Quantum fault-tolerance and locality

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Theory of Complex Quantum Systems

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Talk on joint work with

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arXiv:2402.13863

Colloquium of the Extreme Universe Collaboration
April 19, 2024

Quantum computing long-term vision

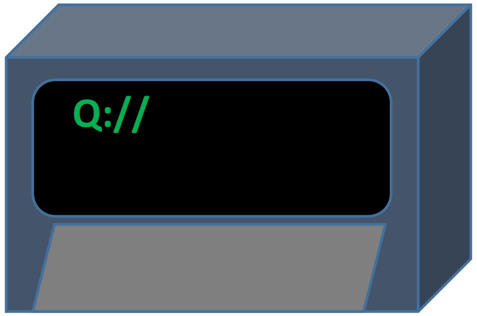
Molecular structure and material science simulations

Quantum walks

Integer factoring

Universal Fault-Tolerant Quantum Computer

Semidefinite programming



Grover search

HHL algorithm
$Ax = b$

Quantum computing 101: Starting the computation

n-qubit **state space** $(\mathbb{C}^2)^{\otimes n} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ (Hilbert space)

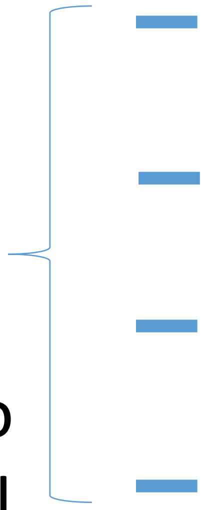
computational basis element $|x\rangle = |x_1\rangle \otimes \cdots \otimes |x_n\rangle$ $x = (x_1, \ldots, x_n) \in \{0,1\}^n$

(tensor product) orthonormal basis element

A system of n qubits can be **initialized** in a basis state $|x\rangle$ for any $x \in \{0,1\}^n$

$|x\rangle$

n-bit input "converted" to computational basis state



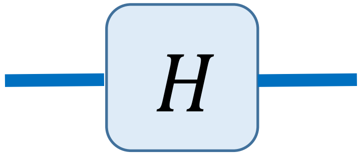
Quantum Computing 101: Quantum gates

n-qubit **evolution/gate** $U: (\mathbb{C}^2)^{\otimes n} \to (\mathbb{C}^2)^{\otimes n}$ linear unitary

$\| U\Psi \| = \| \Psi \|$ for all $\Psi \in (\mathbb{C}^2)^{\otimes n}$

Gates are norm-preserving linear maps (unitary matrices).

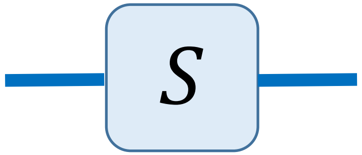
Hadamard



$$|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

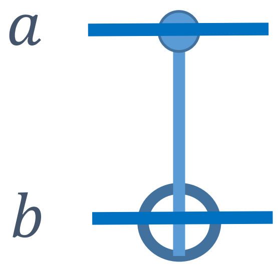
Phase shift



$$|0\rangle \to |0\rangle$$

$$|1\rangle \to i|1\rangle$$

Controlled-NOT



$$|a, b\rangle \to |a, a \oplus b\rangle$$

$|00\rangle \mapsto |00\rangle$

$|01\rangle \mapsto |01\rangle$

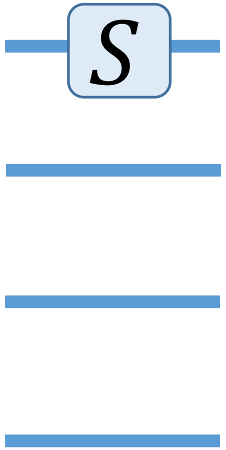
$|10\rangle \mapsto |11\rangle$

$|11\rangle \mapsto |10\rangle$

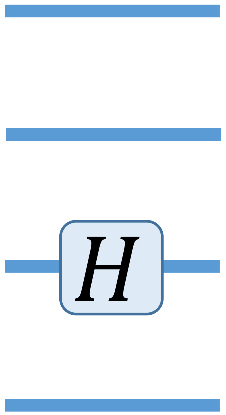
Quantum Computing 101: Quantum gates

Gates can be applied (simultaneously) to (disjoint) subsets of qubits

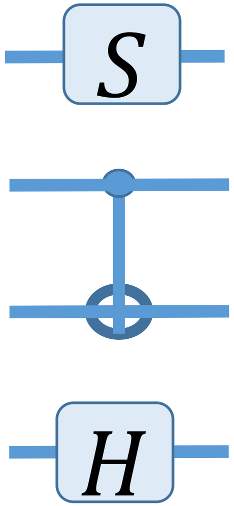
S-gate applied to qubit 1



H-gate applied to qubit 3



(simultaneously applied)
S-gate on qubit 1
CNOT on qubits 2 & 3
H on qubit 4



unitary map

$$S \otimes I \otimes I \otimes I$$

$$I \otimes I \otimes H \otimes I$$

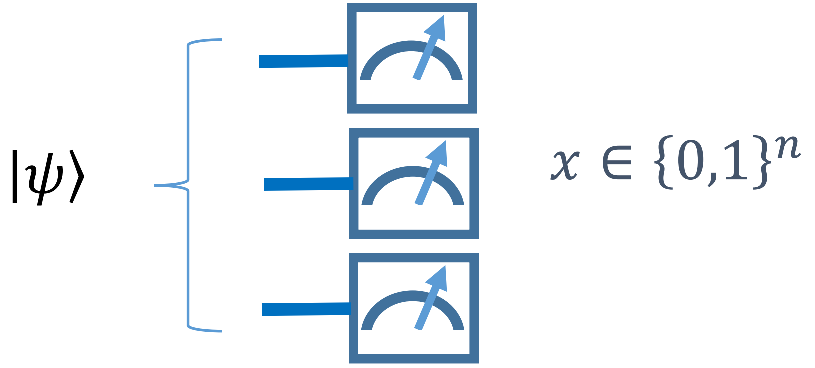
$$S \otimes CNOT \otimes H$$

Quantum computing 101: Measurements

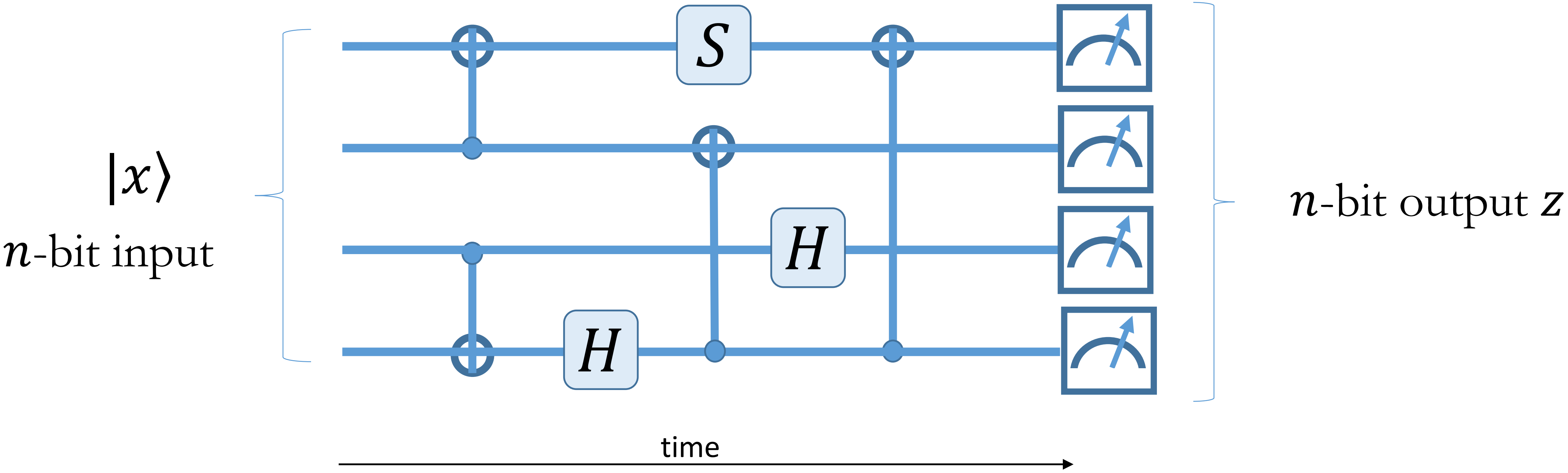
$$\{\text{states on } (\mathbb{C}^2)^{\otimes n}\} \rightarrow \{\text{probability distributions on } \{0,1\}^n\}$$

$$|\psi\rangle = \sum_{x\in\{0,1\}^n} \psi_x |x\rangle \quad \mapsto \quad p_x = |\psi_x|^2$$

