Spectral form factor and eigenstate entanglement entropy in Sachdev-Ye-Kitaev-type models

Second Annual Meeting of the Extreme Universe Collaboration

Kobe International Conference Center

9:00 – 9:40, 28 December 2022

Masaki TEZUKA (B02, Kyoto University)

Contents & list of collaborators

- Sachdev-Ye-Kitaev model
  - Proposal for experiment: PTEP 2017, 083I01 with Ippei Danshita and Masanori Hanada
- **Spectral form factor: structure of eigenstate energies**
  - “Black Holes and Random Matrices”: JHEP 1705(2017)118 with J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, and A. Streicher
  - Scrambling time: JHEP 1807(2018)124 with Hrant Gharibyan, M. Hanada, and S. H. Shenker
- **Sparse SYK**
  - arXiv:2208.12098 (Phys. Rev. B Letter in press) with Onur Oktay, Enrico Rinaldi, M. Hanada and Franco Nori
- Chaotic-integrable transition in SYK4+2
  - PRL **120**, 241603 (2018) with Antonio M. García-García, Bruno Loureiro, and Aurelio Romero-Bermúdez
  - Uniform coupling terms
    - Phys. Lett. B **795**, 230 (2019) and J. Phys. A **54**, 095401 (2021) with Pak Hang Chris Lau, Chen-Te Ma, and Jeff Murugan
- Quantitative analysis of Fock-space localization in SYK4+2
  - Many-body transition point and inverse participation ratio
    - Phys. Rev. Research **3**, 013023 (2021) with Felipe Monteiro, Tobias Micklitz, and Alexander Altland
  - **Entanglement entropy: structure of eigenstates**
    - Phys. Rev. Lett. **127**, 030601 (2021) with F. Monteiro, A. Altland, David A. Huse, and T. Micklitz

Sachdev-Ye-Kitaev model

N Majorana- or Dirac- fermions with all-to-all Gaussian random couplings

[Majorana version]

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a < b < c < d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

[A. Kitaev: talks at KITP (2015)]

cf. SY model [S. Sachdev and J. Ye, PRL 1993]

arXiv:cond-mat/**9212**030 (>1300 citations after 2015)

[Dirac version]

$$\hat{H} = \frac{1}{(2N)^{3/2}} \sum_{ij;kl} J_{ij;kl} \hat{c}_i^{\dagger} \hat{c}_j^{\dagger} \hat{c}_k \hat{c}_l$$

[A. Kitaev's talk]

[S. Sachdev: PRX **5**, 041025 (2015)]

Studied for long time in the nuclear theory context

[French and Wong (1970)][Bohigas and Flores (1971)]

"Two-body Random Ensemble"

Solvable in the large-*N* limit, maximally chaotic, holographic correspondence to 1+1d gravity

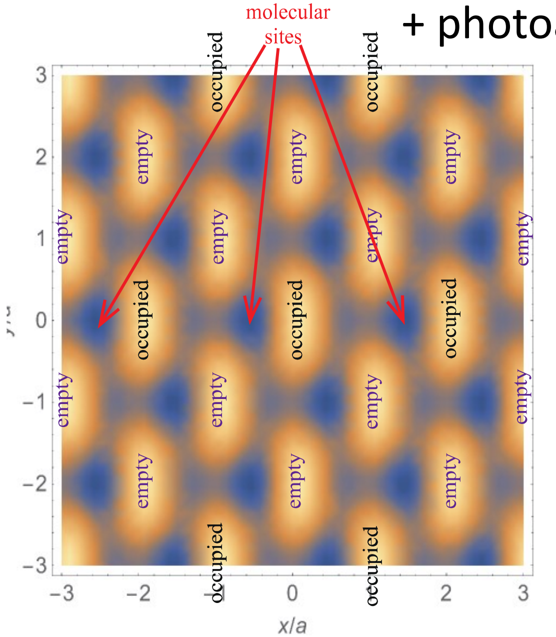
➔ Many variants of the model: bosonic, multiflavor, supersymmetric, nonhermitian, …

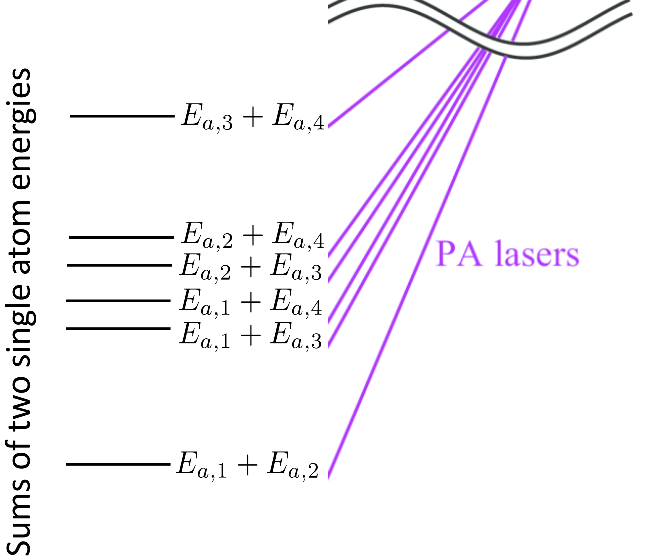
Proposals for experimental realization

(though not the target of the B02 group experiment)

[I. Danshita, M. Hanada, MT: PTEP **2017**, 083I01 (2017)]

Ultracold fermions in optical lattice
+ photoassociation lasers





s: molecular levels

$$\hat{H}_{\mathrm{m}} = \sum_{s=1}^{n_s} \left\{ \nu_s \hat{m}_s^\dagger \hat{m}_s + \sum_{i,j} g_{s,ij} \left(\hat{m}_s^\dagger \hat{c}_i \hat{c}_j - \hat{m}_s \hat{c}_i^\dagger \hat{c}_j^\dagger \right) \right\}.$$

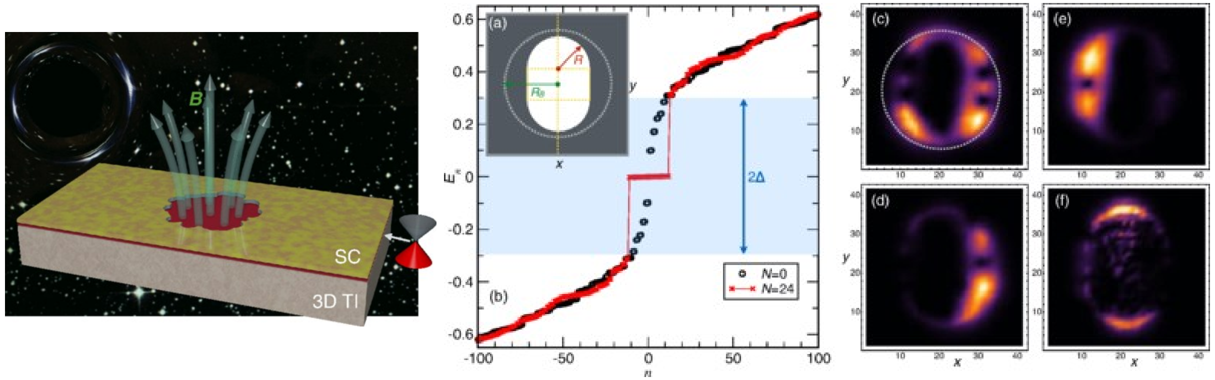
$|\nu_s| \gg |g_{s,ij}|$

$$\hat{H}_{\mathrm{eff}} = \sum_{s,i,j,k,l} \frac{g_{s,ij} g_{s,kl}}{\nu_s} \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l.$$

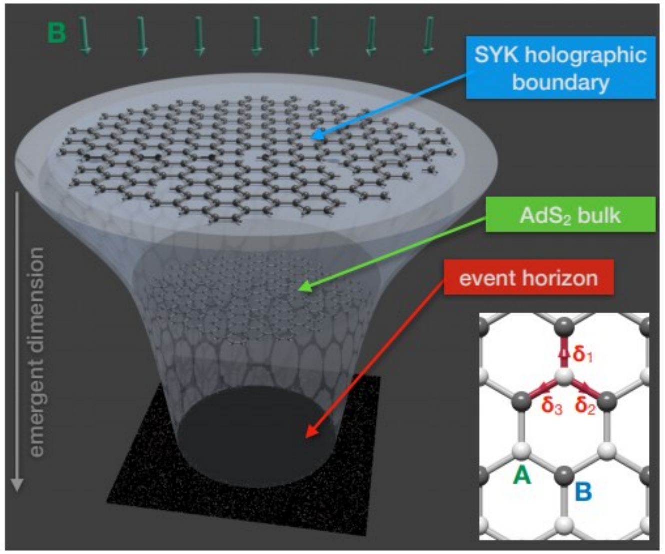
Approximately Gaussian if many levels are used

[D. I. Pikulin and M. Franz, PRX **7**, 031006 (2017)]

N quanta of magnetic flux through a nanoscale hole



[A. Chen, R. Ilan, F. de Juan, D.I. Pikulin, M. Franz, PRL **121**, 036403 (2018)]



Graphene flake with an irregular boundary in magnetic field

NMR experiment for the SYK model

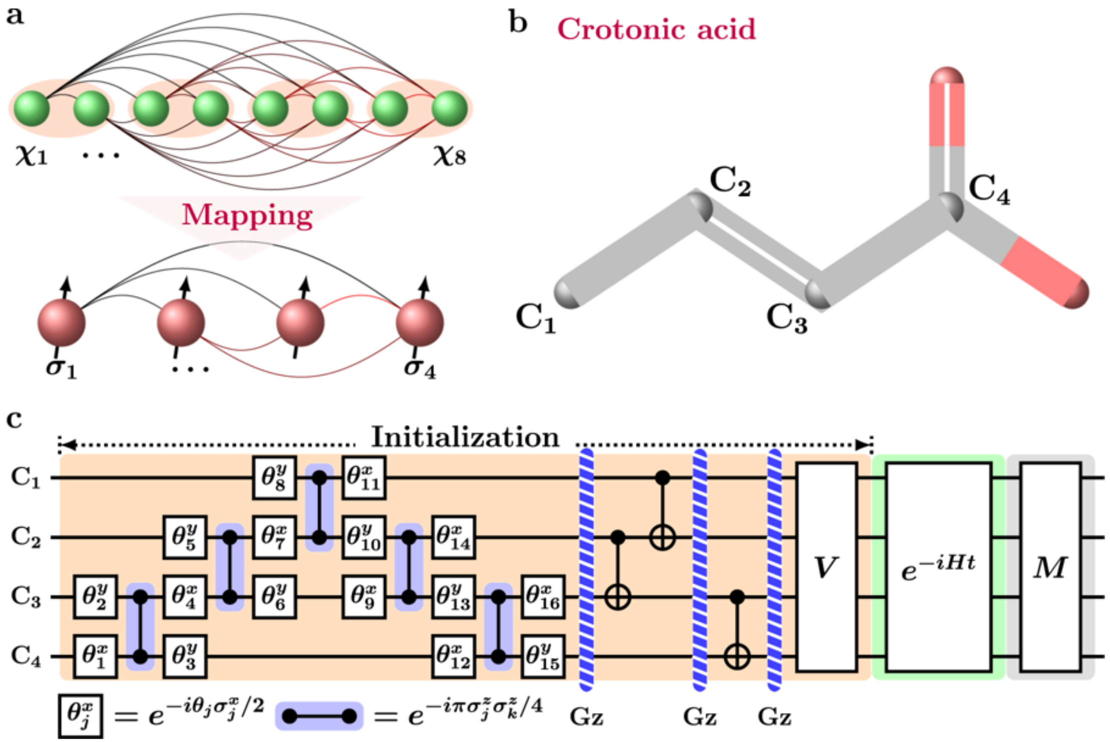
"Quantum simulation of the non-fermi-liquid state of Sachdev-Ye-Kitaev model" Zhihuang Luo, Yi-Zhuang You, Jun Li, Chao-Ming Jian, Dawei Lu, Cenke Xu, Bei Zeng and Raymond Laflamme, npj Quantum Information **5**, 53 (2019)

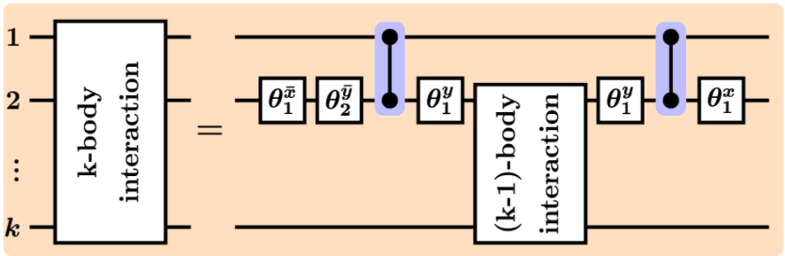
$$H = \frac{J_{ijkl}}{4!}\chi_i\chi_j\chi_k\chi_l + \frac{\mu}{4}C_{ij}C_{kl}\chi_i\chi_j\chi_k\chi_l$$

$$\chi_{2i-1} = \frac{1}{\sqrt{2}}\sigma_x^1\sigma_x^2\cdots\sigma_x^{i-1}\sigma_z^i,\ \chi_{2i} = \frac{1}{\sqrt{2}}\sigma_x^1\sigma_x^2\cdots\sigma_x^{i-1}\sigma_y^i.$$

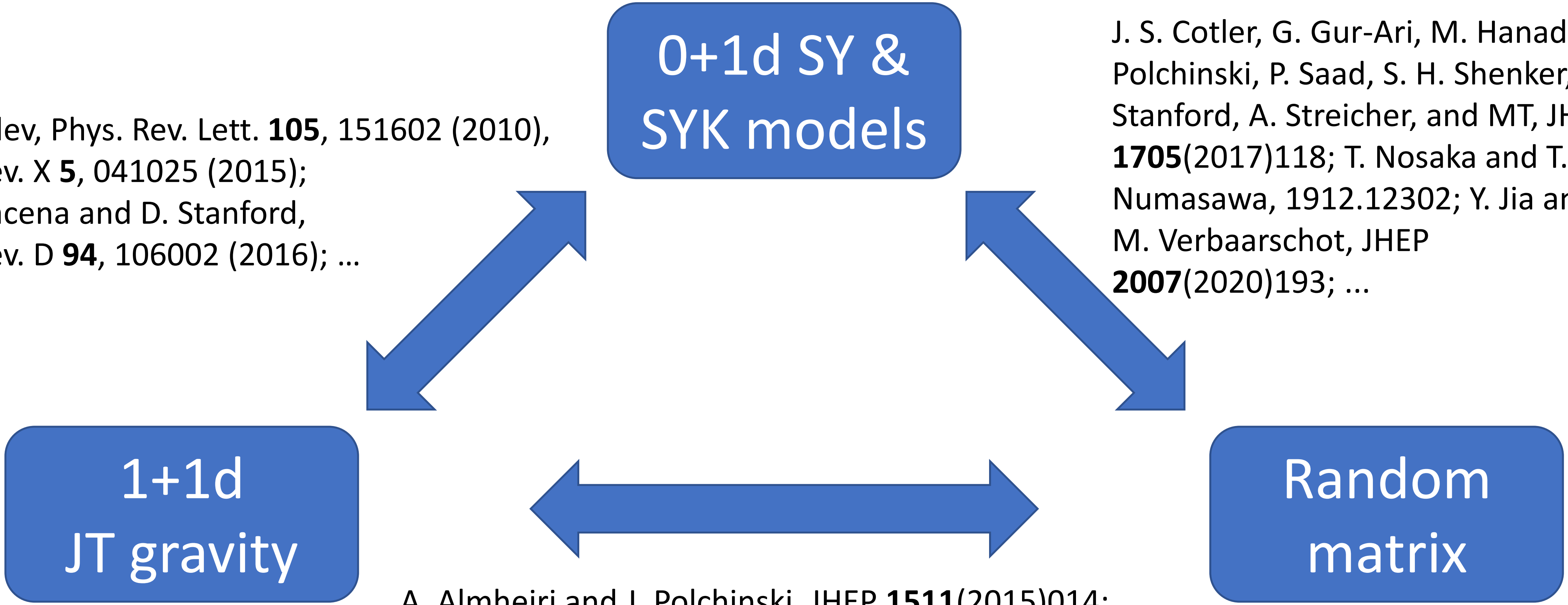
$$H = \sum_{s=1}^{70} H_s = \sum_{s=1}^{70} a^s_{ijkl}\sigma^1_{\alpha_i}\sigma^2_{\alpha_j}\sigma^3_{\alpha_k}\sigma^4_{\alpha_l}$$

$$e^{-iH\tau} = \left(\prod_{s=1}^{70} e^{-iH_s\tau/n}\right)^n + \sum_{s<s'}\frac{[H_s, H_{s'}]\tau^2}{2n} + O(|a|^3\tau^3/n^2),$$





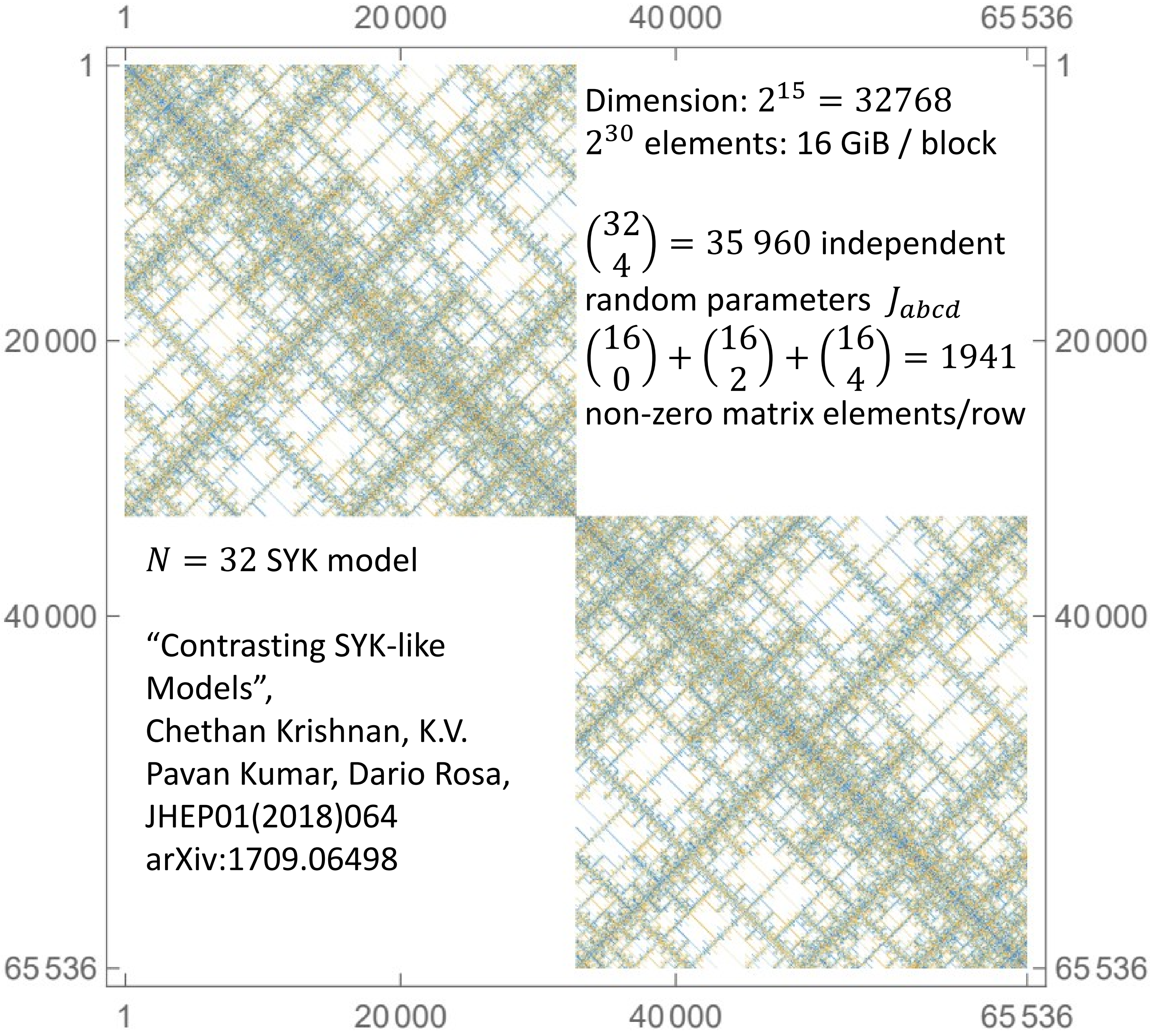
Maximally chaotic systems



S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010), Phys. Rev. X **5**, 041025 (2015);
J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016); …

J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and MT, JHEP **1705**(2017)118; T. Nosaka and T. Numasawa, 1912.12302; Y. Jia and J. J. M. Verbaarschot, JHEP **2007**(2020)193; ...

