A gauge theory of measurement-based quantum computation

Robert Raussendorf, UBC

joint work with Gabriel Wong (Harvard) and Bartek Czech (Tsinghua)

Outline

1. Measurement-based quantum computation (MBQC)
3. Gauge theory of MBQC

Symmetry

2. Computational phases of quantum matter (SPTO)
4. Outlook

Part I:

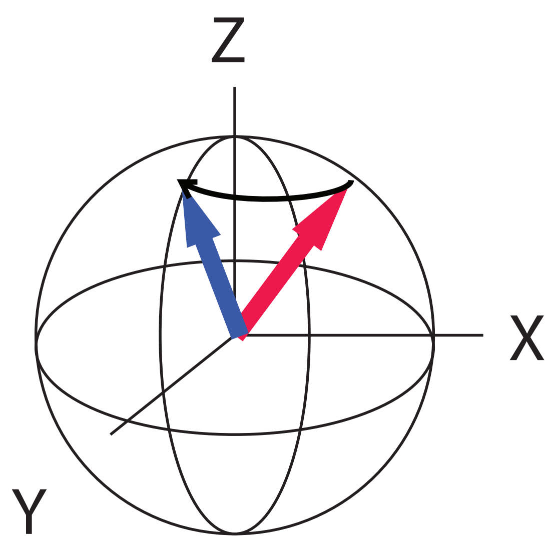
MBQC

The 2D cluster state is a computationally universal “material”

$\boxed{\text{Quantum states can have computational power}}$

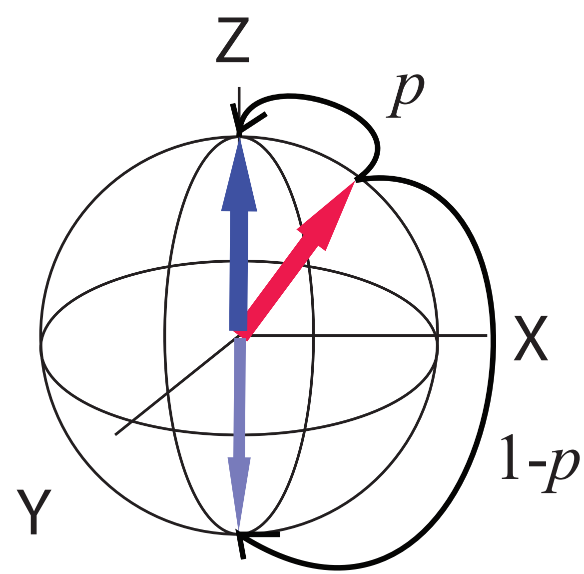
Measurement-based quantum computation

Unitary transformation



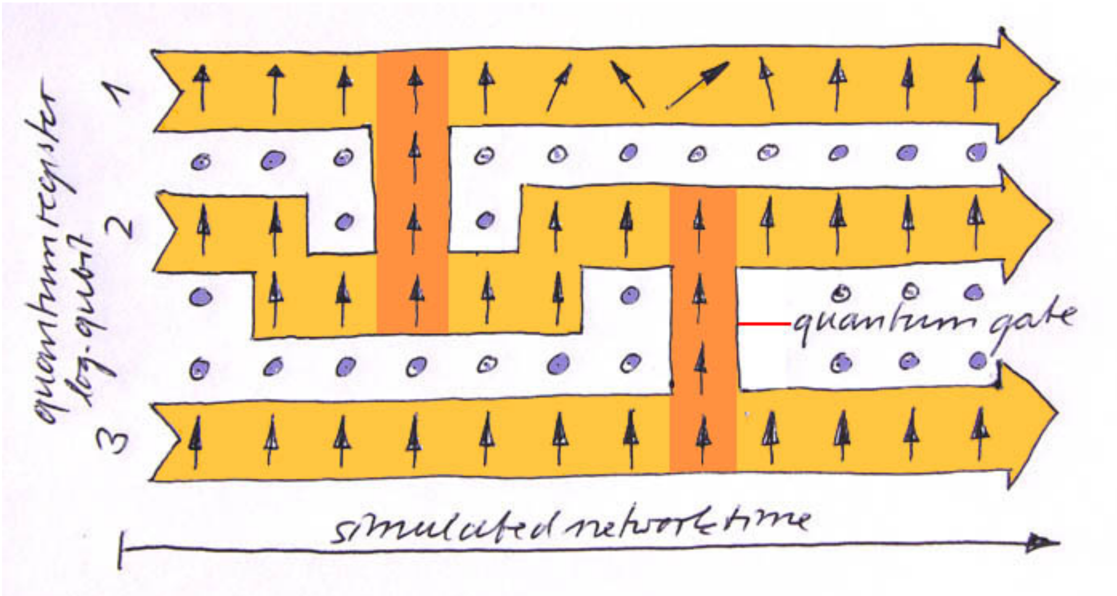
deterministic,
reversible

Projective measurement



probabilistic,
irreversible

Measurement-based quantum computation



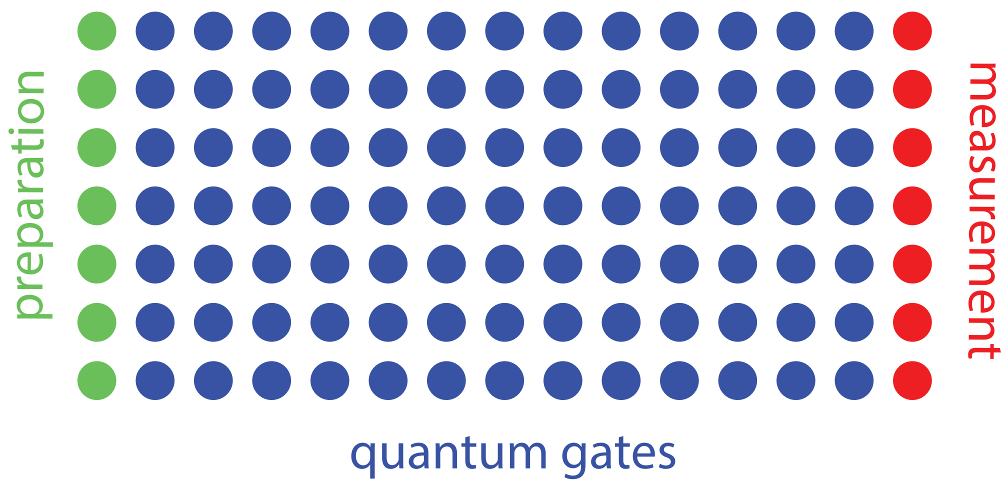
measurement of Z ($\odot$), X ($\uparrow$), $\cos\alpha\, X + \sin\alpha\, Y$ ($\nearrow$)

- Information written onto the resource state, processed and read out by one-qubit measurements only.
- Universal computational resources exist: cluster state, AKLT state.

R. Raussendorf, H.-J. Briegel, Physical Review Letters 86, 5188 (2001).

The MBQC "elevator thought experiment"

Consider a 2D cluster state:

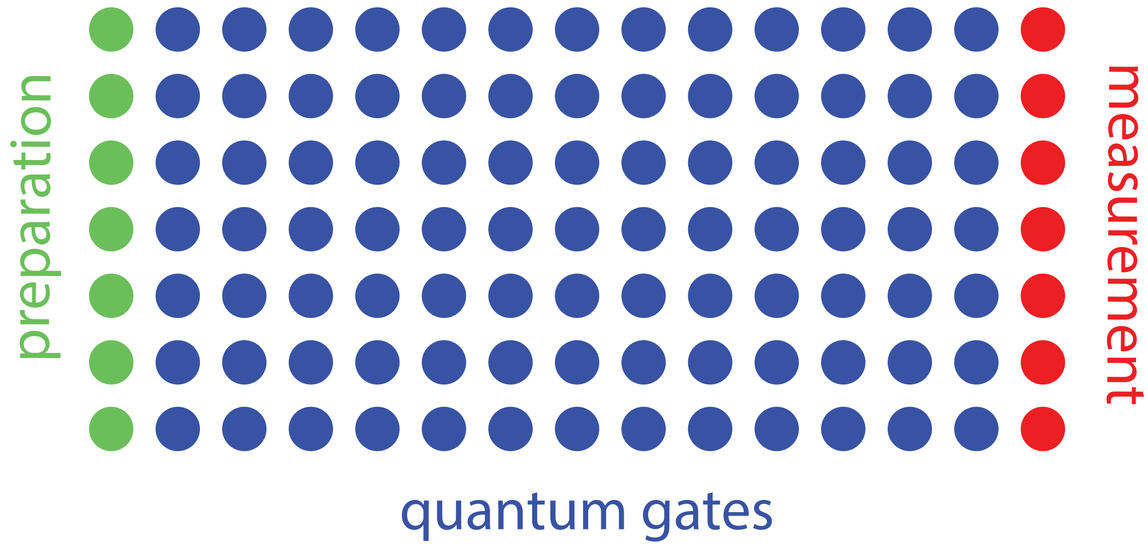


At the end of any MBQC, all that is left is a tensor product state and the measurement record **s**.