Measurement allows one to sample $x \in \{0,1\}^n$ from the probability distribution p_x



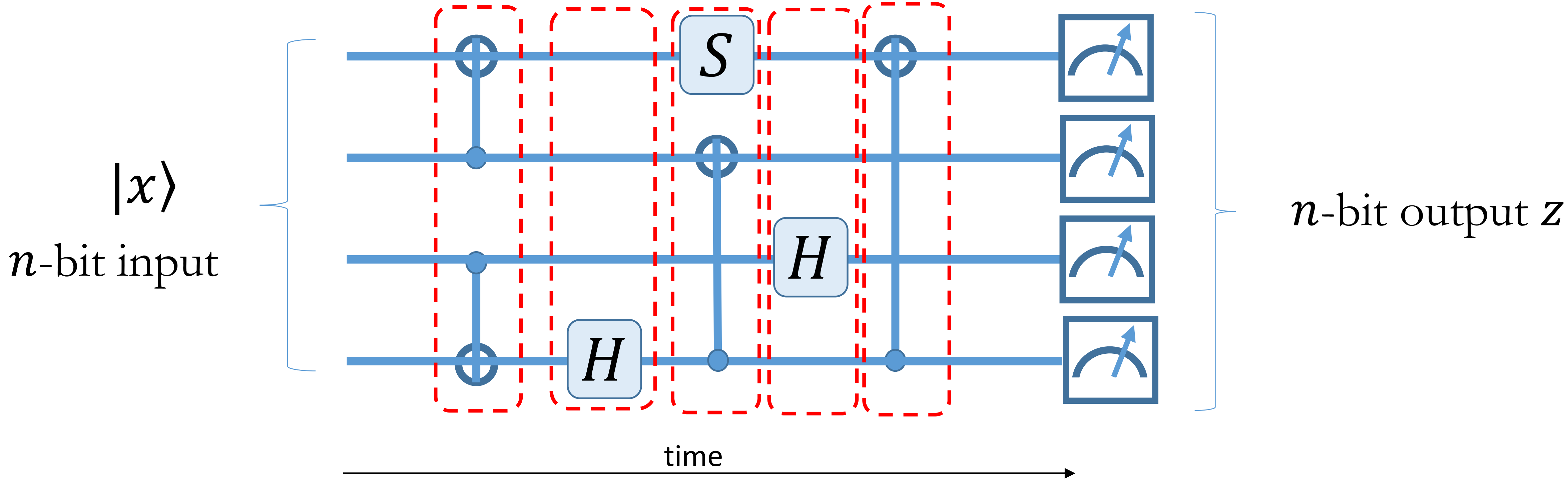
Quantum Computing 101: Quantum circuits



The behavior of the circuit is completely described by the conditional distribution

$$P(z|x) \qquad x, z \in \{0,1\}^n$$

Quantum Computing 101: Quantum circuits



A **depth-d quantum circuit** consists of d time steps.

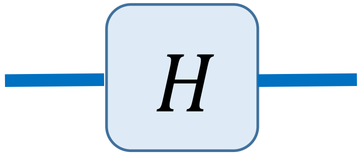
Each time step contains one- and two-qubit gates acting on disjoint subsets of qubits.

Universal gate sets and the Solovay-Kitaev Theorem

Basic question: What is needed to apply a general unitary U on $(\mathbb{C}^2)^{\otimes n}$?

Which gates should we be using?

Hadamard



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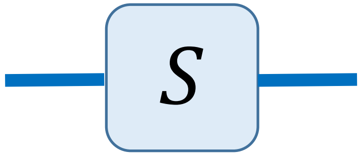

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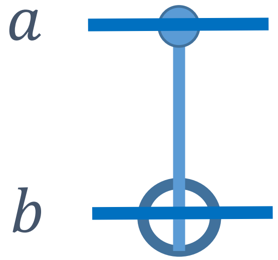

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Universal gate sets and the Solovay-Kitaev Theorem

Basic question: What is needed to apply a general unitary U on $(\mathbb{C}^2)^{\otimes n}$?

Answer: It suffices to be able to apply any gate from a (possibly finite) ***universal gate set.***

Definition: A family $\mathcal{G} = \left\{ U_j \mid U_j \text{ unitary on } (\mathbb{C}^2)^{\otimes n} \right\}$ is **universal** if

$$\langle \mathcal{G} \rangle = \left\{ U_{j_1}^{x_1} \cdots U_{j_M}^{x_M} \mid M \in \mathbb{N}, x_j \in \{\pm 1\} \right\}$$

is dense in the unitary group $\mathrm{U}(2^n)$.

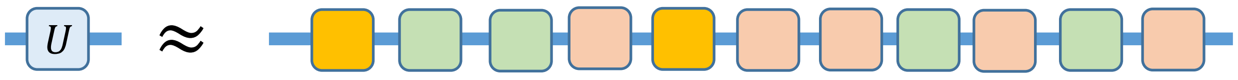
Theorem (Solovay & Kitaev):

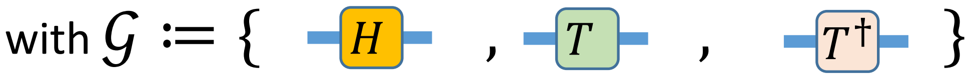
Let $\mathcal{G}$ be a universal gate set for $SU(2)$ closed under taking inverses. There is an algorithm which, given $\varepsilon > 0$ and any input unitary $U \in SU(2)$, outputs a sequence $g_1, \ldots, g_L \in \mathcal{G}$ of length

$$L = O(\log^{\alpha} 1/\varepsilon)$$

where $\alpha \approx 2.7$, such that $\|U - g_1 \cdots g_L\| < \varepsilon$

Furthermore, this algorithm runs in polynomial time in L.





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Examples for $n = 1$ (a qubit)

Universal gate sets:

- $\{R_X(\alpha)\}_{\alpha\in[0,2\pi)} \cup \{R_Z(\beta)\}_{\beta\in[0,2\pi)}$

$$R_Z(\theta) = e^{-i\theta Z}$$
$$R_X(\theta) = e^{-i\theta X}$$

- $\{T, H\}$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad T = \begin{pmatrix} e^{i\pi/8} & 0 \\ 0 & e^{-i\pi/8} \end{pmatrix}$$

Universal gate sets and the Solovay-Kitaev Theorem

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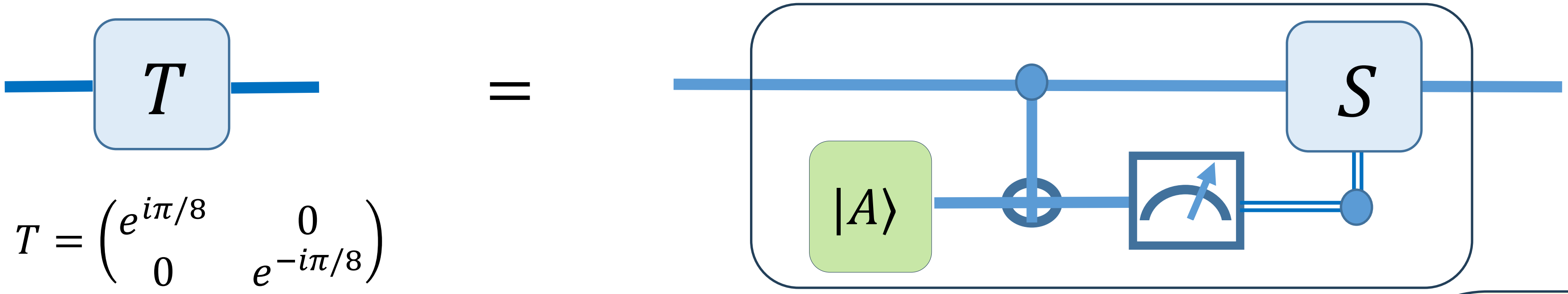
NOT universal :

Pauli group	$\mathcal{C}_1 = \left\langle \{X_j, Y_j, Z_j\}_{j=1}^n \right\rangle$ where $X_j = I \otimes \cdots I \otimes \underset{j}{X} \otimes I \cdots I$	$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Clifford group	$\mathcal{C}_2 = \left\{U \in \mathrm{U}(2^n) \mid U\mathcal{C}_1 U^\dagger \subset \mathcal{C}_1\right\}$	$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

Sergey Bravyi and Alexei Kitaev. Universal quantum computation with ideal Clifford gates and noisy ancillas. Phys. Rev. A, 71:022316, Feb 2005.