A. Almheiri and J. Polchinski, JHEP **1511**(2015)014;
P. Saad, S. H. Shenker, and D. Stanford, arXiv:1903.11115;
D. Stanford and E. Witten, arXiv:1907.03363; …

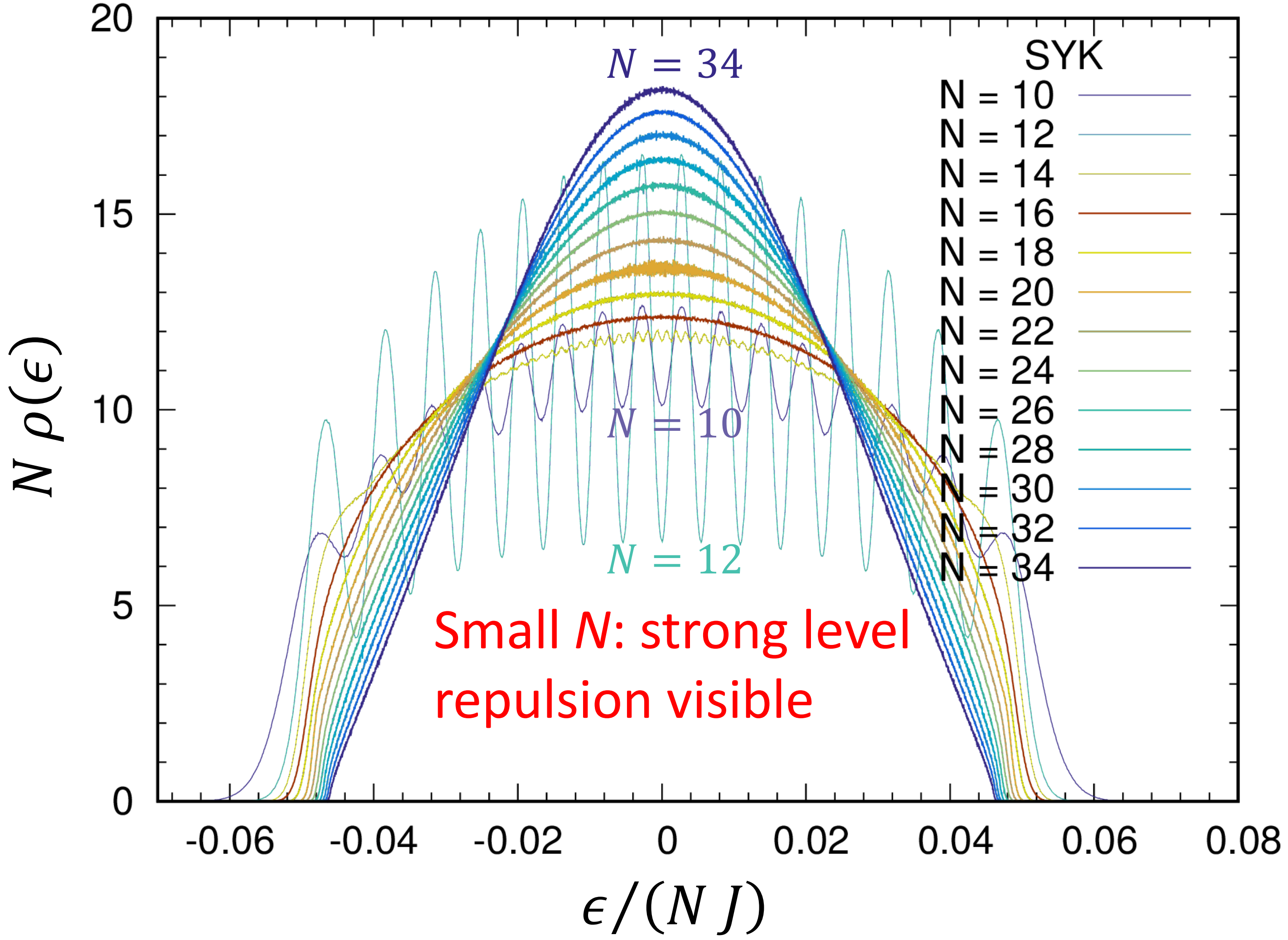


1
20 000
40 000
65 536
Dimension: $2^{15} = 32768$
2^{30} elements: 16 GiB / block
$\binom{32}{4} = 35\,960$ independent
random parameters J_{abcd}
$\binom{16}{0} + \binom{16}{2} + \binom{16}{4} = 1941$
non-zero matrix elements/row
$N = 32$ SYK model
“Contrasting SYK-like
Models”,
Chethan Krishnan, K.V.
Pavan Kumar, Dario Rosa,
JHEP01(2018)064
arXiv:1709.06498

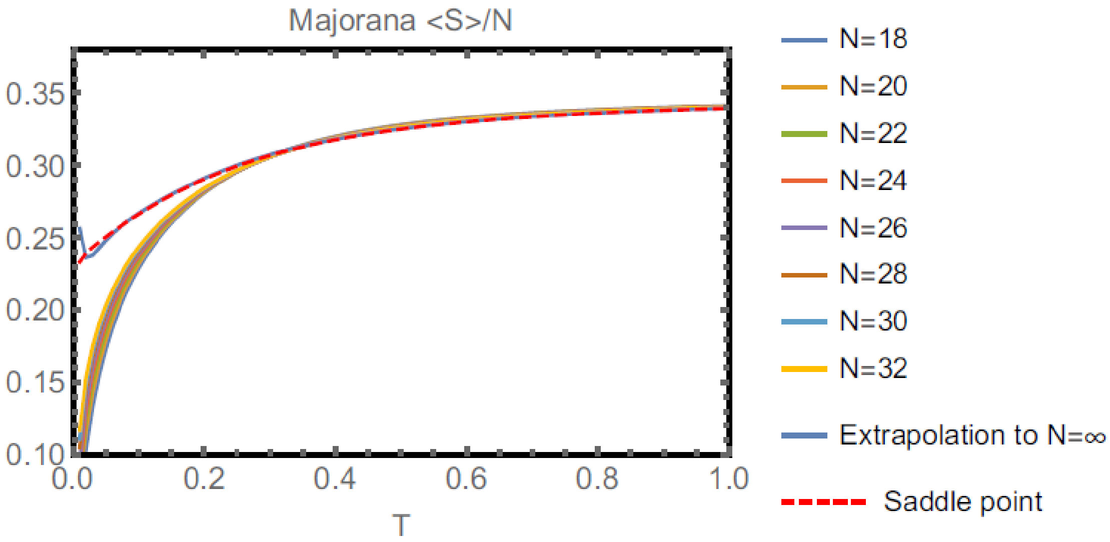
Diagonalization of the Hamiltonian → Eigenvalue spectrum

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a<b<c<d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

J_{abcd} : Gaussian and variance $\sigma^2 = J^2$



Small N: strong level repulsion visible



Entropy extrapolated to large N: finite in the low T limit

cf. Analytical spectral density for large N [A. M. García-García and J. J. M. Verbaarschot: PRD **94**, 126010 (2016), PRD **96**, 066012 (2017)]

Level statistics concepts

- Diagonalize matrices (e.g. SYK Hamiltonians)

→Eigenvalues obtained

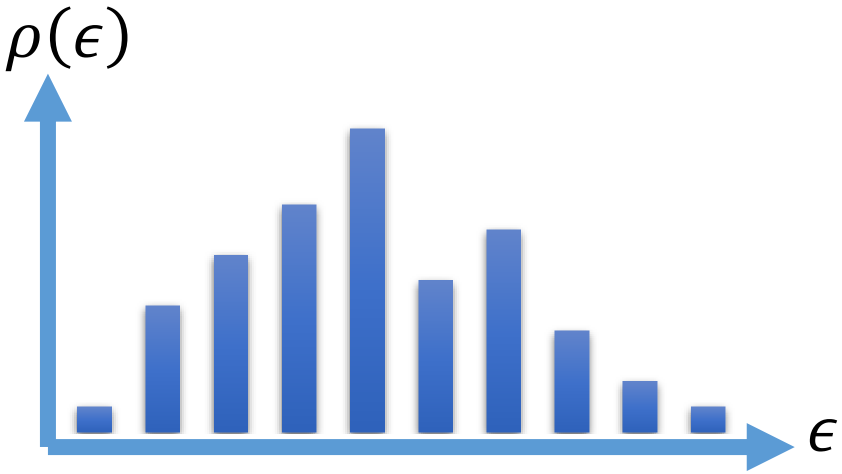
$\{\epsilon_1, \epsilon_2, \ldots\}$ for each sample

How are they distributed / correlated?

Distribution

- Density of states

$$\rho(\epsilon) = \sum_k \delta(\epsilon - \epsilon_k)$$



Correlation

short range

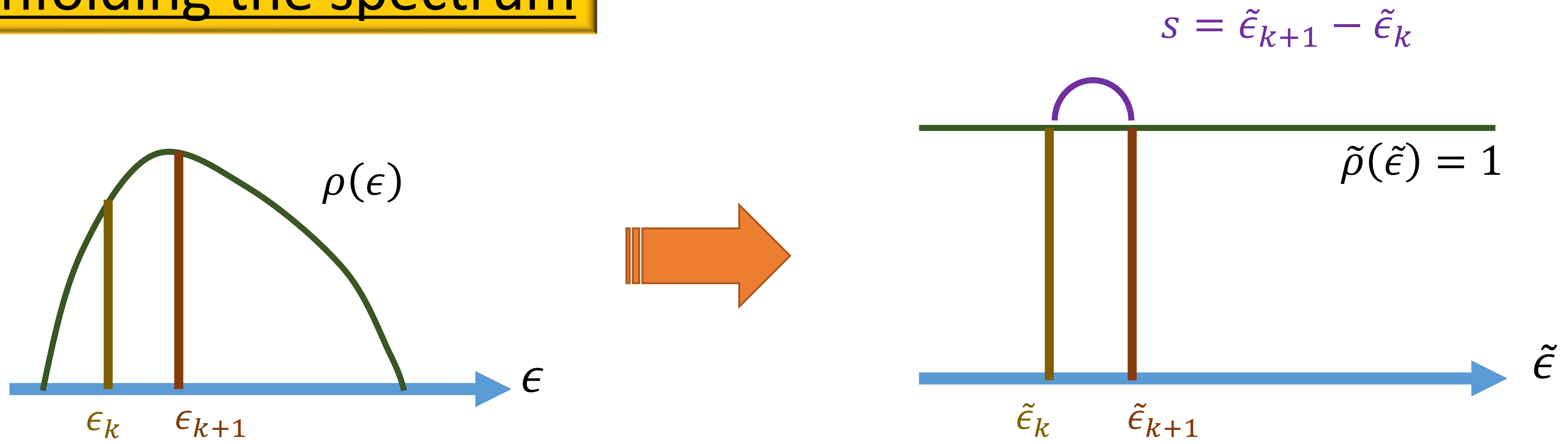
- Gap distribution
- Nearest gap ratio

long(er) range

- Number variance
- Spectral form factor

Normalize the density

Unfolding the spectrum



$$\tilde{\epsilon} = \int_{-\infty}^{\epsilon} \rho(\epsilon')\, d\epsilon'$$

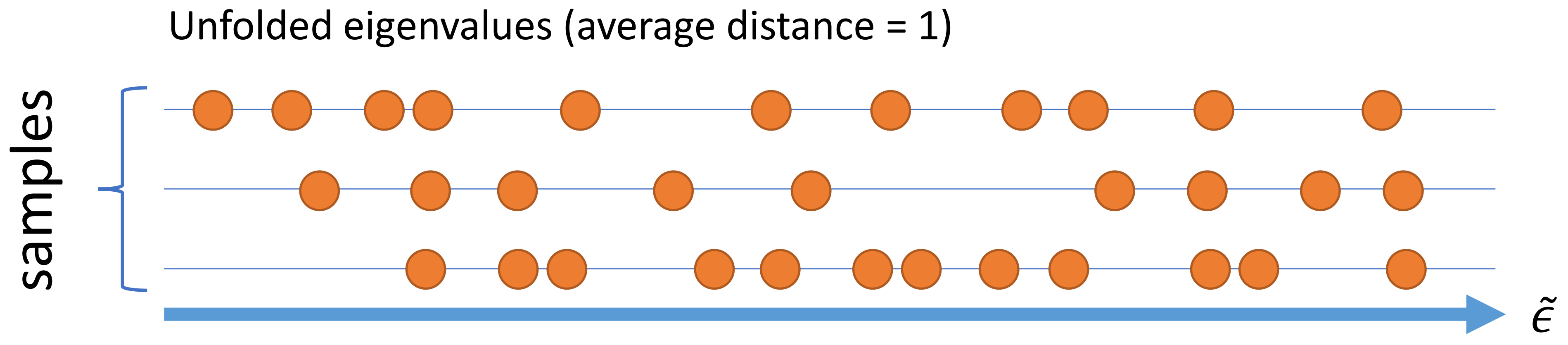
$\rho(\epsilon)$: Smoothed density of states
(in practice: fit with a function easy to integrate)

Detecting short-range correlations

$P(s)$: level spacing distribution

Ratio of consecutive level spacing $\tilde{\epsilon}_{i+1} - \tilde{\epsilon}_i$ to the local mean level spacing Δ
(requires unfolding of the spectrum)
Uncorrelated: Poisson (e^{-s})

$$s = \frac{\tilde{\epsilon}_{i+1} - \tilde{\epsilon}_i}{\Delta}$$



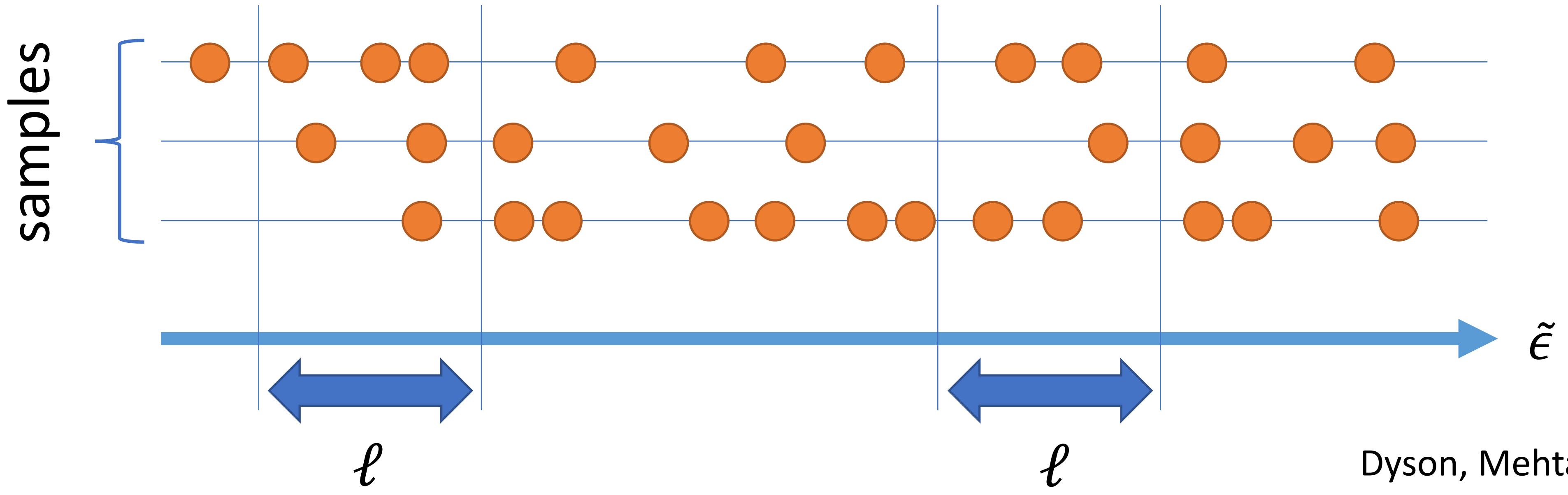
$\langle r \rangle$: average adjacent gap ratio

Average of $\frac{\min(\epsilon_{i+1}-\epsilon_i\,,\,\epsilon_{i+2}-\epsilon_{i+1})}{\max(\epsilon_{i+1}-\epsilon_i\,,\,\epsilon_{i+2}-\epsilon_{i+1})}$ (does not require unfolding)

Longer range: Number variance

How many eigenvalues are actually observed in distance ℓ, where we expect ℓ eigenvalues on average? ➔ variance

Unfolded eigenvalues (average distance = 1)



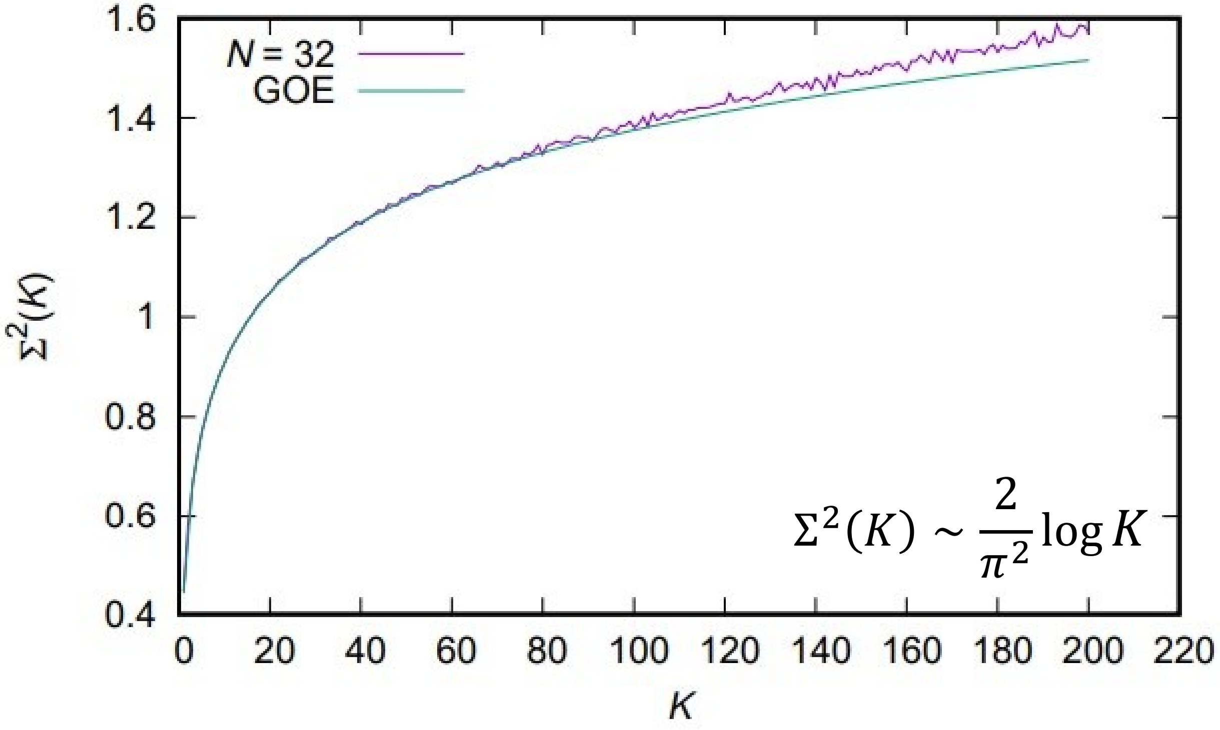
Dyson, Mehta

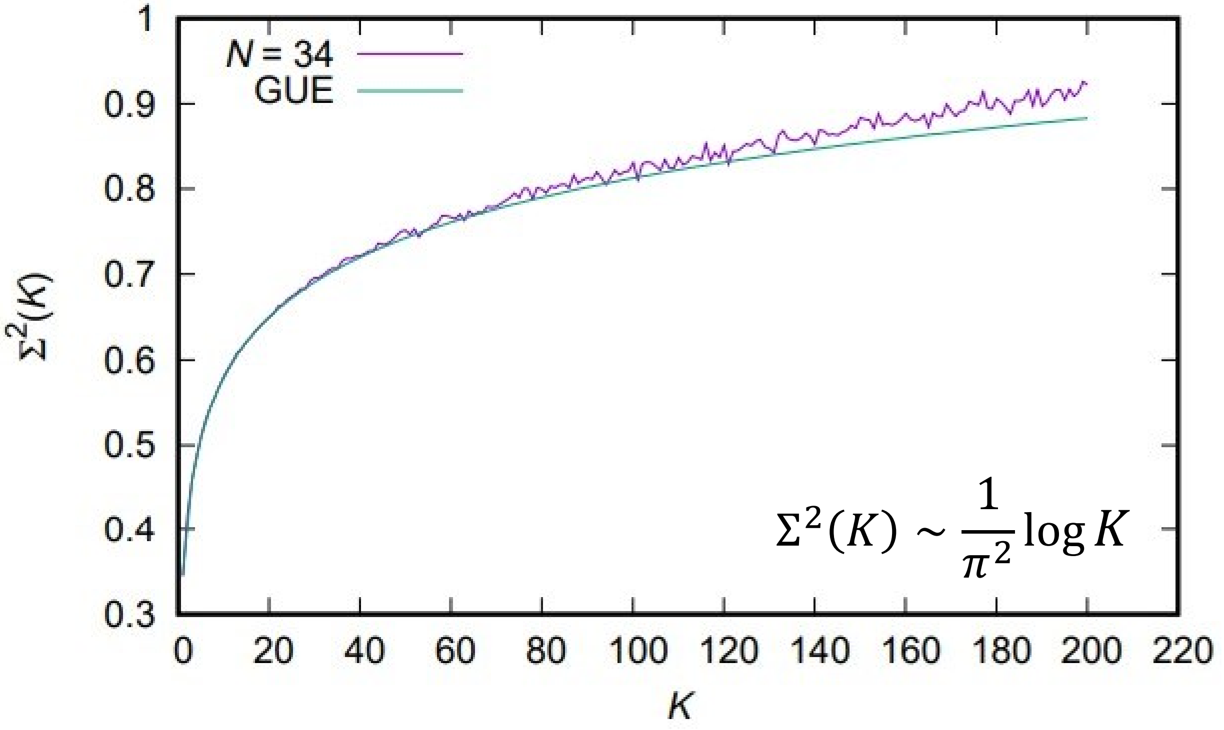
	GOE	GUE	GSE
	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
β	1	2	4

$$\Sigma^2(\ell) = \langle n^2 \rangle - \langle n \rangle^2 = \langle n^2 \rangle - \ell^2 \sim \begin{cases} \dfrac{2}{\beta \pi^2} \log \ell & \text{(Gaussian random matrix)} \\ \ell & \text{(uncorrelated)} \end{cases}$$

Appendix of [Gharibyan, Hanada, Shenker, Tezuka: JHEP07(2018)124]
cf. [S. Torquato, Phys. Rep. **745**, 1 (2018)]

Hyperuniform distribution of eigenvalues





- Shift the origin so that $\langle\epsilon_j\rangle = 0$, rescale so that $\mathrm{Tr}H^2 = \sum_j \epsilon_j^2 = \mathrm{const.}$
- Unfold each sample using the density profile $\langle\rho(E)\rangle$.
- Compute the number variance $\Sigma^2(K) = \langle n^2(E,K)\rangle - \langle n(E,K)\rangle^2 = \langle n^2(E,K)\rangle - K^2$.

