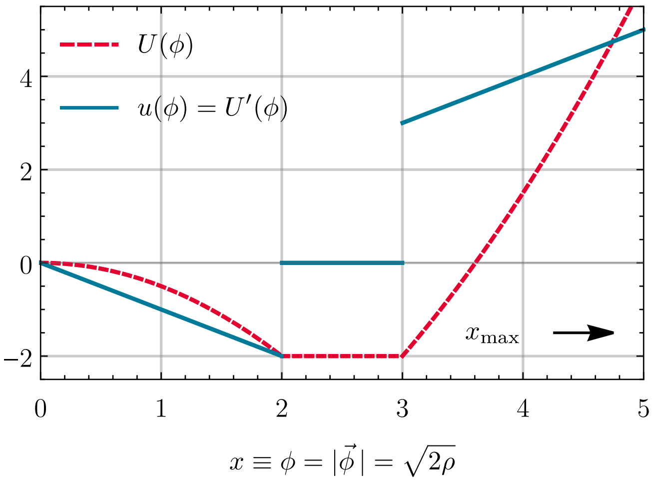
- Choose an non-analytic UV-initial condition with degenerate minima in the symmetry broken phase to make things as challenging as possible for the numerics,

$$U(\vec{\phi}) = \begin{cases} -\frac{1}{2} \, \vec{\phi}^{\,2} \, , & \text{if} \quad \phi \leq 2 \, , \\ -2 \, , & \text{if} \quad 2 < \phi \leq 3 \, , \\ +\frac{1}{2} \left(\vec{\phi}^{\,2} - 13 \right) , & \text{if} \quad 3 < \phi \, . \end{cases}$$

- The symmetry must be restored in the IR (Mermin-Wagner-theorem).
- Numerical tools for RG-flow equations should handle non-analyticities.

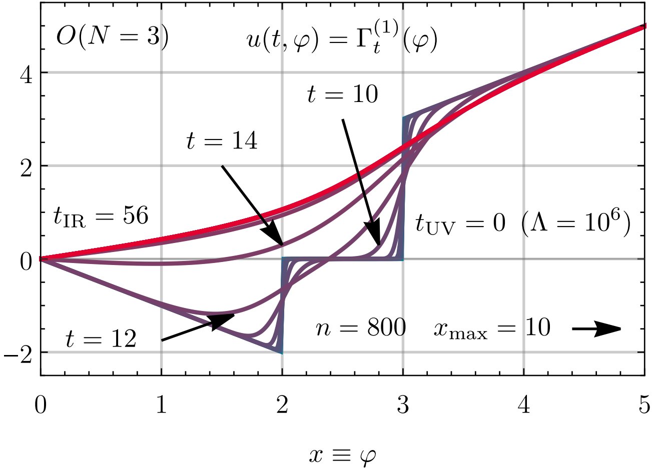
A. Koenigstein, M. J. Steil, et al. in preparation.

Non-analytic UV-initial potential



A. Koenigstein, M. J. Steil, et al. in preparation.

RG-flow of $u(t,\varphi) = \partial_\varphi U(t,\varphi)$



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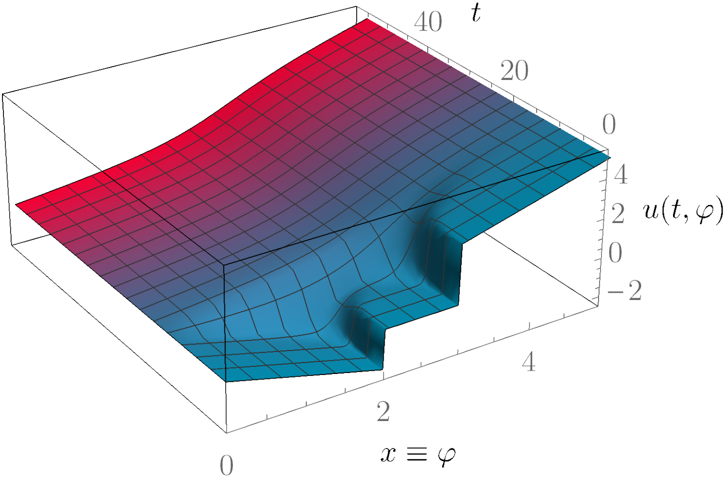
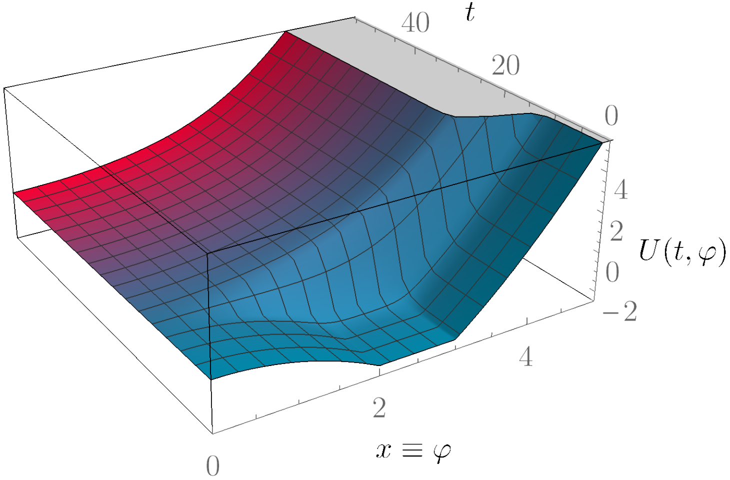
$O(N = 3)$