Magic state injection: useful gates from resource states



$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

$T = \begin{pmatrix} e^{i\pi/8} & 0 \\ 0 & e^{-i\pi/8} \end{pmatrix}$

$|A\rangle = 2^{-1/2}\left(|0\rangle + e^{i\pi/4}|1\rangle\right)$

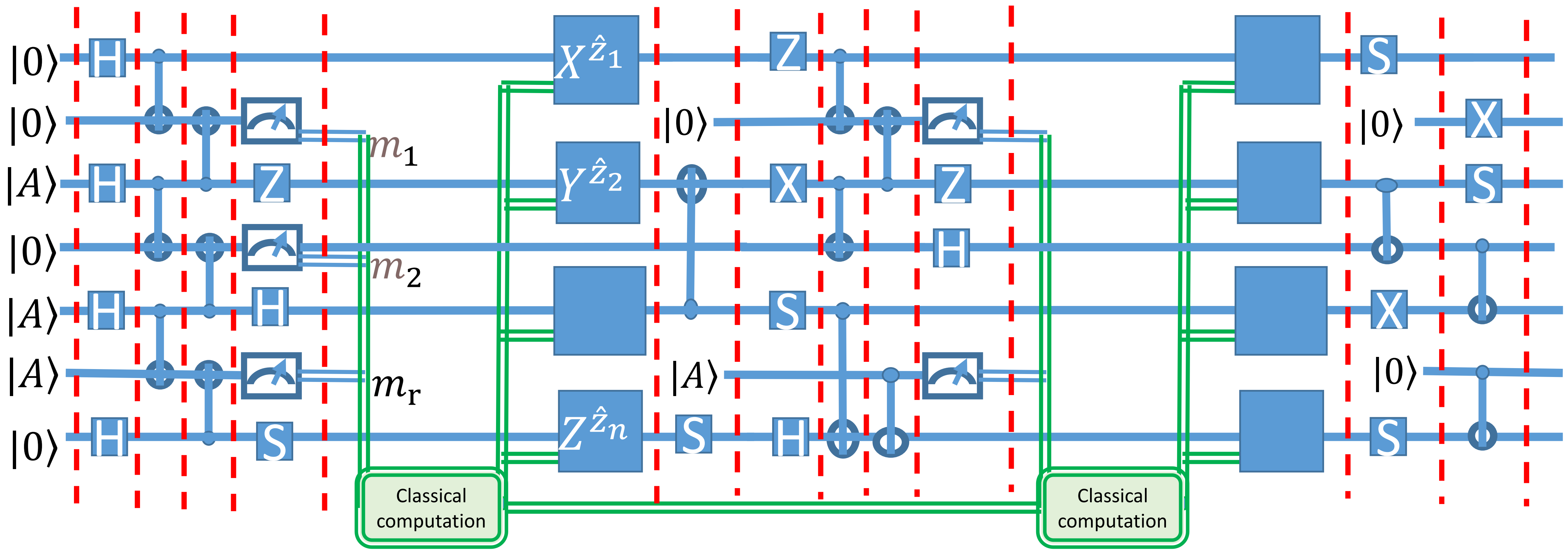
Idea: Implement certain gates by using

- "magic" resource states
- (computational basis) measurements and
- adaptive (but simple) operations (e.g., Clifford unitaries/Paulis)

The following set of one- and two-qubit operations give computational universality:

- State preparation of a single-qubit state $|0\rangle$ or $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi/4}|1\rangle\right)$
- Application of a single- or two-qubit Clifford unitary (possibly the identity)
- Measurement of any qubit in the computational basis

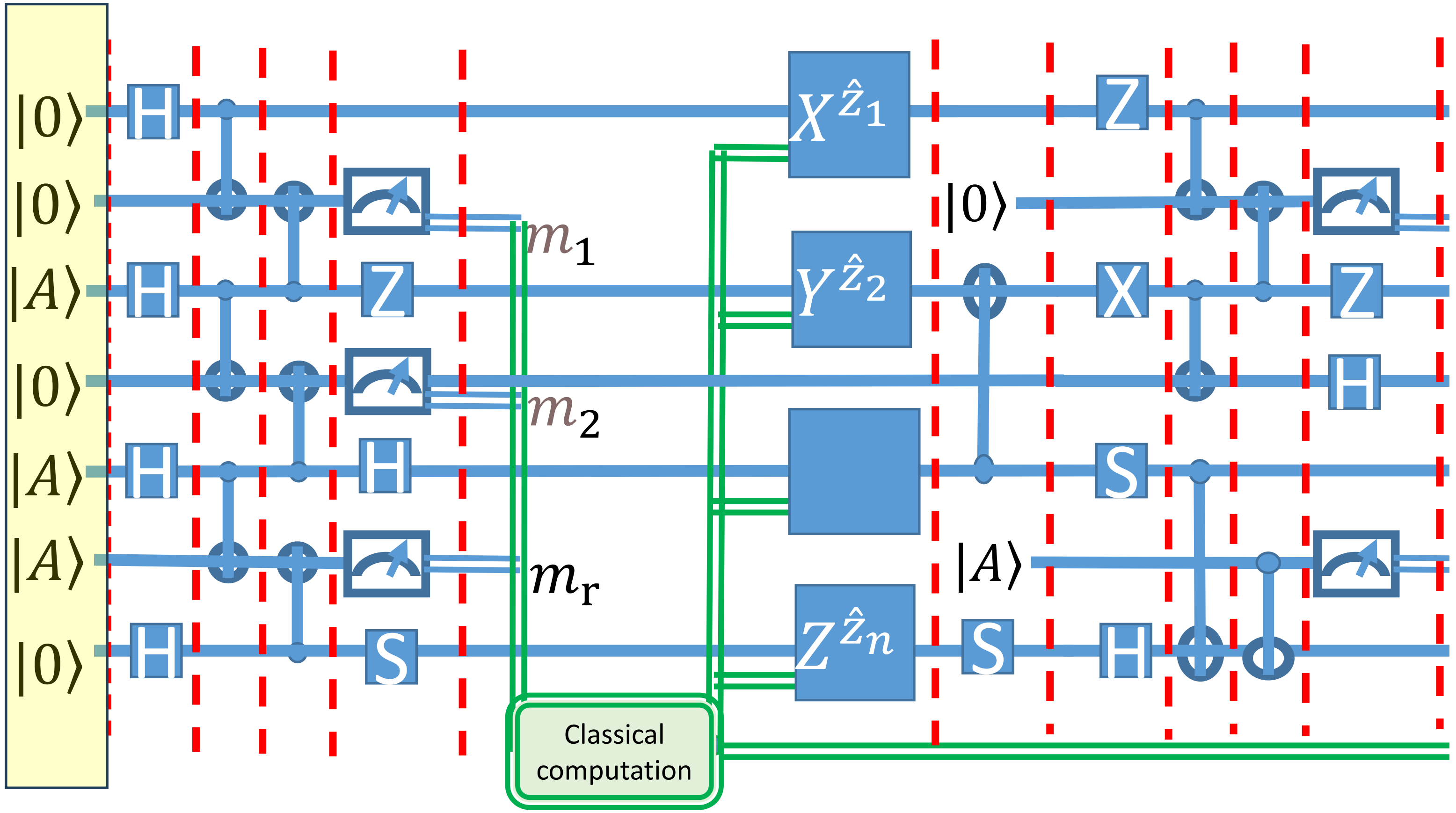
Definition of adaptive circuits and (quantum) circuit depth



Adaptivity: Each two-qubit operation may depend (in an efficiently computable manner) on previous measurement outcomes.

