Entanglement in Gauge Theories, Matrices and Space-Time

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Outline

- Introduction to Entanglement
- Gauge Theories: a) Definition
  b) Operational Meaning
- Matrix Theory and Space-Time Entanglement: a) Definition b) Proposal
- Conclusions

The Entanglement Frontier

- We are in the midst of exploring a new frontier of science today.

- It does Not pertain to:

i) The very small, deep within the atom

ii) The very large, way beyond our galaxy

iii) Or the very many: Atoms, Molecules or Neurons.

The Entanglement Frontier

Instead, it pertains to our growing understanding of Quantum Mechanics. Its true meaning and significance.

Since quantum mechanics provides the bedrock for our understanding of science, this new frontier promises to have important implications for many branches of sciences:

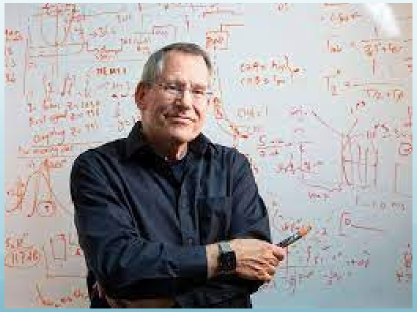
Physics, Chemistry, Computer Science, Other aspects of Engineering, Mathematics.

The Entanglement Frontier

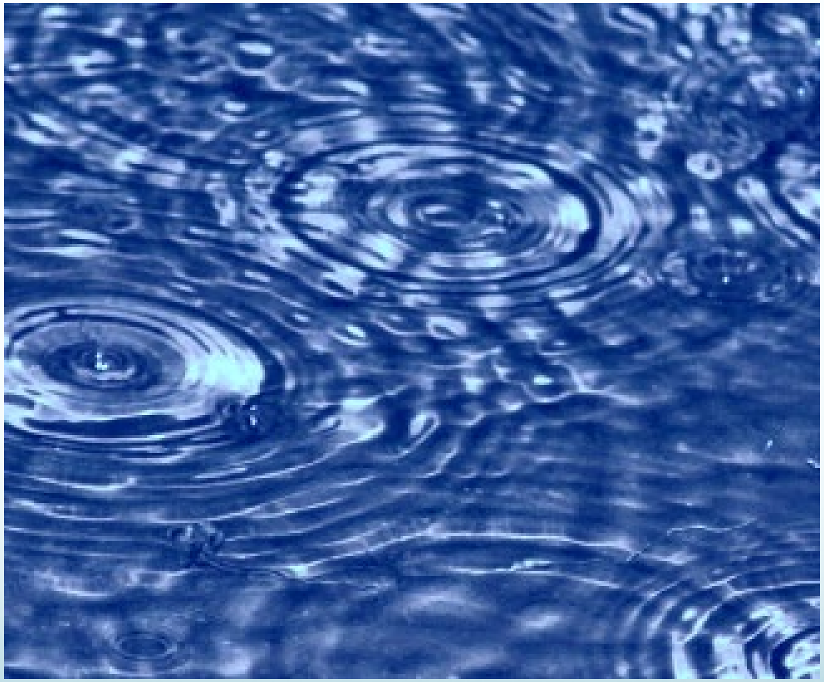
Quantum Mechanics is Strange.

Much of this strangeness has to do with a property called Entanglement which has no classical analogue.

This new frontier is therefore often referred to as the ``Entanglement Frontier'' (Preskill)



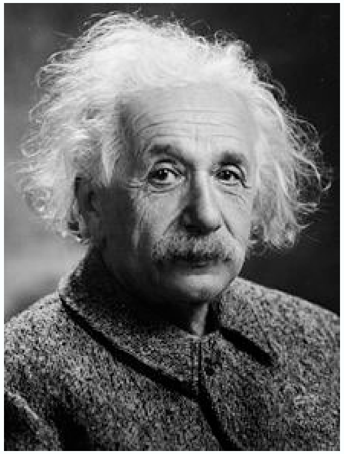
- Of course we all know that the underlying description in Quantum Mechanics is in terms of a wave function (probability amplitude).
- Different paths can interfere with each other as they evolve, like the different ripples of a wave interfere as it propagates.



$$P(x) = |\psi(x)|^2$$

Probability to find the particle at point x

Wave function



Einstein, Podolsky, Rosen

- Actually there is something else which is very strange about Quantum Mechanics. It was Einstein who first put his finger on this “weirdness” and it caused him great disquiet.
- A kind of ``spooky action at a distance”.
- The wave function which describes a system is non-local, even though all consequences are local.
- It is this additional weirdness which Entanglement refers to.

Quantum systems have non-local correlations. These have no Classical Analogue.

Even though all dynamics is local.

These non-local correlations are called entanglement

A quantitative measure is provided by the Entanglement Entropy.

Entangled Spins

$$|\psi> = \frac{1}{\sqrt{2}}[|+->-|-+>]$$

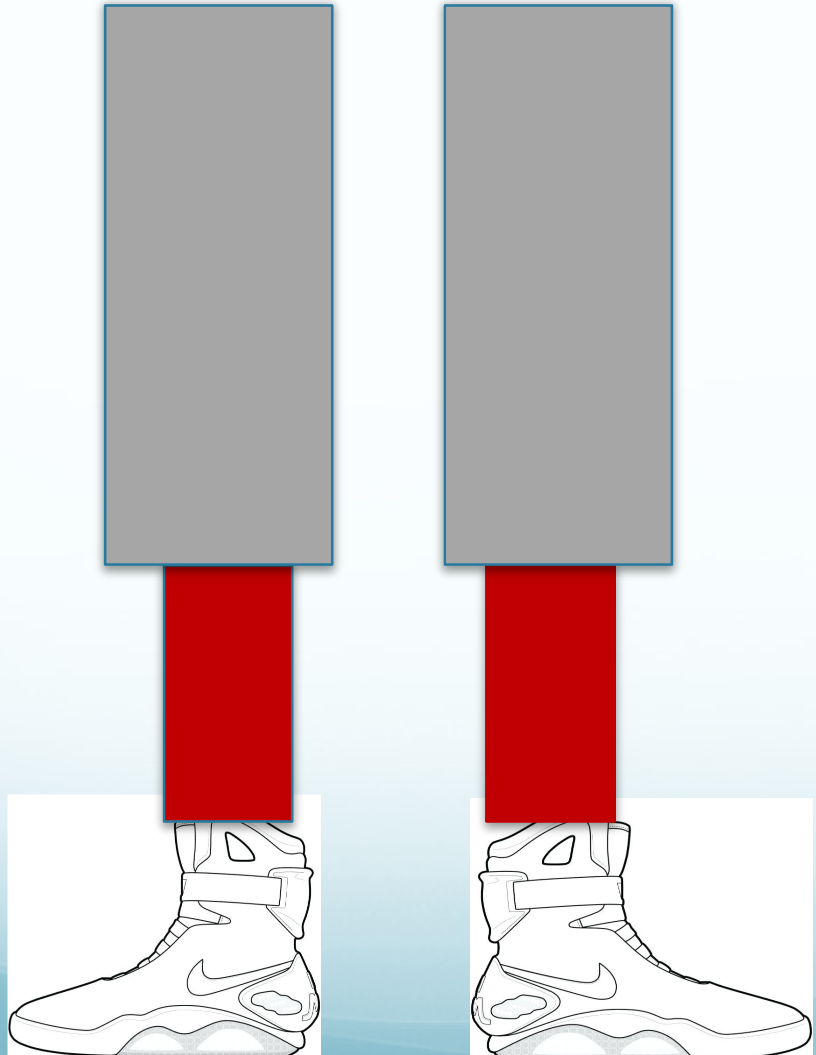
Entangled Spins

The first electron does not have a definite spin in any direction.

Regardless, once one find out its spin along any direction, x,y,z, by doing a measurement one knows the spin of the second electron as well.

This kind of Quantum Correlation is called Entanglement.

Of course correlations can arise in classical systems as well.



Looking at one sock you can tell the colour of the other one.

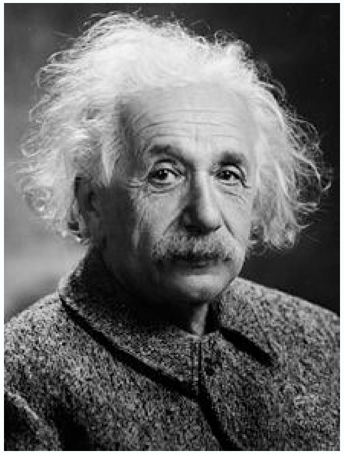
Entangled Spins

But the quantum correlations in the two electron system are different.

The first electron does not have a definite spin in any direction.

Regardless, once one find out its spin along any direction, by doing a measurement, one knows the spin of the second electron with certainty.