- The computational output **o** is extracted from the measurement record $\mathbf{s} = (s_1, s_2, .., s_N)$.
- Individually, each bit s_i of measurement record is completely random; *information is only in the correlations*.

The MBQC “elevator thought experiment”

Consider a 2D cluster state:

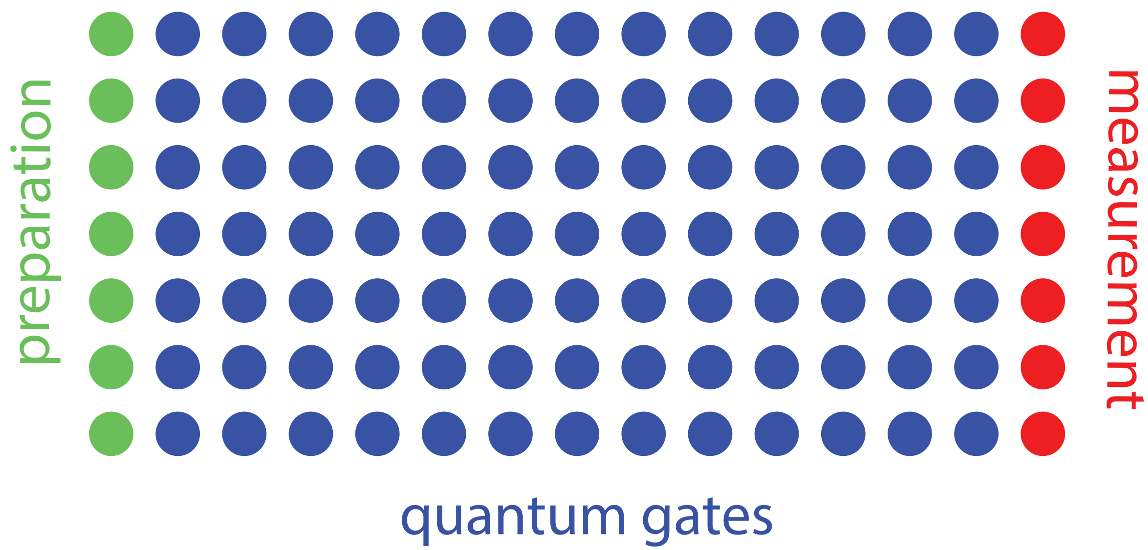


Some cluster qubits simulate the preparation of a quantum register, some its measurement, and some implement quantum gates.

Do these different types of qubits contribute differently to the MBQC output?

The MBQC “elevator thought experiment”

Consider a 2D cluster state:



Some cluster qubits simulate the preparation of a quantum register, some its measurement, and some implement quantum gates.

Do these different types of qubits contribute differently to the MBQC output?

No!

The classical side-processing relations

Every MBQC has a classical side-processing relation that convert measurement record $\mathbf{s}$ into

- computational output $\mathbf{o}$, and
- the choice $\mathbf{q}$ of local measurement bases,

$$\mathbf{o} = Z\mathbf{s} \mod 2, \;\; \mathbf{q} = T\mathbf{s} \mod 2. \tag{1}$$

Every bit of computational output is a parity of individual local measurement outcomes.

What determines the support of these parities?

Which measurement outcomes s_i contribute to what output bit o_j?

The classical side-processing relations

Every MBQC has a classical side-processing relations

$$\mathbf{o} = Z\mathbf{s} \mod 2, \;\; \mathbf{q} = T\mathbf{s} \mod 2.$$

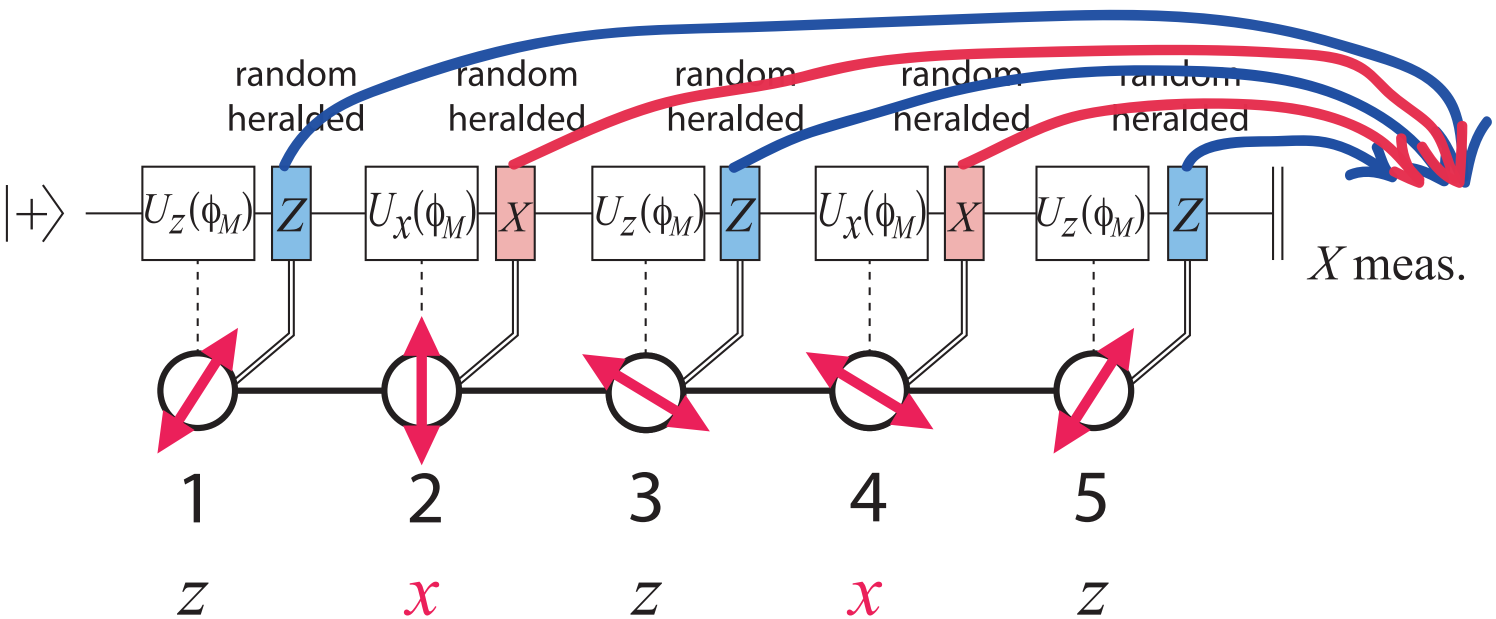
Every bit of computational output is a parity of individual local measurement outcomes.

What determines the support of these parities?

The measurement record **s** is a gauge field, and each bit of the computational output **o** is a holonomy of the gauge field.

How the side-processing comes about

... it describes how to counteract the randomness of measurement outcomes.