(Quantum) circuit depth of Q $:=$ number T of layers.

Definition of adaptive circuits and (quantum) circuit depth



Each layer $\mathcal{M}^{(T)}$ consists in the parallel application of the following one- and two-qubit operations:

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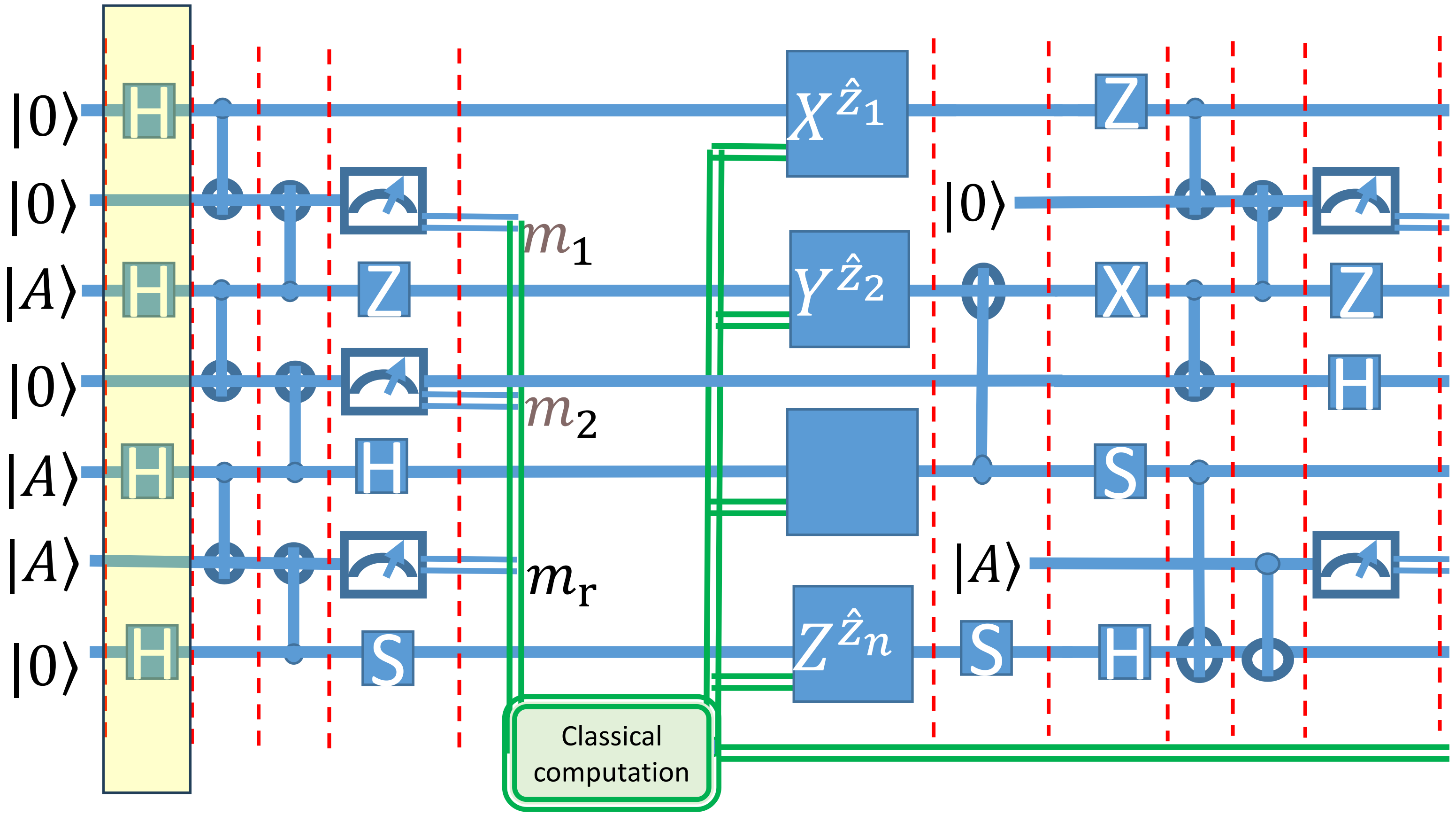

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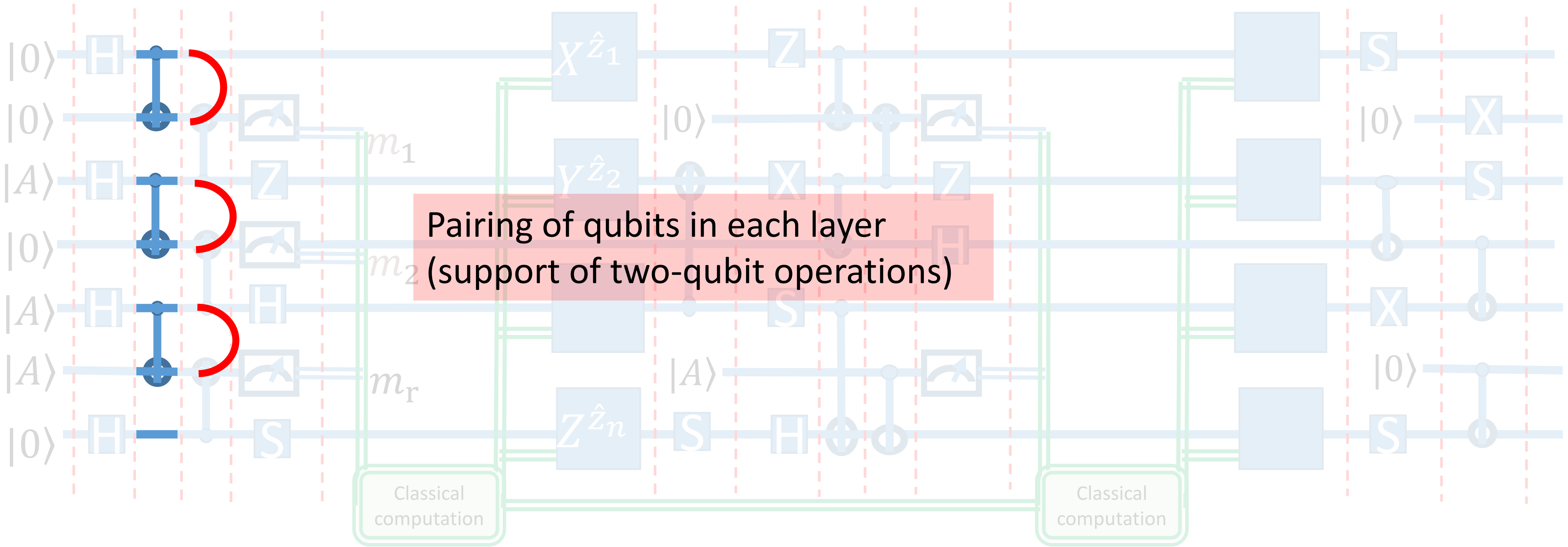