Entanglement characterises this kind of certainty present in the midst of the inherent uncertainty of quantum mechanics.



Einstein,
Podolsky,
Rosen

- All physical effects are local. No signal travels faster than light. Nevertheless our description of a state is inherently non-local.

- Lead Einstein to suggest that there was a more complete description which was classical, local and with additional hidden variables

- John Bell - brilliant paper in 1964.

- Pointed out that Entanglement would lead to experimental consequences which could be used to distinguish quantum theory from any classical theory with extra local hidden variable.

The differences are summarised by
Bell's inequalities.

No classical system can violate them.
The essential point is that probabilities
cannot be negative.

Quantum systems do violate them
for entangled states.

Alain Aspect

Experiments have brilliantly confirmed that Quantum Mechanics is correct and ruled out the hidden variable theories.

Quantifying Entanglement

$$|\psi>=\frac{1}{\sqrt{2}}[|+->-|-+>]$$

The two electrons in the state above are entangled.

The electrons in the state :

$$|\psi>=|++>$$

are also correlated. But they are not entangled. This state does not lead to violations of Bell's inequalities.

How do we quantify the quantum nature of these correlations?

Entanglement Entropy has emerged as an important measure.

$$\rho_{(1)} = Tr_{(2)}|\psi >< \psi|$$

$$= \sum_n < n|\psi >< \psi|n >$$

Operator acting on 1st electron; Called **Density Matrix**

Sum over all possible states for 2nd electron

Entanglement Entropy

$$S = -Tr_{(1)}[\rho_{(1)} \log \rho_{(1)}]$$

For the state:

$$|\psi >= \frac{1}{\sqrt{2}}[|+- > -|-+ >]$$

$$S = log(2)$$

Entanglement Entropy

$$S = -Tr_{(1)}[\rho_{(1)} \log \rho_{(1)}]$$

$$S = log(2)$$

Essentially the two states |+>, |-> are equally probable with probability p=1/2

$$S = -\sum_i p_i \ln(p_i) \quad = log(2)$$

Entanglement Entropy is the von Neumann Entropy of the Density Matrix.

- For the state

$$|\psi >= |++>$$

$$S = 0$$

- The two electrons are still correlated. But not entangled.

- This state will not lead to violations of Bell's inequalities.

- More generally, it is necessary that the state has a non-vanishing entanglement entropy to get a violation of Bell's inequalities.

- This makes entanglement entropy a good measure of ``quantum weirdness"

Entanglement Entropy in Gauge Theories

- Three of the four fundamental forces of nature are described by gauge theories.

- Gravity is also a kind of gauge theory.

- Gauge theories often emerge at low energies in condensed matter systems.

- Thus it is important to understand how to precisely define entanglement entropy in Gauge theories.

Four Fundamental Forces

Strong Interactions

Weak Interactions

Electro-Magnetism

$$SU(3) \times SU(2) \times U(1)$$

Gravity

Entanglement in Gauge Theories

- Turns out to be subtle to define.
- In a spin system the degrees of freedom are local.
- This makes it straightforward to define a measure of the non-local quantum correlations.

Entanglement in Gauge Theories

In contrast, for a gauge theory the degrees of freedom are non-local.

Electric and Magnetic lines cannot abruptly end at the boundary of the region of interest, and must satisfy continuity conditions.

This means the flux cannot be varied inside the region and outside in an independent manner.

Gauss' Law:

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

(Electromagnetism without charge)

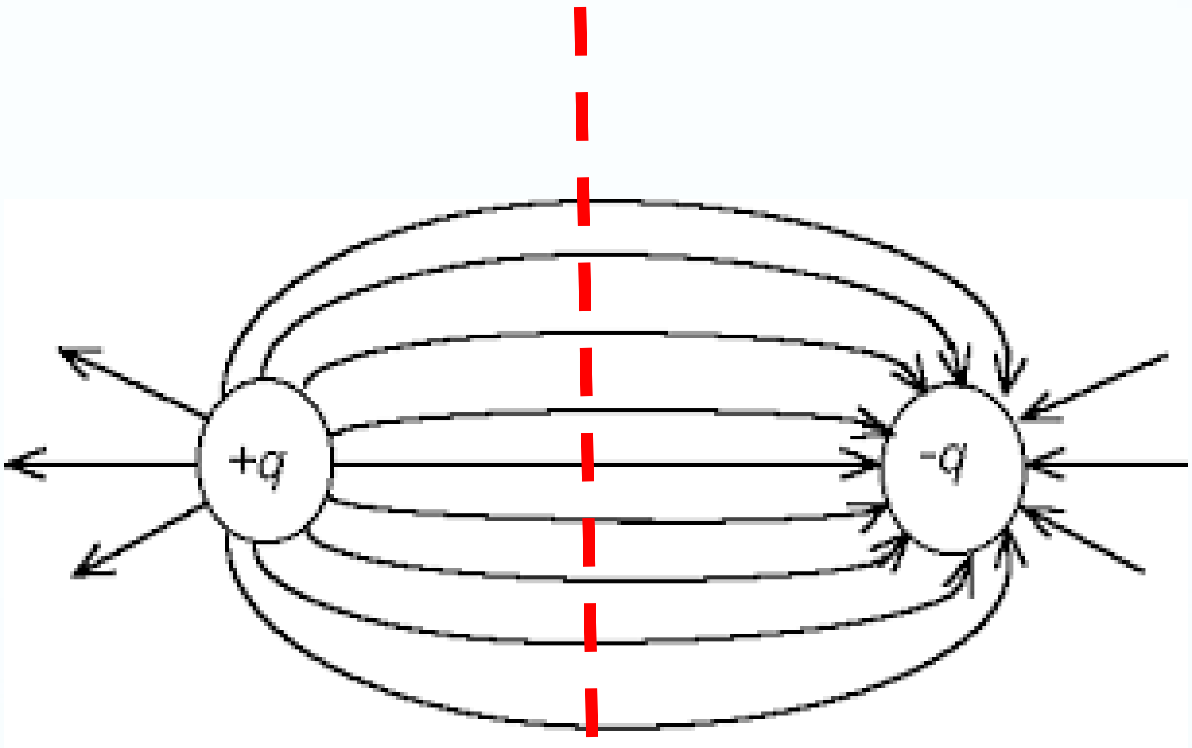


Figure 1.1

Lines of Force cross from on side to another.
The inside and outside are not independent

In the quantum theory (of pure electromagnetism without any charges)

Gauss' Law: $\vec{\nabla} \cdot \vec{E} = 0$

Ensures that the physical states are gauge invariant.

($A_0 = 0$ gauge).

Note that this is different from the two electron system.

$$|\psi>=\frac{1}{\sqrt{2}}[|+->-|-+>]$$

The particular state with S=0 was correlated. But in general the state of one electron could be changed independent of the other.

For example:

$$|\psi >= |++>$$

$$|\psi >= \left(a|+> +b|->\right)|+>$$

The full Hilbert space of the two electrons is therefore a tensor product

$$H = H_1 \otimes H_2$$

The tensor product nature of H then makes it straightforward to define the Entanglement Entropy for a spin system.

In contrast, since the electric and magnetic fluxes cannot be varied independently inside and outside the Hilbert Space of Gauge Invariant states in a Gauge Theory does not admit a tensor product decomposition.

This is the essential complication.

Ronak Soni

Sudip Ghosh

Upamanyu Moitra

JHEP 1509, 069 (2015), JHEP 1601, 136 (2016), JHEP 1702,101 (2017), JHEP 08 (2019) 059

Important Contributions

Casini, Huerta, Rosabal (PRD 89, 2014)

Donnelley (PRD 85, 2012, Class Quant. Grav. 31, 2014,…)

Casini, …

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``On The Entanglement Entropy For Gauge Theories”,
Sudip Ghosh, Ronak Soni and SPT
JHEP 1509, 069 (2015)

``Aspects of Entanglement Entropy For Gauge Theories”,
Ronak Soni, SPT, JHEP 1601, 136 (2016)

``Entanglement Entropy in U(1) Gauge” Theory, Ronak Soni, SPT, JHEP 1702,101 (2017)

``On The Entanglement Entropy For Gauge Theories'',
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JHEP 1509, 069 (2015)

``Aspects of Entanglement Entropy For Gauge Theories'',
Ronak Soni, SPT, JHEP 1601, 136 (2016)

``Entanglement Entropy in U(1) Gauge''
Theory, Ronak Soni, SPT, JHEP 1702,101 (2017)

The approach we took was to embed the physical states in an Extended Hilbert space which include states that do not satisfy the Gauss law constraint.

These additional states are not Gauge Invariant.

The extended Hilbert space now has a tensor product decomposition giving rise to a Entanglement Entropy.

