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Pathology in WKB wave function for tunneling assisted by gravity

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There are several exotic tunneling processes that can be realized only by incorporating the effect of gravity.
Here, we point out that we encounter difficulties in constructing WKB wave function, once we try to describe quantum fluctuations around the dominant escape path.
We present examples of pathology in

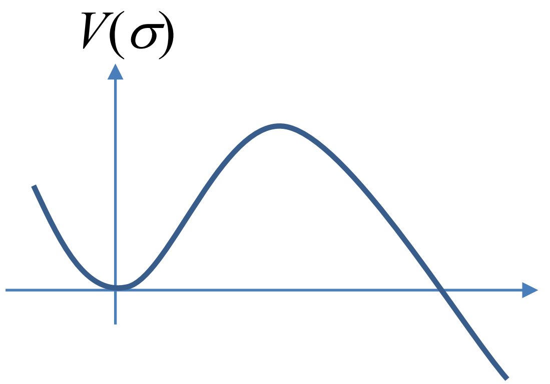
- the true vacuum decay/upward tunneling,
- the false vacuum decay catalyzed by a black hole and
- the tunneling with black hole/wormhole production.

WKB wave function

We start with discussion 2 degrees of freedom system.

$$H = \frac{1}{2}\pi_\sigma^2 + U(\sigma) + \frac{1}{2}\pi_\phi^2 + \frac{1}{2}m^2(\sigma)\phi^2$$

$$H|\Psi\rangle = E|\Psi\rangle$$



WKB wave function neglecting ϕ.

$$\Psi_0(\sigma) = \langle \sigma | \Psi \rangle = \frac{1}{\sqrt{\pi_\sigma(\sigma)}} \exp\left[-\int^\sigma \pi_\sigma(\sigma') d\sigma' \right]$$

$$\pi_\sigma(\sigma) = \sqrt{2(U(\sigma) - E_0)}$$

$$\frac{d^2\sigma}{d\tau^2} = \frac{dU(\sigma)}{d\sigma}$$

⇒ Solution of Euclidean EOM connecting the initial and final states (Instanton) ⇔ WKB wave function for σ

$$\tau = \int^\sigma \frac{d\sigma'}{\sqrt{2(U(\sigma') - E_0)}}$$

⇒ We can parametrize the tunneling trajectory (=dominant escape path) by τ instead of σ.

Now, we take into account ϕ.

$$\Psi = \Psi_0(\sigma)\Phi(\phi, \tau(\sigma))$$

Gaussian ansatz for ϕ:

$$\Phi(\phi,\tau) = \langle \sigma(\tau), \phi | \Psi \rangle = \frac{e^{\delta E \tau}}{\sqrt{K(\tau)}} \exp\left[-\frac{1}{2} \frac{d \log K}{d\tau} \phi^2 \right]$$

$$\left[\frac{d^2}{d\tau^2} + m^2(\sigma(\tau)) \right] K = 0$$

K is the growing mode function.

$$\frac{d \log K}{d\tau} \to m(0) \quad \text{for} \quad \tau \to -\infty \qquad \Rightarrow \qquad \delta E := E - E_0 = m(0)$$

Extension to the field theory

$$\Rightarrow \qquad \Phi(\phi(\mathbf{x}),\tau) \approx \exp\left[-\frac{1}{2} \int d^3x\, d^3y \frac{\sqrt{\gamma(\mathbf{x},\tau)}}{N(\mathbf{x},\tau)} \sum_i \frac{dK_i(\mathbf{x},\tau)}{d\tau} K_i^{-1}(\mathbf{y},\tau) \phi(\mathbf{x}) \phi(\mathbf{y}) \right]$$

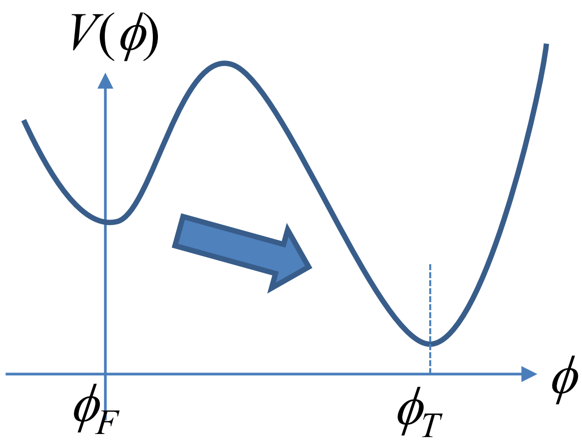
$$\phi(\mathbf{x}) = \sum_j K_i(\mathbf{x},\tau) \phi_j$$

$$\Phi(\phi(x),\tau) \approx \exp\left[-\frac{1}{2} \int d^3x \frac{\sqrt{\gamma(\mathbf{x},\tau)}}{N(\mathbf{x},\tau)} \sum_{i,j} \frac{dK_i(\mathbf{x},\tau)}{d\tau} K_j(\mathbf{x},\tau) \phi_i \phi_j \right]$$

$K_i(\mathbf{x},\tau) = 0$ or $\dfrac{dK_i(\mathbf{x},\tau)}{d\tau} = 0$ implies breakdown of WKB wave function.