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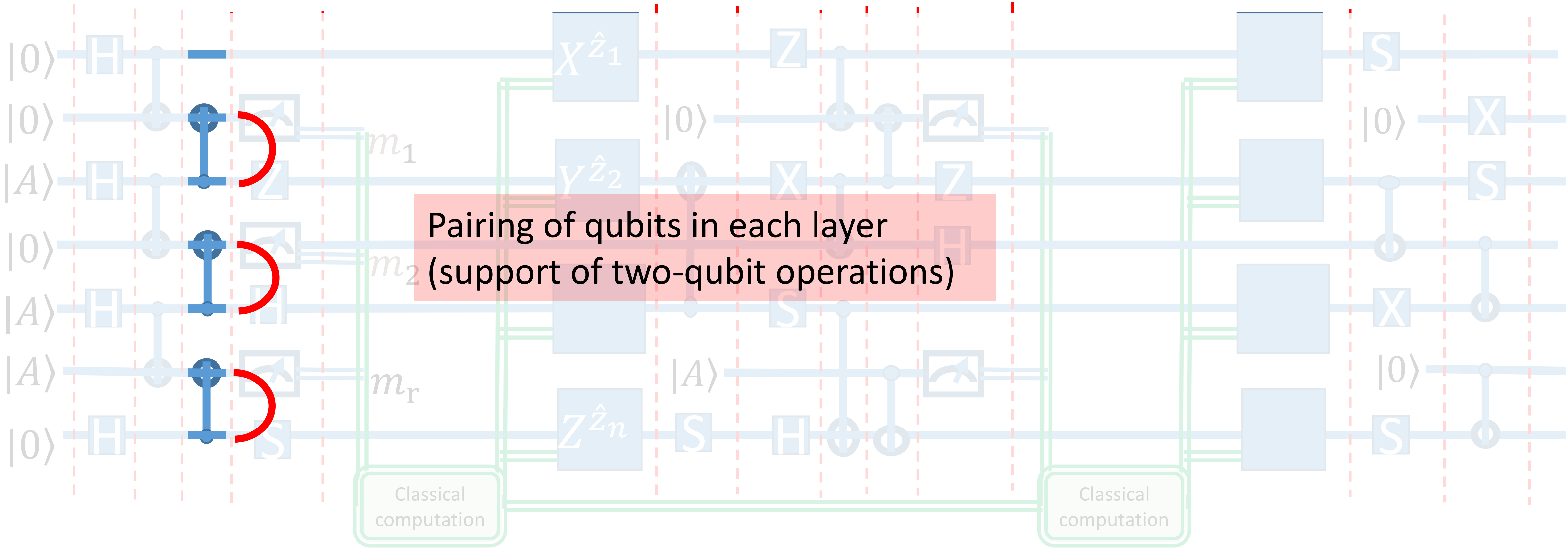

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(Quantum) circuit depth of Q $:=$ number T of layers.

Definition of adaptive circuits and (quantum) circuit depth

Consider a n-qubit circuit $\mathcal{Q} = \mathcal{M}^{(T)} \circ \cdots \circ \mathcal{M}^{(1)}$

composed of T layers $\mathcal{M}^{(1)}, \dots, \mathcal{M}^{(T)}$

That is, there are for each $t \in [T]$:

(i) a pairing $\left\{\left(i_r^{(t)}, j_r^{(t)}\right)\right\}_{r=1}^{k}$ of the set of qubits $[n] := \{1, \dots, n\}$

(ii) two-qubit operations $\left\{\mathcal{M}^{(t,r)}\right\}_{r=1}^{k}$ such that $\mathcal{M}^{(t)} = \bigotimes_{r=1}^{k} \mathcal{M}^{(t,r)}_{Q_{i_r}^{(t)} Q_{j_r}^{(t)}}$

Each layer $\mathcal{M}^{(T)}$ consists in the parallel application of the following one- and two-qubit operations:

- State preparation of a single-qubit state $|0\rangle$ or $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi/4}|1\rangle\right)$
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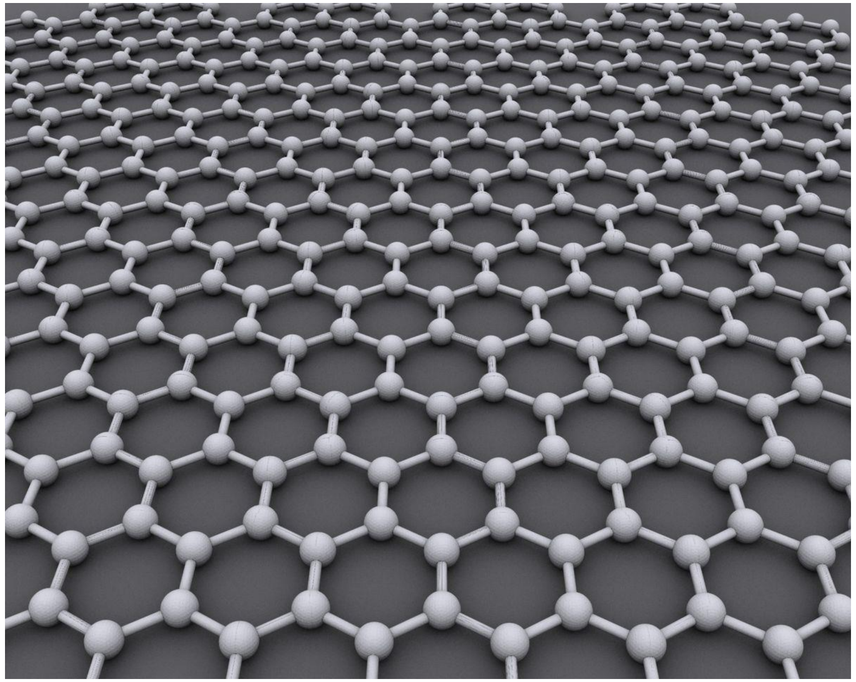
Adaptivity: Each two-qubit operation may depend (in an efficiently computable manner) on previous measurement outcomes.

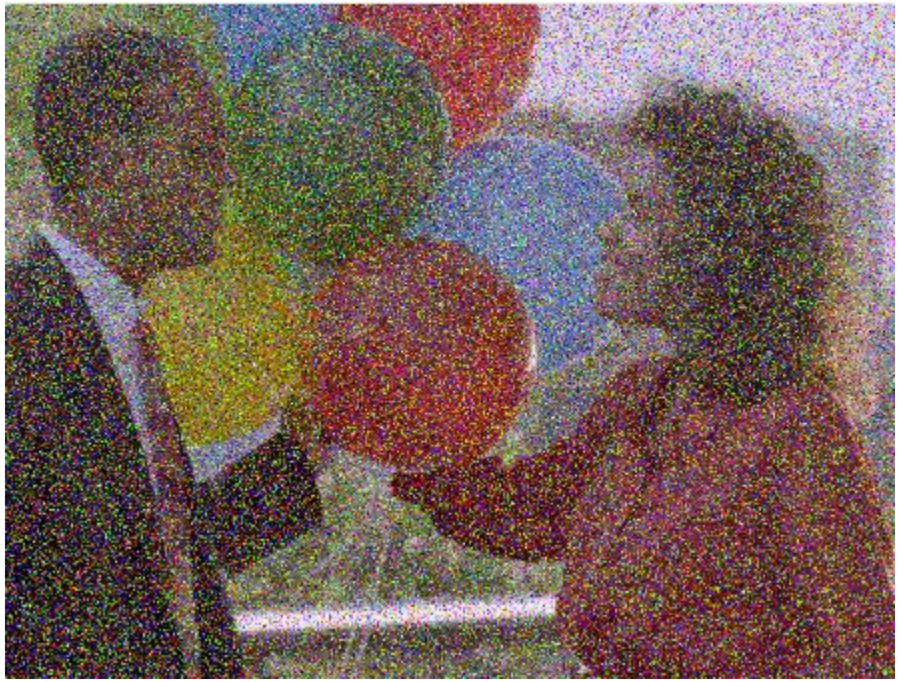
(Quantum) circuit depth of $\mathcal{Q}$ $:=$ number T of layers.