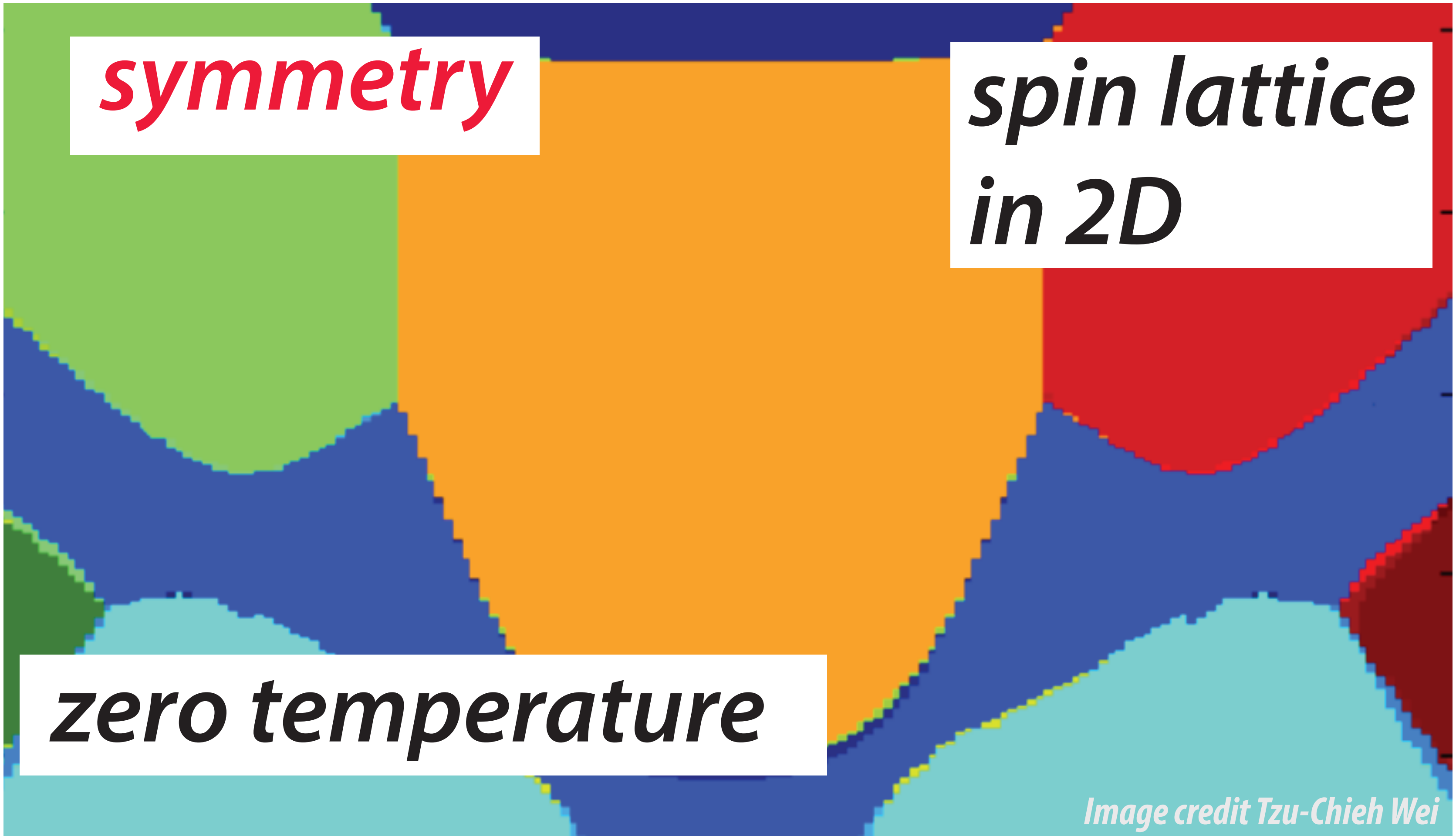
- Propagate the random heralded *byproduct operators* forward in time past the end of the computation.

Part II:

A short history of

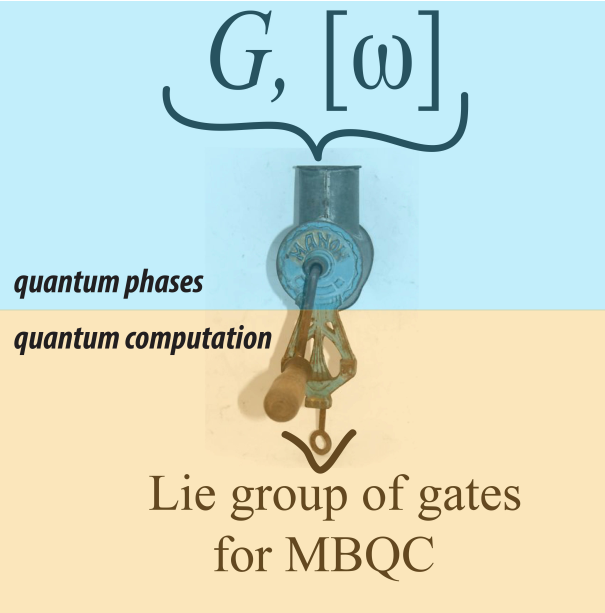
"computational phases of quantum matter"

Symmetry-protected phases of quantum matter



... all across support quantum computation.

Motivation: MBQC and symmetry

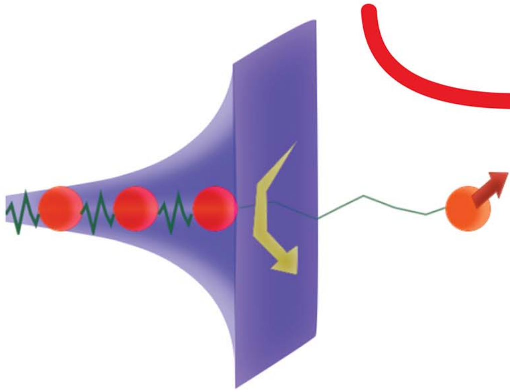


Can MBQC schemes be classified by symmetry, in a similar way as, say, elementary particles can?

If so, does this have a bearing on quantum algorithms?

I. Symmetry protects computation

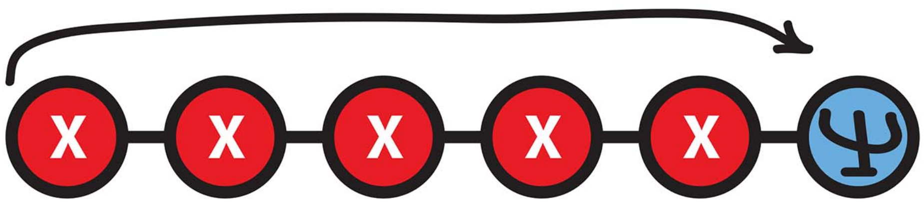
we observe low-maintenance features of the ground-code MQC in that this computation is doable without an exact (classical) description of the resource ground state as well as without an initialization to a pure state. It



It turns out these features are deeply intertwined with the physics of the 1D Haldane phase (cf. Fig. 1), that is well characterized as the symmmetry-protected topological order in a modern perspective [6, 7]. We believe our approach must bring the study of MQC, conventionally based on the analysis of the model entangled states (e.g., [1, 8, 9]), much closer to the condensed matter physics, which is aimed to describe characteristic physics based on the Hamiltonian.

A. Miyake, Phys. Rev. Lett. 105, 040501 (2010).

II. Symmetry-protected wire in MBQC



- Computational wire persists throughout symmetry-protected phases in 1D.
- Imports group cohomology from the classification of SPT phases.

D.V. Else, I. Schwartz, S.D. Bartlett and A.C. Doherty, PRL 108 (2012).

F. Pollmann *et al.*, PRB B 81, 064439 (2010); N. Schuch, D. Perez-Garcia, and I. Cirac, PRB 84, 165139 (2011); X. Chen, Z.-C. Gu, and X.-G. Wen, PRB 83, 035107 (2011).

Under the hood ..

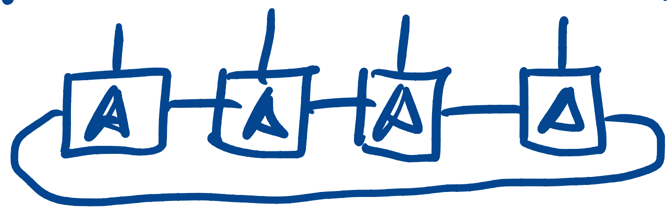
Theorem. Consider a symmetry-protected phase characterized by a finite abelian symmetry group and a maximally non-commutative cohomology class $[\omega]$. In the symmetry-respecting basis $\mathcal{B} = \{|i\rangle\}$, the MPS tensor of a given state in the SPT phase described by $[\omega]$ has the form

$$A[i] = B_i \otimes A_{\text{junk}}[i].$$

The byproduct operators B_i are all unitary and constant throughout the phase.