False vacuum decay with gravity

$$S_E=\int d^4x\sqrt{g}\left[-\frac{1}{2\kappa}R+\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+V(\phi)\right]$$

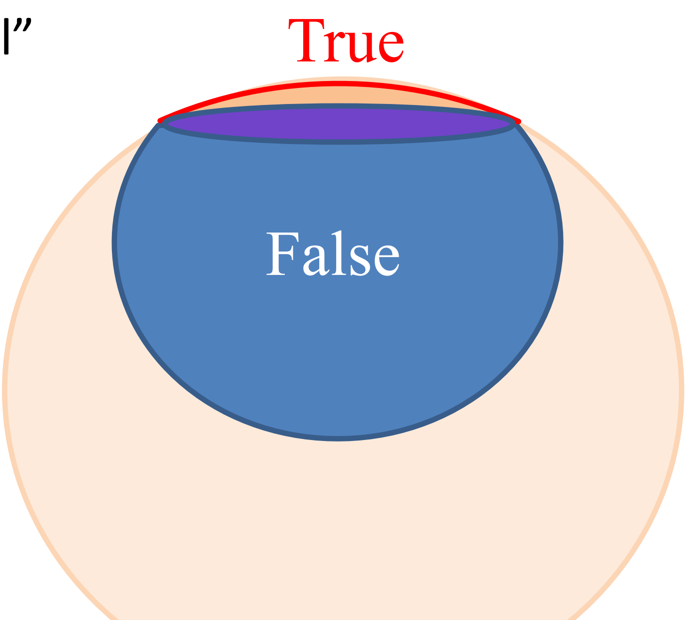


$$ds^2=N^2d\tau^2+a^2\left(d\chi^2+\sin^2\chi\left(d\theta^2+\sin^2\theta\,d\varphi^2\right)\right)$$

$$\begin{cases}\ddot{\phi}+\left(3\frac{\dot{a}}{a}-\frac{\dot{N}}{N}\right)\dot{\phi}=N^2\frac{dV}{d\phi}\\ \dot{a}^2-\frac{\kappa}{6}a^2\dot{\phi}^2=N^2\left(1-\frac{\kappa}{3}a^2V(\phi)\right)\end{cases}$$

Thin wall approximation

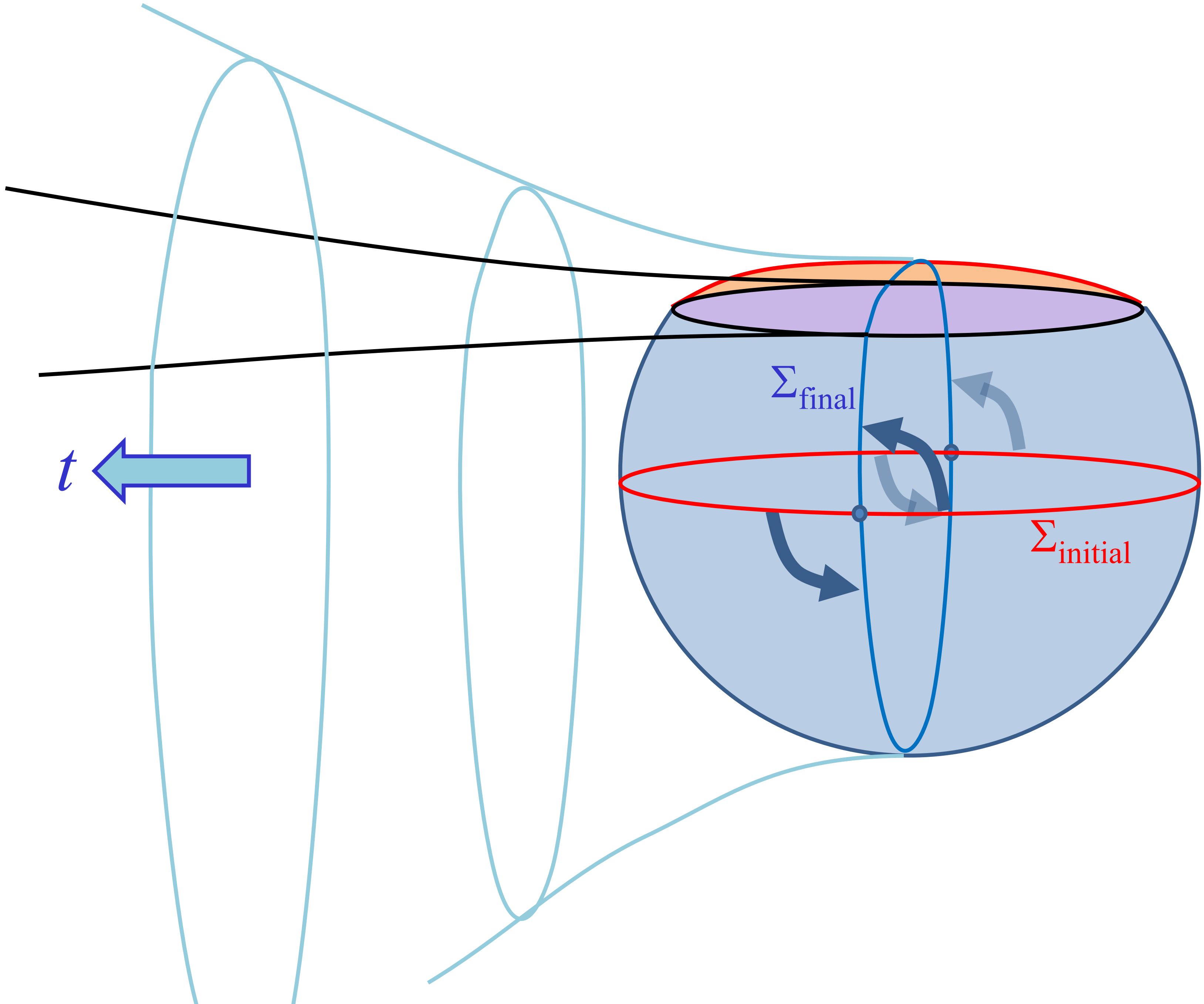
= "ϕ changes only in an infinitesimally thin wall"