$n(E,K)$: number of levels in $[E, E+K\bar{\Delta}]$

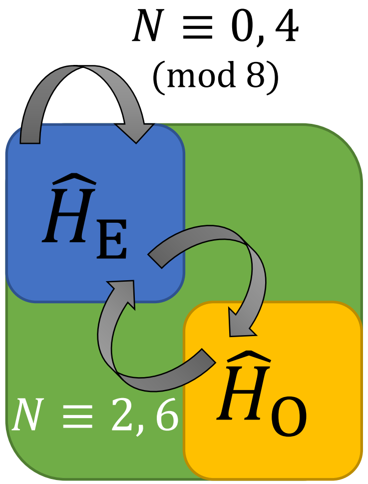
N mod 8 classification of Majorana $\mathrm{SYK}_{q=4}$

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1\le a<b<c<d\le N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

SPT phase classification for class BDI, 1D:
$\mathbb{Z} \rightarrow \mathbb{Z}_8$ due to interaction
[L. Fidkowski and A. Kitaev, PRB 2010, PRB 2011]

Introduce $N/2$ complex fermions $\hat{c}_j = \frac{(\hat{\chi}_{2j-1}+\mathrm{i}\hat{\chi}_{2j})}{\sqrt{2}}$

$\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$ respects the complex fermion parity
Even ($\hat{H}_\mathrm{E}$) and odd ($\hat{H}_\mathrm{O}$) sectors: $L = 2^{N/2-1}$ dimensions



N mod 8	0	2	4	6
η	-1	+1	+1	-1
$\hat{X}^2$	**+1**	+1	**-1**	-1
$\hat{X}$ maps H_E to	H_E	H_O	H_E	H_O
Class	**AI**	**A+A**	**AII**	**A+A**
Gaussian ensemble	**GOE ($\mathbb{R}$)**	**GUE ($\mathbb{C}$)**	**GSE ($\mathbb{H}$)**	**GUE ($\mathbb{C}$)**

$$\hat{X} = \hat{K}\prod_{j=1}^{N/2}(\hat{c}_j^\dagger + \hat{c}_j) \qquad \hat{X}\hat{c}_j\hat{X} = \eta\hat{c}_j^\dagger \qquad [\hat{X},\hat{H}] = 0$$

[Y.-Z. You, A. W. W. Ludwig, and C. Xu, PRB **95**, 115150 (2017)];
[F. Sun and J. Ye, PRL **124**, 244101 (2020)] for generic q and SUSY cases; …

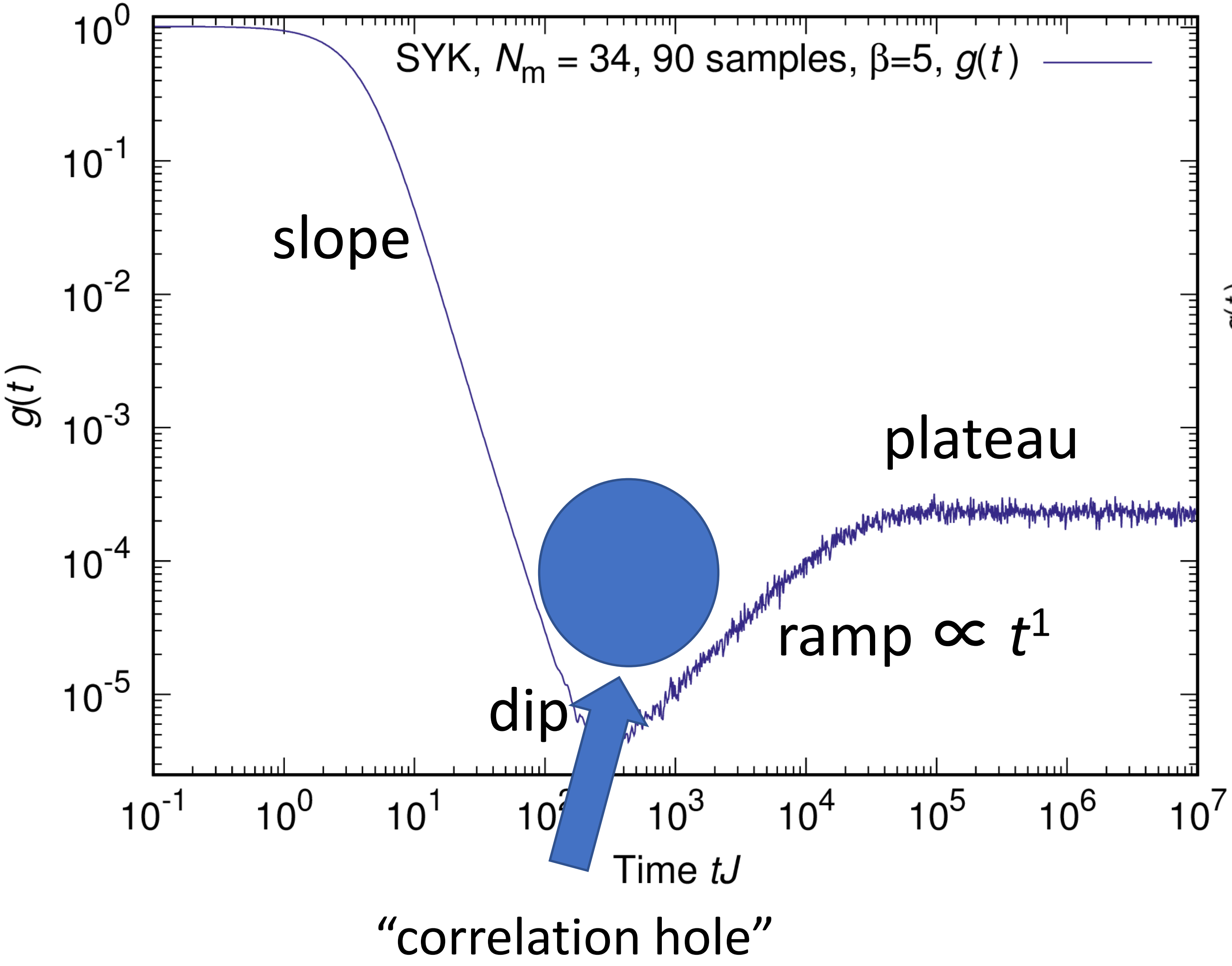
Also see [A. M. Garcia-Garcia, L. Sa, J. J. M. Verbaarschot, PRX **12**, 021040 (2022)] for classification of non-hermitian SYK:
19 out of 38 [Kawabata-Shiozaki-Ueda-Sato] classes identified

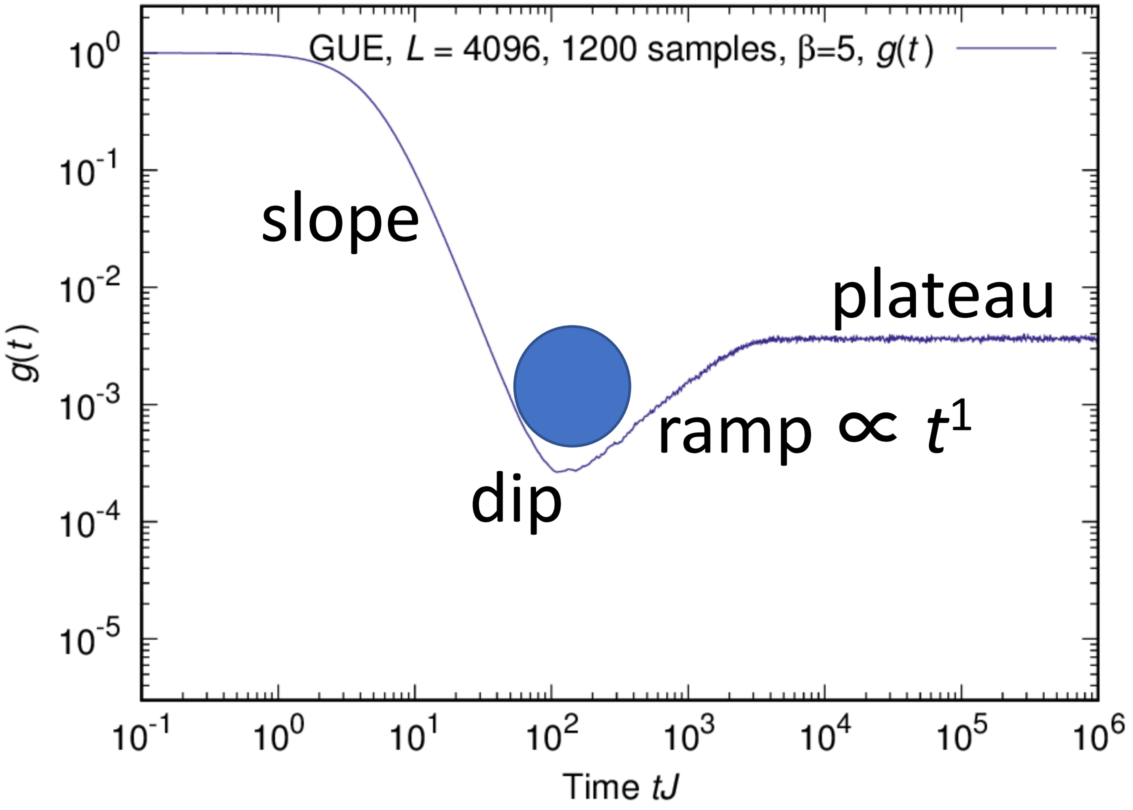
SYK: sparse matrix, but energy spectral statistics strongly resemble that of the corresponding (dense) Gaussian ensemble

[Cotler, …, MT, JHEP 2017]

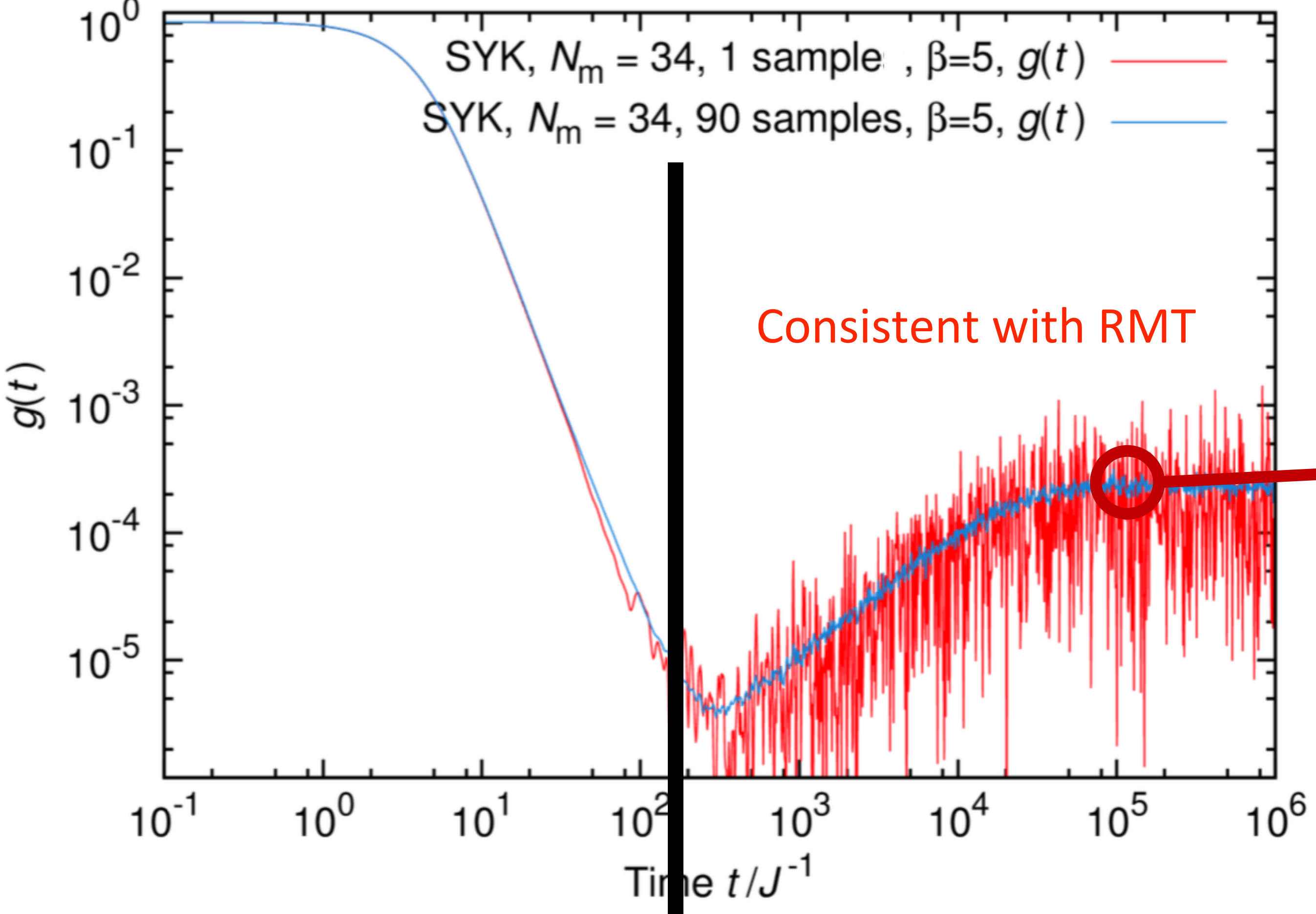
Spectral form factor

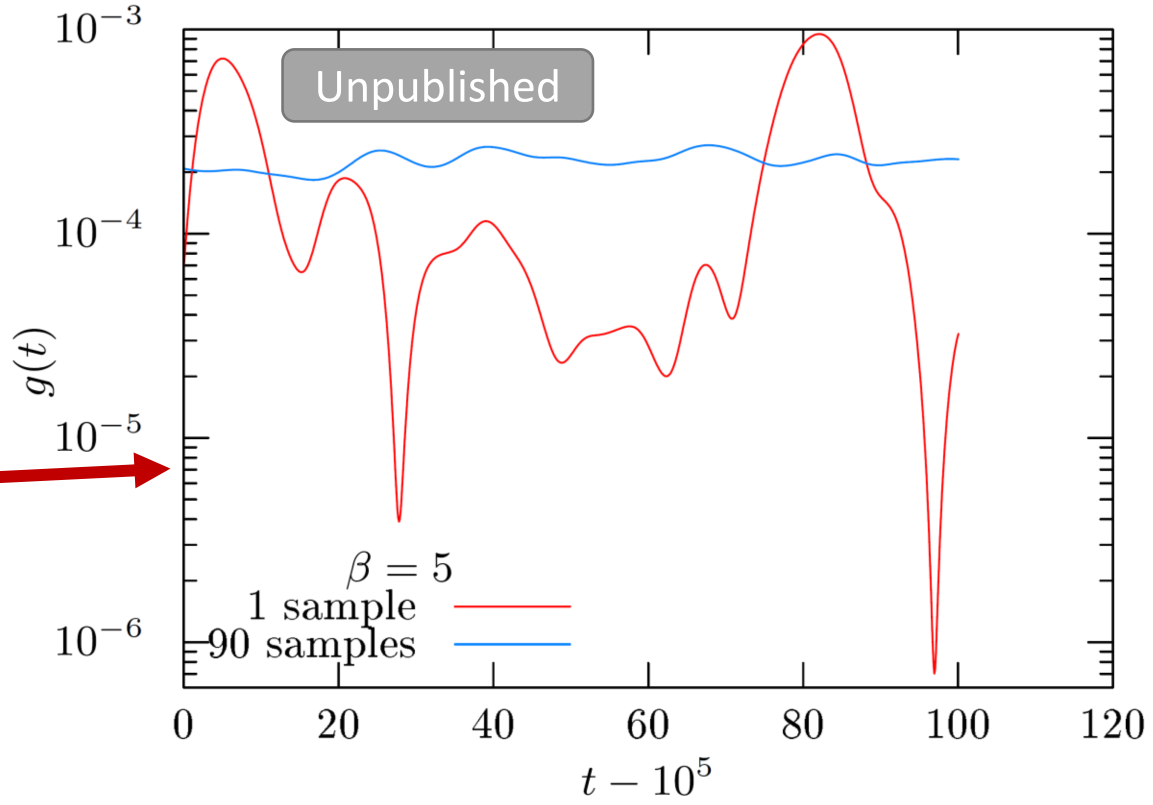
$$g(\beta, t) = \frac{\langle |Z(\beta, t)|^2 \rangle_{\{J\}}}{\langle Z(\beta) \rangle_{\{J\}}^{\ 2}} \qquad Z(\beta, t) = Z(\beta + \mathrm{i}t) = \mathrm{Tr}\left(\mathrm{e}^{-\beta \hat{H} - \mathrm{i}\hat{H}t}\right)$$





(Non-)self-averaging





1 sample = many samples
Self-averaging

$O(1)$ variance for 1 sample
Not self-averaging

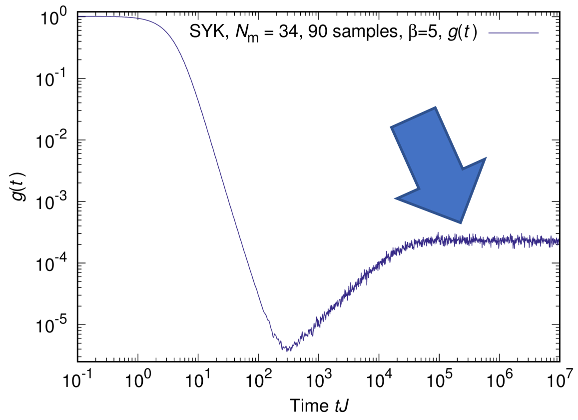
$1/N$ expansion

$1/K \sim e^{-N}$ expansion

$K = 2^{N/2}$

Plateau height determined by $Z(\beta)$

$$g(\beta,t)=\frac{\langle|Z(\beta,t)|^2\rangle_J}{\langle Z(\beta)\rangle_J^{\;2}}$$



For each sample, consider the long time average of

$$\left|\frac{Z(\beta,t)}{Z(\beta,t=0)}\right|^2=\sum_{m,n}\mathrm{e}^{-\beta(E_m+E_n)}\mathrm{e}^{\mathrm{i}(E_m-E_n)t}$$

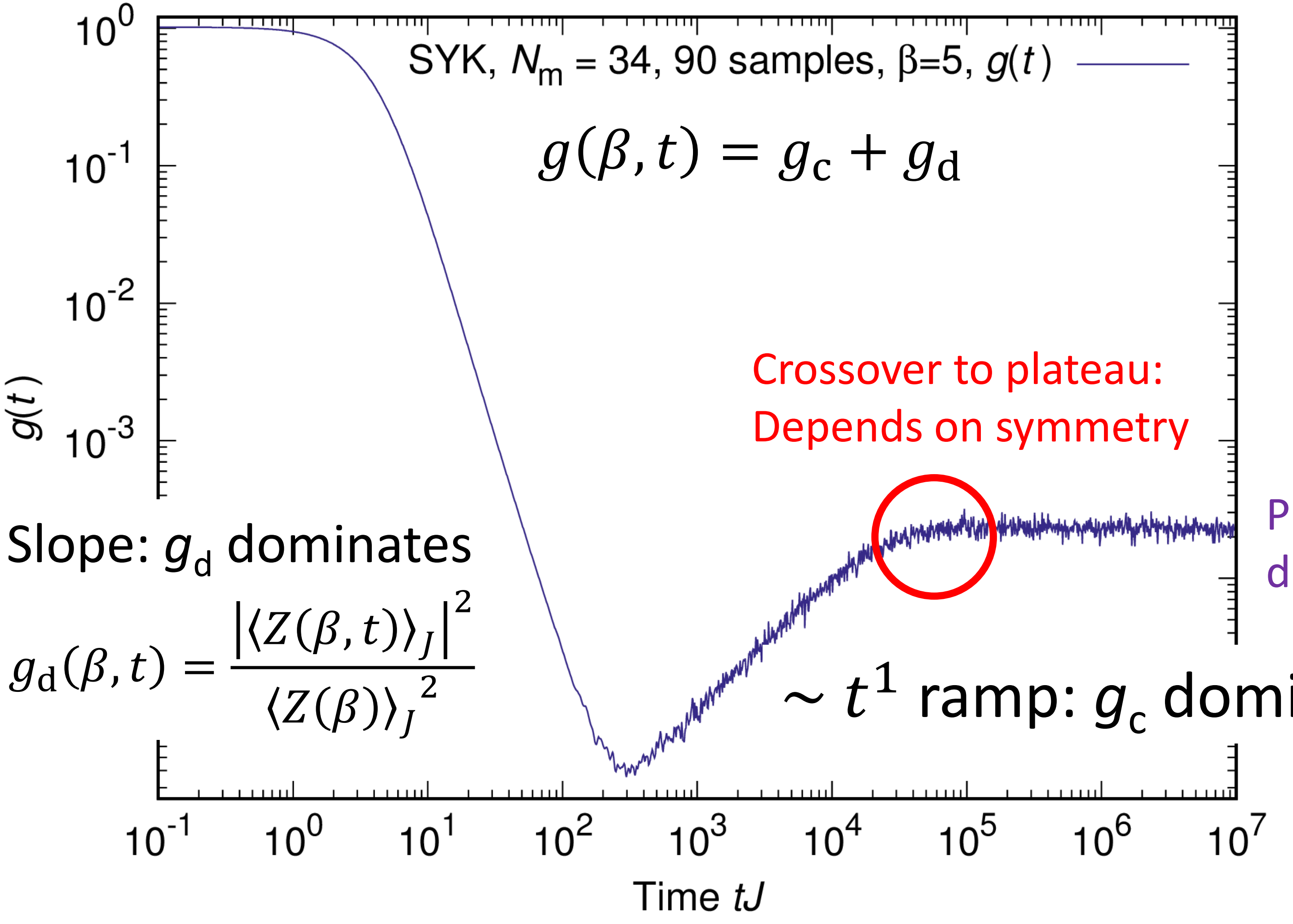
$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\left|\frac{Z(\beta,t)}{Z(\beta,t=0)}\right|^2=\frac{\sum_E N_E^{\;2}\mathrm{e}^{-2\beta E}}{Z(\beta,t=0)^2}=N_E\frac{Z(2\beta)}{Z(\beta)^2}$$

(if degeneracy of E: N_E is independent of E)

Because $Z\sim\mathrm{e}^{aS}$ $(a>0)$, long-time average will be $\sim\mathrm{e}^{-aS}$ (non-perturbative in 1/N)

Slope-dip-ramp-plateau structure

Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, and MT, JHEP **1705**(2017)118



$$g(\beta,t) = g_c + g_d$$

Crossover to plateau:
Depends on symmetry

Slope: g_d dominates

$$g_d(\beta,t) = \frac{\left|\langle Z(\beta,t)\rangle_J\right|^2}{\langle Z(\beta)\rangle_J^{\ 2}}$$

$\sim t^1$ ramp: g_c dominates

$$Z(\beta,t) = \mathrm{Tr}\left(e^{-\beta\hat{H}-i\hat{H}t}\right)$$

$$g_c(\beta,t) = \frac{\langle|Z(\beta,t)|^2\rangle_J - \left|\langle Z(\beta,t)\rangle_J\right|^2}{\langle Z(\beta)\rangle_J^{\ 2}}$$