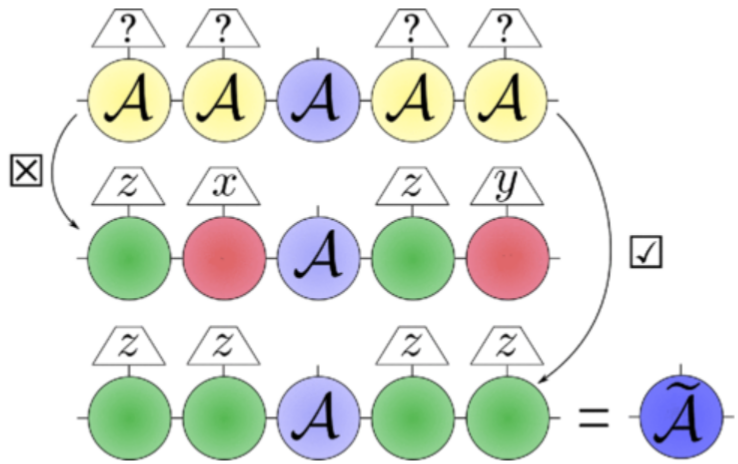
D.V. Else, I. Schwartz, S.D. Bartlett and A.C. Doherty, PRL 108 (2012).

Resource state $|\phi\rangle$:



MBQC on $|\phi\rangle$

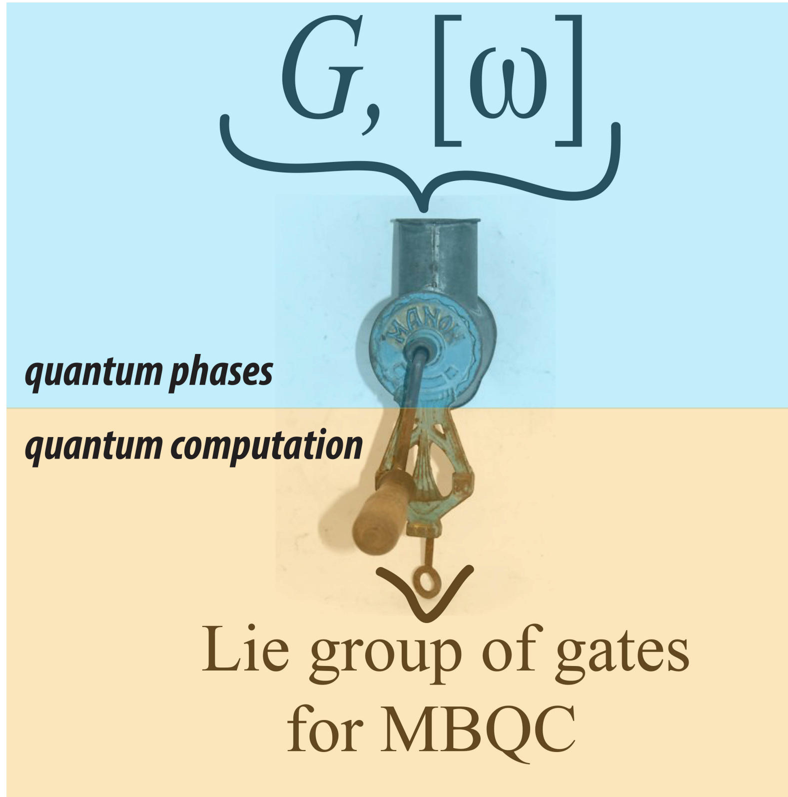
III. First quantum computational phase



- 1-qubit universal MBQC on a chain of spin-1 particles protected by an S_4 symmetry.

J. Miller and A. Miyake, Phys. Rev. Lett. 114, 120506 (2015).

IV. The SPT⇒MBQC meat grinder



A classification of MBQC schemes by symmetry in 1D.

A. Prakash and T.-C. Wei, Phys. Rev. A (2016).

RR, A.Prakash, D.-S. Wang, T.-C. Wei, D.T. Stephen, Phys. Rev. A (2017).

Under the hood ..

cohomology class $[\omega]$

$\downarrow$ The given SPT phase determines the byproduct operators [Else et al., RRL 2012]

B_i

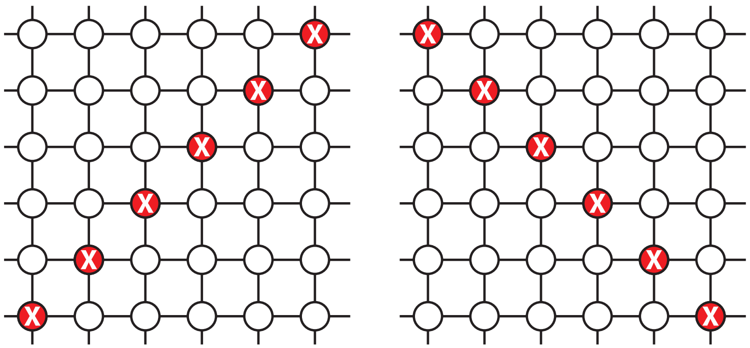
$\downarrow$ The byproduct operators determine MBQC computational power [RR et al., RRA 2017]

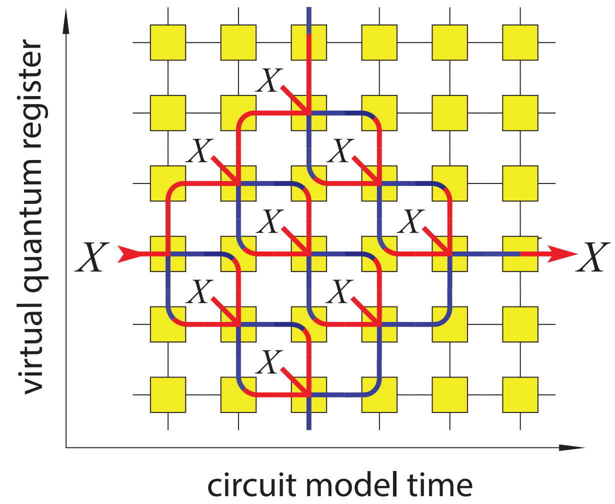
set of gates

Byproduct operators: *What entered as a nuisance that could fortunately be dealt with, becomes the central object governing MBQC computational power.*

V: Computationally universal SPT phase in 2D

- The symmetries of the phase are





- The 2D cluster state is inside the phase

Result. For a spin-1/2 lattice on a torus all ground states in the 2D cluster phase, except a possible set of measure 0, are universal resources for measurement-based quantum computation.

R. Raussendorf, C. Okay, D.S. Wang, D.T. Stephen, H.P. Nautrup, *A computationally universal quantum phase of matter*, Phys. Rev. Lett. 122, 090501 (2019).

Part III:

A gauge theory of MBQC

Motivation

The discussion of computational phases of quantum matter is ongoing. It has so far taught us

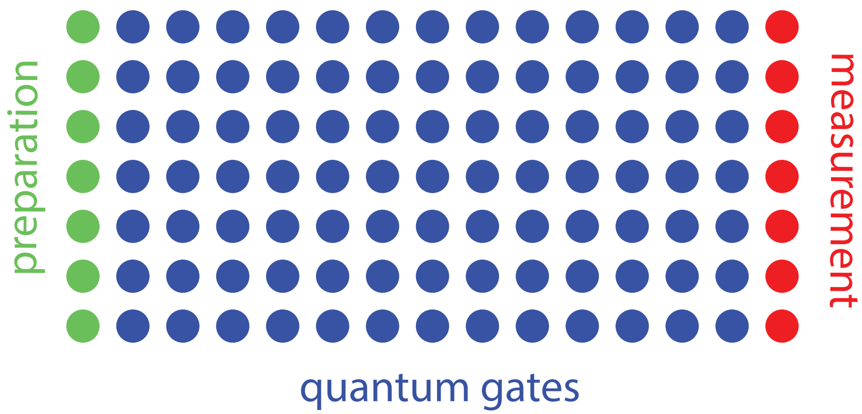
- The computational capability of MBQC schemes is determined and classified by symmetry.

- To harness this capability, however, a breaking of symmetry is required, namely through the local measurements.

We want a complete picture of the role of symmetry in MBQC.
Gauge symmetry seems part of that.