$$\left(\frac{1}{N}\frac{da}{d\tau}\right)^2=1-a^2\hat{H}_\alpha^2 \qquad \hat{H}_\alpha^2:=\frac{\kappa V(\phi_\alpha)}{3}$$

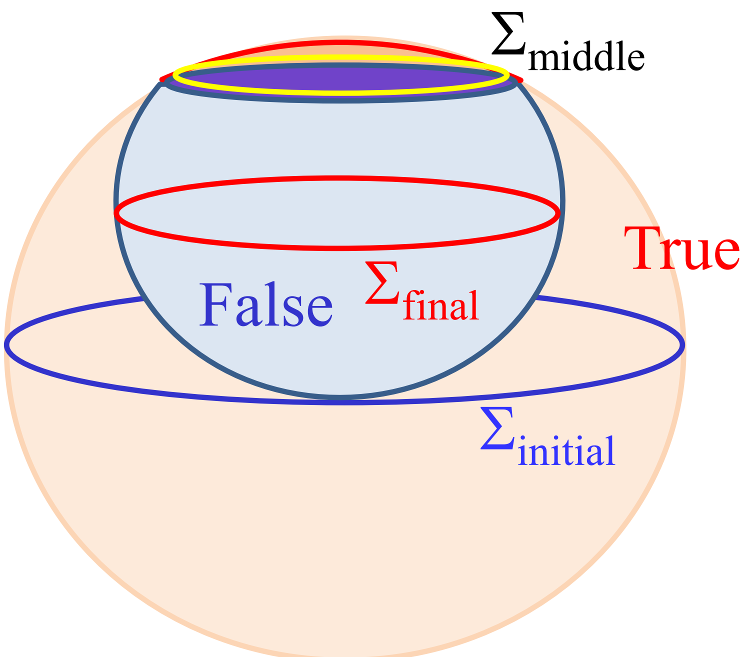
$$a(\tau)=a_\alpha(\tau):=\hat{H}_\alpha^{-1}\sin\left(\hat{H}_\alpha\int^\tau N(\tau')d\tau'\right)$$

$$\dot{\phi}(\tau_{\rm i})=\dot{\phi}(\tau_{\rm f})=0$$



Σ_{final}
Σ_{initial}
t

Simplified version of upward tunneling



Pure de Sitter in true vacuum
⇒ pure de Sitter in false vacuum without formation of bubble

Instanton is not given as a single manifold.
Two different manifolds are switched when the time slice reaches Σ_{middle}.

Breakdown of the WKB approximation

Perturbation mode equation, assuming a massless scalar field here.

$$\ddot{K}_L + 3H_\alpha \dot{K}_L - \frac{L(L+2)}{a_\alpha^2} K_L = 0 \qquad H_\alpha = -\sqrt{\frac{1}{a_\alpha^2} - \hat{H}_\alpha^2} \qquad \hat{H}_\alpha^2 := \frac{\kappa V(\phi_\alpha)}{3}$$

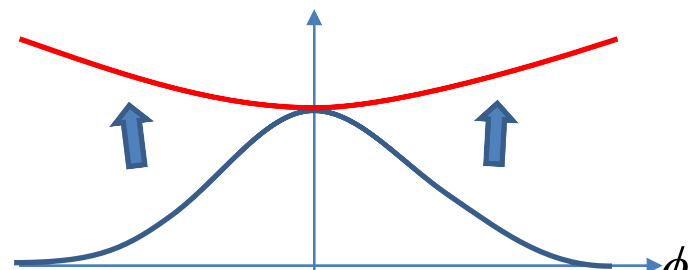
Assuming, $K_L \sim \exp(\lambda\tau)$, the equation around the wall becomes

$$\Rightarrow \quad \lambda^2 + 3H_{w\alpha}\lambda - p^2 \approx 0 \quad \Rightarrow \quad \lambda_{\alpha\pm} = \frac{-3H_{w\alpha} \pm \sqrt{9H_{w\alpha}^2 + 4p^2}}{2} = \pm p - \frac{3}{2} H_{w\alpha}$$

$$p^2 := \frac{L(L+2)}{a_{w\alpha}^2}$$

After passing Σ_{middle}, we connect the mode function at the wall.