A. Koenigstein, M. J. Steil, et al. in preparation.

RG-flow of $U(t, \varphi)$

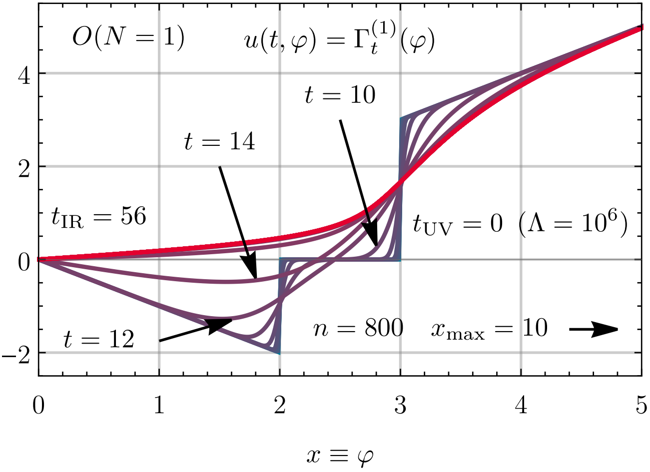
$O(N = 3)$



A. Koenigstein, M. J. Steil, et al. in preparation.

RG-flow of $u(t, \varphi) = \partial_\varphi U(t, \varphi)$

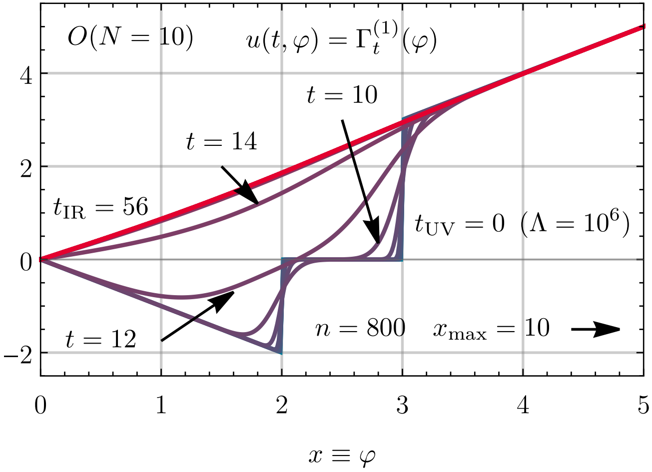
- $O(N = 1) \longrightarrow$ only radial σ-mode $\longrightarrow$ purely diffusive flow.



A. Koenigstein, M. J. Steil, et al. in preparation.

RG-flow of $u(t, \varphi) = \partial_\varphi U(t, \varphi)$

- $O(N = 10) \longrightarrow$ nine π-modes $\longrightarrow$ strong advection, little diffusion.



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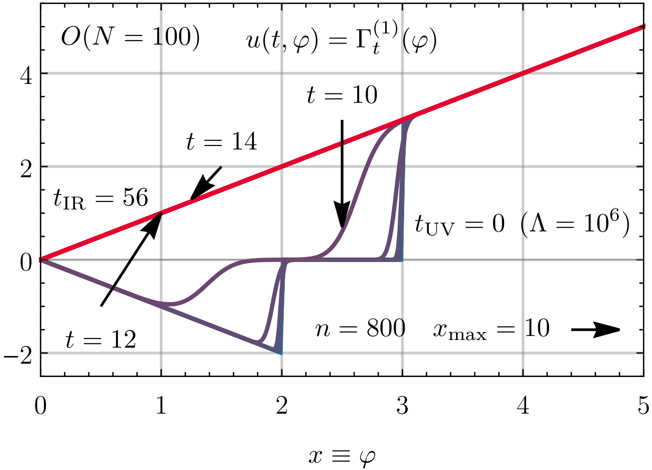
RG-flow of $u(t,\varphi) = \partial_\varphi U(t,\varphi)$

