Creases, corners and caustics: non-smooth structures on horizons

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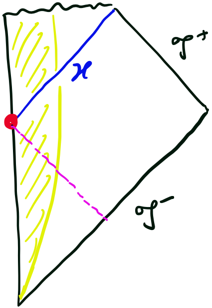
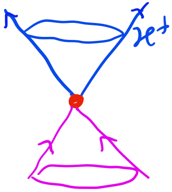
Introduction

Every point of an event horizon $\mathcal{H}$ belongs to a null geodesic that lies within $\mathcal{H}$. These geodesics are the *generators* of $\mathcal{H}$.

A generator cannot have a future endpoint, i.e., it cannot leave $\mathcal{H}$ to the future.

Generators can have *past* endpoints:





Horizon non-smoothness

We assume that spacetime is smooth.

Theorem: $\mathcal{H}$ is an achronal continuous hypersurface.
(Achronal: no two points of $\mathcal{H}$ are timelike separated.)

$\mathcal{H}$ is *not smooth* except in very special cases e.g. a time-independent black hole.

What is the nature of the non-smoothness of $\mathcal{H}$?

There exist examples of spacetimes for which $\mathcal{H}$ is non-differentiable on a dense set (Chrusciel & Galloway 96) but these are "very artificial"

Theorem (Beem & Krolak 97):

- $\mathcal{H}$ is differentiable at p iff p lies on *exactly one* generator
- A point lying on more than one generator is an endpoint (converse untrue)

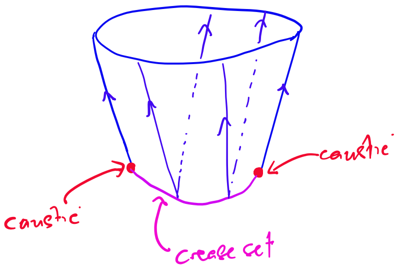
Let $\mathcal{H}_{\mathrm{end}}$ be the set of past endpoints of horizon generators.

Siino & Koike 04: classification of points of $\mathcal{H}_{\mathrm{end}}$ assuming a particular definition of *genericity*

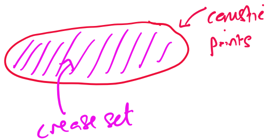
In explicit examples of gravitational collapse or black hole mergers, $\mathcal{H}_{\rm end}$ consists of a 2d spacelike *crease set* where *pairs* of generators enter $\mathcal{H}$, together with its boundary, which is a line of *caustic points* (where "infinitesimally nearby generators intersect")

(Hughes *et al* 94, Shapiro *et al* 95, Lehner *et al* 99, Husa & Winicour '99, Hamerly & Chen 10, Cohen *et al* 11, Emparan & Martinez 16, Bohn *et al* 16, Emparan *et al* 17)

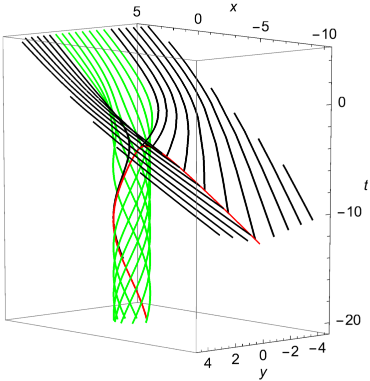
In 2+1 dimensions:

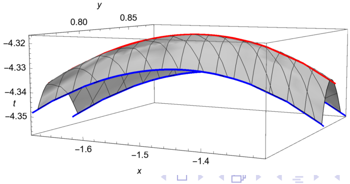


Asymmetric gravitational collapse in 3+1 dimensions:



Non-axisymmetric black hole merger (Emparan *et al* 17):





What features of these spacetimes lead to this simple structure for $\mathcal{H}_{\rm end}$?

What other structures are possible?

Assumptions

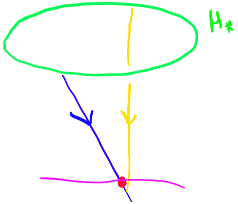
- Spacetime is *globally hyperbolic*
- $\mathcal{H}$ is *smooth at late time*: there exists a Cauchy surface Σ to the future of $\mathcal{H}_{\rm end}$ such that $H_\star \equiv \Sigma \cap \mathcal{H}$ is smooth

(No assumptions about equations of motion.)

We show that $\mathcal{H}_{\rm end}$ is the past *null cut locus* of $H_\star$.

Null cut locus

A null geodesic emitted orthogonally to $H_\star$ cannot be deformed to a timelike curve from $H_\star$ *locally*. A *null cut point* is the first point along a such a null geodesic beyond which it can be deformed into a timelike curve. The *null cut locus* of $H_\star$ is the set of all null cut points.



In Riemannian geometry a cut locus can be very complicated (e.g. fractal). But it can be decomposed into parts with simpler structure (Itoh & Tanaka 1998). We obtained a Lorentzian analogue of this decomposition.