Wave function becomes unnormalizable



$$\begin{cases} K_{L,T} = \exp(\lambda_{T+}\tau) \\ K_{L,F} = c_1 \exp(\lambda_{F+}\tau) + c_2 \exp(\lambda_{F-}\tau) \end{cases}$$

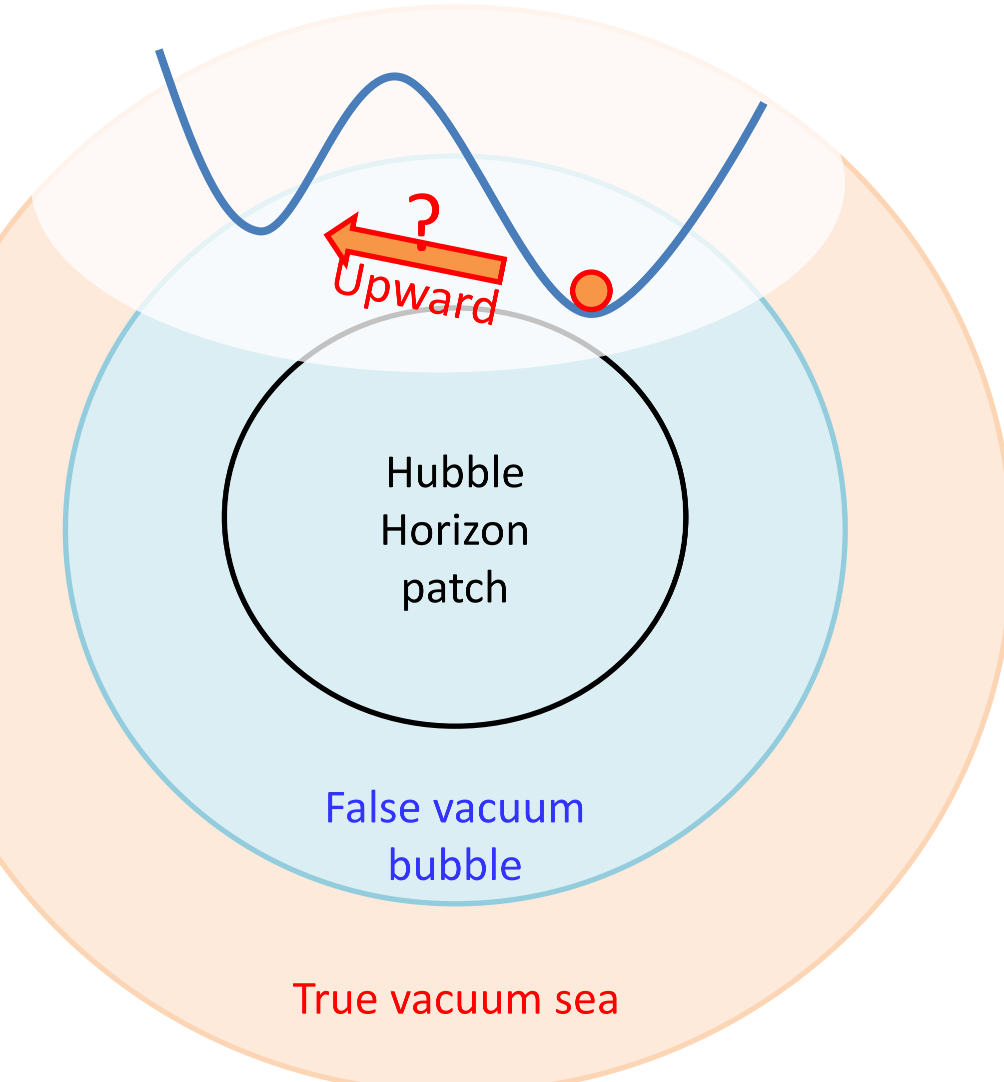
$$\Rightarrow \begin{cases} 1 = c_1 + c_2 \\ \lambda_{T+} = c_1\lambda_{F+} + c_2\lambda_{F-} \end{cases}$$

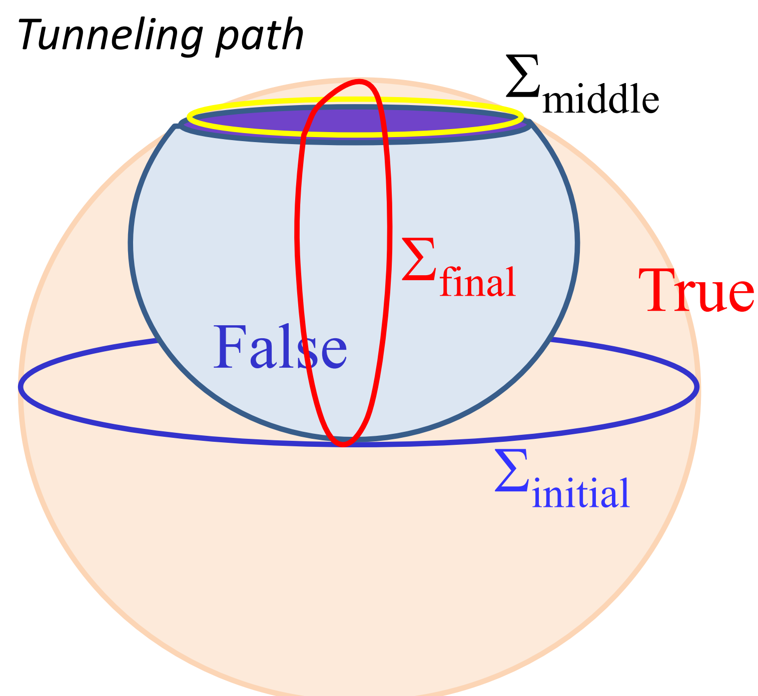
dK_L vanishes at $\tau = \tau_c$.