The result for the Entanglement Entropy is gauge invariant.

Method works for gauge theories in general, including Abelian and Non-Abelian ones, with and without matter.

Agrees with the replica trick.

The algebra of gauge invariant operators has a non-trivial center.

This means the Hilbert space is a sum over tensor products.

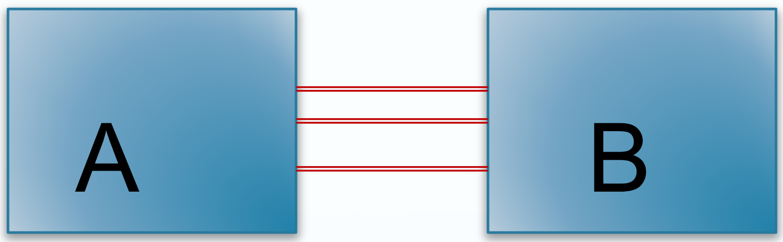
This structure generalizes for the non-Abelian case too.

$$H = \sum_i H_i^{(A)} \otimes H_i^{(A^c)}$$

(Casini, Huerta, Rosabal)

For a Gauge Theory:

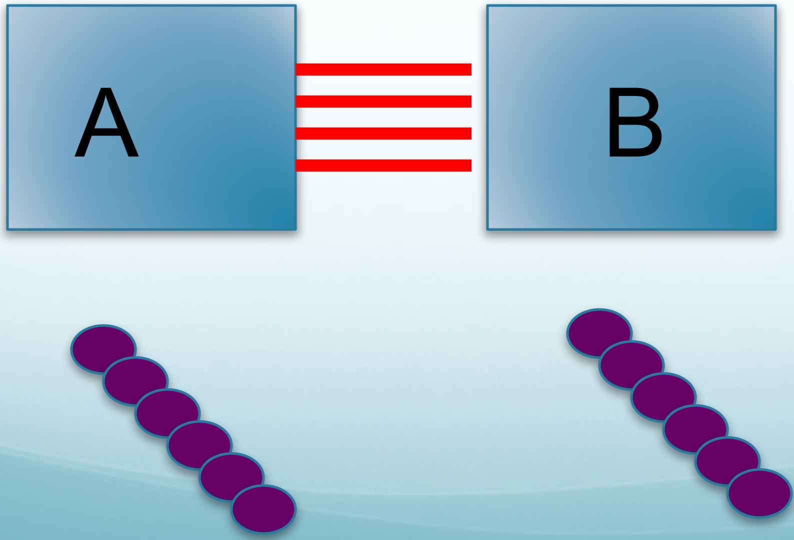
All the entanglement cannot be distilled



Soni, ST,
Acoleyen et.al.,

N entangled Bel pairs

Entanglement Distillation :



2N unentangled qubits

The von Neumann entropy for the density matrix in gauge theories has a distillable and non-distillable piece.

The distillable amount tells you how many Bell pairs you can extract.

The non-distillable amount is also interesting. In some cases (e.g.Toric Code) it is a measure of `topological” properties of the state.
(Topological Entanglememnt Entropy)

Matrix Theory Entanglement

$$S = \frac{N}{2(g_s N) l_s} \int dt [\sum_{i=1}^{9} Tr(D_t X^I)^2 - \frac{1}{l_s^4} \sum_{I \neq J = 1}^{9} Tr([X^I, X^J]^2)]$$

+ fermions

$$D_t X^I = \partial_t X^I + i[A_t, X^I]$$

$X^1, X^2 \cdots, X^9$: 9 NXN matrices

$O(N^2)$: degrees of freedom.

Matrix Theory

$U(N)$ Gauge Theory

A_t : Gauge Field

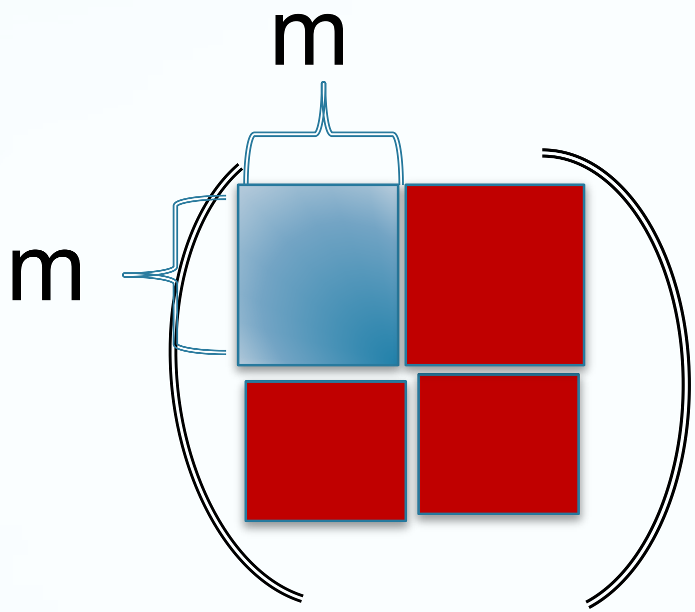
Action Invariant under $X^I \to U X^I U^{-1}$

$$A_t \to iU\partial_t U^{-1} + U A_t U^{-1}$$

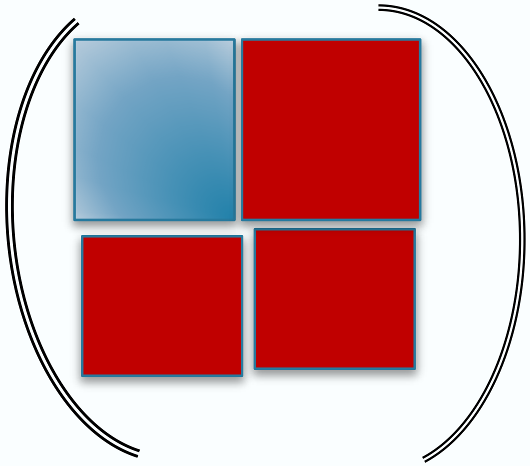
$O(N^2)$ Colour degrees of freedom

Roughly speaking, we want to ask how entangled are some of the degrees of freedom of the matrices with others?

Gauge Symmetry makes this question complicated!



Keep $m \times m$ block for all matrices and ask what is the entanglement entropy of the resulting density matrix?



NXN Matrix

U(N) gauge transformation will mix the blue block with the rest.

So what is the gauge invariant way to ask this question?

This is the question we will address.

Some References:

1) Mazenc and Ranard (hep-th/1910.07449)

2) S. Das, A. Kashyap, G. Mandal, ST, (hep-th/2004.00613)

3) S. Das, S. Liu, A. Kashyap, G. Mandal, ST, (hep-th/2011.13857)

4) H. Hampapura, J. Harper, A. Lawrence, (hep-th/2012.15683)

Context:

Why is this kind of Matrix Model interesting?