Any point in a null cut locus is either a caustic or lies on at least 2 generators (Beem & Ehrlich 81, Kemp 84, Kupeli 85). So we can classify points of $\mathcal{H}_{\rm end}$ as follows:

- caustic points
- non-caustic points
  - normal crease points: lie on exactly 2 generators
  - normal corner points: lie on exactly 3 generators
  - points on ≥ 4 generators

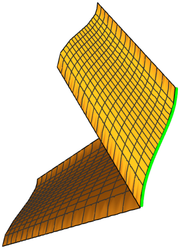
We prove:

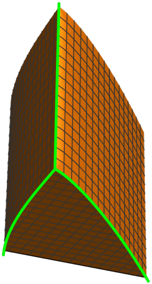
- (a) Normal crease points form a 2d spacelike *crease submanifold*
- (b) Normal corner points form a 1d spacelike *corner submanifold*
- (c) All other points form a set of (Hausdorff) dimension ≤ 1

Creases and corners

Normal crease points form a 2d spacelike *crease submanifold*.
Normal corner points form a 1d spacelike *corner submanifold*

Consider $\Sigma \cap \mathcal{H}$ for some Cauchy surface Σ. Creases are lines at which two smooth sections of horizon meet. Corners are points at which three smooth sections of horizon meet.

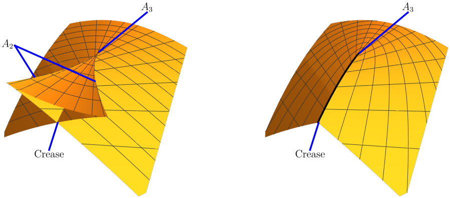




Application: Bousso entropy bound (99)

A *lightsheet* is a non-expanding null hypersurface ending at caustic set. Consider entropy S crossing lightsheet emanating orthogonally from a 2d spacelike surface Σ of area A. Conjecture: $S \leq A/4G\hbar$.

Proofs: (a) for matter possessing a local entropy current obeying reasonable conditions (Flanagan, Marolf & Wald 99); (b) for quantum fields (Bousso *et al* 14)



Terminate lightsheet at null cut locus of Σ? (Tavakol & Ellis 99). Our results apply to a general null cut locus. Combining with the above proofs gives

$$S \leq (A - 2A_{\text{crease}})/4G\hbar$$

.

Perestroikas

Let τ be a time function and Σ_τ denote a Cauchy surface of constant τ

$\Sigma_\tau \cap \mathcal{H}$ is the "horizon at time τ". This will have some arrangement of creases, corners and caustics.

As τ varies, this arrangement may undergo a qualitative change at a critical value of τ. We call this a *perestroika* (restructuring).

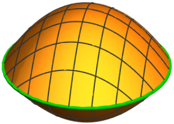
A crease perestroika occurs at a time τ for which Σ_τ is tangent to the crease submanifold.

Near the point of tangency, $\mathcal{H}$ is (part of) the union of two intersecting null hypersurfaces. By introducing Riemannian normal coordinates around this point we can determine the exact local behaviour of $\mathcal{H}$.

There are three qualitatively different possibilities. Shift τ so that perestroika occurs at $\tau = 0$.

Flying saucer

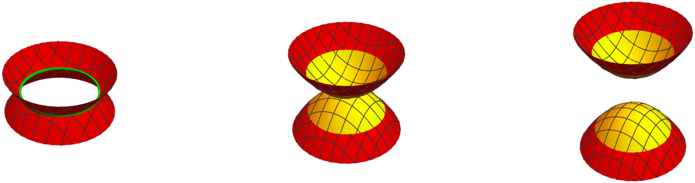
This perestroika describes the nucleation of a component of $\mathcal{H}$ in generic gravitational collapse



Length of crease and angle at crease scale as $\sqrt{\tau}$, area scales as τ

Collapse of hole in horizon

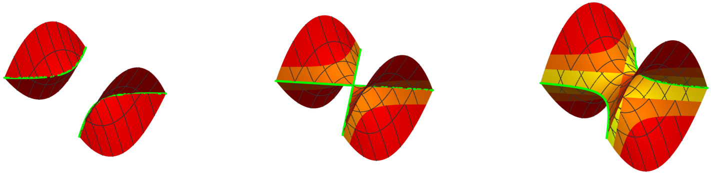
In examples of gravitational collapse or a black hole merger, some choices of time function give a brief period where horizon has toroidal topology (Hughes *et al* 94, Siino 97, Cohen *et al* 11, Bohn *et al* 16). The "hole in the torus" collapses superluminally. The collapse is described by a perestroika:



Length of crease and angle at crease scale as $\sqrt{-\tau}$.

Black hole merger

This perestroika describes the merger of two (locally) disconnected sections of horizon e.g. two merging black holes.



$\tau < 0$ $\tau = 0$ $\tau > 0$

Angle at creases scales as $\sqrt{|\tau|}$

Genericity/stability

Which features of $\mathcal{H}_{\rm end}$ are stable under small perturbations?

e.g. spherically symmetric collapse: $\mathcal{H}_{\rm end}$ is a single (caustic) point. If we perturb spacetime then non-trivial crease submanifold is present, so original structure of $\mathcal{H}_{\rm end}$ is unstable/non-generic.