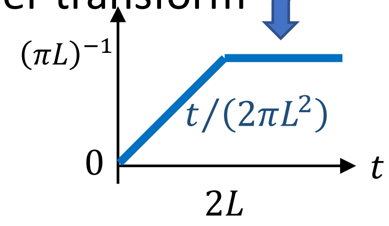
$$\sim \iint d\lambda_1 d\lambda_2 \langle\delta\rho(\lambda_1)\delta\rho(\lambda_2)\rangle e^{it(\lambda_1-\lambda_2)}$$

$$\rho(\lambda) = \sum_j \delta(\epsilon_j - \lambda)$$

Plateau height:
determined by degeneracy

$$R(\lambda) = \langle\delta\rho(\lambda_1)\delta\rho(\lambda_1-\lambda)\rangle = -\frac{\sin^2 L\lambda}{(\pi L\lambda)^2} + \frac{1}{\pi L}\delta(\lambda)$$

Fourier transform



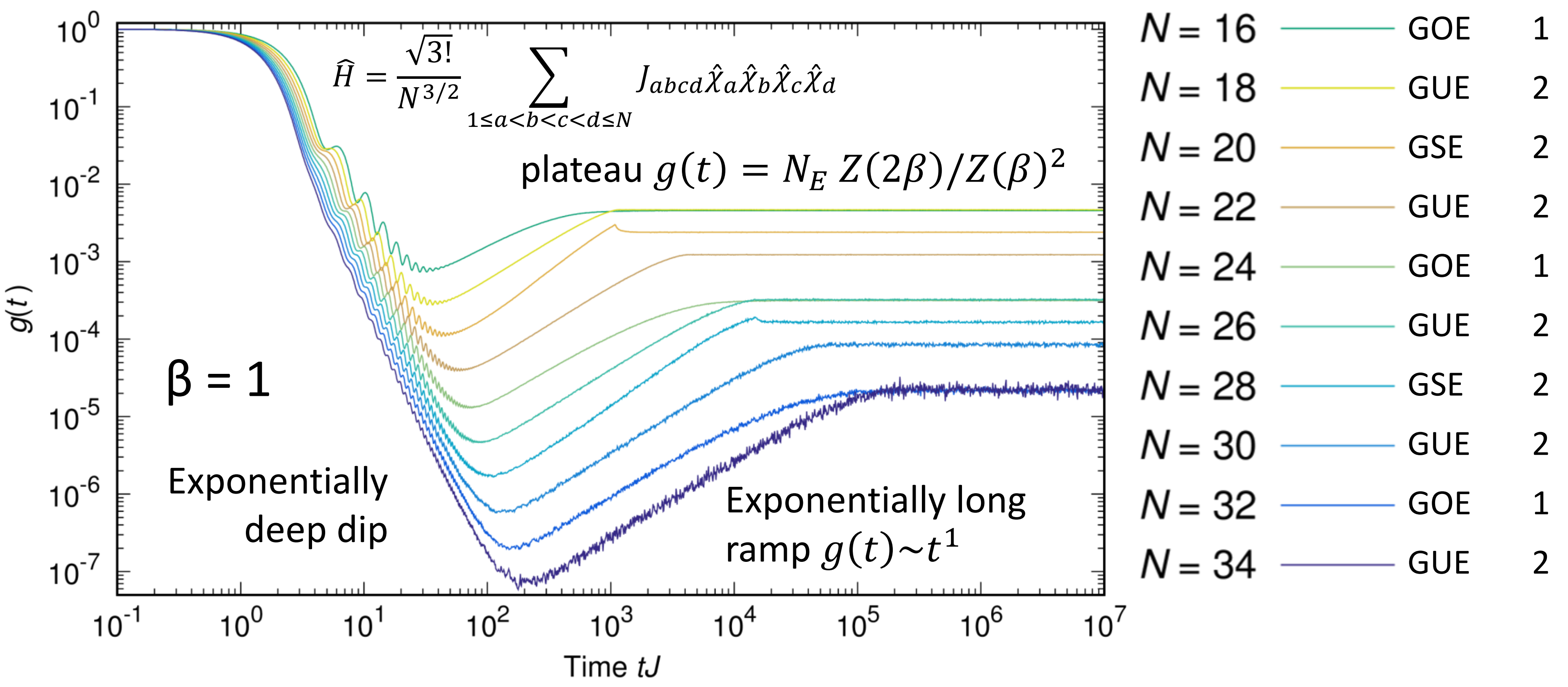
Random matrix theory
(GUE)

Exponentially long $\sim t^1$ ramp

➔ Rigid spectrum of the Sachdev-Ye-Kitaev model

$g(t)$: Dependence on N (nonperturbative in $1/N$)

Cotler et al., JHEP **1705**(2017)118



Classification of SPT order in class BDI: reduced from Z to Z_8 by interaction

[L. Fidkowski and A. Kitaev: PRB **81**, 134509 (2010); PRB **83**, 075103 (2011)]

Many-body level statistics 🡐 corresponding (dense) random matrix ensemble

$N_\chi \pmod 8$	0	1	2	3	4	5	6	7
qdim	1	$\sqrt{2}$	2	$2\sqrt{2}$	2	$2\sqrt{2}$	2	$\sqrt{2}$
lev. stat.	GOE	GOE	GUE	GSE	GSE	GSE	GUE	GOE

[Y.-Z. You, A. W. W. Ludwig, and Cenke Xu, PRB **95**, 115150 (2017)]

Correlation functions

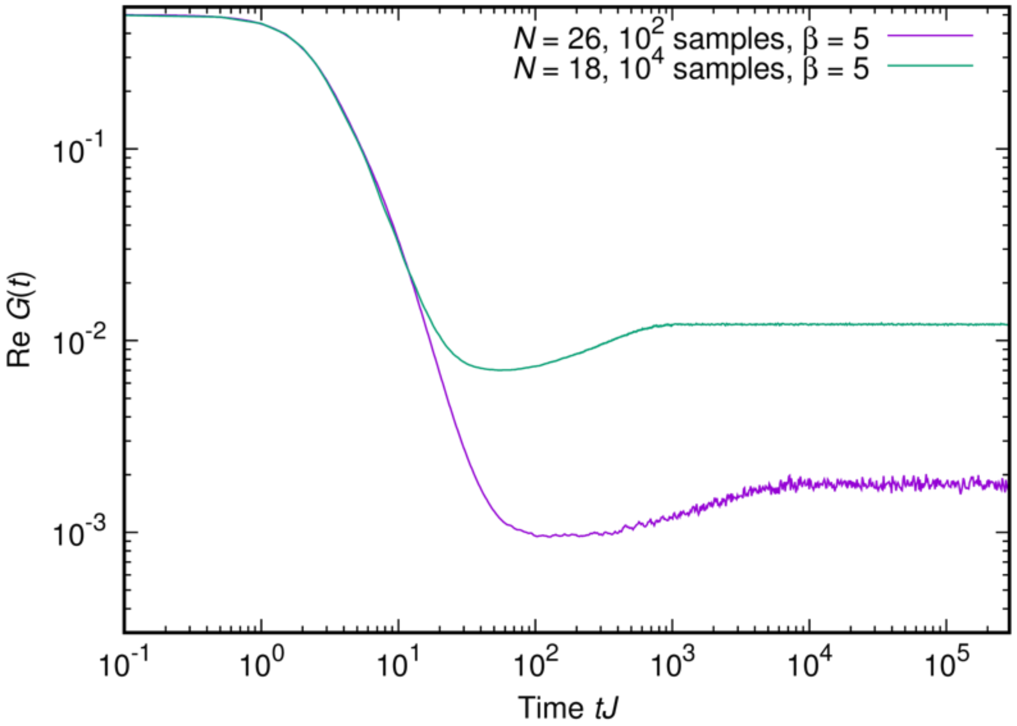
$N \equiv 0 \pmod 8$: X maps e⇔e, o⇔o and $X^2 = 1$ (no degeneracy)
$N \equiv 2 \pmod 8$: X maps e⇔o, $\langle \text{even}|\chi|\text{odd}\rangle$ finite
$N \equiv 4 \pmod 8$: X maps e⇔e, o⇔o and $X^2 = -1$ (degeneracy)
$N \equiv 6 \pmod 8$: X maps e⇔o but $\langle \text{even}|\chi|\text{odd}\rangle = 0$

$$G(t) = \langle \chi_a(t)\chi_a(0)\rangle$$

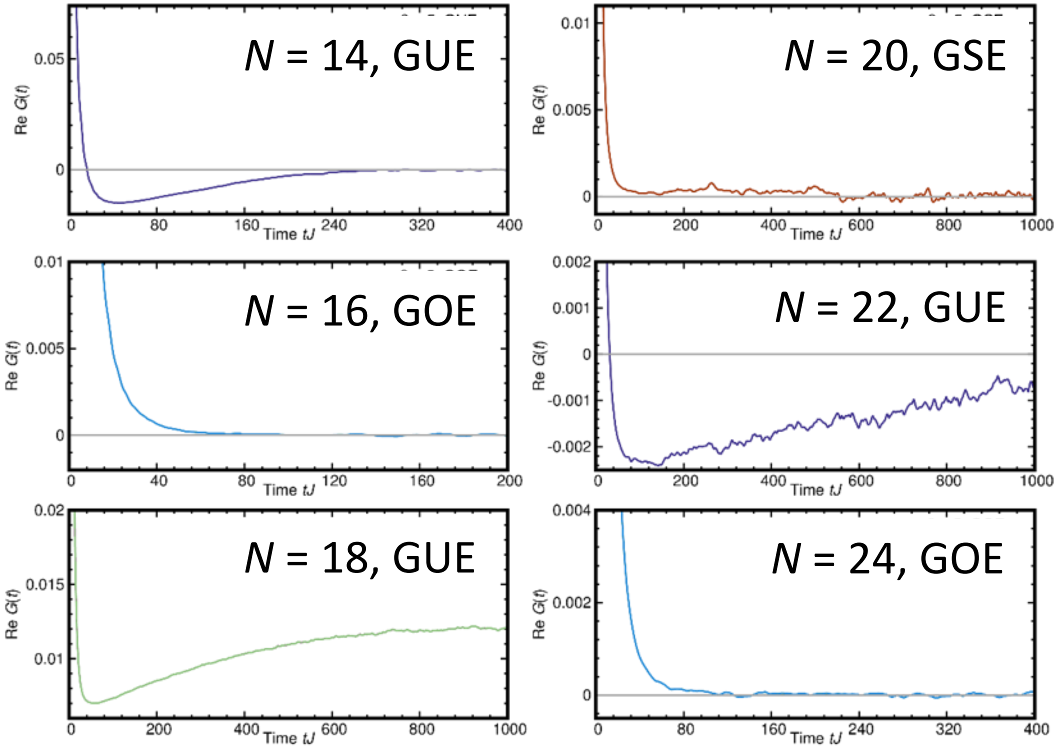
$N \equiv 2 \pmod 8$: dip-ramp-plateau similar to $g(\beta, t)$

$$g(\beta,t) \sim \left|\frac{Z(\beta,t)}{Z(\beta,t=0)}\right|^2 = \frac{1}{Z(\beta,t=0)^2}\sum_{m,n} e^{-\beta(E_m+E_n)} e^{i(E_m-E_n)t}$$

$$G(t) = \langle \hat{\chi}_a(t)\hat{\chi}_a(0)\rangle_\beta = \frac{1}{Z(\beta,t=0)}\sum_{m,n} e^{-\beta E_m}|\langle m|\hat{\chi}_a|n\rangle|^2 e^{i(E_m-E_n)t}$$

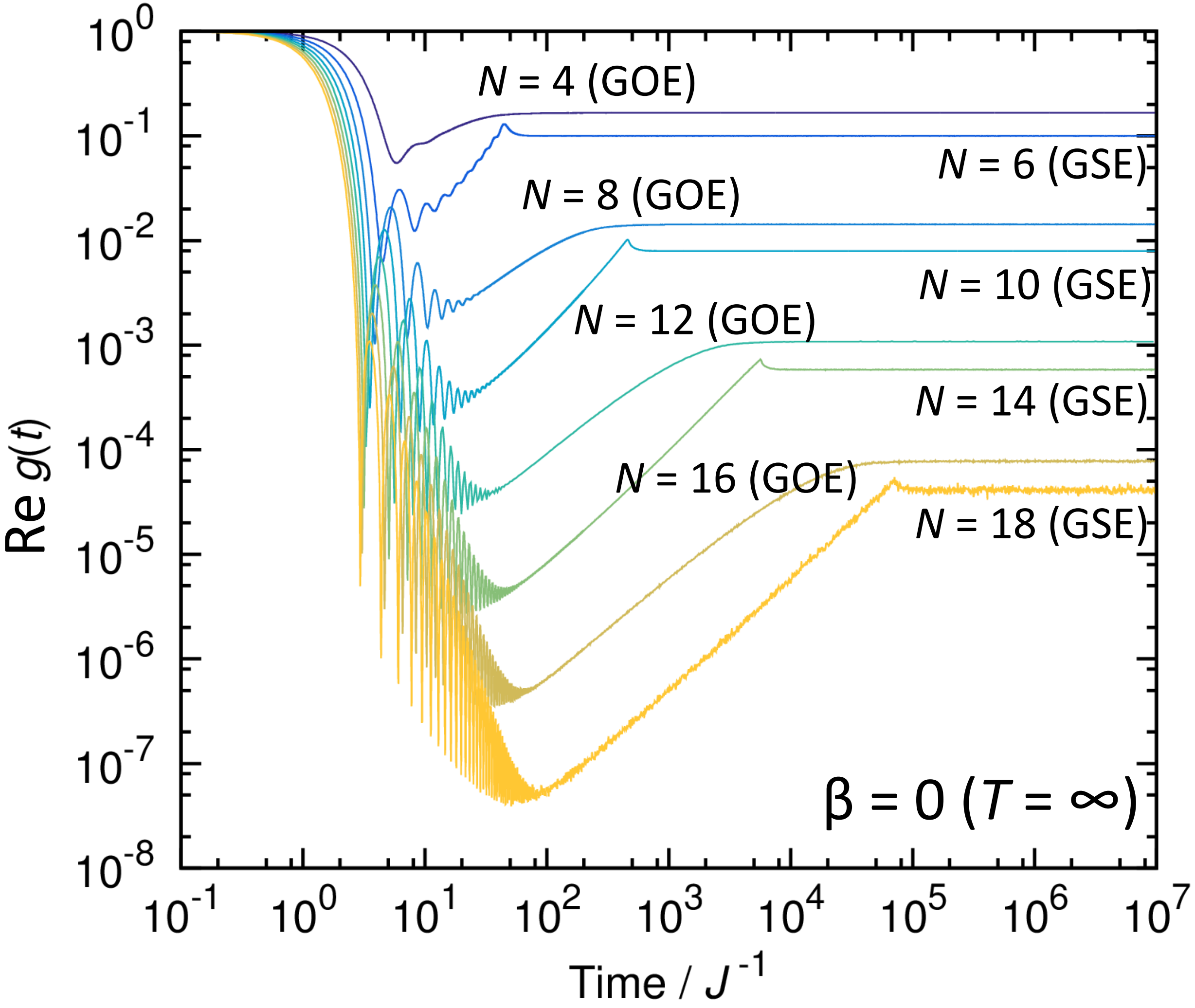


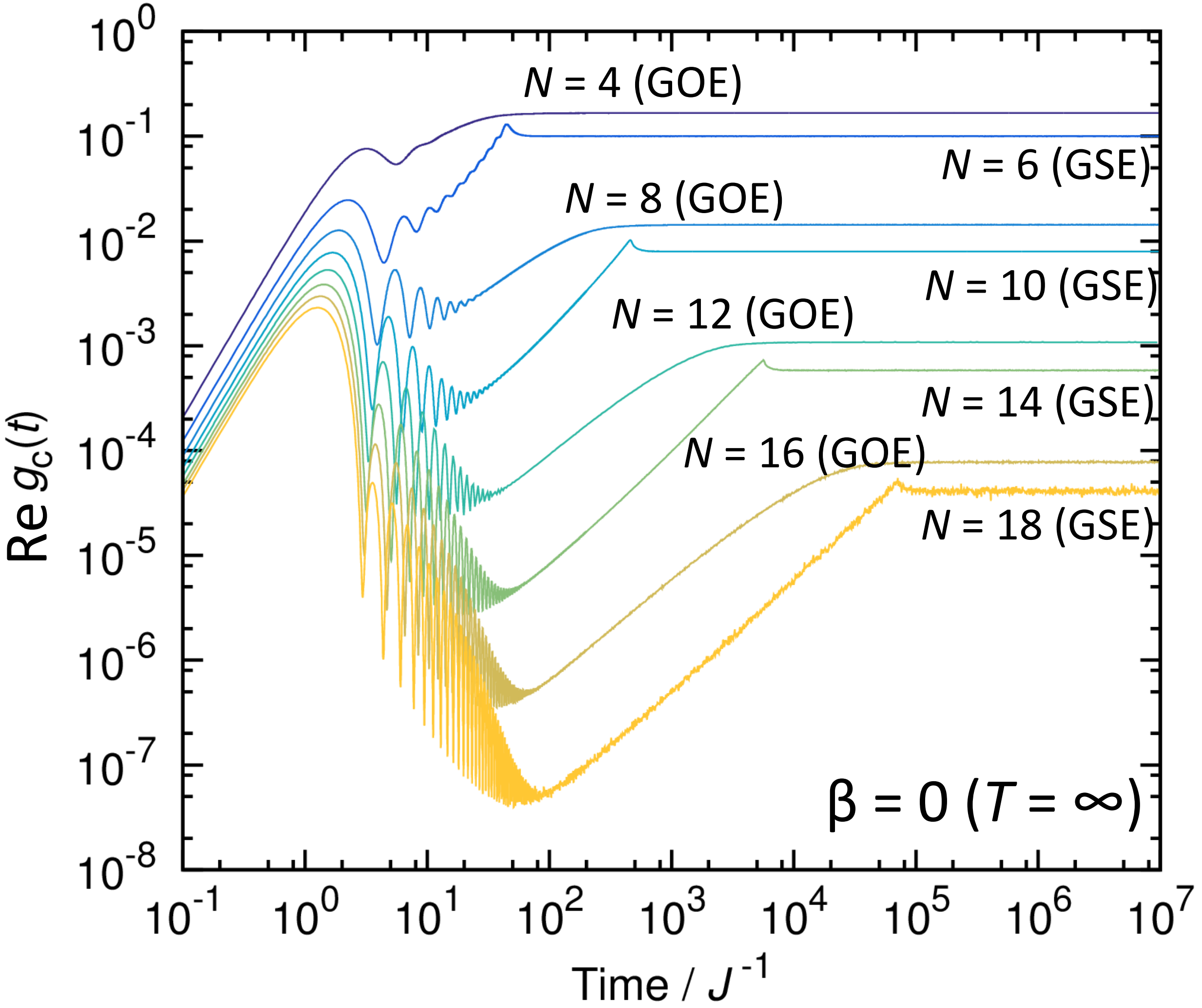
$$X = K\prod_{j=1}^{N/2}\left(\hat{c}_j^\dagger + \hat{c}_j\right),\ \hat{c}_j = \frac{(\chi_{2j-1} + i\chi_{2j})}{\sqrt{2}}$$



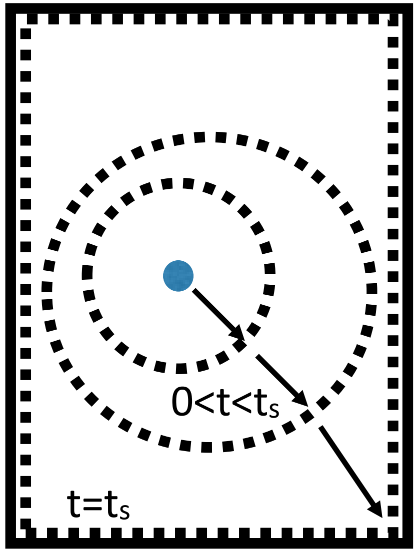
Unpublished

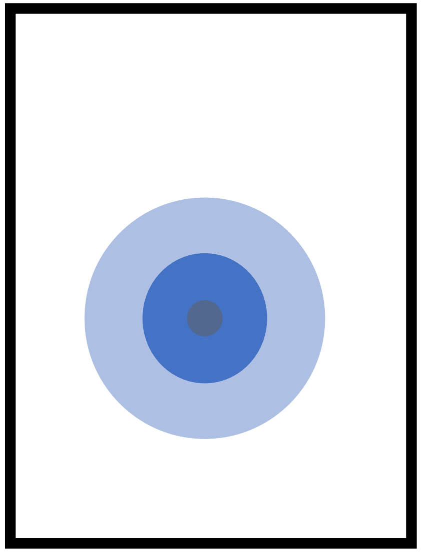
Dirac fermions





Dip time: scrambling or diffusion?