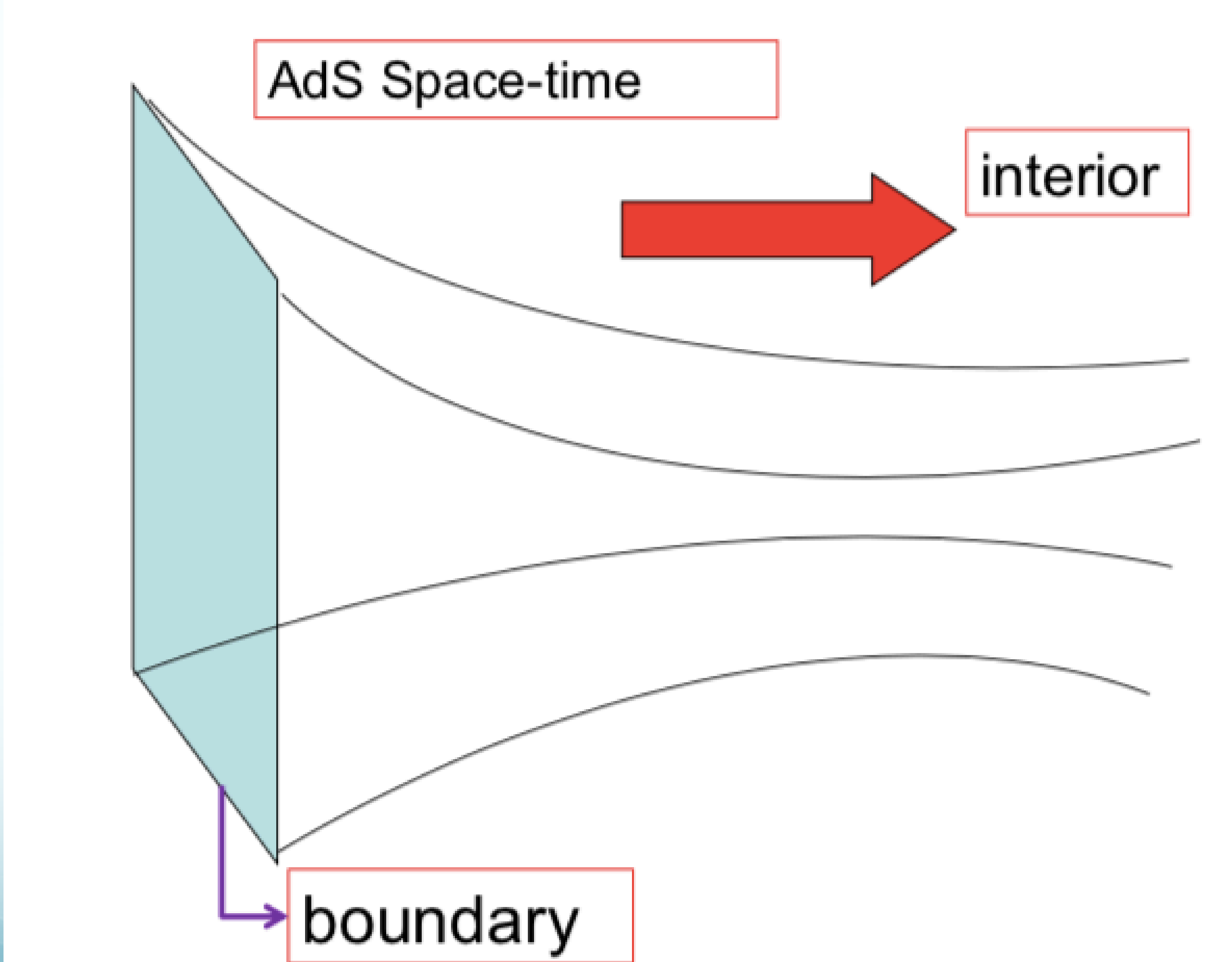
The AdS/CFT correspondence and related developments in the past two decades have shown that such models (and their higher dimensional analogues) describe spacetimes which are smooth in appropriate limits.

Juan Maldacena, …

A well known example of AdS/CFT:

The N=4 supersymmetric Yang Mills Theory (CFT) in 3+1 dimensions is equivalent to String Theory in 5 Dimensional Anti-de Sitter Spacetime $AdS_5 \times S^5$

(In the matrix theory, the boundary theory only lives in time, not along any spatial directions, so this is a higher dimensional analogue).



CFT (Conformal Field Theory)

How did this correspondence come about?

- D-branes, which are soliton- like objects in string theory, give rise to 10 dimensional solutions of (super) gravity.
- Their low energy excitations can be described in two ways, as a gauge theory, or as excitations in a gravity theory.
- The equivalence between these two descriptions gives rise to the correspondence.

For D0 branes (Near Horizon limit)

$$ds^2_{string} = -H_0(r)^{-1/2}dt^2 + H_0(r)^{1/2}[dx_1^2 + \cdots + dx_9^2]$$

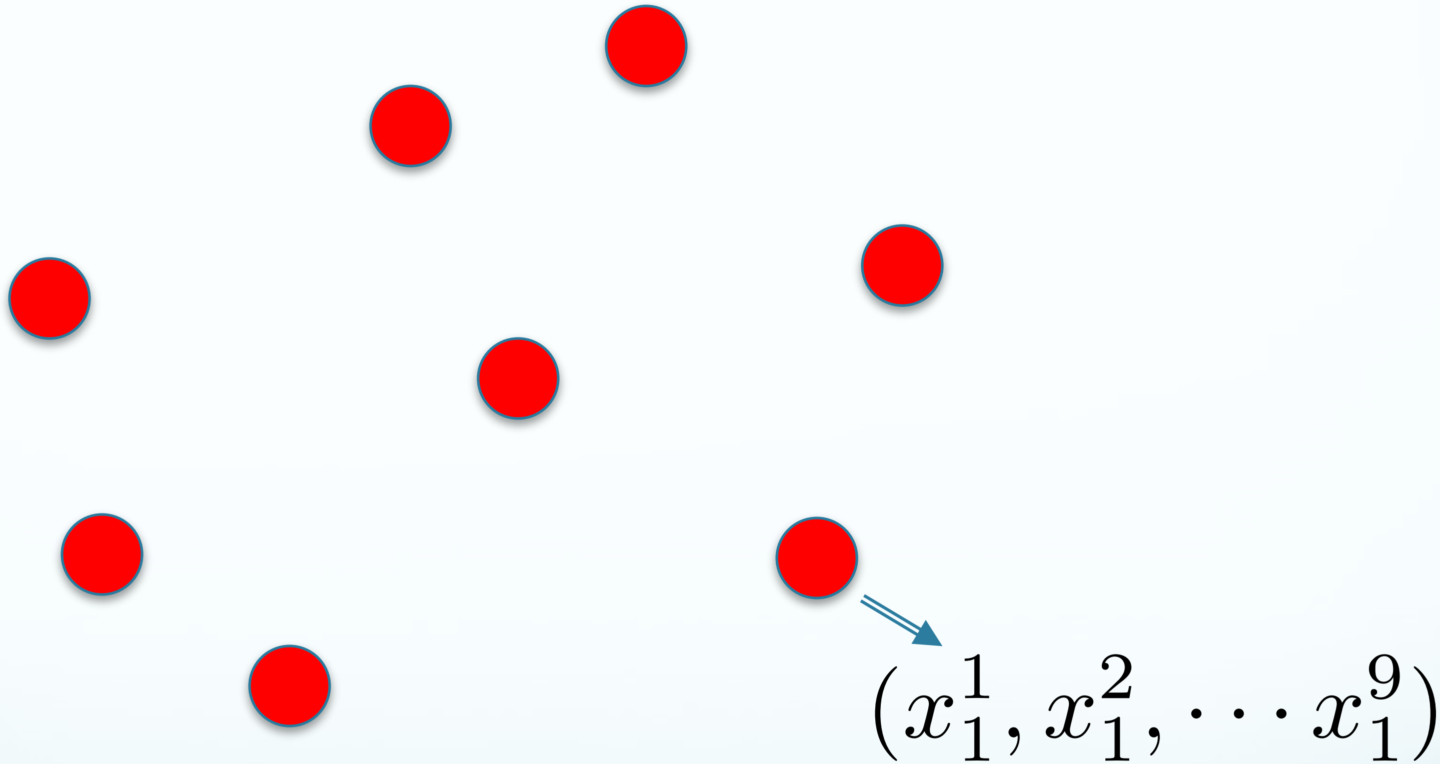
$$H_0(r) = \frac{R^7}{r^7}$$

$$r^2 = x_1^2 + x_2^2 + \cdots x_9^2$$

$$e^{-2\phi} = g_s^{-2} H_0(r)^{-3/2}$$

$$R^7 = \frac{(2\pi)^7}{7\Omega_8} l_s^7 (g_s N).$$

Coulomb Branch



D0 branes in Bulk (in Coulomb branch) One to one correspondence with the vacuua in Matrix Model.

Coulomb Branch in Matrix Theory

Moduli space of vacua (Coulomb branch)
: All 9 matrices commute.

$$X^1 = diag(x_1^1, x_2^1, \cdots x_N^1),$$
$$X^2 = diag(x_1^2, x_2^2, \cdots x_N^2),$$

$(x_1^1, x_1^2, \cdots x_1^9)$: location of first D0 brane etc

Matrix Theory Entanglement

This definition is actually motivated from the bulk (the gravity side).

Suppose we are interested in the bulk region $x^1 > a$ and want to ascribe an entropy to it.

Look at corresponding constraint in $X^1 > a$ boundary

Entanglement Entropy in Matrix Theory

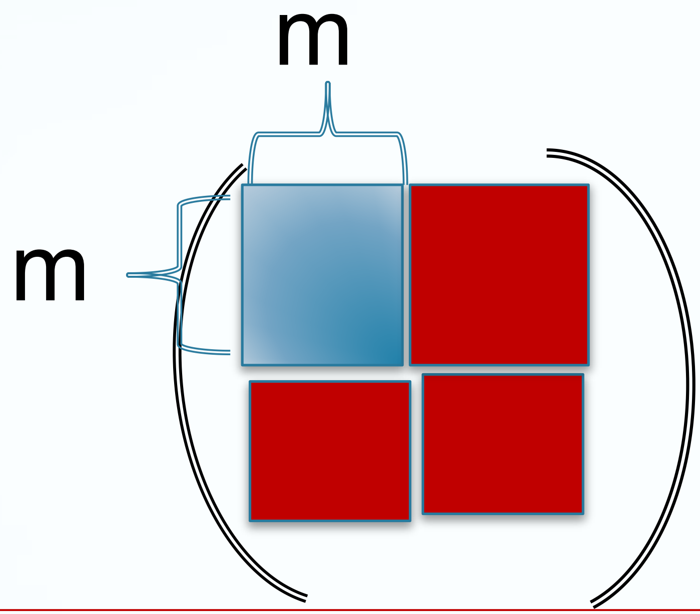
Let us go to a gauge where X^1 is diagonal

$$X^1 = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix}$$

Arrange eigenvalues in descending order

$$\lambda_1 > \lambda_2 \cdots > \lambda_{N-1} > \lambda_N$$

Let's say first m eigenvalues meet the condition $\lambda > a$



Keep $m \times m$ block for all matrices and compute the density matrix for these degrees of freedom.