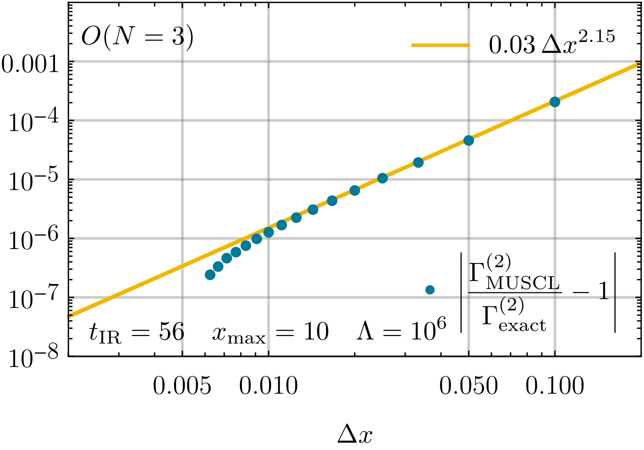
- $O(N = 100) \longrightarrow \approx$ large-N $\longrightarrow$ completely advection dominated.



A. Koenigstein, M. J. Steil, et al. in preparation.

Error-scaling tests for spatial resolution

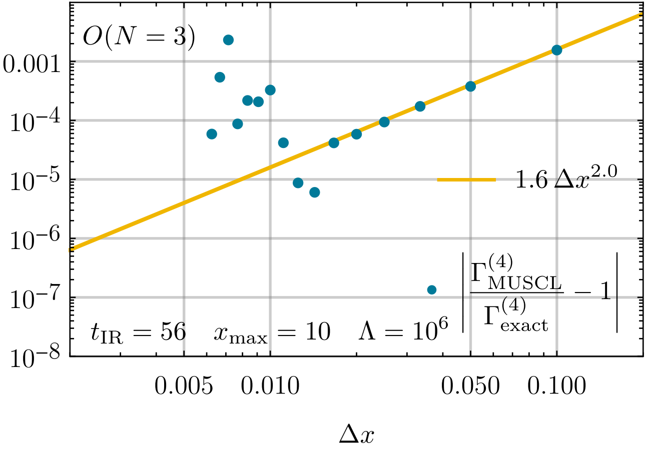
- Rel. error of the two-point function $\Gamma^{(2)}$ depending on the spatial resolution Δx.



A. Koenigstein, M. J. Steil, et al. in preparation.

Error-scaling tests for spatial resolution

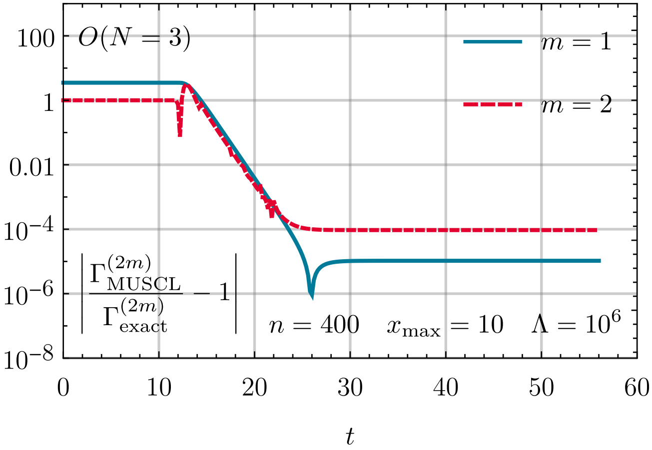
- Rel. error of the four-point function $\Gamma^{(4)}$ depending on the spatial resolution Δx.



A. Koenigstein, M. J. Steil, et al. in preparation.

Tests for UV- and IR-cutoff scales/freeze-out scale

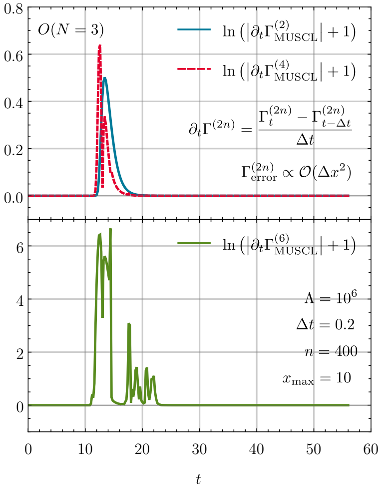
- Rel. error of the two- and four-point function $\Gamma^{(2m)}$ depending on RG-time.