Main problem addressed in this talk:

How to deal with real-world constraints

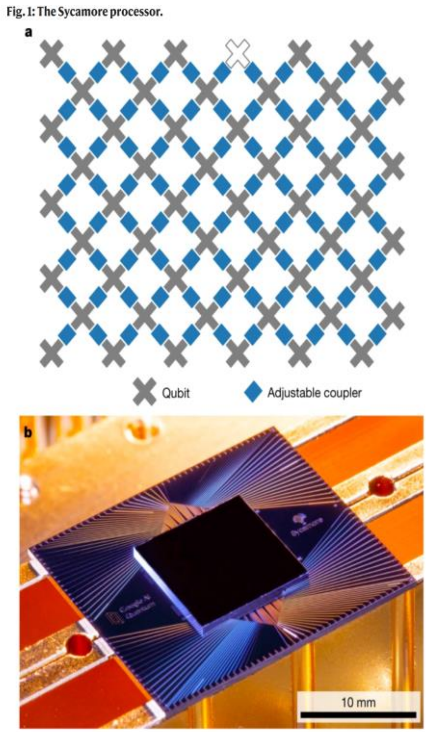


Src: wikipedia



Src: people.math.sc.edu/Burkardt/c_src/image_denoise/image_denoise.html

Real-world obstacle I:
Hardware architectures and geometric locality



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Frank Arute, Kunal Arya, Ryan Babbush, Dave Bacon, Joseph C. Bardin, Rami Barends, Rupak Biswas, Sergio Boixo, Fernando G. S. L. Brandao, David A. Buell, Brian Burkett, Yu Chen, Zijun Chen, Ben Chiaro, Roberto Collins, William Courtney, Andrew Dunsworth, Edward Farhi, Brooks Foxen, Austin Fowler, Craig Gidney, Marissa Giustina, Rob Graff, Keith Guerin, ... John M. Martinis + Show authors

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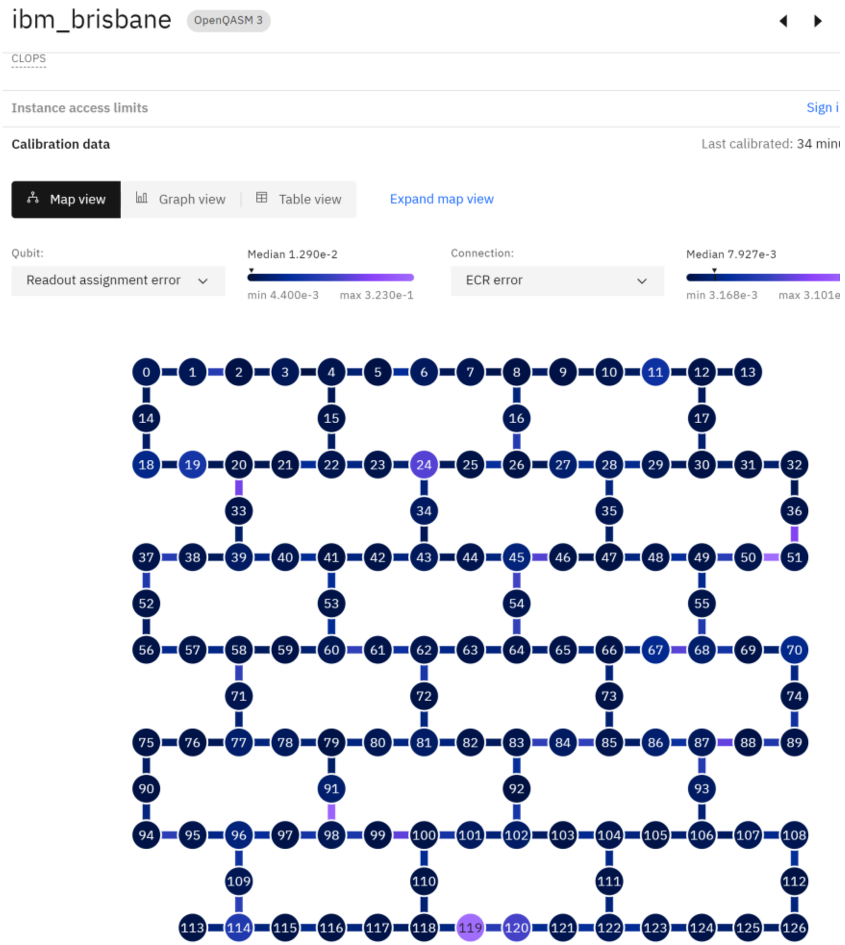
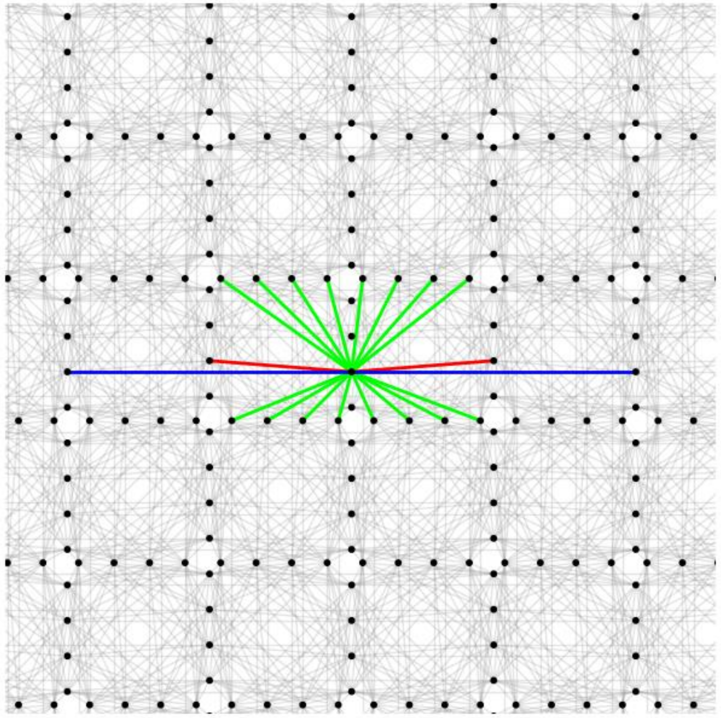