Note: in general the other matrices, $X^2, \cdots X^9$ will not be diagonal in this gauge.

But the Hilbert space (in this sector) still admits a tensor product decomposition for all matrix degrees of freedom.

By tracing over the remaining matrix elements we can then obtain a density matrix ρ_m

(This is version 1 of the proposal)

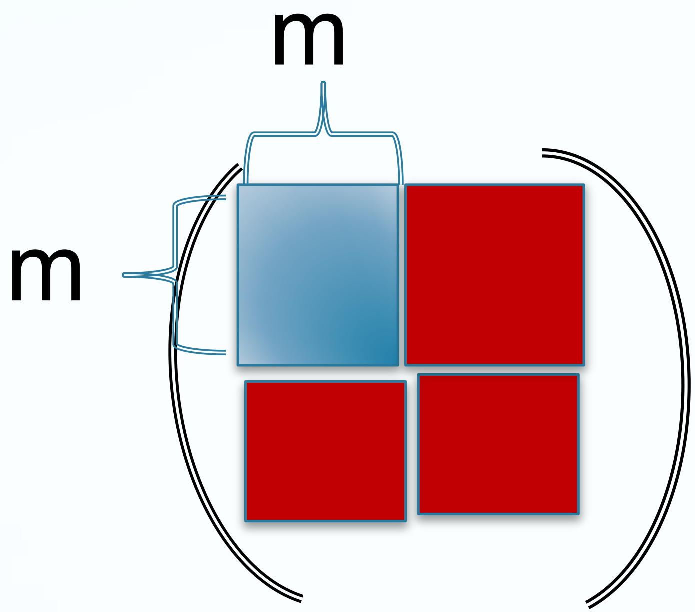
Entanglement in the m th sector:

$$S_m = -Tr_{(m)} \rho_m log(\rho_m)$$

Full entanglement given by a sum over all sectors

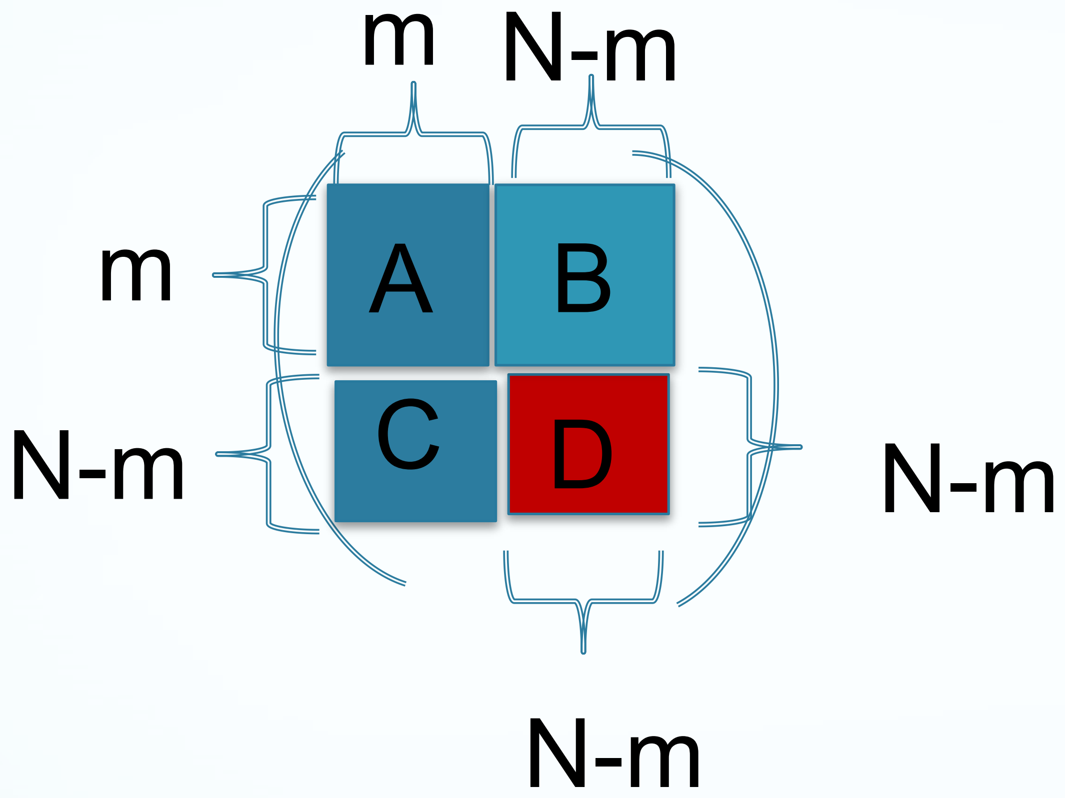
$$\boxed{\text{Target space entanglement}} \qquad S_T = \sum_{m=0}^{m=N} S_m$$

Include possibility that no eigenvalues lies in region of interest



Version 1

Keep $m \times m$ block for all matrices and compute its entanglement entropy.



Version 2

Keep blocks A, B, C and remove D – which is of size (N-m)X(N-m)- for all matrices.

For either version we now have to sum over the various sectors: $m = 0, 1, \ldots, N$

Full Target Space Entanglement is the sum over the contributions from the various sectors.

Matrix Theory Entanglement

We have used a particular gauge to define the entropy.

But in fact it can be given a gauge invariant meaning using suitable projection operators and the resulting gauge invariant subalgebra.

(There are two versions of our definition as we will see shortly)

Matrix Entanglement

Definition can be generalised for any constraint involving the matrices

$$X^I, I = 1 \cdots 9$$

As operators X^I commute, any constraint of the form $F(X^I) > 0$ can be dealt with by diagonalising the constraint and then obtaining the entanglement sector by sector.

E.g. $$F(X^I) = \sum_{I=1}^{9} (X^I)^2 > R^2$$

Target Space Entanglement

This definition is actually motivated from the bulk (the gravity side).

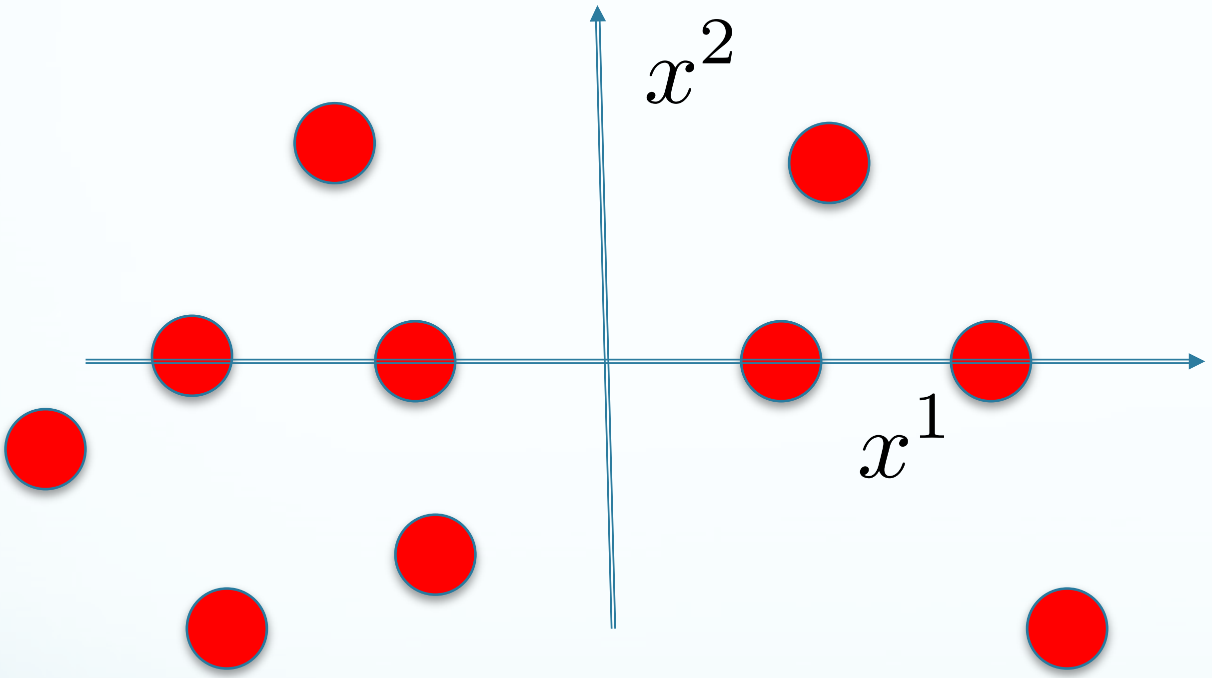
Suppose we are interested in the bulk region $x^1 > a$ and want to ascribe an entropy to it.