Result #1

Recall the MBQC elevator thought experiment:

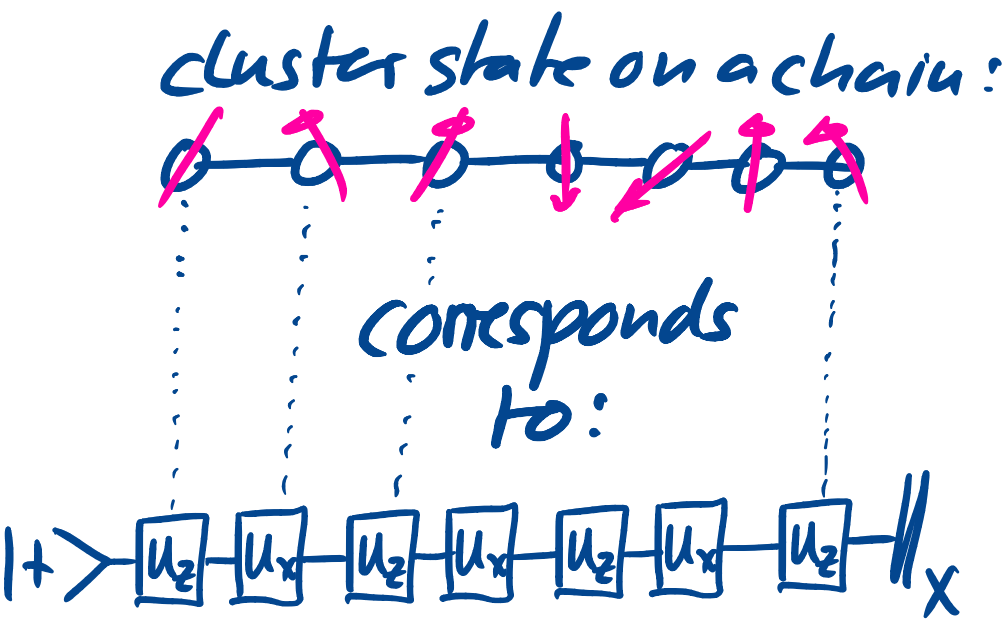


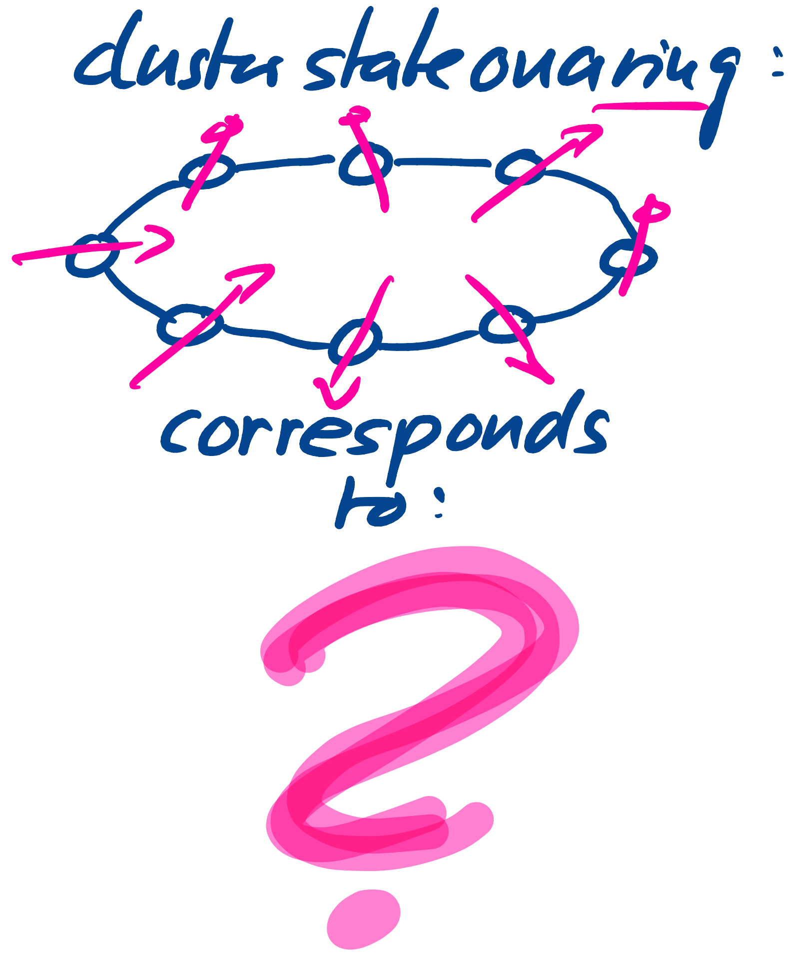
All cluster qubits contribute in the same fashion to the computational output, namely through parities.

What determines the support of these parities?

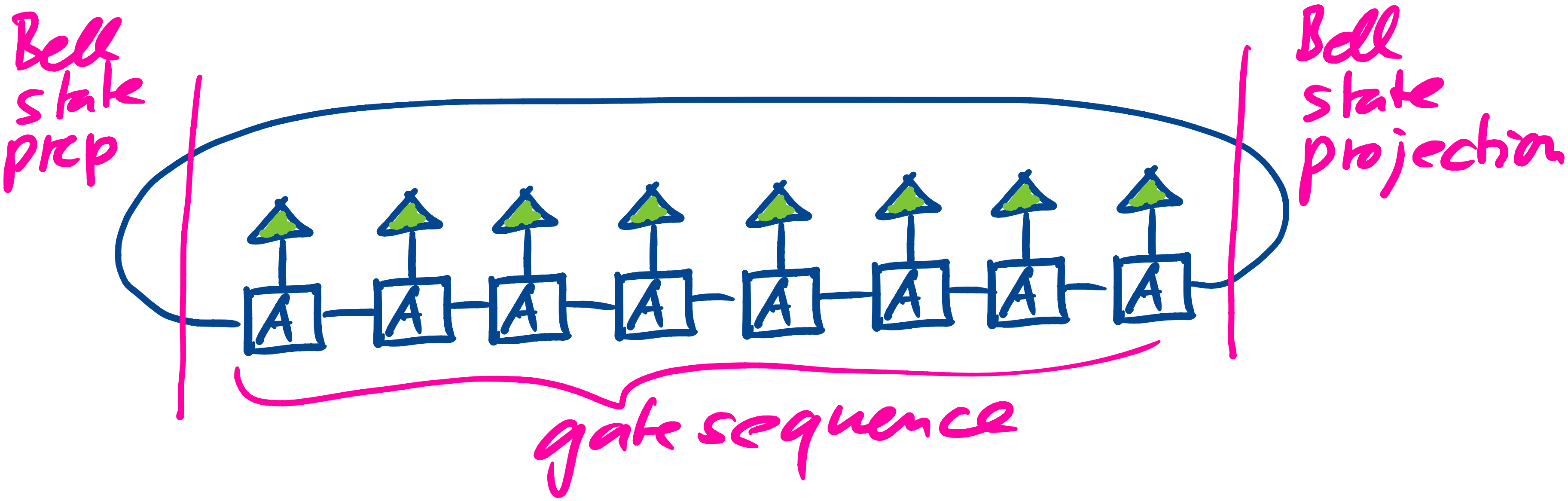
Answer: *The measurement record* **s** *can be regarded as a gauge field, and the computational output* **o** *corresponds bit-wise to holonomies of that gauge field.*