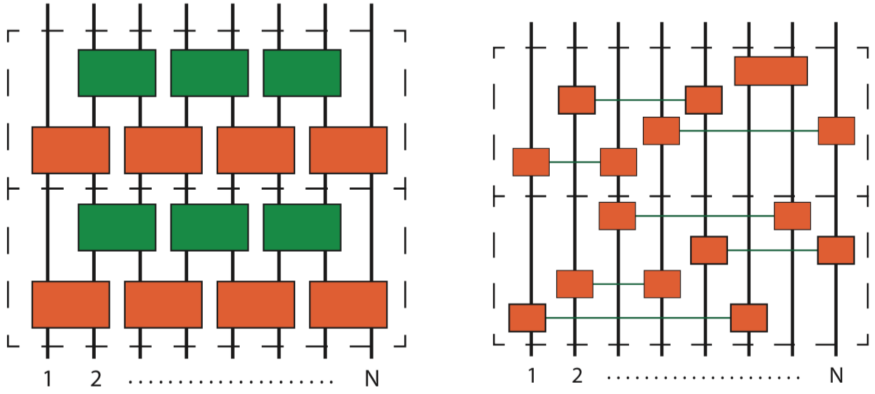


- RMT Universality observed after t_{ramp}
- Relationship to scrambling and diffusion?
- Our results: dip time determined by diffusion, not by scrambling

G. Gharibyan, M. Hanada, S. H. Shenker, and MT, JHEP 1807(2018)124 (arXiv:1803.08050)

We examined:

- Fixed Hamiltonian: SYK, randomly coupled spins (RCQ), XXZ spin chain
  - RCQ: both all-to-all and geometrically local
- Random dynamics: RCQ, XXZ
- Known case: band matrix (single particle hopping)



Sparse (or pruned) SYK

$$\hat{H} = \sum_{a<b<c<d} x_{abcd} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d \,, x_{abcd} = \begin{cases} 1 & \text{(probability } p) \\ 0 & \text{(probability } 1-p) \end{cases}, P(J_{abcd}) = \frac{\exp\left(-\frac{J_{abcd}^2}{2J^2}\right)}{\sqrt{2\pi J^2}}$$

$K_{\text{cpl}} = \binom{N}{4} p$: Number of non-zero x_{abcd}

$K_{\text{cpl}} \sim \mathcal{O}(1) N$ enough for
- Random matrix-like behavior
- Large entropy per fermion at low T !

$$p \sim \frac{4!}{N^3} = \mathcal{O}(N^{-3})$$

- Talk by Brian Swingle at Simons Center (18 September 2019)
- “Sparse Sachdev-Ye-Kitaev model, quantum chaos and gravity duals” A. M. García-García, Y. Jia, D. Rosa, J. J. M. Verbaarschot, Phys. Rev. D **103**, 106002 (2021)
- “A Sparse Model of Quantum Holography” S. Xu, L. Susskind, Y. Su, and B. Swingle, arXiv:2008.02303
- “Spectral Form Factor in Sparse SYK models” E. Cáceres, A. Misobuchi, and A. Raz, JHEP **2208**, 236 (2022)

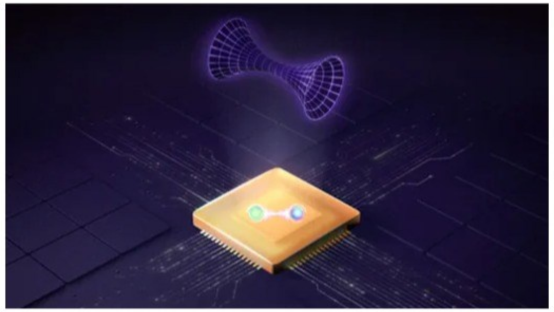
Article | Published: 30 November 2022

Traversable wormhole dynamics on a quantum processor

Daniel Jafferis, Alexander Zlokapa, Joseph D. Lykken, David K. Kolchmeyer, Samantha I. Davis, Nikolai Lauk, Hartmut Neven & Maria Spiropulu

Nature 612, 51–55 (2022) | Cite this article

Quanta Magazine (30 November 2022)



QUANTUM GRAVITY

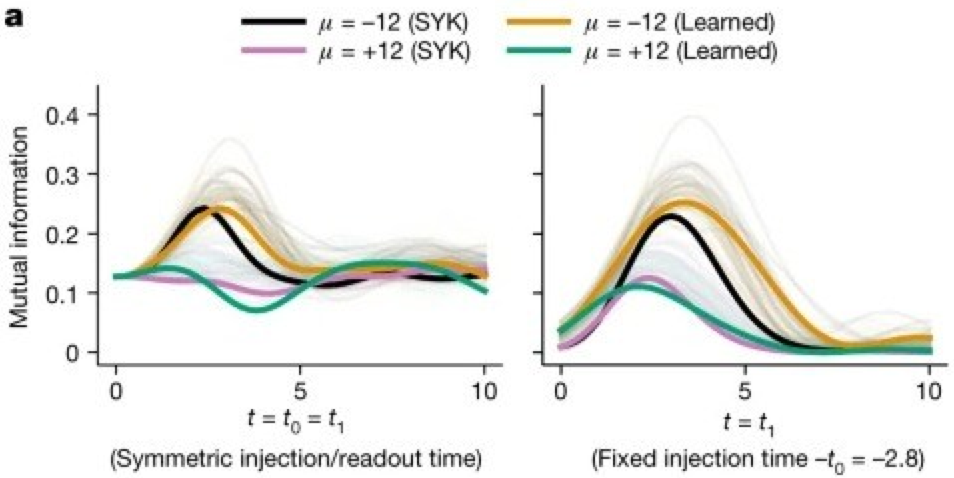
Physicists Create a Wormhole Using a Quantum Computer

By NATALIE WOLCHOVER | NOVEMBER 30, 2022 |

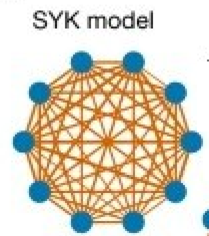
3 |

The unprecedented experiment explores the possibility that space-time somehow emerges from quantum information.

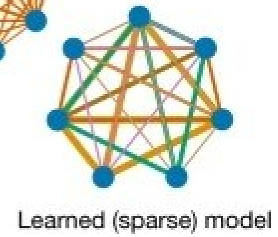
Fig. 2: Learning a traversable wormhole Hamiltonian from the SYK model.



b



$$H_{\mathrm{L,R}} = \sum_{1\le j_1<\cdots<j_q\le N} J_{j_1\ldots j_q}\psi_{\mathrm{L,R}}^{j_1}\cdots\psi_{\mathrm{L,R}}^{j_q},$$



Machine learning

$$H_{\mathrm{L,R}} = -0.36\psi^1\psi^2\psi^4\psi^5 + 0.19\psi^1\psi^3\psi^4\psi^7 - 0.71\psi^1\psi^3\psi^5\psi^6$$
$$+0.22\psi^2\psi^3\psi^4\psi^6 + 0.49\psi^2\psi^3\psi^5\psi^7,$$

➔ Realized on the Google Sycamore processor (nine-qubit circuit of 164 two-qubit, 295 single-qubit gates)

M. Tezuka, O. Oktay, E. Rinaldi, M. Hanada, and F. Nori, arXiv:2208.12098 (PRB Letter in press)

Sparse (or pruned) SYK with interaction $= \pm 1$

$$\hat{H} = C_{N,p} \sum_{1 \le a < b < c < d \le N} x_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d \,, x_{abcd} = \begin{cases} 1 & (\text{probability } p/2) \\ -1 & (\text{probability } p/2) \\ 0 & (\text{probability } 1-p) \end{cases}$$

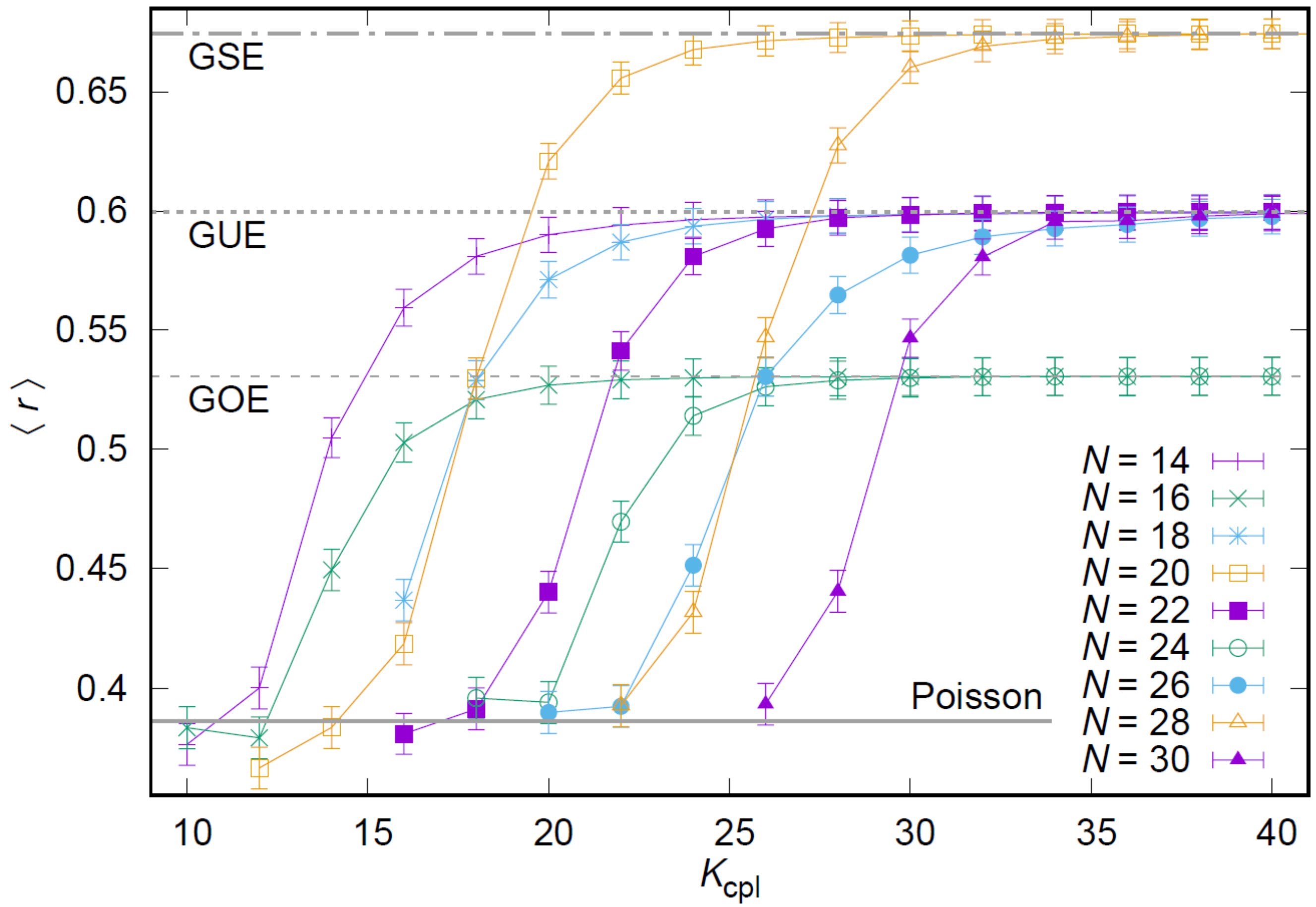
Random-matrix statistics for $K_{\text{cpl}} = \binom{N}{4} p \gtrsim N$.

cf. Non-Gaussian disorder average [T. Krajewski, M. Laudonio, R. Pascalie, and A. Tanasa, PRD **99**, 126014 (2019)]; Kitaev's talk (2015)

x_{abcd} can be taken to be +1 at finite $p \ll 1$ (unary sparse SYK, see appendix of 2208.12098), however at $p = 1$, the model is not chaotic [P. H. C. Lau, C.-T. Ma, J. Murugan, and MT, J. Phys. A. **54**, 095401 (2021)]

M. Tezuka, O. Oktay, E. Rinaldi, M. Hanada, and F. Nori, arXiv:2208.12098 (PRB Letter in press)

$\langle r \rangle$ as a function of K_{cpl}: approach RMT value



Neighboring gap ratio

$$r = \frac{\min(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}{\max(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}$$

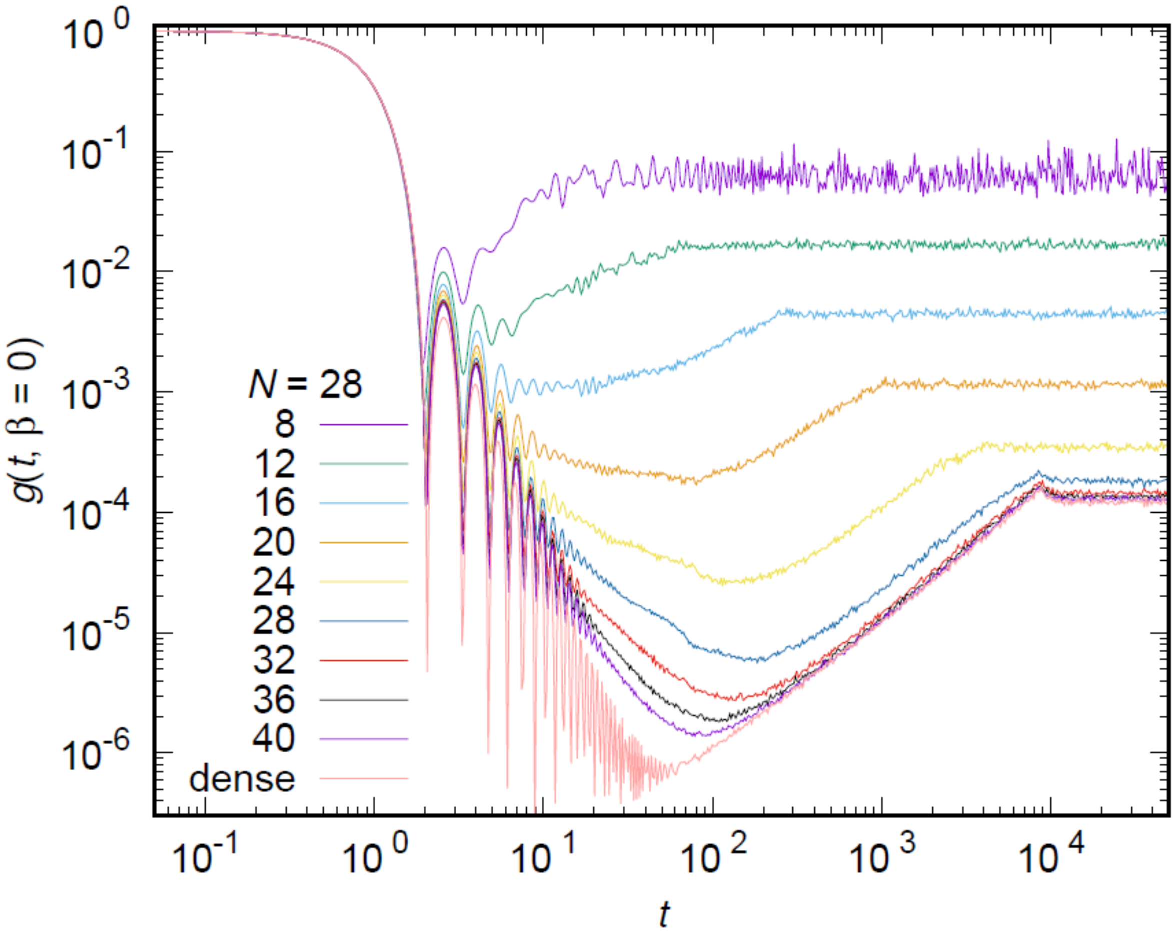
	Uncorrelated	GOE	GUE	GSE
$\langle r \rangle$	2log 2 – 1 = 0.38629…	0.5307(1)	0.5996(1)	0.6744(1)

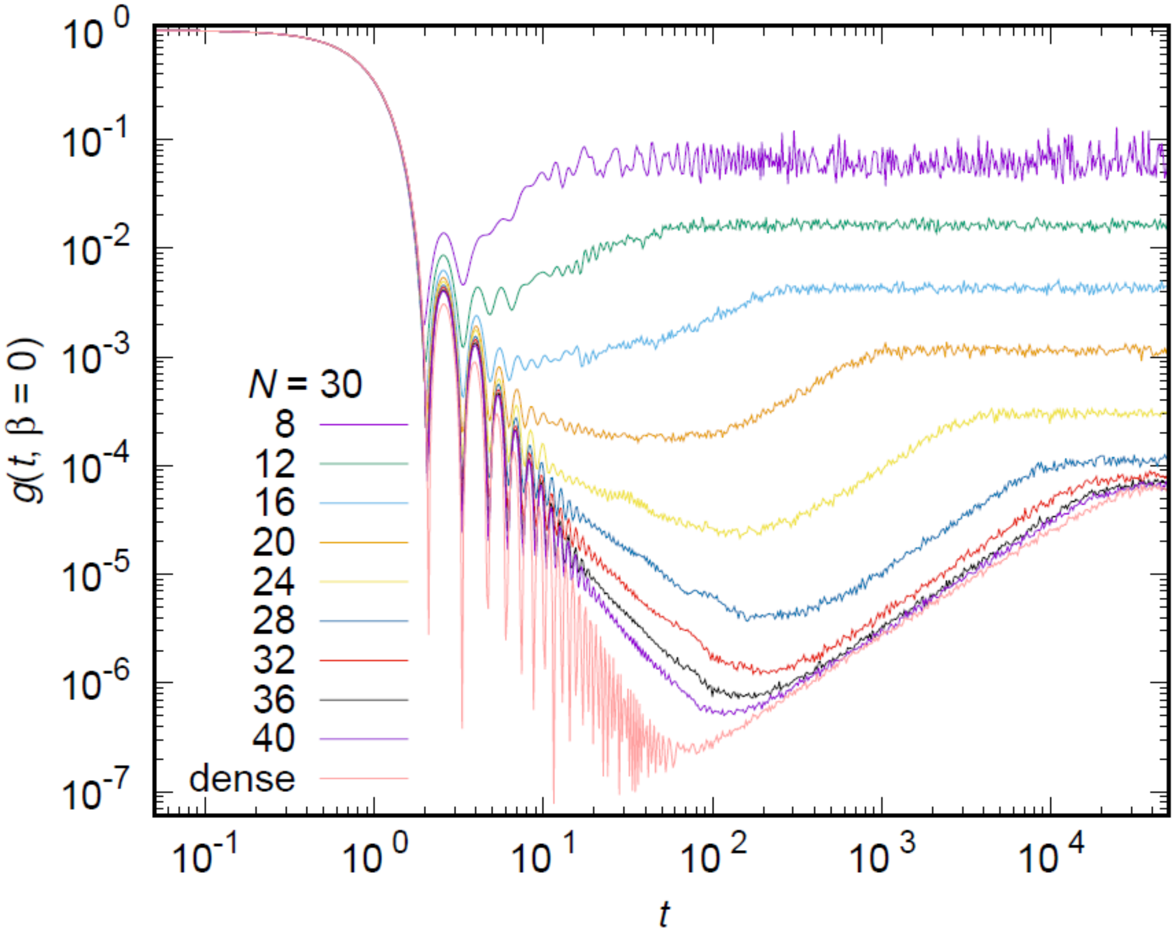
[Y. Y. Atas *et al.* PRL 2013]

M. Tezuka, O. Oktay, E. Rinaldi, M. Hanada, and F. Nori, arXiv:2208.12098 (PRB Letter in press)

Spectral form factor

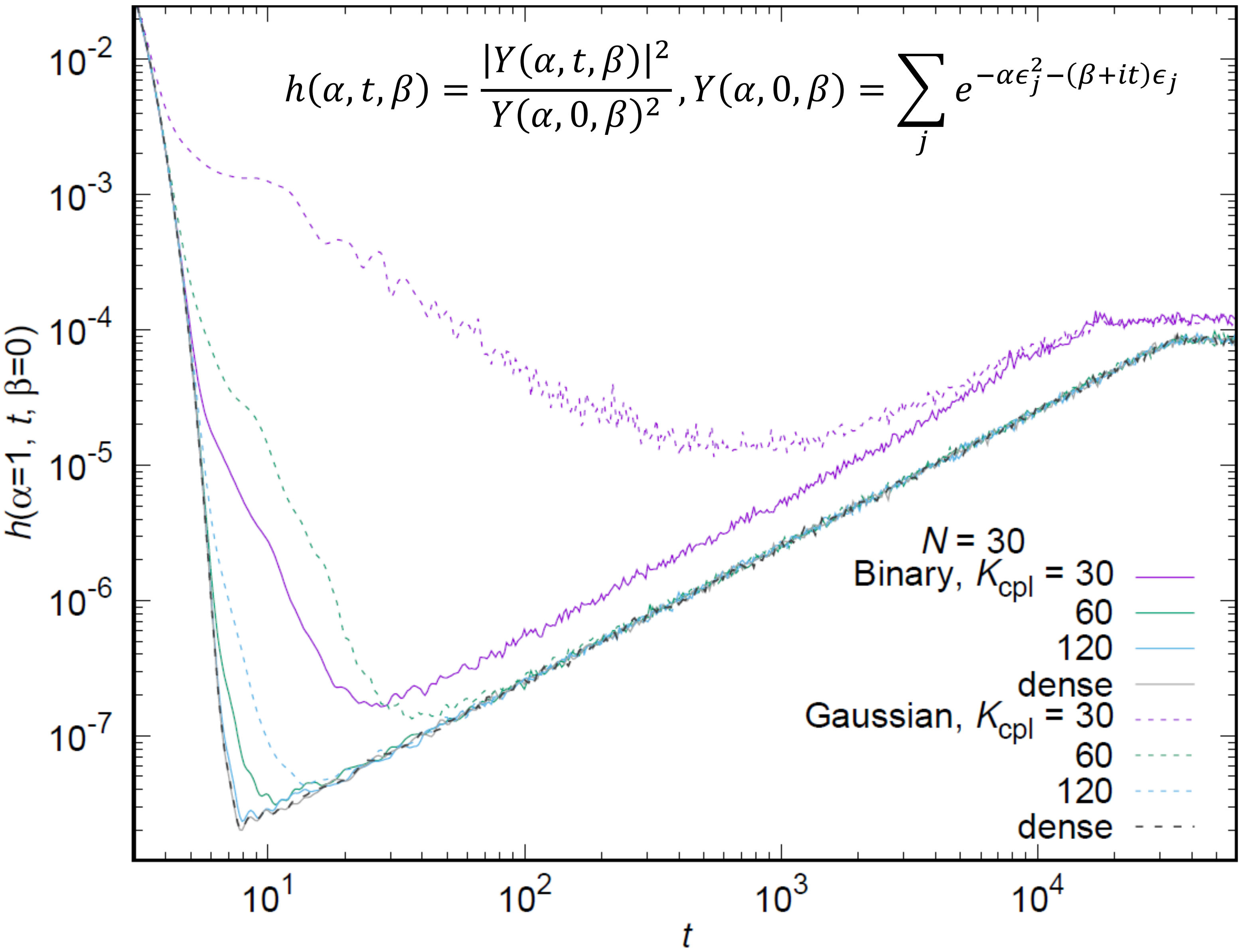
Clear ramp for $K_{\mathrm{cpl}} \gtrsim N$, coincides with the dense SYK as $N \to$ large





M. Tezuka, O. Oktay, E. Rinaldi, M. Hanada, and F. Nori, arXiv:2208.12098 (PRB Letter in press)

Modified SFF (focus on band center)



- Rigidity comparable to Gaussian-coupling model with twice as large K_{cpl}

A. M. García-García, A. Romero-Bermúdez, B. Loureiro, and MT, Phys. Rev. Lett. **120**, 241603 (2018)

Also see our reply [PRL **126**, 109102 (2021)] to the comment by J. Kim and X. Cao [PRL **126**, 109101 (2021)]

SYK_{4+2}

Q.: Minimum requirements for chaotic behavior? (→ gravity interpretation?)
Study a simple model with analytical + numerical methods

$$\hat{H} = \underbrace{\sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d}_{\mathrm{SYK}_4} + i\underbrace{\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b}_{\mathrm{SYK}_2}$$