Look at corresponding target space $X^1 > a$ constraint in boundary

In the bulk it is difficult to make the idea of entanglement precise.

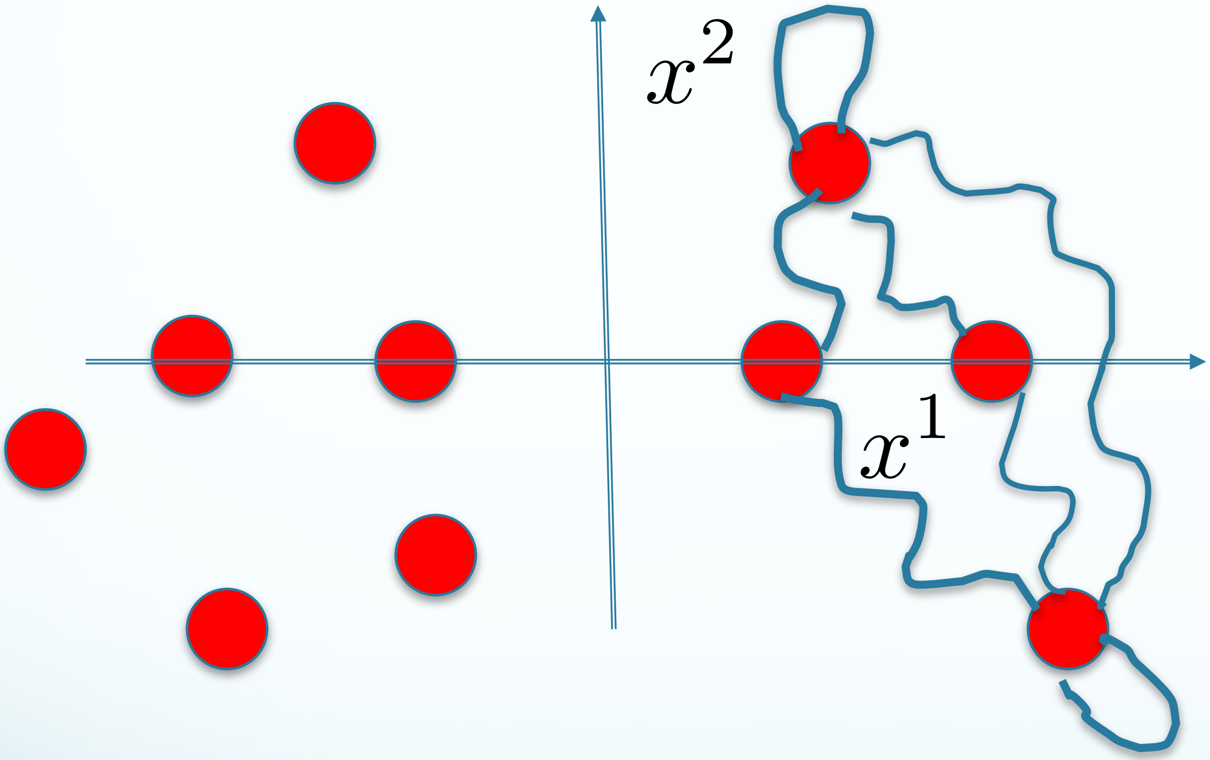
But we do have a precise definition of a related quantity in the Matrix theory.

Physical Picture: $x^1 > 0$



4 D0 branes to meet constraint.
(m=4 sector)

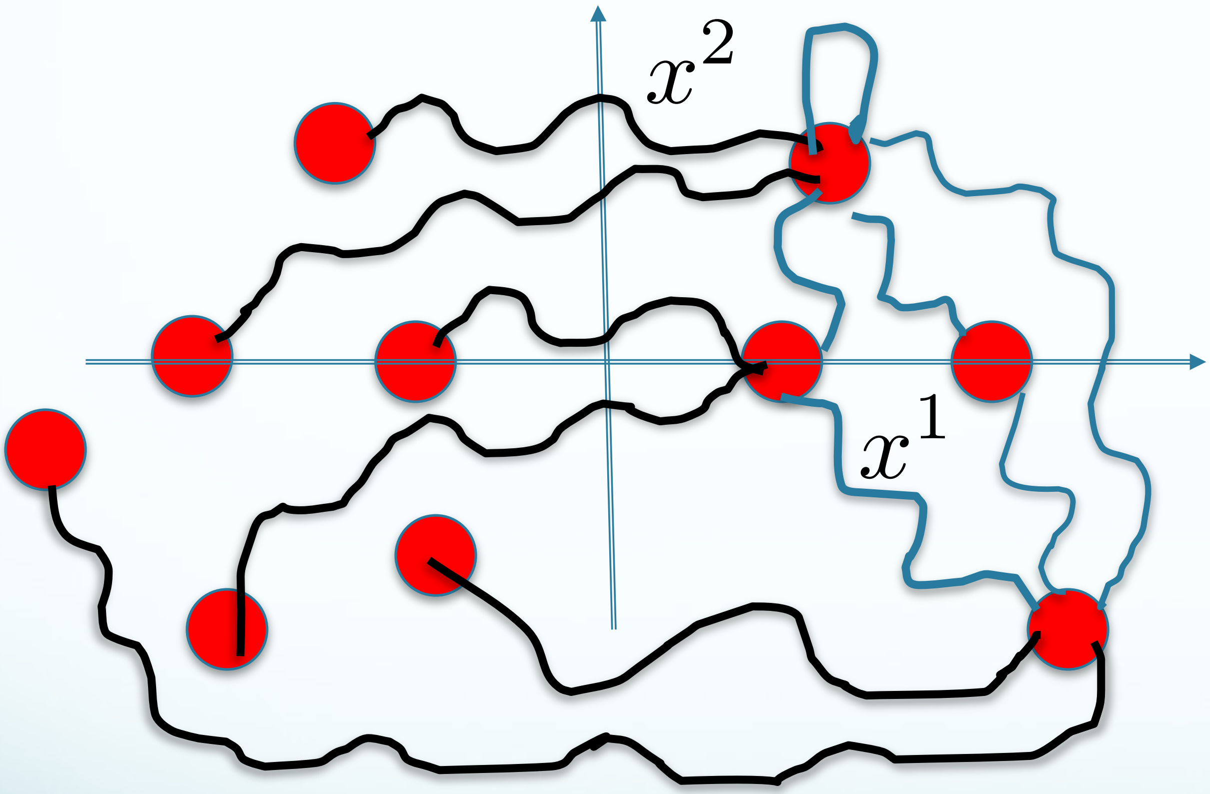
<u>Version 1:</u> $x^1 > 0$



4 D0 branes meet constraint. Keep all degrees of freedom associated with them denoted by blue strings.

Version 2:

$$x^1 > 0$$



In addition also keep black strings running between the 4 D0 branes and the rest.

On the bulk side the entanglement corresponds to strings stretching between the D branes.

On the boundary side it corresponds to ``colour" degrees of freedom of $O(N^2)$.

In fact there are two versions of our definition as the bulk pictures show.

Version 1

m values

N-m values

Wave function $$\Psi(x^a, x^\alpha, Y_{ab}, Y_{a\alpha}, Y_{\alpha a}, Y_{\alpha\beta})$$

$$\rho_m(x^a, Y_{ab}, x'^a, Y'_{ab}) =$$

$$\int_{\bar{A}} \Psi^*(x^a, x^\alpha, Y_{ab}, Y_{a\alpha}, Y_{\alpha a}, Y_{\alpha\beta})\Psi(x'^a, x^\alpha, Y'_{ab}, Y_{a\alpha}, Y_{\alpha a}, Y_{\alpha\beta})dx^\alpha dY_{a\alpha}dY_{\alpha a}dY_{\alpha\beta}$$

m th sector density matrix

$\bar{A}$ Complement of the region of interest, $x^1 < 0$

It is tempting to speculate that the answer in the matrix theory will agree with the bulk area (in Planck units)

$$S_{TS} = \frac{A}{4G_N} \qquad G_N \to N^2$$

Connection with bulk area is a proposal. Remains to be checked.