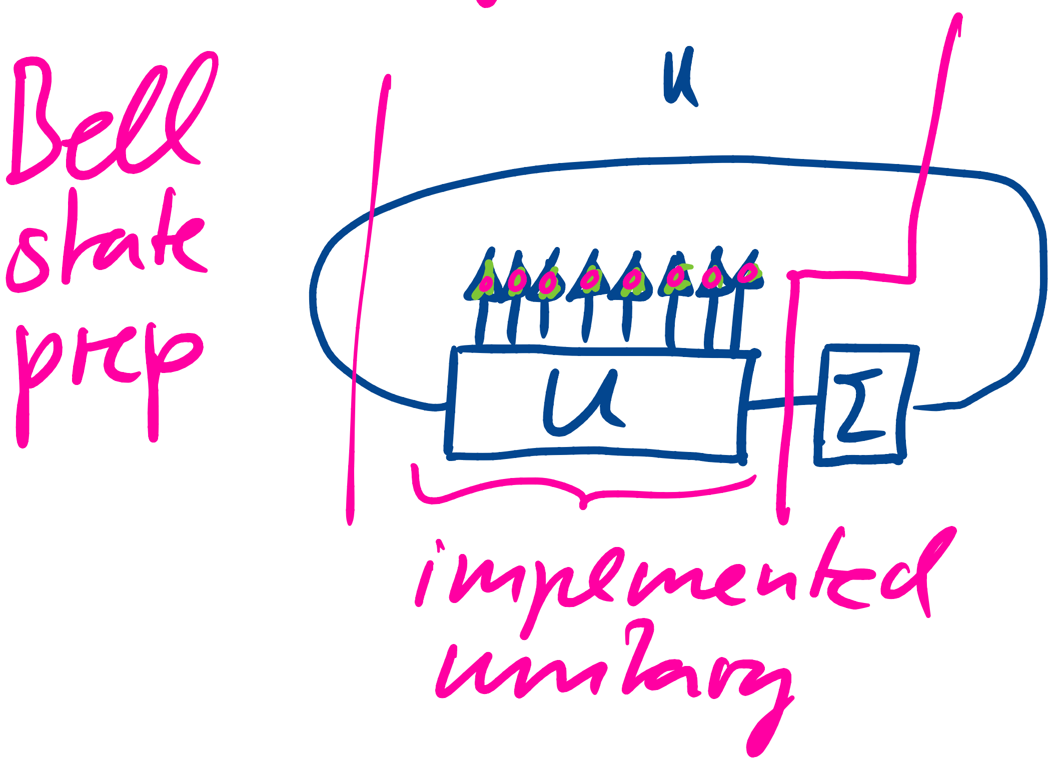
Slightly modified setting—dodging boundaries





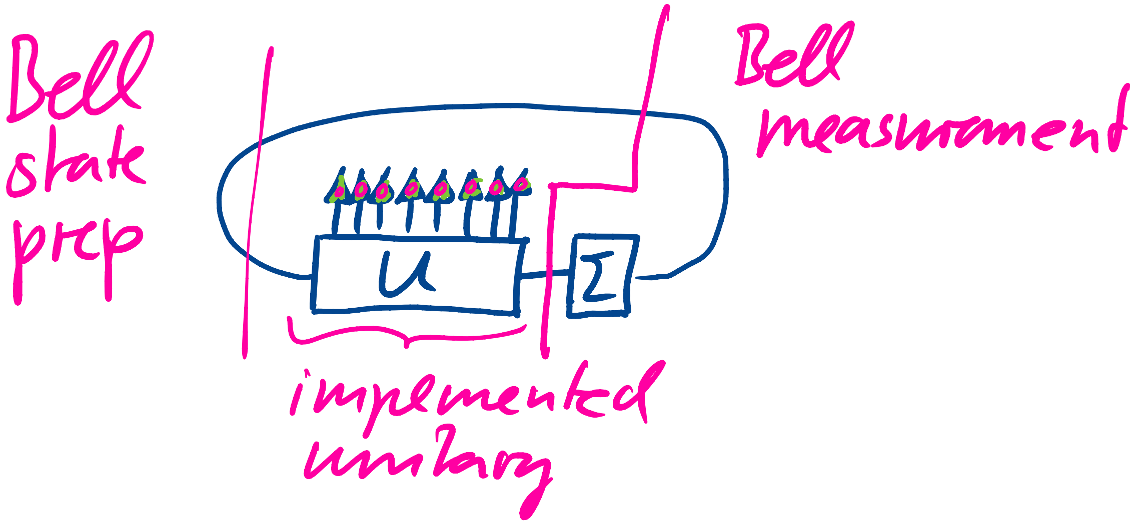
What is implemented on the cluster ring?





$$\underbrace{\Sigma = X^{o_1} Z^{o_2}}_{\text{byproduct op.}}$$

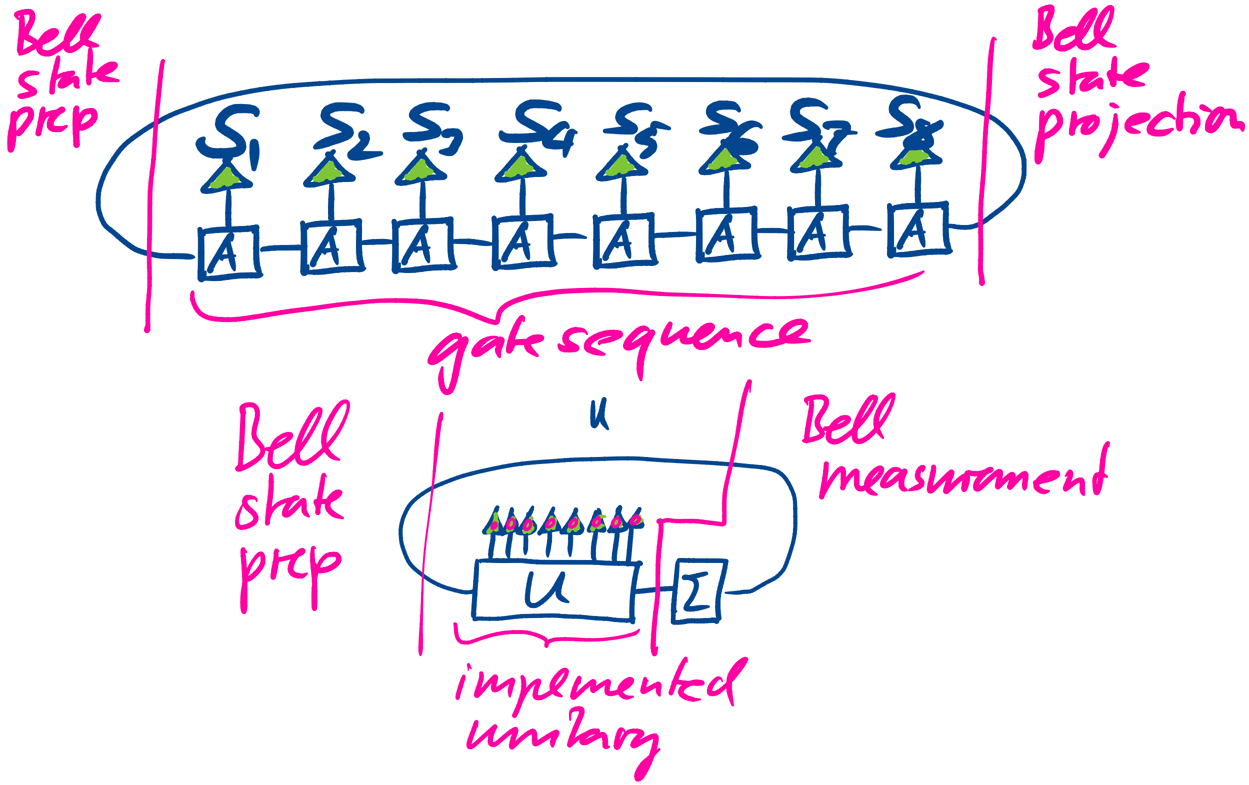
What do the two output bits represent?