$$\Rightarrow \quad c_2 \approx \frac{3}{4p}(H_{wT} - H_{wF}) \quad \Rightarrow \quad \tau_c = \frac{1}{\lambda_{F+} - \lambda_{F-}} \log\left(-\frac{c_2}{c_1}\right) \approx \frac{1}{2p} \log\left[\frac{3}{4p}(H_{wF} - H_{wT})\right] < 0$$

Other pathological examples

Upward tunneling





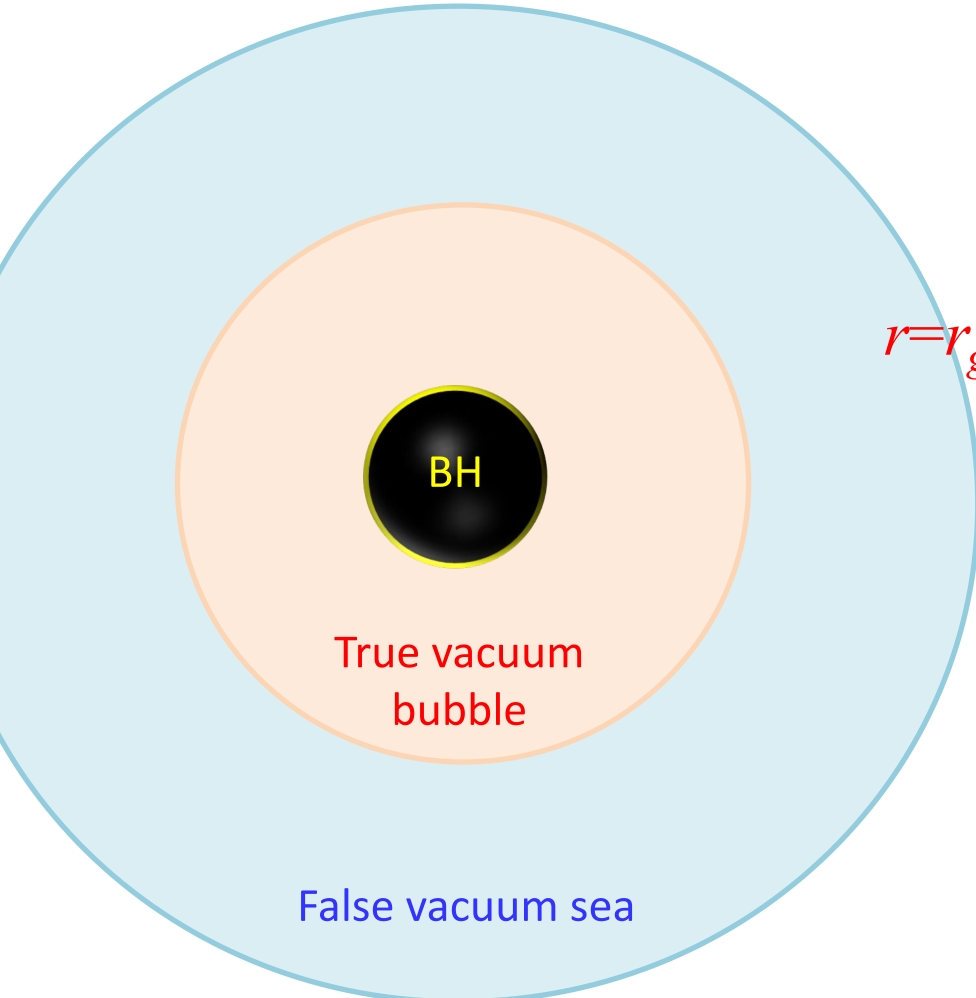
This process also requires the flip of the sign of the lapse function.