Gaussian random couplings

J_{abcd}: average 0, standard deviation $\frac{\sqrt{6}J}{N^{3/2}}$

K_{ab}: average 0, standard deviation $\frac{K}{\sqrt{N}}$

$J = 1$: unit of energy

Normalization here: $\{\hat{\chi}_a, \hat{\chi}_b\} = \delta_{ab}$

SYK_4 as unperturbed Hamiltonian,
K controls the strength of SYK_2 (one-body random term, solvable)

Here we take (GUE) $N \equiv 2,6 \pmod 8$

Both terms respect charge parity in complex fermion description

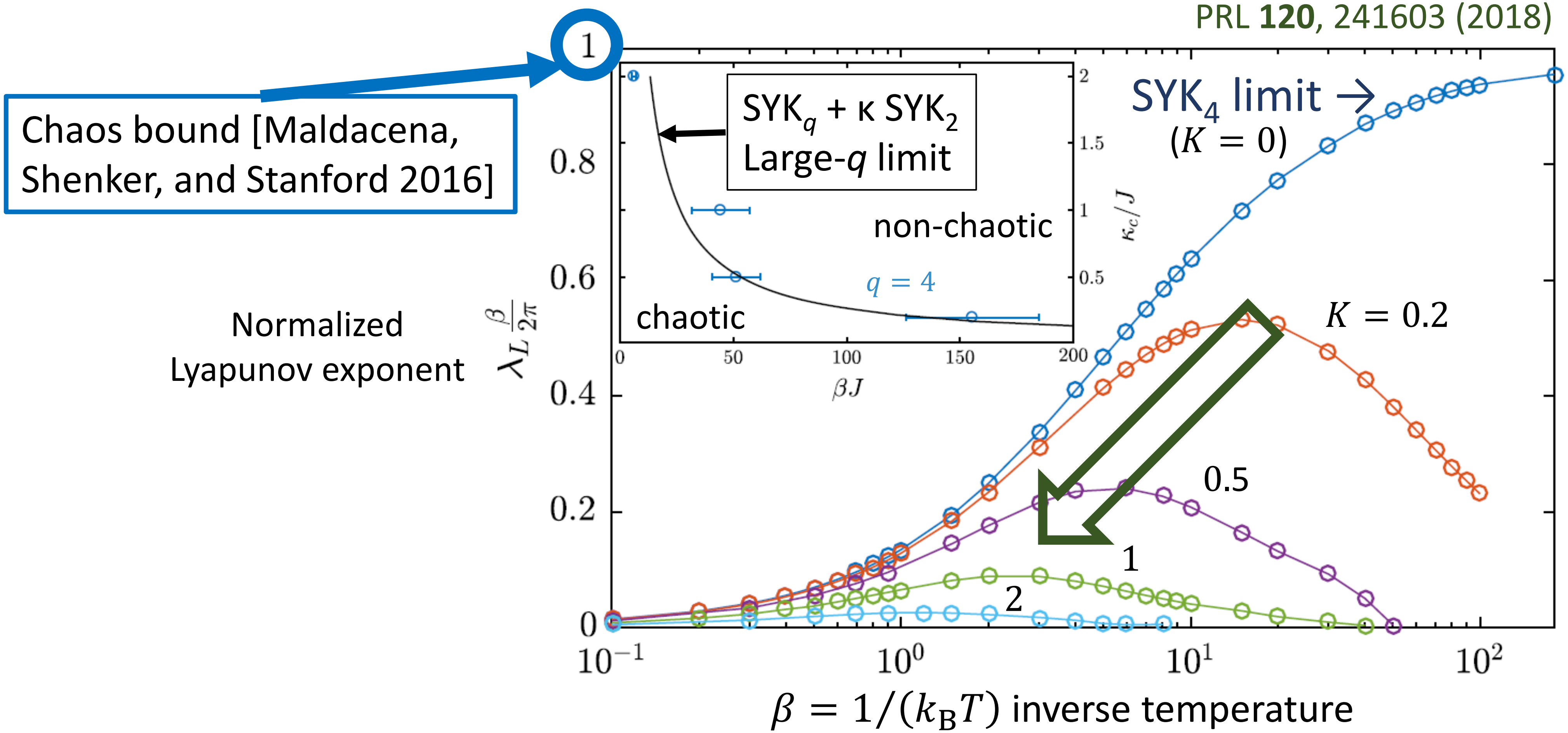
➔ Full numerical exact diagonalization (ED) of $2^{N/2-1}$-dimensional matrix, $N \lesssim 34$ possible

Large-N calculation for Out-of-Time Order Correlator (OTOC)

$$\hat{H} = \overbrace{\sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d}^{\text{SYK}_{q=4}} + i \overbrace{\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b}^{\text{SYK}_2}$$

K_{ab}: standard deviation $\frac{K}{\sqrt{N}}$

PRL **120**, 241603 (2018)



Deviation from the chaos bound as SYK$_2$ component is introduced

PRL **120**, 241603 (2018)

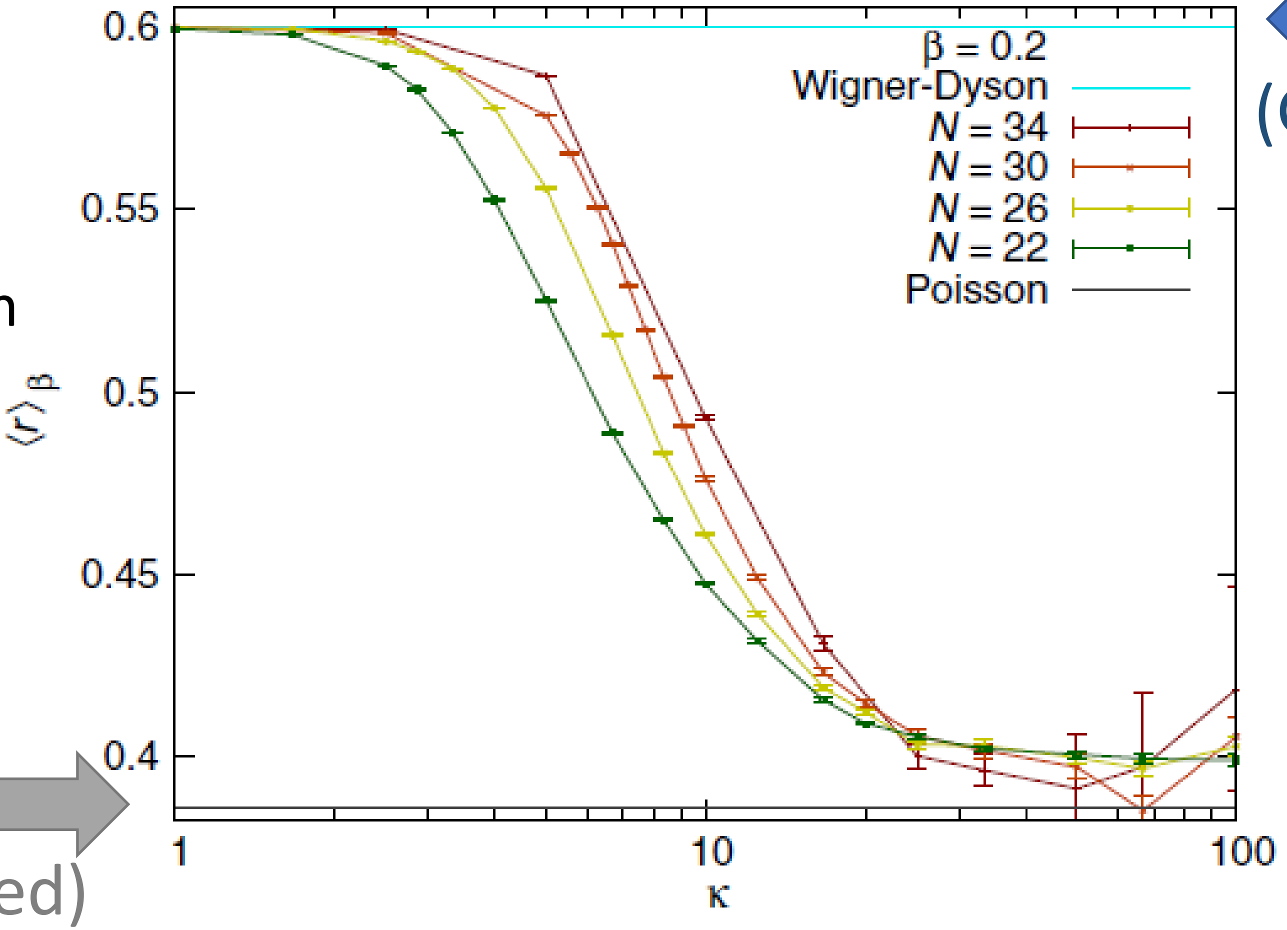
$\mathrm{SYK}_{q\geq 4} + \mathrm{SYK}_2$: breakdown of chaos

$$\hat{H} = \overbrace{\sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d}^{\mathrm{SYK}_4} + i\overbrace{\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b}^{\mathrm{SYK}_2}$$

K_{ab}: standard deviation = $\kappa/\sqrt{N}$

GUE (Gaussian Unitary Ensemble)

Averaged ratio between neighboring energy level separations



Poisson (uncorrelated)

➔ Understood as localization of the many-body wave function in Fock space

F. Monteiro, T. Micklitz, MT, and A. Altland, Phys. Rev. Research **3**, 013023 (2021)

Fock-space localization: choice of basis

$$\mathrm{SYK}_4 + \delta\,\mathrm{SYK}_2$$

$$\hat{H} = -\sum_{1\le a<b<c<d}^{N=2N_{\mathrm{D}}} J'_{abcd}\hat{\psi}_a\hat{\psi}_b\hat{\psi}_c\hat{\psi}_d + i\sum_{1\le a<b}^{N} K_{ab}\hat{\psi}_a\hat{\psi}_b$$

Block-diagonalize the SYK_2 part
(the skew-symmetric matrix (K_{ab}) has eigenvalues $\pm v_j$)

$$\hat{H} = -\sum_{1\le a<b<c<d}^{2N_{\mathrm{D}}} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + i\sum_{1\le j\le N}^{2N_{\mathrm{D}}} v_j\hat{\chi}_{2j-1}\hat{\chi}_{2j}$$

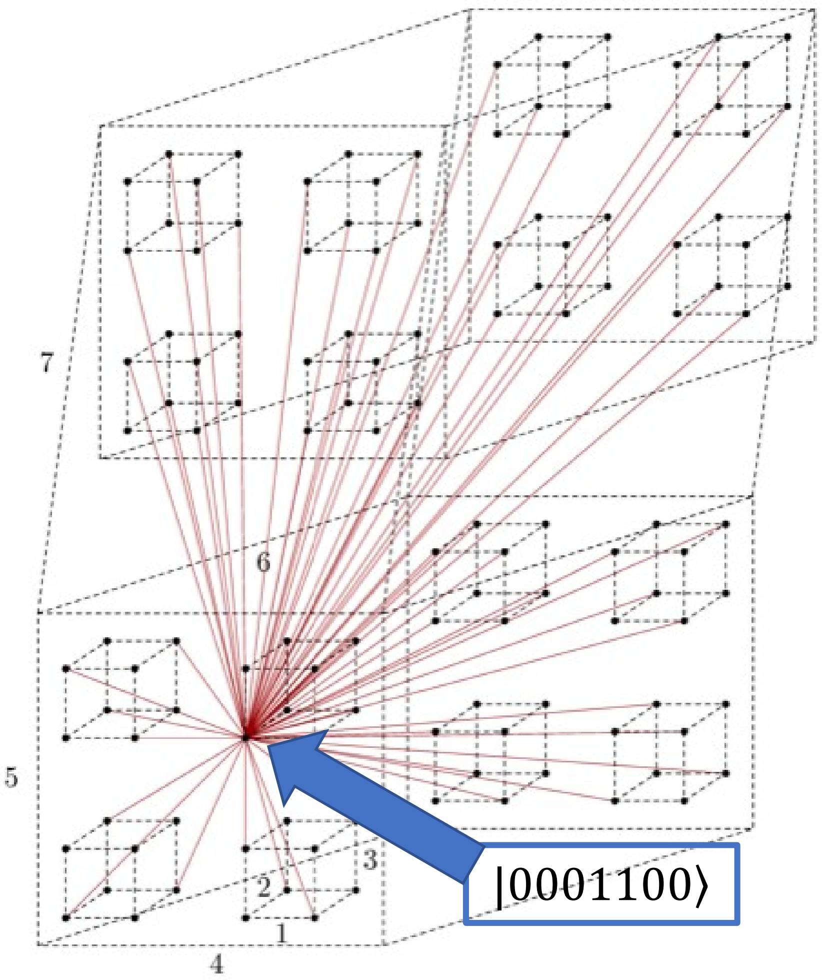
Normalization of J_{abcd}, v_j :
SYK_4 bandwidth = 1,
Width of v_j distribution = δ

We choose $\{\hat{\psi}_a,\hat{\psi}_b\} = \{\hat{\chi}_a,\hat{\chi}_b\} = 2\delta_{ab}$ as the normalization for the $N = 2N_{\mathrm{D}}$ Majorana fermions. For $\hat{c}_j = \frac{1}{2}\left(\hat{\chi}_{2j-1} + \mathrm{i}\hat{\chi}_{2\mathrm{j}}\right)$ we have $\{\hat{c}_i,\hat{c}_j^\dagger\} = \delta_{ij}$.

F. Monteiro, T. Micklitz, MT, and A. Altland, Phys. Rev. Research **3**, 013023 (2021)

Fock-space localization: choice of basis

$N = 2N_\mathrm{D} = 14$: $2^7 = 128$ states



Basis diagonalizing the complex fermion number operators $\hat{n}_j = \hat{c}_j^\dagger \hat{c}_j$ → Sites: the 2^{N_D} vertices of an N_D-dim. hypercube.

$$\hat{c}_j = \frac{1}{2}(\hat{\chi}_{2j-1} + \mathrm{i}\hat{\chi}_{2\mathrm{j}})$$

$$\hat{H} = -\sum_{1 \le a<b<c<d}^{2N_\mathrm{D}} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + i\sum_{1\le j\le N}^{N_\mathrm{D}} v_j \hat{\chi}_{2j-1}\hat{\chi}_{2j}$$

$$= -\sum_{1 \le a<b<c<d}^{2N_\mathrm{D}} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + \sum_{1\le j\le N}^{N_\mathrm{D}} v_j(2\hat{n}_j - 1)$$

Each term of SYK_4 connects vertices with distance = 0, 2, 4.

For $N = 14$, each vertex is directly connected with
1 (distance=0, itself) + 21 (distance=2) + 35 (distance=4)
vertices out of the possible $2^N = 128$ (64 per parity).

Diagnostic quantities: Moments of wave functions and spectral two-point correlation function

- Moments of eigenstate wave functions

$$I_q = \nu^{-1} \sum_{n,\psi} \left\langle |\langle \psi | n \rangle|^{2q} \delta(E_\psi) \right\rangle_J$$

with average density of states at band center

$$\nu = \nu(E \simeq 0), \nu(E) = \sum_{\psi} \left\langle \delta(E - E_\psi) \right\rangle_J$$

➔ Parametrizes localization, allows comparison with numerics

$I_2 = \nu^{-1} \sum_{n,\psi} \left\langle |\langle \psi | n \rangle|^4 \delta(E_\psi) \right\rangle_J$:

inverse participation ratio (IPR), $\frac{1}{D} \leq I_2 \leq 1$

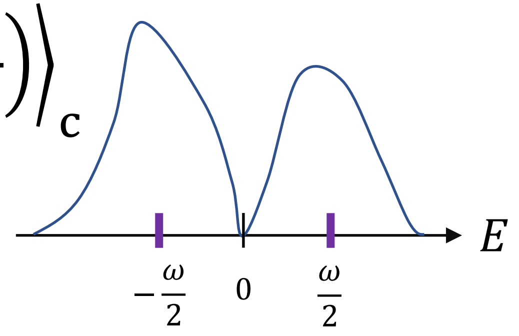
Equal weights

Single non-zero element

D: dimension of $\{|n\rangle\} = 2^{N-1}$

- Spectral two-point correlation function

$$K(\omega) = \nu^{-2} \left\langle \nu\left(\frac{\omega}{2}\right) \nu\left(-\frac{\omega}{2}\right) \right\rangle_c$$



c: connected part

$\langle AB \rangle_c = \langle AB \rangle_J - \langle A \rangle_J \langle B \rangle_J$

➔ Reflects level repulsion if the spectrum is random matrix-like

We calculate these quantities for large N and compare against numerical results

Analytical results

Method: Exact matrix integral representation; mapping to a supersymmetric sigma model; saddle point equations; effective medium approximation

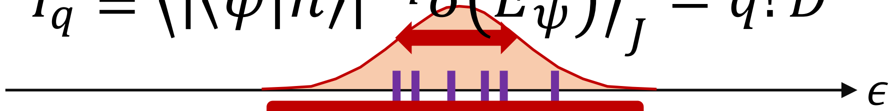
PRR 3, 013023 (2021)

$(N_\mathrm{D} = \frac{N}{2}, c = \mathcal{O}(1), D = 2^{N_\mathrm{D}-1})$

I

- I: Avg. density of states (ADoS) at band center $\nu(\epsilon = 0) = cD$
- $I_q \equiv \langle |\langle\psi|n\rangle|^{2q} \delta(E_\psi) \rangle_J = q!\, D^{1-q}$

Fully delocalized



SYK4 bandwidth=1

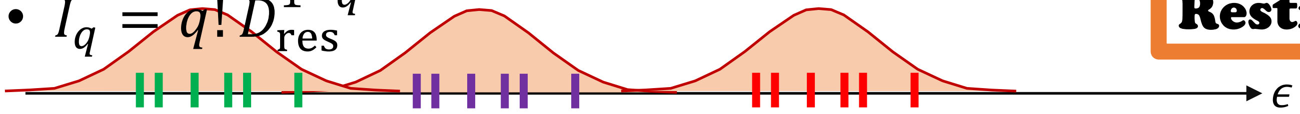
SYK_2 energy distribution $\sim \sqrt{N_D}\delta$

$N_\mathrm{D}^{-1/2}$

II

- II: ADoS $\nu = \frac{cD}{\sqrt{N_\mathrm{D}}\delta}$, spread of wave functions $D_\mathrm{res} \simeq \frac{D}{\sqrt{N_\mathrm{D}}\delta}$
- $I_q = q!\, D_\mathrm{res}^{1-q}$

Restricted



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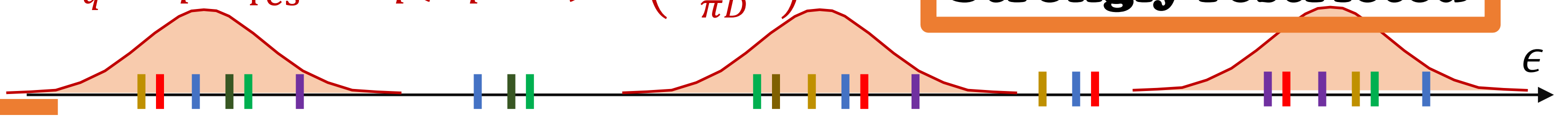
III

- III: ADoS $\nu = \frac{cD}{\sqrt{N_\mathrm{D}}\delta}$, spread of wave functions $D_\mathrm{res} \simeq \frac{D}{\sqrt{N_\mathrm{D}}\delta^2}$
- $I_q = q!\, D_\mathrm{res}^{1-q} = q(2q-3)!!\left(\frac{4\sqrt{N}\delta^2}{\pi D}\right)^{q-1}$

Strongly restricted



δ_c

IV

$\delta_\mathrm{c} = \frac{N_\mathrm{D}^2}{4\sqrt{3}} \log N_\mathrm{D}$ for large N

- IV: All eigenstates localized to $\mathcal{O}(1)$ sites

δ: typical fermion mass $\propto$ SYK_2 coefficient

Eigenenergy spectral statistics (for odd N case for simplicity)

$\widetilde{K}(s) = 1 - \frac{\sin^2 s}{s^2} + \delta\left(\frac{s}{\pi}\right)$,

$s = \pi\omega\nu$ in I, II, III :

agrees with Gaussian Unitary Ensemble (GUE)

IV: **Poisson statistics**

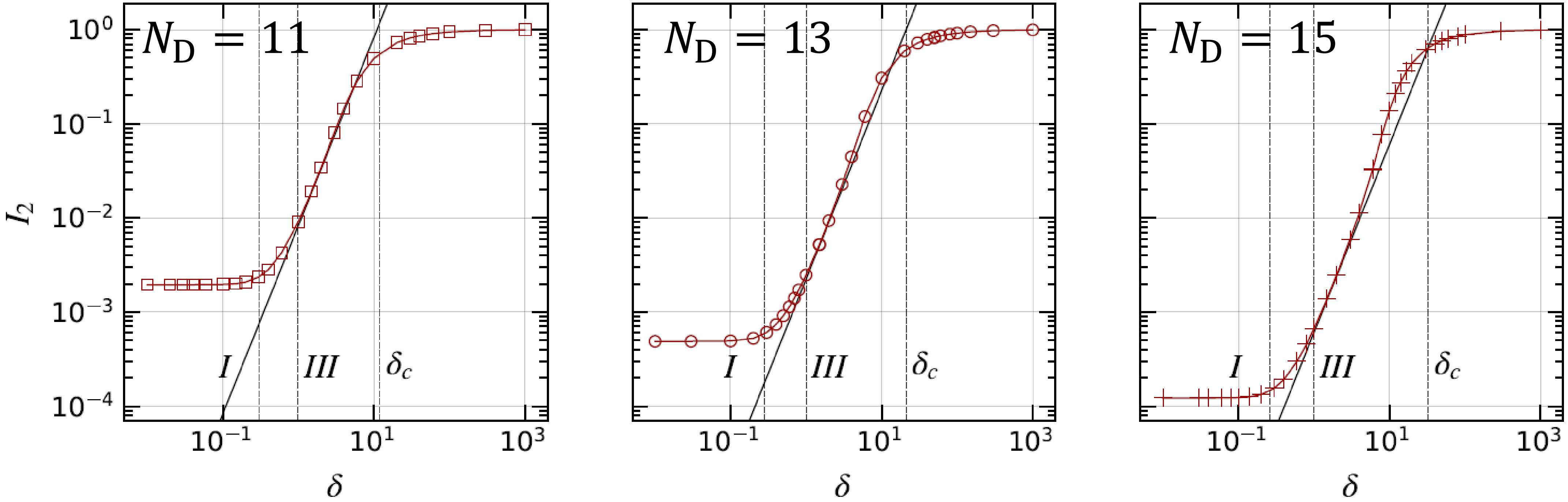
PRR 3, 013023 (2021)