$$U = \alpha I + \beta X + \gamma Z + \delta Y.$$

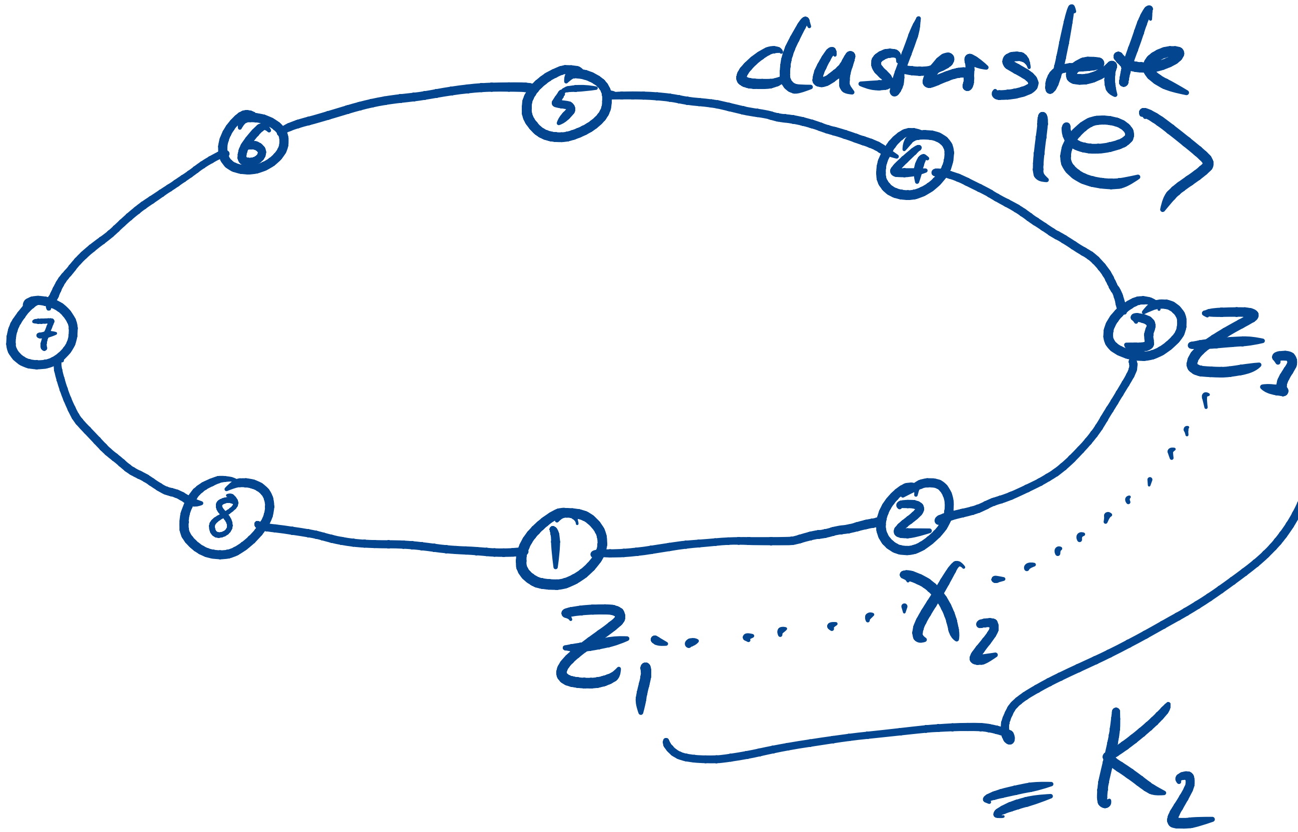
outcome	occurs with probability
00	$\lvert\alpha\rvert^2$
01	$\lvert\beta\rvert^2$
10	$\lvert\gamma\rvert^2$
11	$\lvert\delta\rvert^2$

How are the two output bits obtained from s?



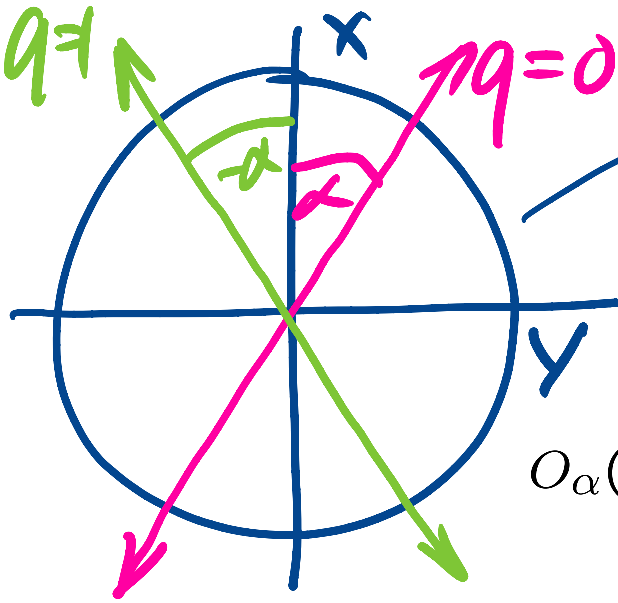
$$o_1 = \sum_{i \text{ even}} s_i \mod 2, \quad o_2 = \sum_{i \text{ odd}} s_i \mod 2. \qquad (2)$$

Recap: cluster state stabilizer



$$\boxed{K_j|\mathcal{C}\rangle = |\mathcal{C}\rangle}$$

Pauli/ stabilizer action on s and q



X/Y equator of the Bloch sphere

Local observables measured:

$$O_\alpha(q) = \cos\alpha\, X + (-1)^q \sin\alpha\, Y.$$

X flips q

$$\boxed{\begin{aligned} X^\dagger O_\alpha(q) X &= O_\alpha(q \oplus 1), \\ Z^\dagger O_\alpha(q) Z &= -O_\alpha(q). \end{aligned}}$$

Z flips s

Gauge transformations

Recall:

$$\begin{aligned} X^\dagger O_\alpha(q) X &= O_\alpha(q \oplus 1), \\ Z^\dagger O_\alpha(q) Z &= -O_\alpha(q). \end{aligned}$$

flips q
flips s

and:

$$K_j = Z_{j-1} X_j Z_{j+1}$$

$$\langle \mathcal{C} | O^{(i)}_{\alpha_i}(q_i) | \mathcal{C} \rangle = \langle \mathcal{C} | K_j^\dagger O^{(i)}_{\alpha_i}(q_i) K_j | \mathcal{C} \rangle, \ \ \forall i, j$$

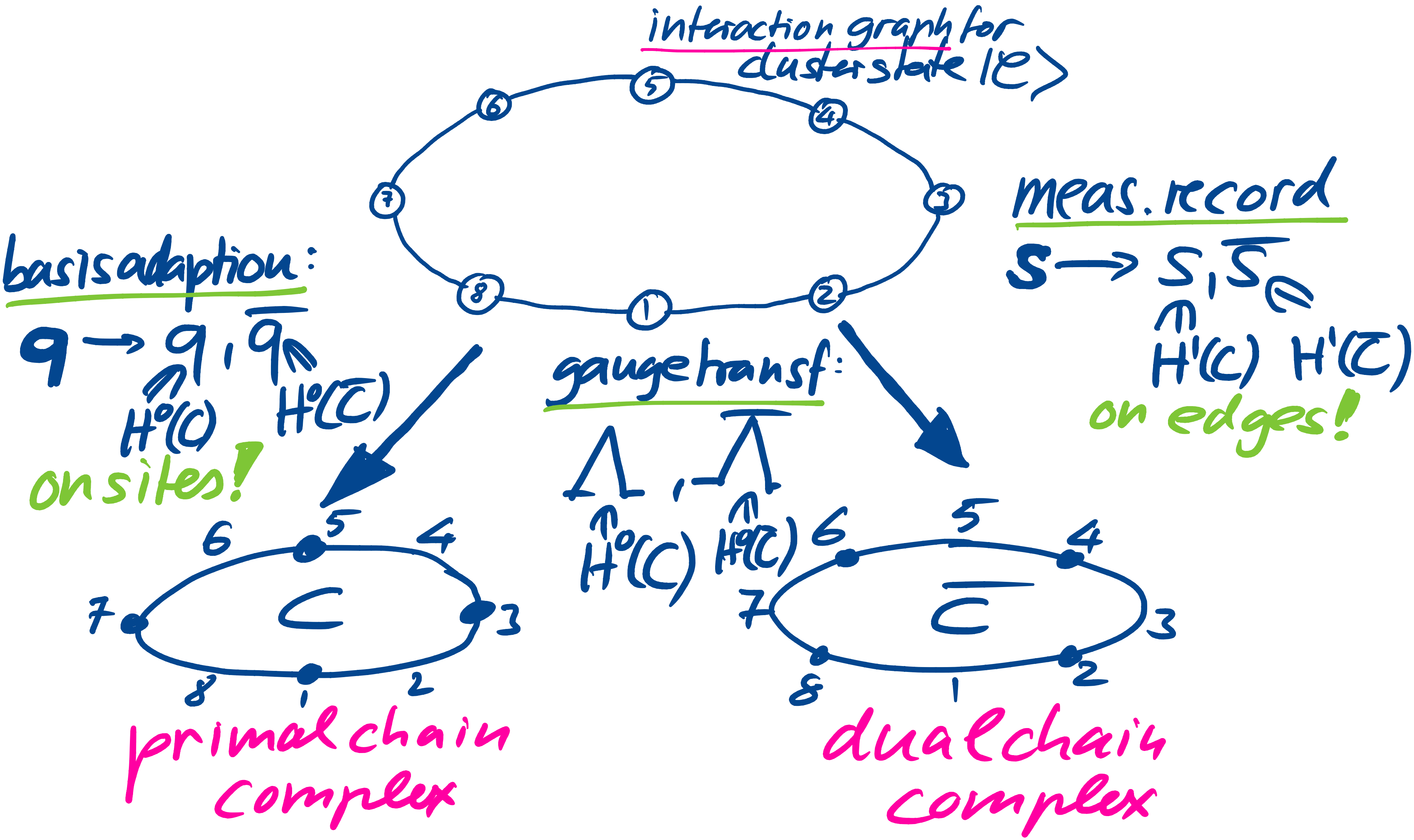
insertion of K_j has no effect!