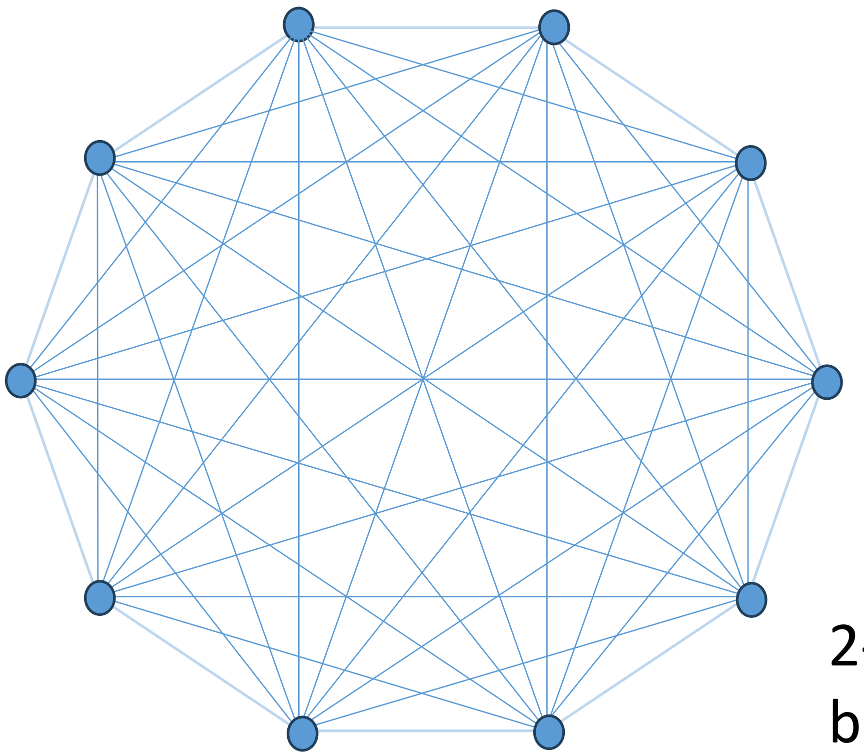
D:WAVE



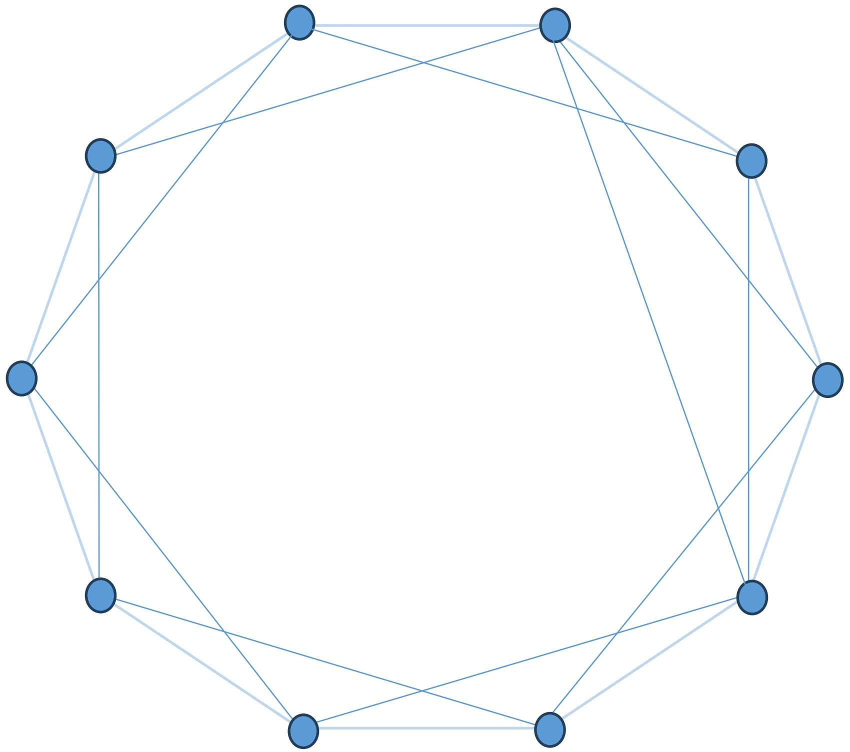
Two-qubit operations applicable only to ***neighboring pairs of qubits*** on a graph!

Real-world obstacle I:
Geometric locality: (gate) connectivity

full connectivity



limited connectivity



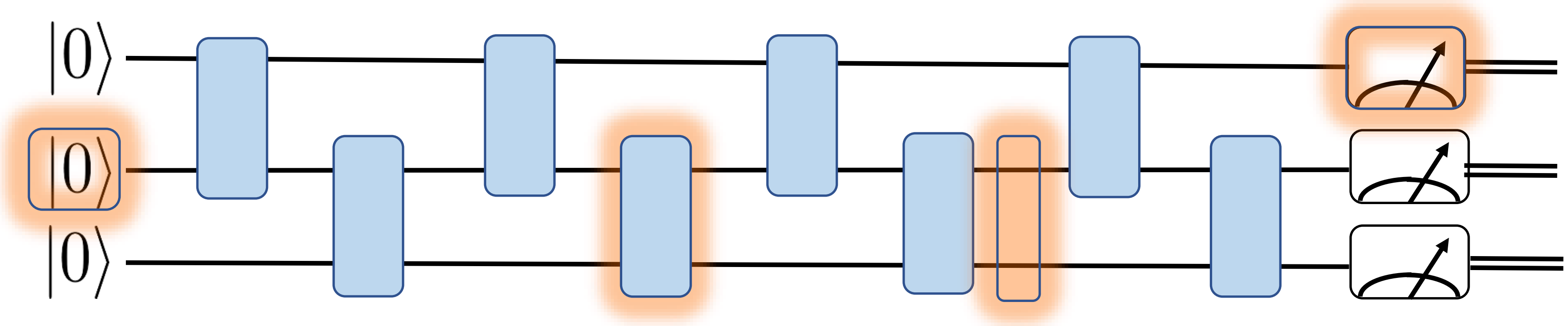
2-qubit operations between....

...... any pair of qubits

...... e.g., nearest and next-to-nearest neighboring qubits

Real-world obstacle II: Noisy building blocks

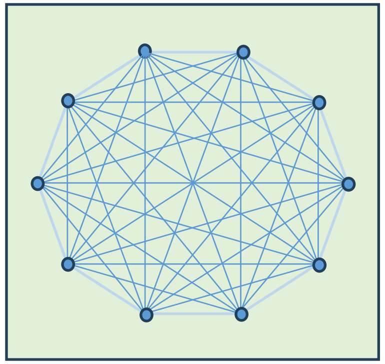
Errors can affect all involved operations: preparation, storage, gates and readout.



This talk: How to use noisy, local operations

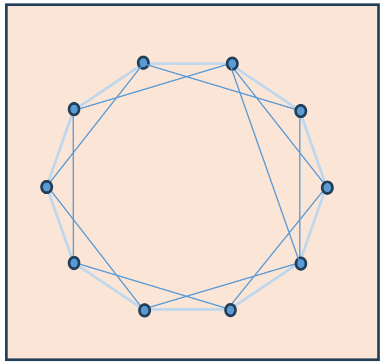
noisy qubits/ operations

ideal qubits/ operations



Fully connected ideal device

How to use a **limited-connectivity, noisy device** to simulate an **ideal, fully-connected device?**



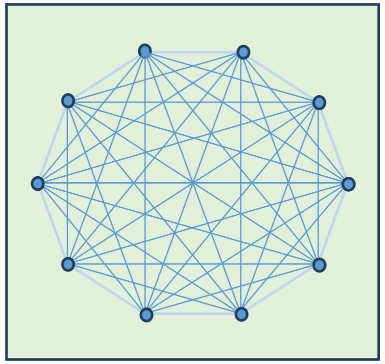
Low-connectivity noisy device

- How many (additional) qubits are needed?
- What is the time required/blow-up in quantum circuit depth?