Inverse participation ratio for Regime III

IPR $I_2 =$ average of $\sum_n |\langle\psi|n\rangle|^4$ for normalized ψ, $\frac{1}{D} \leq I_2 \leq 1$

Equal weights

Single non-zero element

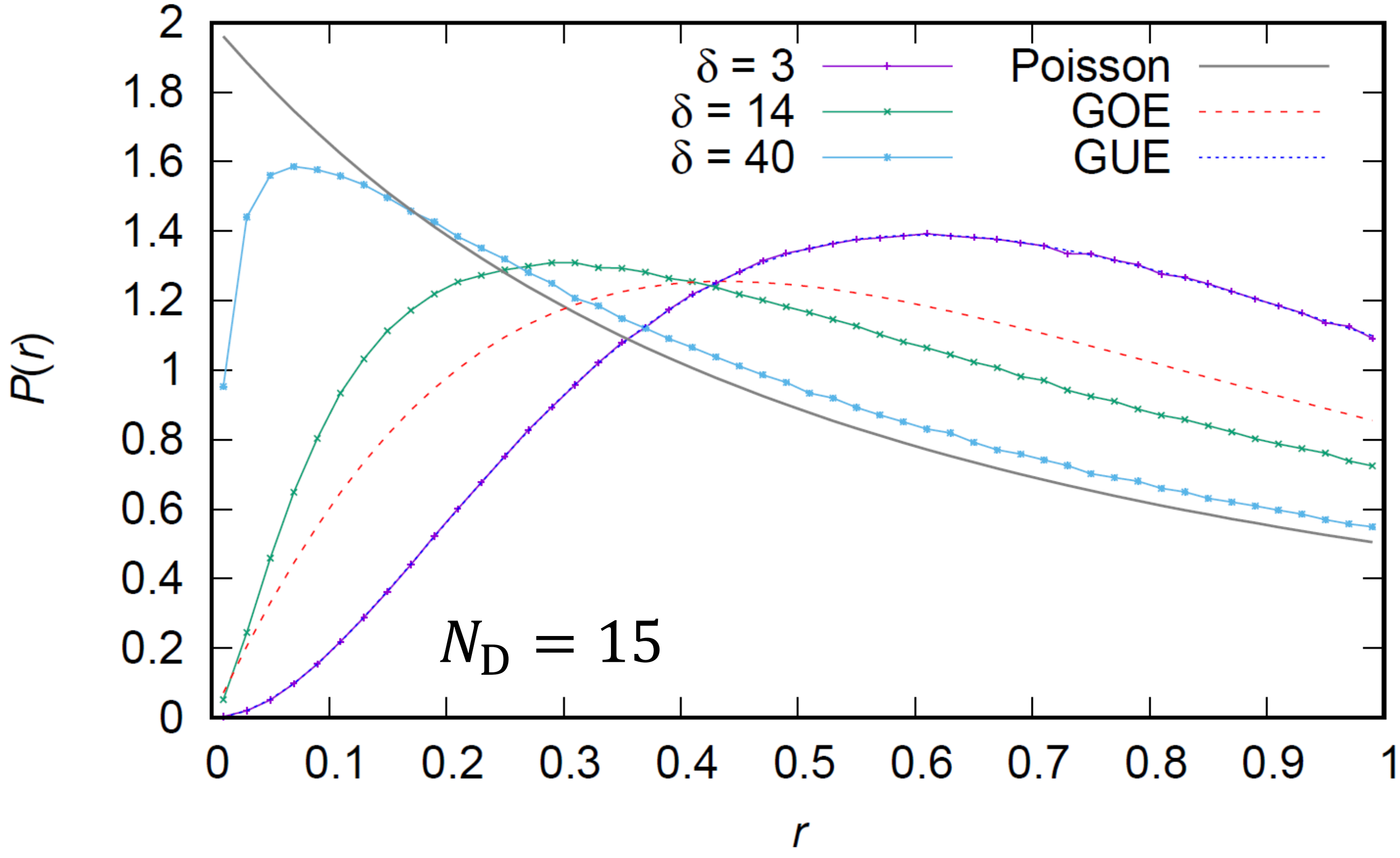


$$I_q = \frac{q(2q-3)!!}{\delta^{2(1-q)}}\left(\frac{\pi D}{4\sqrt{N_D}}\right)^{1-q} = q(2q-3)!!\left(\frac{4\sqrt{N_D}\delta^2}{2^{N-1}\pi}\right)^{q-1} \text{ in III}$$

Central 1/7 of the energy spectrum

PRR 3, 013023 (2021)

Spectral statistics: gap ratio distribution



(Analytical prediction: $\delta_{\mathrm{c}} = \frac{Z}{\sqrt{2\rho}} W(2Z\sqrt{\pi}) = 38.47$)

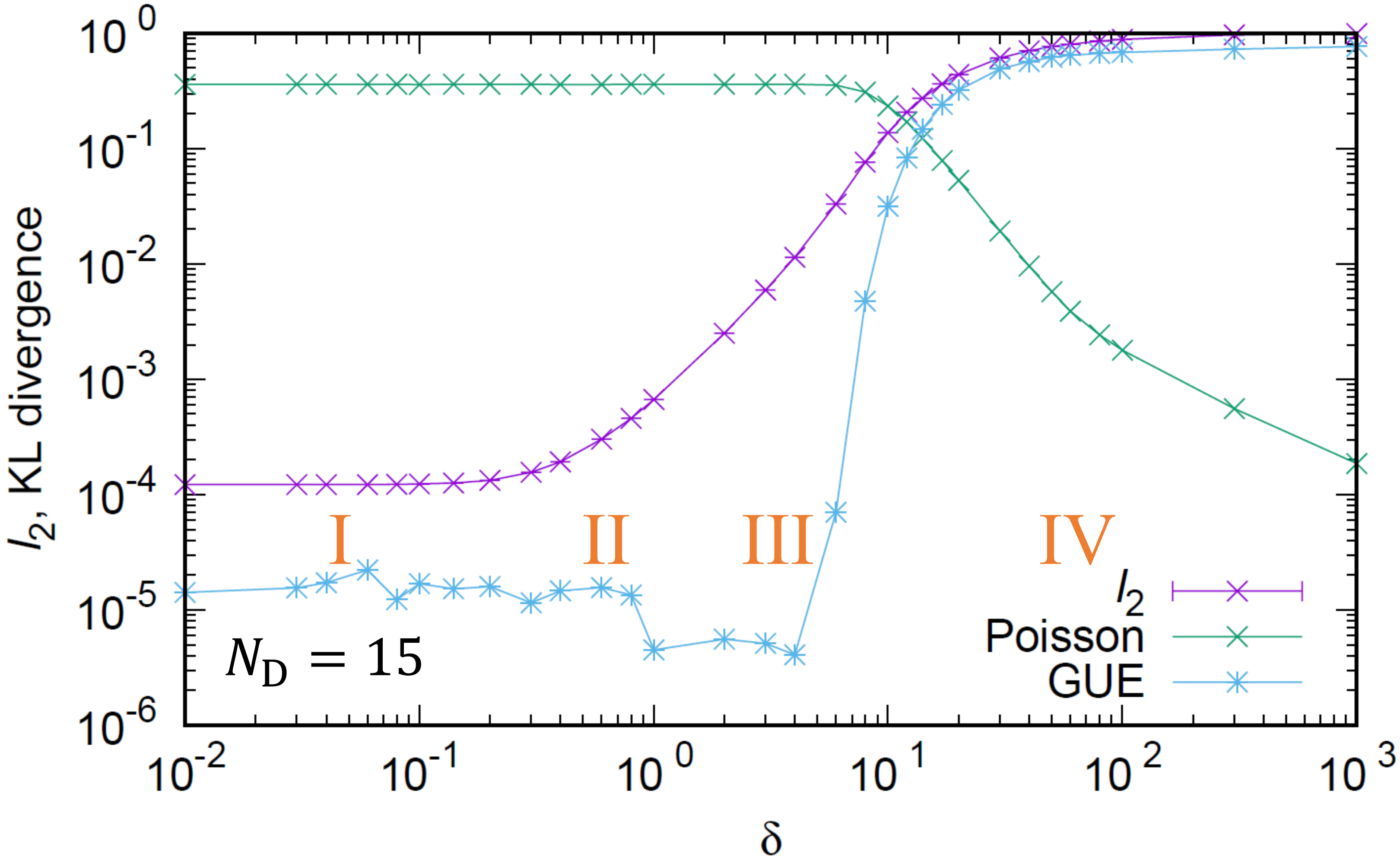
Measure difference by Kullback-Leibler (KL) divergence:

$$D_{\mathrm{KL}}(P||Q) = \sum_x P(x) \log \frac{P(x)}{Q(x)}.$$

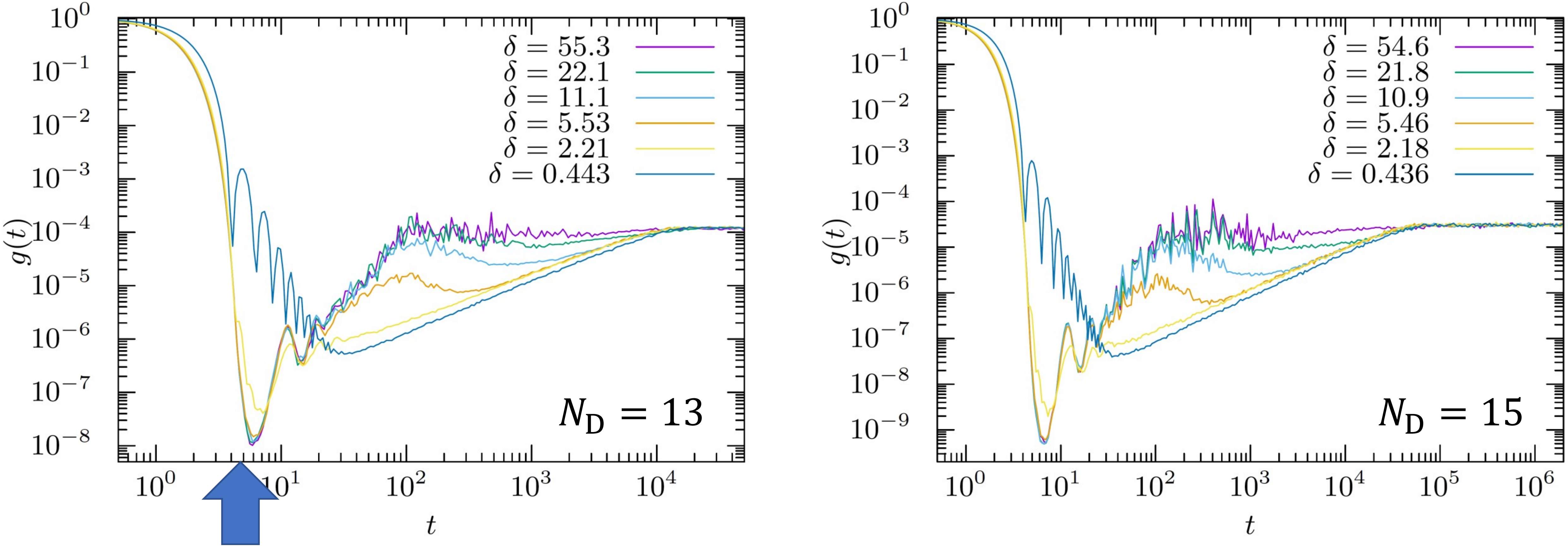
$$r = \frac{\min(E_{i+1} - E_i\,,\ E_{i+2} - E_{i+1})}{\max(E_{i+1} - E_i\,,\ E_{i+2} - E_{i+1})}$$

PRR 3, 013023 (2021)

Departure from random matrix $P(r)$ occurs **after** IPR (I_2) has grown significantly



SYK_{4+2}: spectral form factor

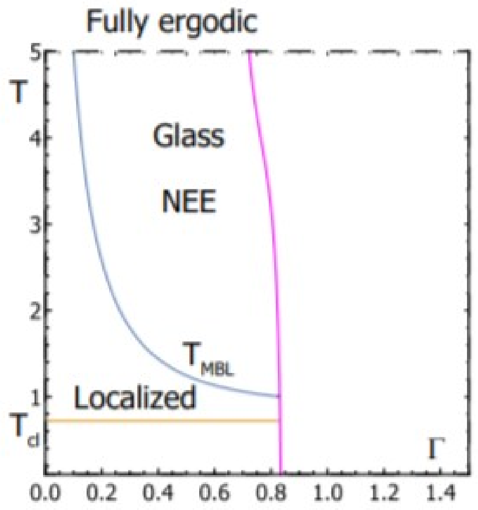


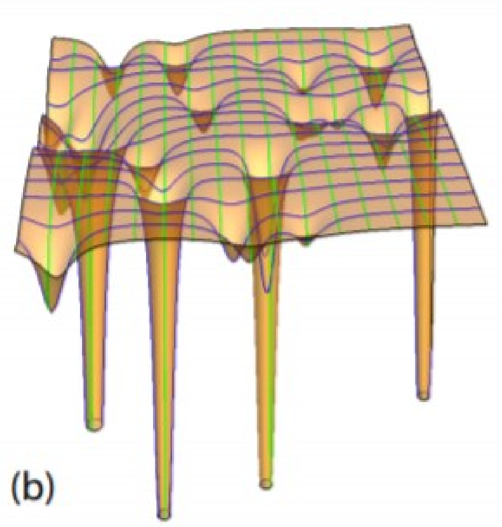
This dip (not directly followed by ramp) appears for SYK2 (+ uniform SYK4).
see 1812.04770 and 2003.05401 for detailed discussion

$\hat{H} = (\cos\theta)\hat{H}_{SYK_4} + (\sin\theta)\hat{H}_{SYK_2}, \delta = \tan\theta$
1.57×10^7 eigenvalues (1920 samples for $N_D = 13$)

Physics just outside MBL (regions II & III)?

- Thermal phase smoothly connected to extended states (as those in translationally invariant models)?
- Non-ergodic extended (NEE) states discussed for several models
  (Bethe lattice, random regular graphs, disordered Josephson junction chains, …)



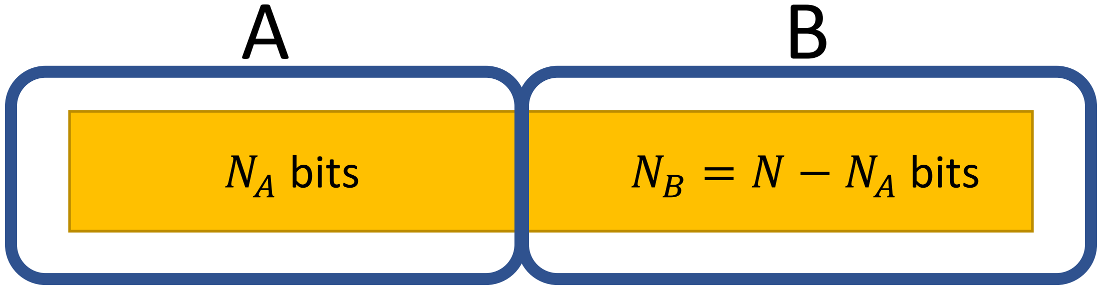


"golf course" potential energy landscape

"Non-ergodic extended phase of the Quantum Random Energy Model"
[L. Faoro, M. V. Feigel'man, L. Ioffe, Ann. Phys. **409**, 167916 (2019)]

F. Monteiro, MT, A. Altland, D. A. Huse, and T. Micklitz, PRL **127**, 030601 (2021)

Entanglement entropy for eigenstates



Zero-energy eigenstate $|\psi\rangle$, density matrix $\rho = |\psi\rangle\langle\psi|$

Reduced density matrix $\rho_A = \mathrm{tr}_B\rho$

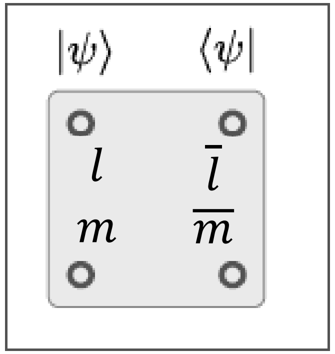
Entanglement entropy $S_A = -\mathrm{tr}_A(\rho_A \ln \rho_A)$

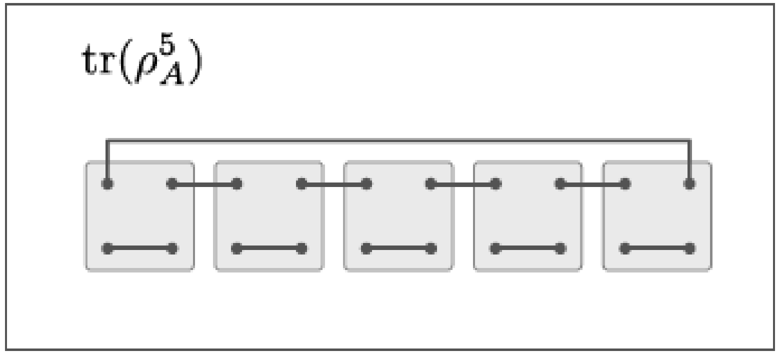
Replica method: Evaluate disorder averaged moments $M_r = \langle \mathrm{tr}_A(\rho_A^r)\rangle$, $S_A = -\partial_r M_r|_{r=1}$.

Fock space $\mathcal{F} = \mathcal{F}_A \otimes \mathcal{F}_B$

$$n = (l, m)$$

$$\mathcal{N} = (n^1, n^2, \dots, n^r), \mathcal{N}_A = (l^1, l^2, \dots, l^r), \mathcal{N}_B = (m^1, m^2, \dots, m^r)$$





$$\rho_A^r = \sum_{\substack{l^1,\dots,l^r \\ m^1,\dots,m^r}} \psi^{(l^1,m^1)}\,\overline{\psi}^{(l^2,m^1)}\,\psi^{(l^2,m^2)}\,\overline{\psi}^{(l^3,m^2)} \cdots \psi^{(l^r,m^r)}\,\overline{\psi}^{(l^1,m^r)}$$

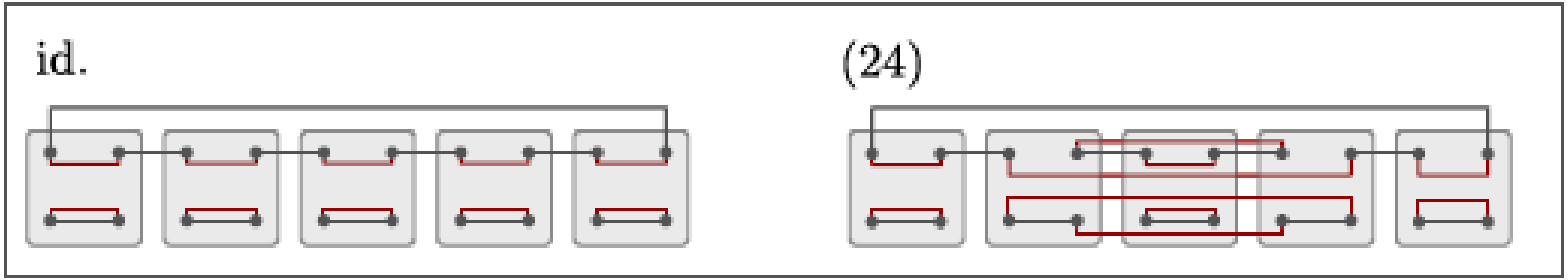
Evaluation of power of reduced density matrix

$$\rho_A^r = \sum_{\substack{l^1,\dots,l^r \\ m^1,\dots,m^r}} \underbrace{\psi^{(l^1,m^1)}}_{n^1} \underbrace{\overline{\psi}^{(l^2,m^1)}}_{\overline{n}^1} \underbrace{\psi^{(l^2,m^2)}}_{n^2} \underbrace{\overline{\psi}^{(l^3,m^2)}}_{\overline{n}^2} \cdots \psi^{(l^r,m^r)} \overline{\psi}^{(l^1,m^r)}$$

For this sum to survive disorder averaging,

$\mathcal{N} = (n^1, n^2, \dots, n^r)$ and $\overline{\mathcal{N}} = (\overline{n}^1, \overline{n}^2, \dots, \overline{n}^r)$ should be equal as sets,

$\mathcal{N}^i = \overline{\mathcal{N}}^{\sigma(i)}$



$n^1 = \overline{n}^1, n^2 = \overline{n}^2, n^3 = \overline{n}^3, n^4 = \overline{n}^4, n^5 = \overline{n}^5$

$n^1 = \overline{n}^1, \boldsymbol{n^2 = \overline{n}^4}, n^3 = \overline{n}^3, \boldsymbol{n^4 = \overline{n}^2}, n^5 = \overline{n}^5$

$$M_r = \langle \mathrm{tr}_A(\rho_A^r) \rangle = \sum_{\sigma} \sum_{\mathcal{N}} \prod_{i=1}^{r} \left\langle \left| \psi_{n^i} \right|^2 \right\rangle \delta_{\mathcal{N}_A, (\sigma\circ\tau)\mathcal{N}_A}\, \delta_{\mathcal{N}_B, \sigma\mathcal{N}_B}$$

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Regime I: maximally random case

$D_{A(B)} = 2^{N_{A(B)}-1}$

$M_r = \langle \mathrm{tr}_A(\rho_A^r) \rangle,\ S_A = -\partial_r M_r|_{r=1}$

Uniform distribution of wave functions, $\nu_n = \nu$

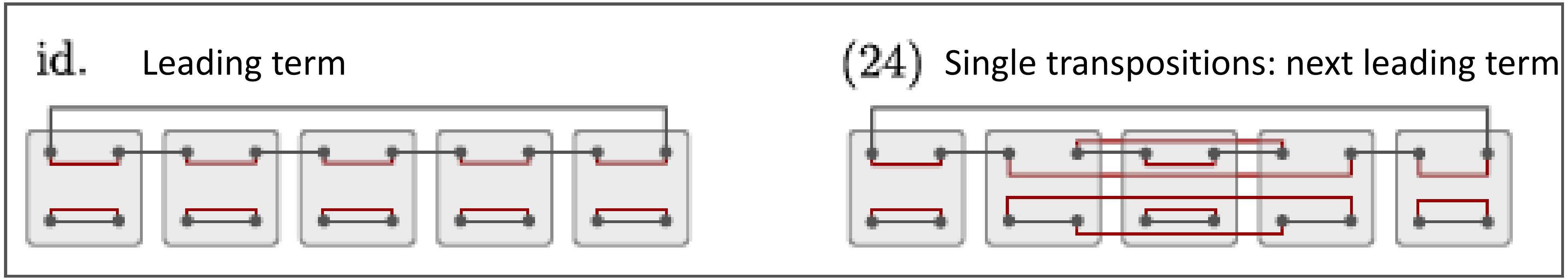
$$M_r \approx D_A^{1-r} + \binom{r}{2} D_A^{2-r} D_B^{-1}$$