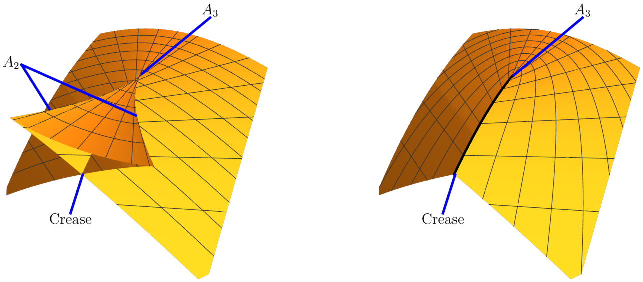
Siino & Koike 04: classification of points of $\mathcal{H}_{\rm end}$ assuming a particular mathematical notion of genericity

- Non-caustic points of double, triple, quadruple self-intersection of $\mathcal{H}$
- Lines of caustic points "of type A_3"

But: how to relate this notion of genericity to genericity w.r.t. perturbations of metric?

Generic caustic point: A_3

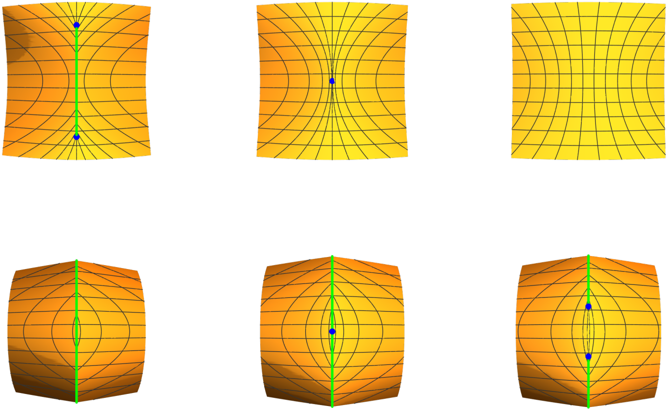
A_3 caustic points form spacelike lines. A horizon cross-section generically has isolated A_3 caustic points. If we extend generators beyond their past endpoints we obtain the swallowtail:



Why can't an A_2 caustic occur on $\mathcal{H}$? Would violate achronality!

A_3 perestroikas

Occur when Σ_τ is tangent to A_3 line.



$\tau < 0$ $\tau = 0$ $\tau > 0$

Crease contribution to black hole entropy

Old idea: some/all of black hole entropy is entanglement entropy of quantum fields across horizon (Bombelli *et al* 86, Srednicki 93, Susskind & Uglum 94). Flat space entanglement entropy exhibits novel features in the presence of a crease (Casini & Huerta 06, Hirata & Takayanagi 06, Klebanov *et al* 12, Myers & Singh 12)

Suggests that a crease might contribute to black hole entropy as

$$\frac{1}{\sqrt{G\hbar}}\int_{\mathrm{crease}} f(\Omega)dl$$

where Ω is angle at crease and $f \propto 1/\Omega$ as $\Omega \to 0$. This is subleading compared to Bekenstein-Hawking entropy $A/4G\hbar$: how to test this idea?

Consider "hole in the horizon" perestroika: this term remains finite and non-zero as $\tau \to 0-$, so discontinuous at $\tau = 0$. Consistency with second law implies that residue of f at $\Omega = 0$ must be non-positive.

Gauss-Bonnet term in entropy

A "Gauss-Bonnet" term in gravitational action is topological in 4d but contributes to black hole entropy (Jacobson & Myers 93, Iyer & Wald 94)

$$S_{\rm GB} = \gamma \int_H d^2x \sqrt{\mu} R[\mu]$$

On smooth horizon $S_{\rm GB} = 4\pi\gamma\chi$ where χ is Euler number of H.

For non-smooth horizon, "regulate" $S_{\rm GB}$, defining via a limit of smooth surfaces to obtain same result. $S_{\rm GB}$ is discontinuous in black hole formation or merger, so only $\gamma = 0$ is consistent with 2nd law (Sarkar & Wall 11)

But: does $S_{\rm GB}$ actually need regulating? No: integral is well-defined for creases, corners and A_3 caustics. No longer topological, continuous in black hole formation/merger.

Still find $\gamma = 0$ if no "higher order" terms in entropy but γ unconstrained if such (EFT) terms are present.

Summary

We've studied the endpoint set of the generators of a horizon that is smooth at late time

Structure of endpoint set: crease submanifold, corner submanifold, 1d set (similarly for a general null cut locus). Examples of corners? Applications of crease submanifold?

Perestroikas describe qualitative changes in the structure of the horizon as a function of time: crease perestroikas describe black hole nucleation and black hole merger; caustic perestroikas describe decay of crease via (dis)appearance of pairs of caustic points

There are open issues concerning the correct notion of genericity for caustics and the classification of caustics in curved spacetime

Creases may contribute to black hole entropy. 4d Gauss-Bonnet term may not be dead.