Impressive advances in numerical work on D0 brane system make one hopeful…

Boundary Quantum Mechanics

$$S = \frac{N}{2(g_s N) l_s} \int dt [\sum_{i=1}^{9} Tr(D_t X^I)^2 - \frac{1}{l_s^4} \sum_{I \neq J = 1}^{9} Tr([X^I, X^J]^2)]$$

+ fermions

$$D_t X^I = \partial_t X^I + i[A_t, X^I]$$

$X^1, X^2 \cdots, X^9$: NXN matrices

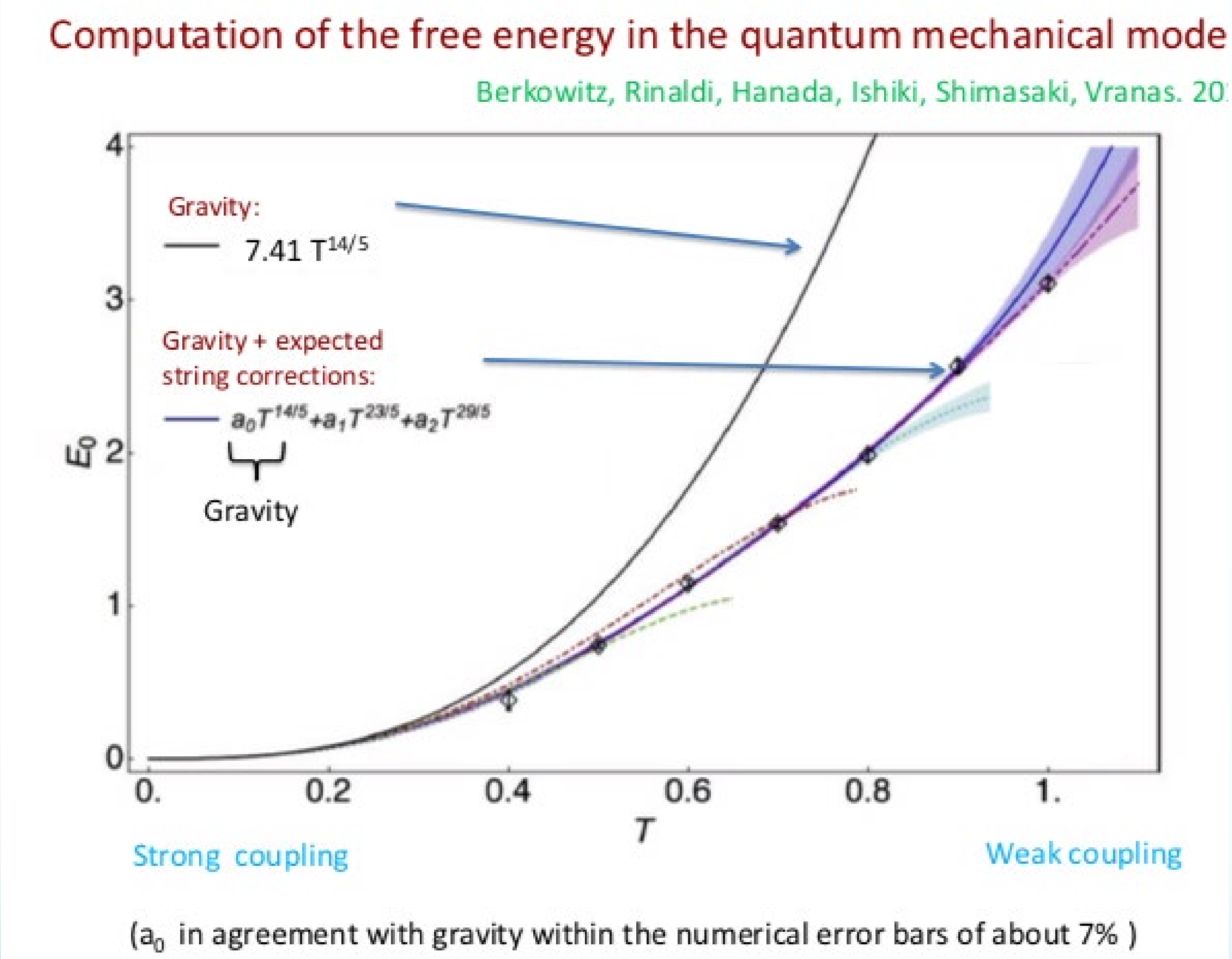
$O(N^2)$: degrees of freedom.

At finite temperature: Black Hole

$$\frac{E}{\Lambda} = 7.41 N^2 \left(\frac{T}{\Lambda}\right)^{14/5}$$

Agrees with Numerics:

Berkowiz, Rinaldi, Hanada, Ishiki, Shimasaki, Vranas, 2016



(slide taken from J. Maldacena, ICTS lecture 2018)

Boundary Quantum Mechanics
Characterised by one energy scale:
Λ

$$H = \frac{(g_s N)^{1/3}}{2 l_s} \mathrm{Tr} \left[\frac{1}{N} \sum_{I=1}^{9} (\tilde{P}^I)^2 + N \sum_{I \neq J = 1}^{9} [\tilde{X}^I, \tilde{X}^J]^2 \right] + \text{fermions}$$

$$X^I = (g_s N)^{1/3} l_s \tilde{X}^I \qquad P^I = \frac{1}{(g_s N)^{1/3} l_s} \tilde{P}^I$$

$$\Lambda = \frac{(g_s N)^{1/3}}{l_s}$$

Ryu Takayanagi Formula

For the higher dimensional cases, where the boundary theory lives along some spatial directions (eg along the 3 spatial directions for 3+1 dimensional CFT):

One can ask how entangled are the degrees of freedom in some region, along these spatial directions, with the rest?

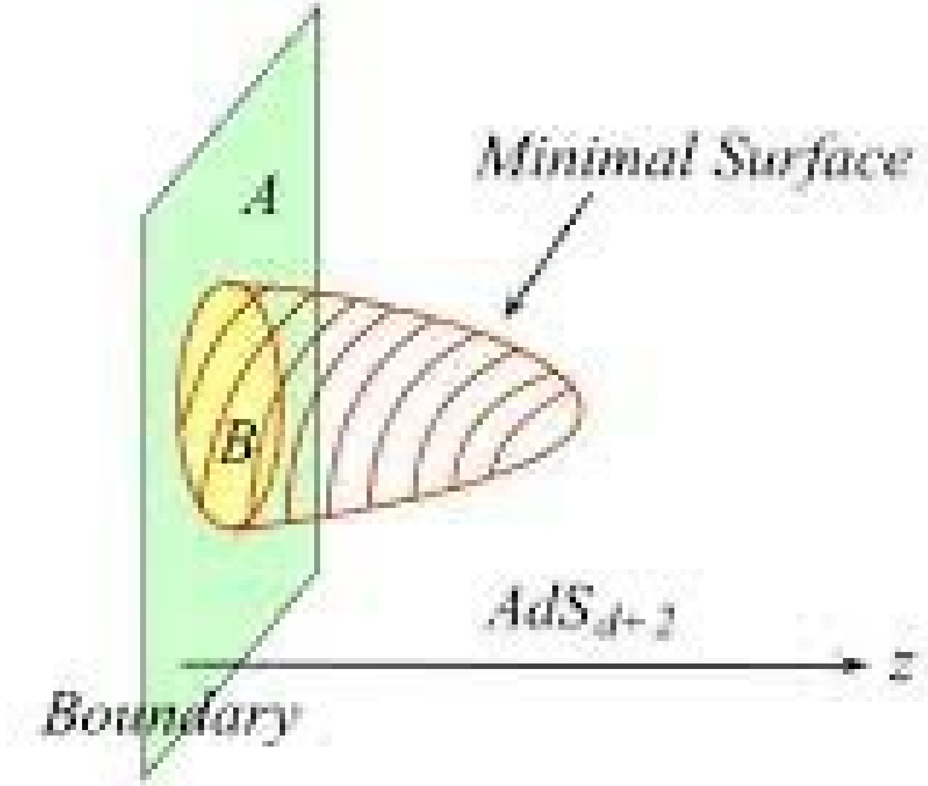


Figure 3: The holographic calculation of entanglement entropy via AdS/CFT.

Entanglement of region B on boundary with A is calculated from the Area of the Minimal surface.

$$S_{EE} = \frac{A_{min}}{4G_N}$$

Remarkable Result of Ryu and Takayanagi!

This is ``base space” entanglement.

We have been interested here in entanglement along the Target Space, in which the boundary theory does not extend.

The D0 branes (or Matrix Mechanics) only lives in time and no spatial directions. So the 9 matrices, $X^1, X^2, \cdots X^9$ are target space directions.

Connection to the RT formula remains to be understood.