Thus we have the following equivalence/ gauge transformations

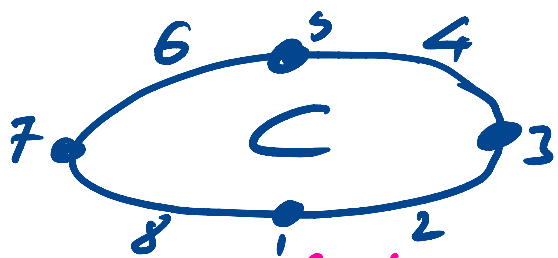
$$K_j : \begin{array}{rcl} s_{j-1} & \mapsto & s_{j-1} \oplus 1, \\ s_{j+1} & \mapsto & s_{j+1} \oplus 1, \\ q_j & \mapsto & q_j \oplus 1. \end{array} \qquad (3)$$

- Those equivalence transformations are local.
- For $j = 2, .., 2N-1$ those equivalence transformations preserve the classical processing relations.

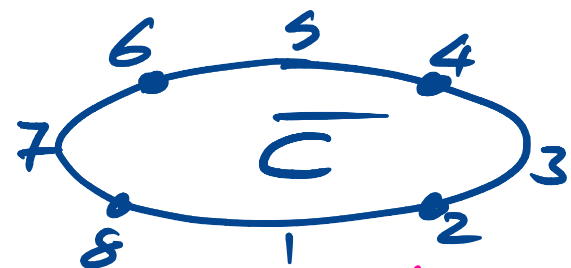
Cohomological interpretation



Gauge transformations in cohomological form



primal chain complex



dual chain complex

standard form

cohomological form

$$K_j : \begin{array}{rcl} s_{j-1} & \mapsto & s_{j-1} \oplus 1, \\ s_{j+1} & \mapsto & s_{j+1} \oplus 1, \\ q_j & \mapsto & q_j \oplus 1. \end{array}$$

$$\begin{array}{ll} s & \mapsto s \oplus d\Lambda, \\ q & \mapsto q \oplus \Lambda, \\ \overline{s} & \mapsto \overline{s} \oplus d\overline{\Lambda}, \\ \overline{q} & \mapsto \overline{q} \oplus \overline{\Lambda}. \end{array}$$

Gauge-invariant quantities

Find a 1-chain e such that, for all λ

$$s(e) \mapsto [s \oplus d\Lambda](e) = s(e).$$

Requires

$$d\Lambda(e) = 0.$$

Hence,

$$0 = d\Lambda(e) = \Lambda(\partial e), \ \ \forall \Lambda,$$

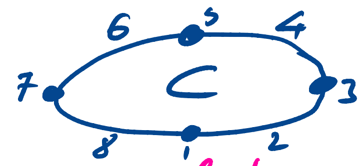
and thus

$$\partial e = 0.$$

The 1-chain e is, in fact, a 1-cycle.

- $s(e)$ is the corresponding holonomy.
- $s(e)$ is gauge invariant.
- Have the same invariant $\overline{s}(\overline{e})$ on the dual complex $\overline{C}$.

Gauge-invariant quantities



primal chain complex

dual chain complex

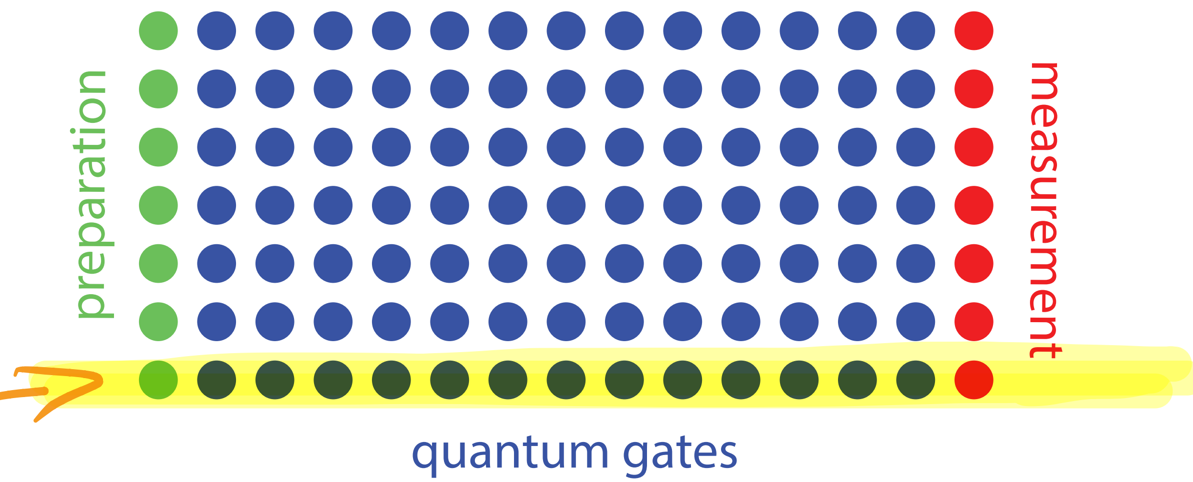
- $s(e)$ is gauge invariant.
- Have the same invariant $\overline{s}(\overline{e})$ on the dual complex $\overline{C}$.

These holonomies have previously been identified as the computational output. Recall:

$$\underbrace{o_1 = \sum_{i \text{ even}} s_i \mod 2}_{s(e)}, \quad \underbrace{o_2 = \sum_{i \text{ odd}} s_i \mod 2.}_{\overline{s}(\overline{e})}$$

The MBQC "elevator thought experiment"

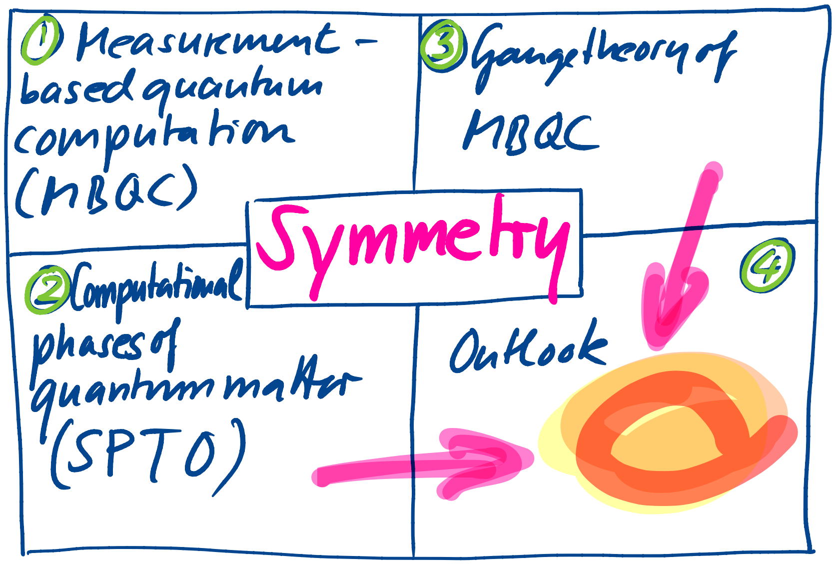
Consider a 2D cluster state:



Every bit of computational output is a parity of individual local measurement outcomes.

In 1D at least, these parities represent holonomies of a gauge field.

Outlook



* Extend gauge-theoretical description from fixed-point state to surrounding SPT phase
  - Problem: Can no longer use the stabilizers
  - Good sign: classical processing relations extend into the SPT phase unchanged

Summary

- In 1D so far, MBQC can be understood in terms of gauge theory, with the measurement record as the gauge field.

- In that picture, the MBQC computational output arises as holonomies of the gauge field.

- Extension to computational phases of quantum matter is suggestive.

Joint work with Gabriel Wong (Harvard)
and Bartek Czech (Tsinghua)