Up to single transpositions

Difference from the thermal value $S_{\mathrm{th}} = \ln D_A$

$$S_A - S_{\mathrm{th}} = -\frac{D_A}{2D_B}$$

Exponentially small if $N_A \ll N_B$;
S_A very close to the thermal value



uniform

$$M_r = \langle \mathrm{tr}_A(\rho_A^r) \rangle = \sum_{\sigma} \sum_{\mathcal{N}} \prod_{i=1}^{r} \left\langle \left|\psi_{n^i}\right|^2 \right\rangle \delta_{\mathcal{N}_A,(\sigma\circ\tau)\mathcal{N}_A} \delta_{\mathcal{N}_B,\sigma\mathcal{N}_B}$$

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Regimes II and III: reduced effective dimension

- **Assume ergodicity** within **energy-shell** and calculate S_A
- Energy shell: extended cluster of resonant sites (width κ) embedded in the Fock space
- Neighboring sites of n: energy $v_m = v_n \pm \mathcal{O}(\delta)$, much more likely to be in the same shell because $\delta \ll \Delta_2 = \sqrt{N_\mathrm{D}}\delta$

Additional assumptions

- Exponentially large number of sites → self averaging (sum over site energies = average over approx. Gaussian distributed contributions of subsystem energies to the total energy)
- Total energy $E \sim E_A + E_B$

➔ Up to single transpositions (justified in $1 \ll N_A \ll N_\mathrm{D}$ & replica limit):

$$S_A - S_\mathrm{th} = -\frac{1}{2}\ln\left(\frac{N_\mathrm{D}}{N_B}\right) + \frac{N_A}{2N_D} - \sqrt{\frac{N_\mathrm{D}}{2N_A}\frac{D_A}{2D_B}}$$

in Regimes II, III ($\frac{1}{\sqrt{N_\mathrm{D}}} \ll \delta < \delta_\mathrm{c} \sim N_\mathrm{D}^2 \ln N_\mathrm{D}$)

$$S_A - S_\mathrm{th} = -\frac{D_A}{2D_B}$$

in Regime I

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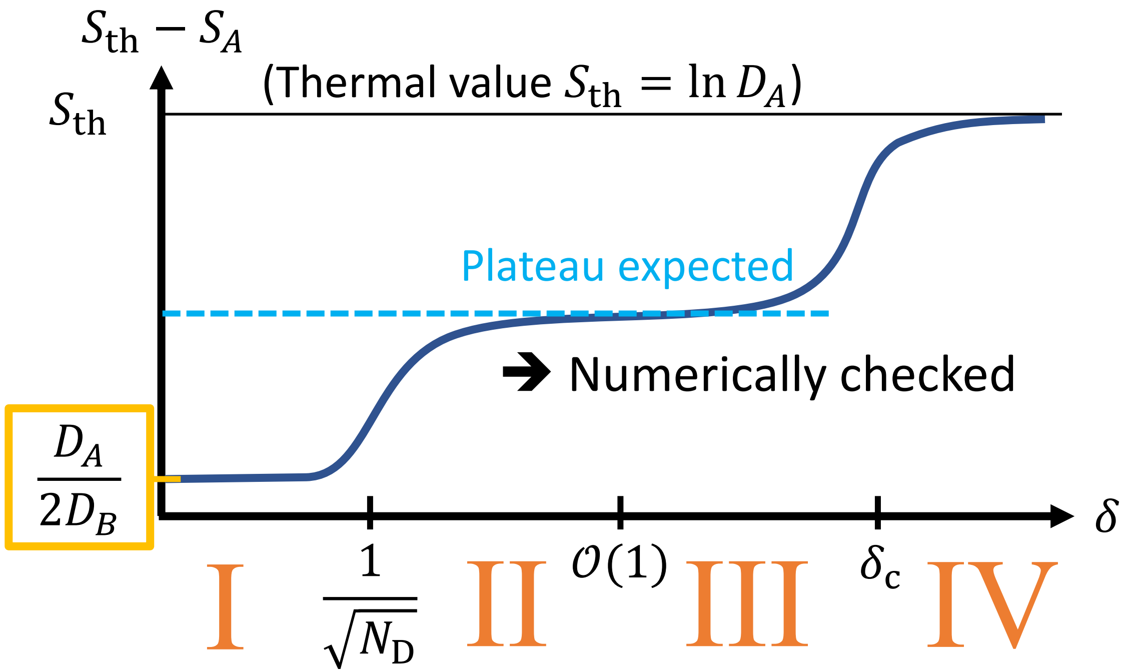
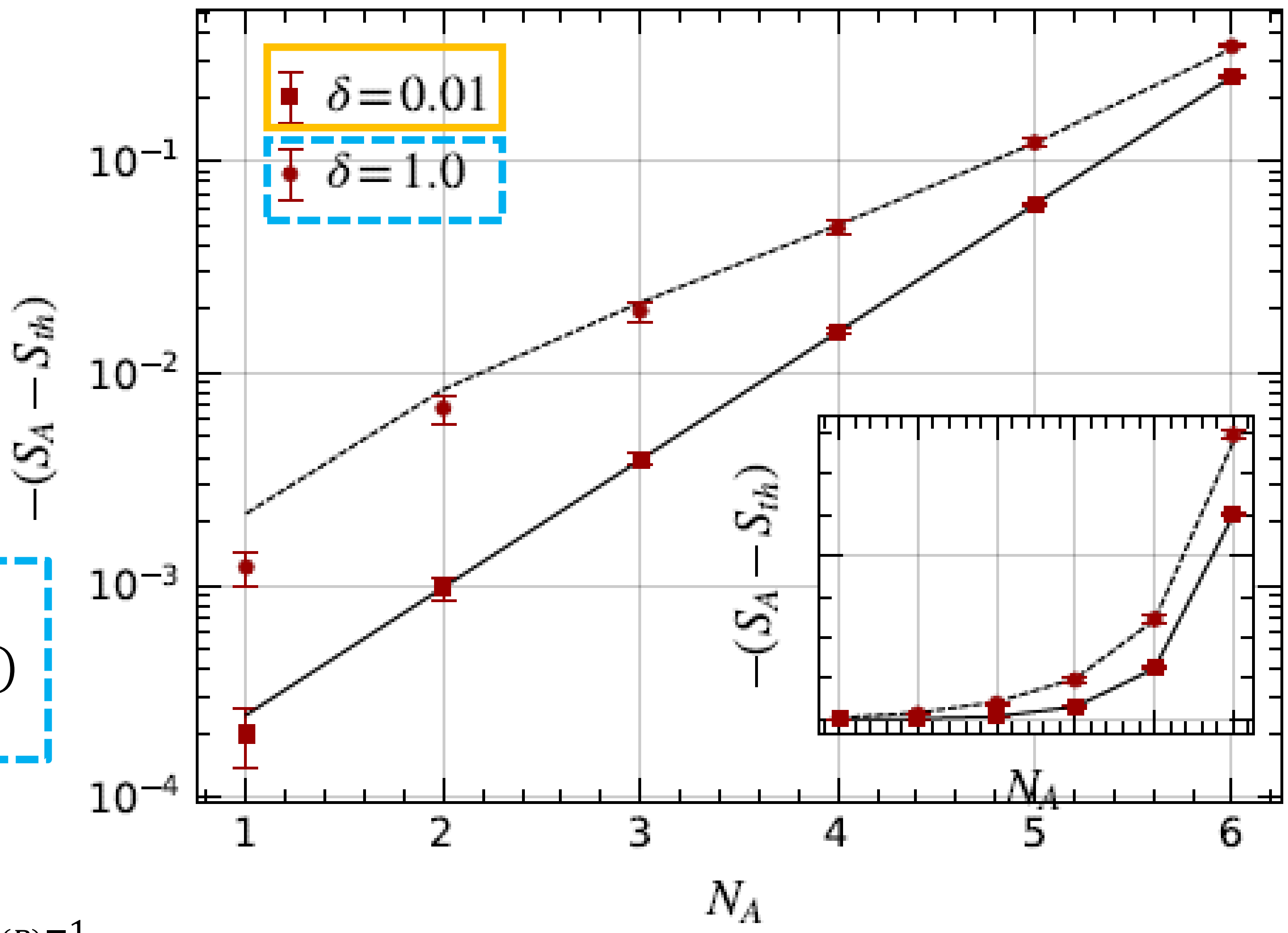
Offset from the thermal value



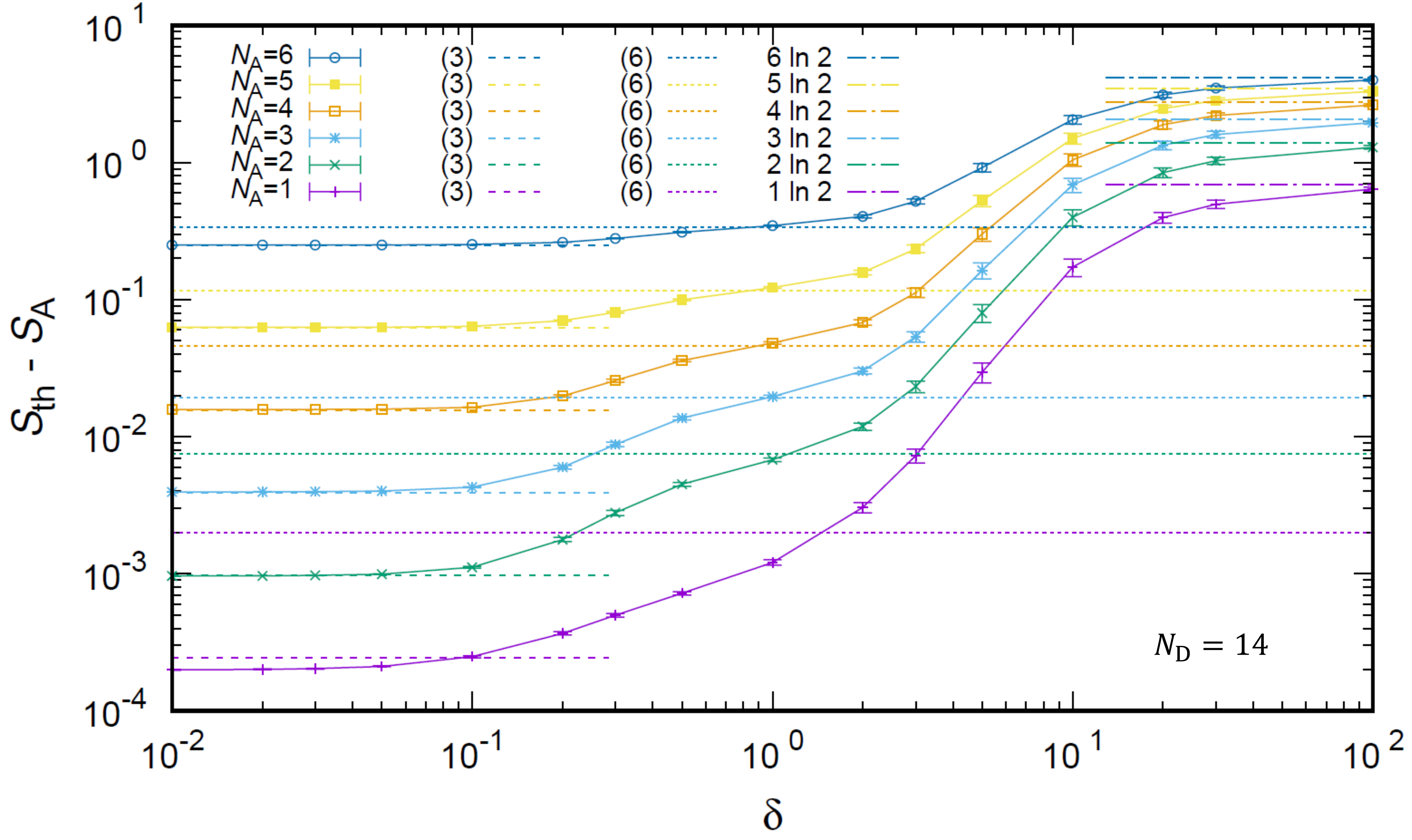
$N_{\text{D}} = 14$ ($N = 28$ Majorana fermions)



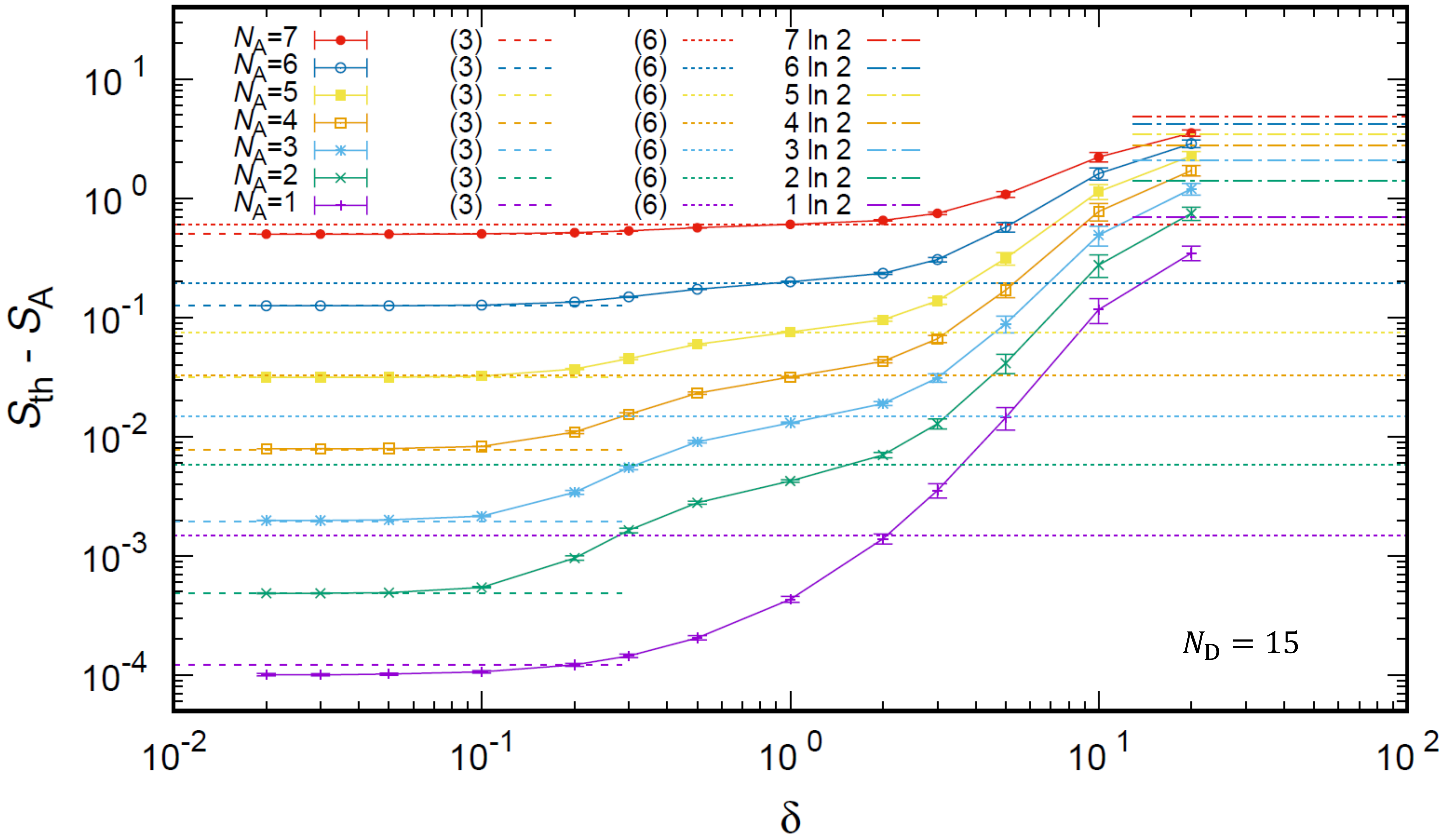
$$S_A - S_{\text{th}} = -\frac{1}{2}\ln\left(\frac{N_{\text{D}}}{N_B}\right) + \frac{N_A}{2N_D} - \sqrt{\frac{N_{\text{D}}}{2N_A}\frac{D_A}{2D_B}} \quad (< 0)$$

in Regimes II, III ($\frac{1}{\sqrt{N_{\text{D}}}} \ll \delta < \delta_{\text{c}} \sim N_{\text{D}}^2 \ln N_{\text{D}}$)

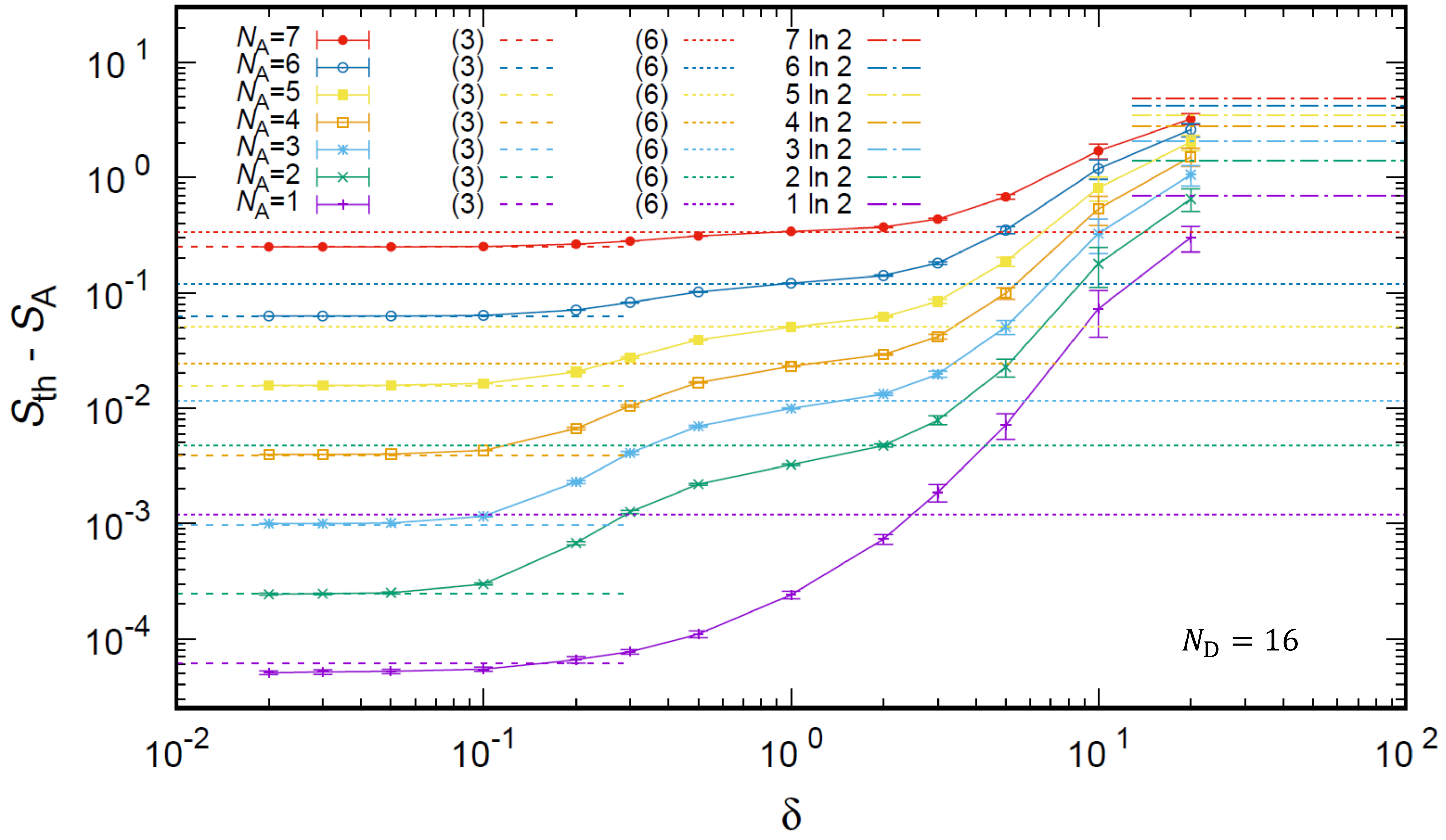
$D_{A(B)} = 2^{N_{A(B)}-1}$



N_A=6
N_A=5
N_A=4
N_A=3
N_A=2
N_A=1
(3)
(6)
6 ln 2
5 ln 2
4 ln 2
3 ln 2
2 ln 2
1 ln 2
S_{th} - S_A
δ
$N_D = 14$



N_A=7
N_A=6
N_A=5
N_A=4
N_A=3
N_A=2
N_A=1
(3)
(6)
7 ln 2
6 ln 2
5 ln 2
4 ln 2
3 ln 2
2 ln 2
1 ln 2
S_{th} - S_A
δ
$N_D = 15$



N_A=7
N_A=6
N_A=5
N_A=4
N_A=3
N_A=2
N_A=1
(3)
(6)
7 ln 2
6 ln 2
5 ln 2
4 ln 2
3 ln 2
2 ln 2
1 ln 2
S_{th} - S_A
δ
$N_D = 16$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
10^1
10^2

Slide to ask questions to other researchers

- Binary sparse SYK: useful?
  - Holography?
  - Quantum computation?
  - Quantum cryptography?
- SFF vs OTOC
- SFF in bosonic systems
- Open quantum many-body systems

Summary

- Sachdev-Ye-Kitaev model
  - $N \bmod 8$ periodicity for Majorana SYK_4
- **Spectral form factor**
  - Exponentially long ramp in SYK [JHEP 1705(2017)118]
- **Sparse SYK**
  - Binary sparse SYK: SFF $\approx$ dense SYK with $\sim 4N$ (± 1) couplings [arXiv:2208.12098 (Phys. Rev. B Letter in press)]
- Chaotic-integrable transition in SYK4+2
  - SYK_2 term suppresses chaotic behavior [PRL **120**, 241603 (2018)]
- Quantitative analysis of Fock-space localization in SYK4+2
  - Many-body transition point and inverse participation ratio [Phys. Rev. Research **3**, 013023 (2021)]
  - **Entanglement entropy**
    - Plateau consistent with ergodicity within energy shells [Phys. Rev. Lett. **127**, 030601 (2021)]