Summary

- Entanglement is a key to understanding the strangeness in quantum mechanics, and unlocking the full potential of quantum techniques and technology.

- We discussed how to define and make precise the meaning of entanglement in gauge theories

Summary

- And also for Matrix models of the kind which are promising in our attempts to understand how space and time arise.

Much excitement lies ahead!

"About your cat, Mr. Schrödinger—I have good news and bad news."

Thank You!

tifr

राष्ट्र की सेवा में परमाणु
प ऊ वि
DAE
ATOMS IN THE SERVICE OF THE NATION

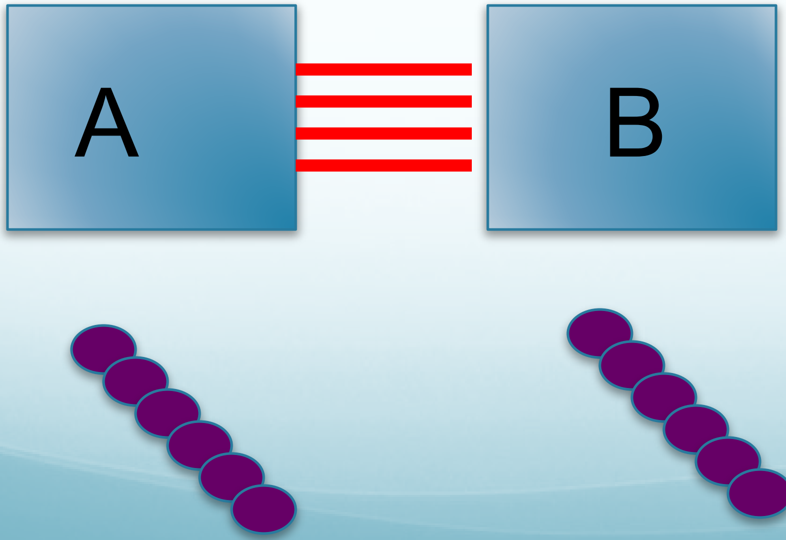
Infosys Foundation

Government of India
Department of Science & Technology
Ministry of Science & Technology

Banyan Tree In TIFR Mumbai

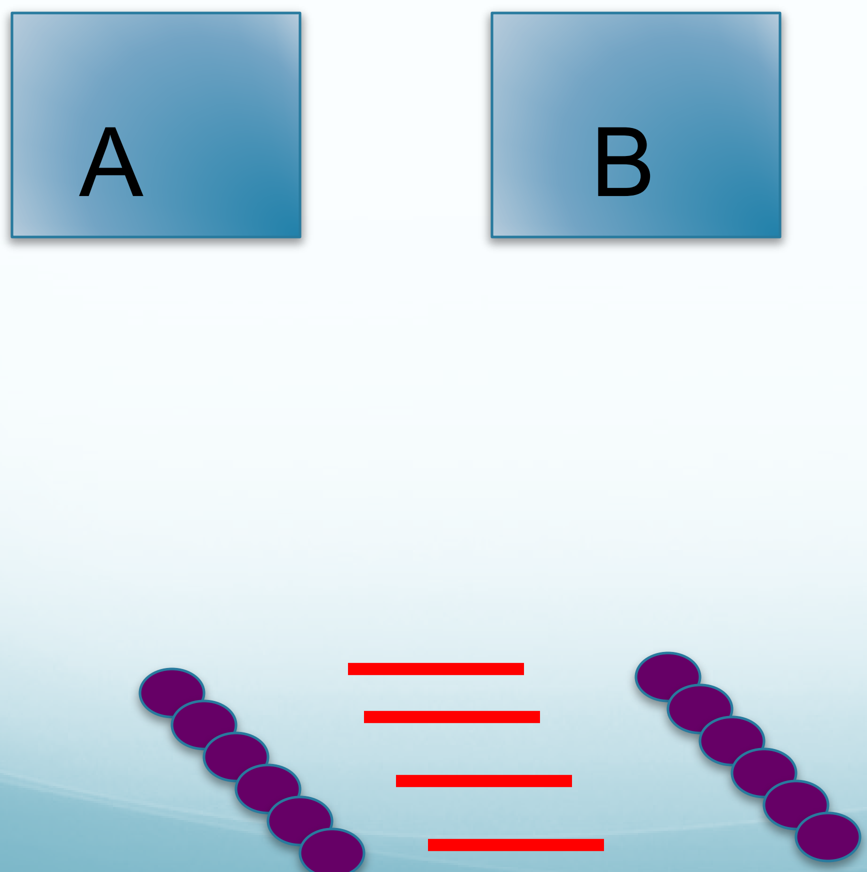
Thank you!

Entanglement Distillation :



2N unentangled qubits

To finally arrive at the situation:



N entangled Bell pairs

Only this term can be extracted

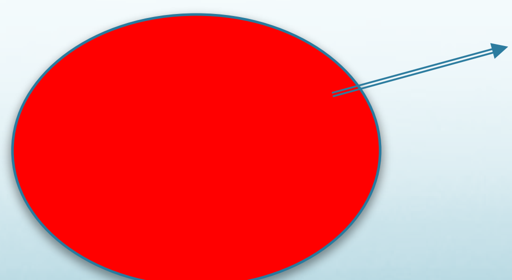
$$S_{EE} = -\sum_i p_i \log p_i \; \overbrace{- \sum_i Tr_i p_i \bar{\rho}_i \log \bar{\rho}_i}$$

$$+\sum_i p_i \left(\sum_a \log d_a^i\right)$$

Target Space Entanglement

Often for a field theory in $R^d \times T$
One considers a subregion of R^d

And asks how entangled are the degrees of freedom inside with those outside.

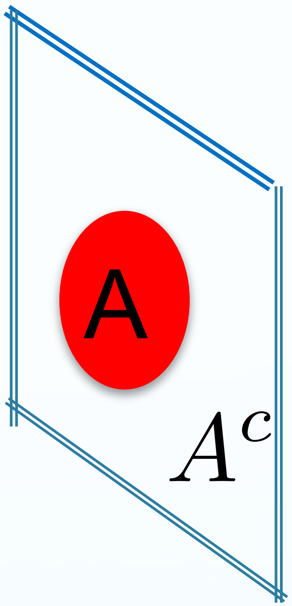


Spatial region in R^d

Ryu Takayanagi Formula

For the higher dimensional cases, where the boundary theory lives along some spatial directions (eg along the 3 spatial directions for 3+1 dimensional CFT):

One can ask how entangled are the degrees of freedom in some region along these spatial directions with the rest?



How entangled is A with. A^c ?

We will be interested here in the entanglement along the target space.

E.g AdS_5 X S^5 has 9 spatial directions. The field theory lives in 3 of them. The remaining 6 directions are target space directions.

(Experts: Adjoint scalar fields take values along the target space directions)

The matrix model we started with has no spatial directions. Its gravity dual is a solution in IIA string theory in 9+1 dimensions.

All of the 9 spatial directions then are target space directions in this case, with no base space directions.

This makes the matrix model a convenient system for our purposes.

For target space entanglement

Define projector:

$$P^1 = \int_{x>0} dx \ \delta(x\mathbf{I} - X^1)$$

For target space constraint $x^1 > 0$

Version 1:

Project all operators by acting with this projector.

Then take a trace to obtain gauge invariant operators.

$$X^I \to (X^I)^{P_1} = P^1 X^I P^1, \; \Pi_J \to (\Pi_J)^{P_1} = P_1 \Pi_J P^1$$

$$Tr((X^I)^{P_1}, ...(\Pi_J)^{P_1}, \cdots)$$

Version 1:

This gives rise to a sub algebra.

Entangelment entropy is associated with this sub algebra.

Similarly for version 2.

Subalgebra defined With an appropriately chosen projector.

Bulk Entangelment : Proposal

Bulk Entangelment Entropy for region

$$f(x^i) > 0$$

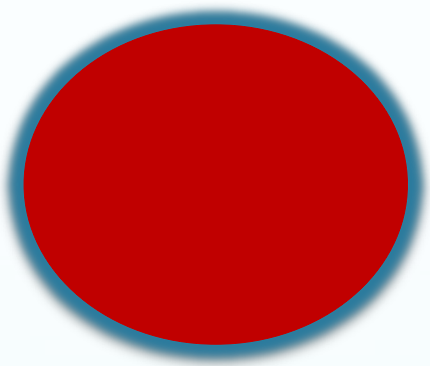
Is given by Target space entanglement associated with constraint $f(X^I) > 0$

And this bulk entanglement saturates the Beckenstein-Hawking bound (for a general surface).

Proposal for Bulk Entanglement

And $$S_{Bulk} = \frac{A_\partial}{4G_N} = S_T$$

Target space entanglement



Boundary with area A_∂

(In one of the two versions of our proposal for S_T)