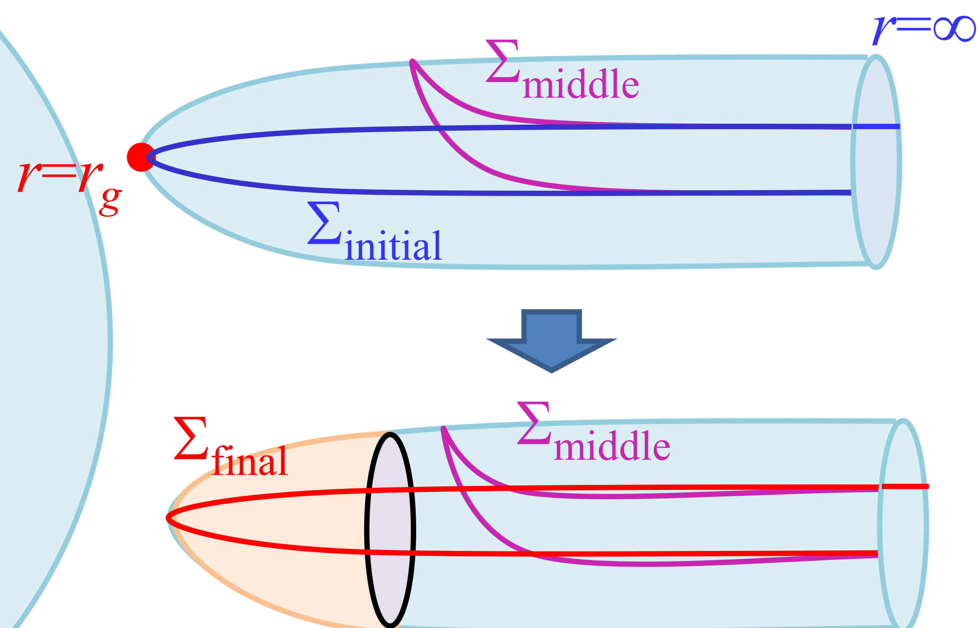
Other pathological examples

False vacuum decay catalyzed by a black hole

Tunneling path

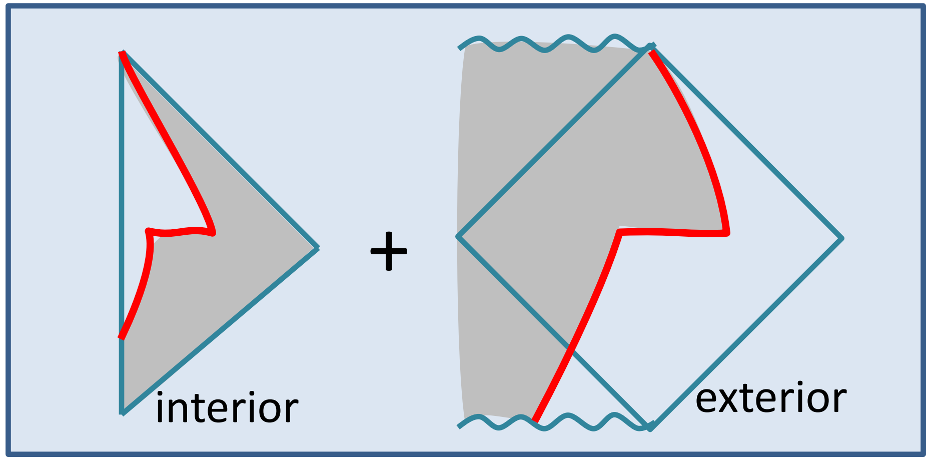
$$ds^2 = \left(1 - \frac{r_g}{r}\right) d\tau^2 + \left(\frac{r}{r - r_g}\right) dr^2 + r^2 d\Omega^2$$