A. Koenigstein, M. J. Steil, et al. in preparation.

Tests for UV- and IR-cutoff scales/freeze-out scale

- Changing rates of the n-point functions $\partial_t \Gamma^{(2n)}$ depending on RG-time.



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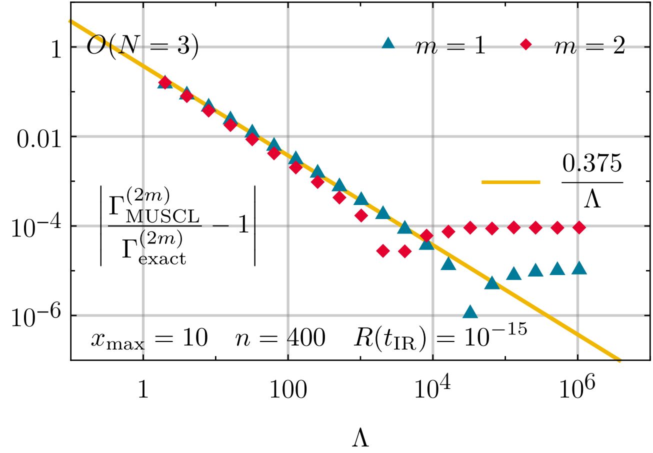
Error-scaling test for (too small) UV-cutoffs

- Rel. error of the two- and four-point functions $\Gamma^{(2n)}$ depending on the choice of Λ.



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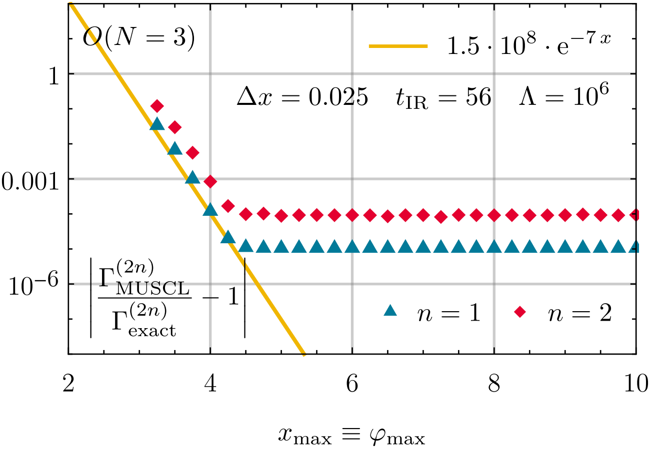
- Rel. error of the two- and four-point functions $\Gamma^{(2n)}$ depending on the size of the computational domain x_{max}.



A. Koenigstein, M. J. Steil, et al. in preparation.

Conclusion and Outlook

Conclusions: Zero-dimensional field theory

- It is perfectly suited as a pedagogical introduction to QFTs.
- Can be used to understand different methods and tools in QFTs.
- The quality and limits of mathematical/numerical methods in QFTs can be studied.
- It is mathematically well defined and n-point-correlation function can be calculated easily to extremely high precision.

Conclusions: FRG and fluid dynamics

- FRG-flow equations can be recast into conservative from?!
- The fluid-dynamical analogy allows for a completely new interpretation of the RG-flow and its contributions as advection, diffusion or source terms.
- Highly developed numerical methods from computational fluid dynamics ensure numerically correct solutions and convergence against exact results within a truncation.
- Study shock- and rarefaction waves in RG-flow equations in the context of phase transitions.

Conclusions: Precision tests for FRG in $d = 0$

- Absence of truncation errors in $d = 0$ allows for quantitative analysis of errors in the solution of FRG flow equations.
- Zero dimensional theories as benchmark problems for FRG flow equations allow for analysis of
  - error scaling with spatial discretization (Δx),
  - error scaling with computational interval truncation (x_{max}),
  - suitable UV initial scale Λ and IR cutoff t_{IR}.
- Provides insight into possible sources of errors in the numerical treatment of higher dimensional models.

Outlook

- Publication of current results for the $O(N)$ model in $d = 0$.
- Study of zero dimensional theory including Grassmann numbers.
- Application of finite volume methods to FRG flow equations of higher dimensional models.
- Study conceptual relations between FRG flow equations and fluid dynamics.