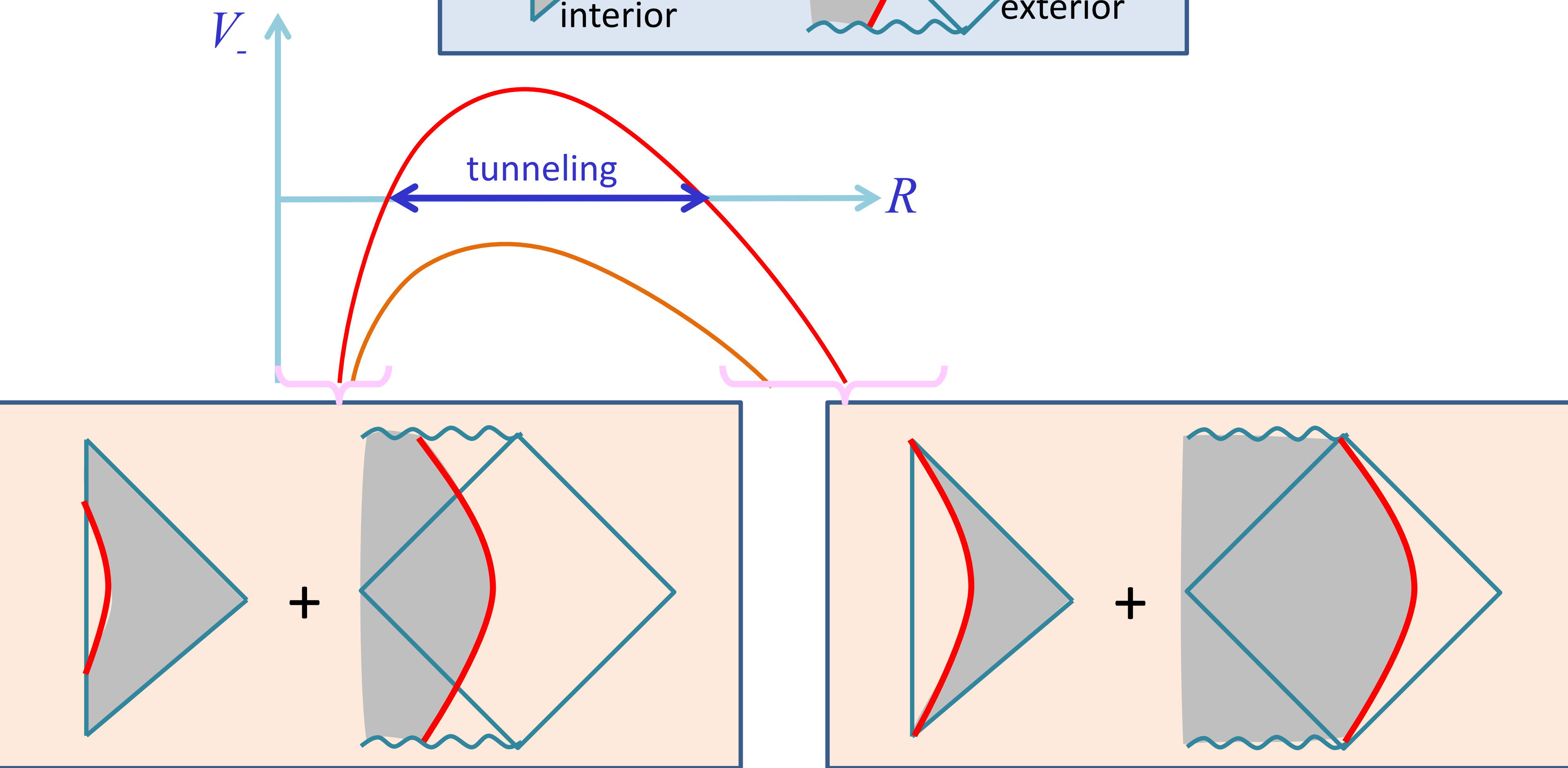




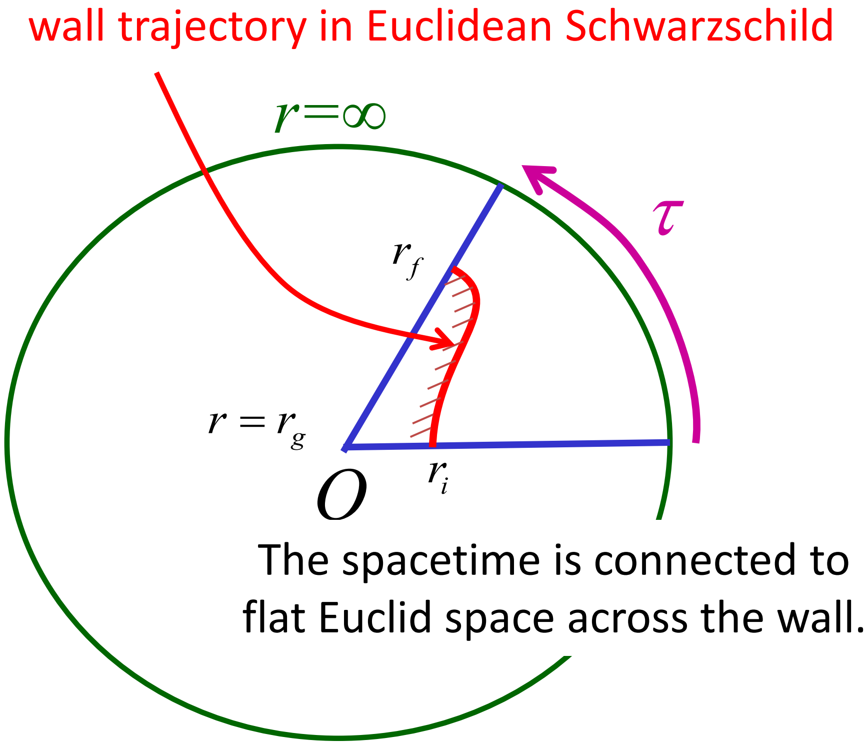
Tunneling of spherical bubble wall

Standard process:
Small bubble branch
→ large bubble branch

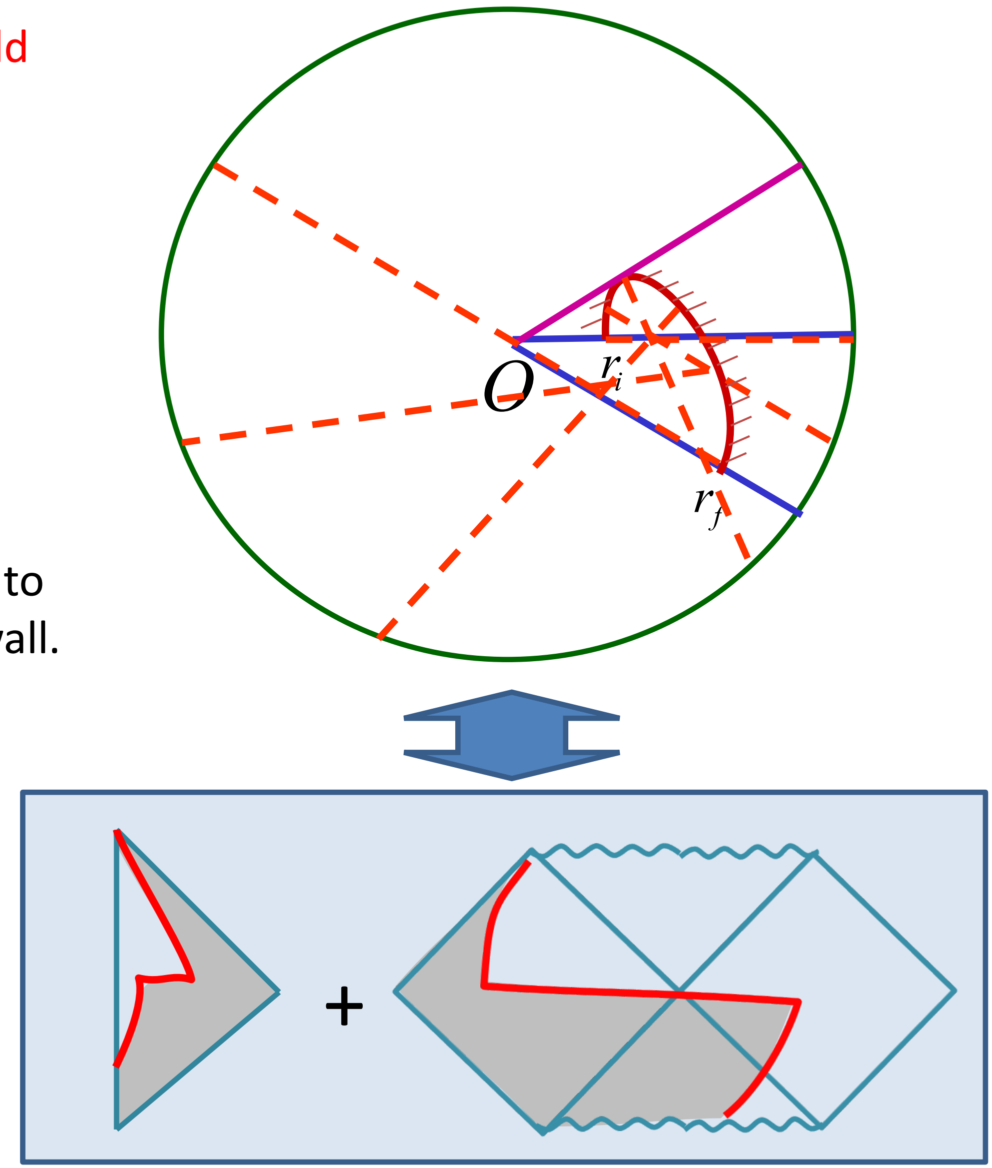


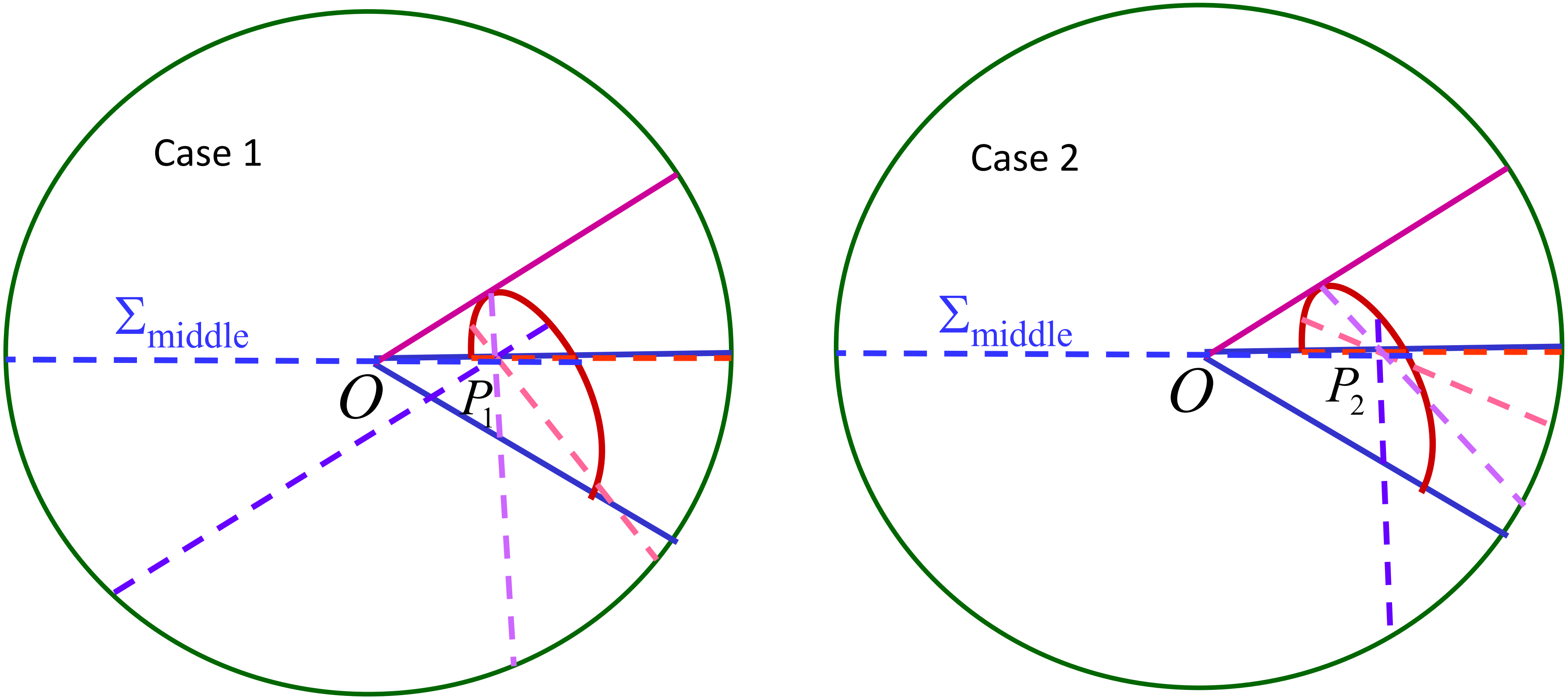


Normal tunneling



Wormhole generation





We focus on the segment between P_1 and P_2 on $\Sigma_{\rm middle}$

Case 1: mode functions should be growing in the anti-clockwise direction.

Case 2: mode functions should be growing in the clockwise direction.

→ The mode functions K_i depend on the foliation.

Summary

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