How to use noisy operations

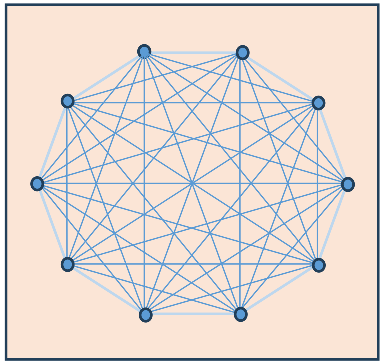
noisy qubits/ operations

ideal qubits/ operations



Fully connected ideal device

Fault-tolerance construction

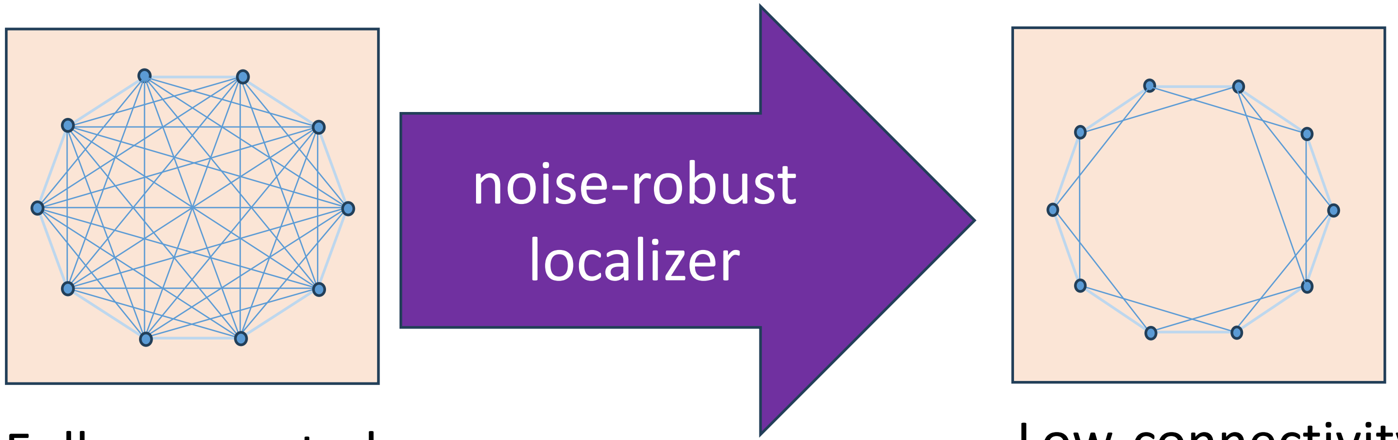


Fully connected noisy device

ignoring locality considerations

How to use noisy, local operations instead of noisy, (general) operations

noisy qubits/
operations

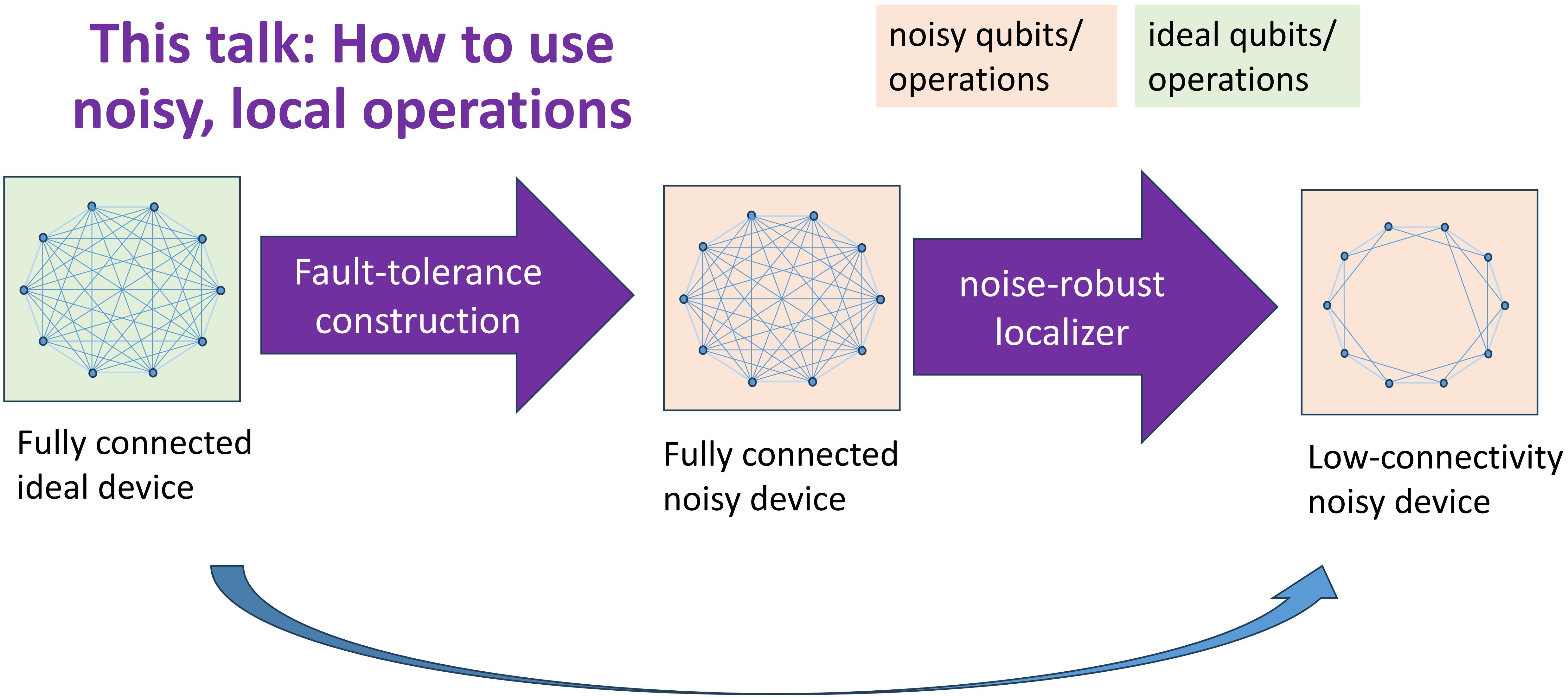


Fully connected noisy device

Low-connectivity noisy device

ignoring the problem of simulating an ideal (noise-free) device

This talk: How to use noisy, local operations



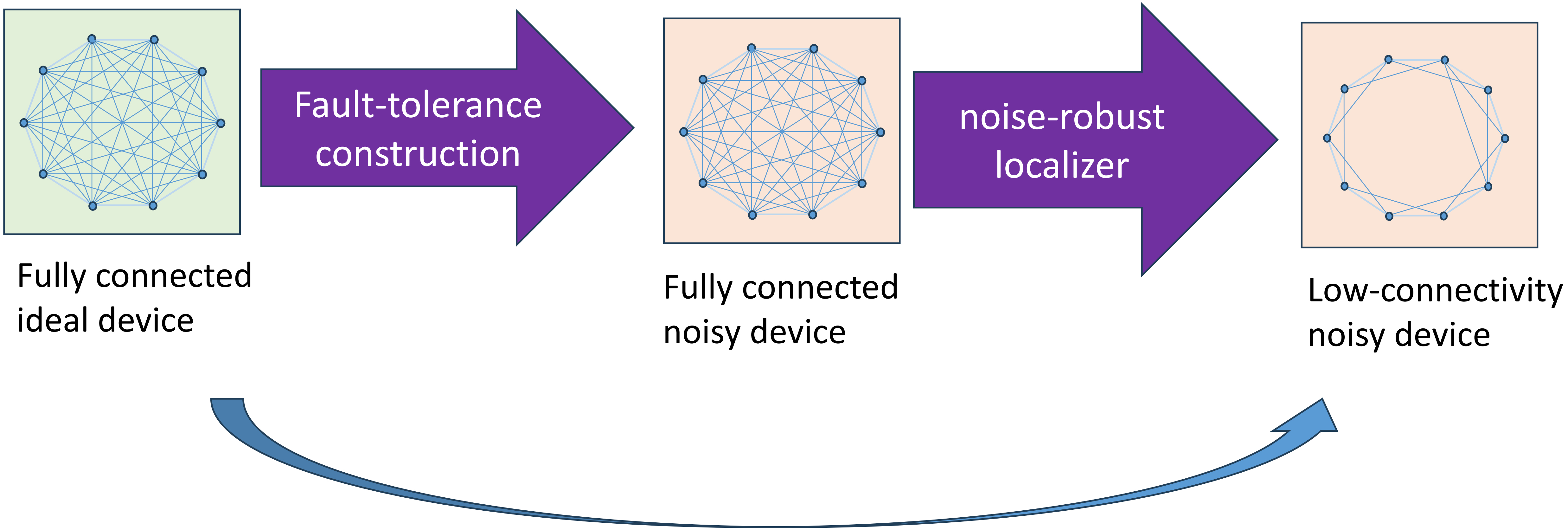
Main conceptual consequence:

Fault-tolerance (threshold) considerations and locality restrictions can be analyzed separately.

This talk: How to use noisy, local operations

noisy qubits/ operations

ideal qubits/ operations

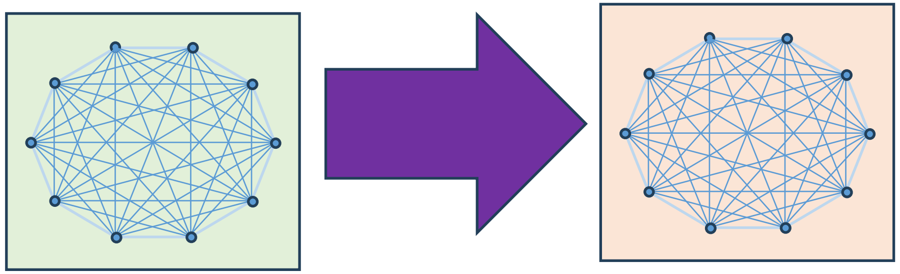


Main consequence:

Overhead-efficient fault-tolerance constructions incorporating locality constraints.

Fault-tolerance construction of Yamasaki and Koashi

Can the (ideal) circuit $\mathcal{Q}_{\text{ideal}}$ be simulated using noisy operations?



Theorem [1] There is a threshold error strength $p_0 > 0$ such that for large n and $\varepsilon \in (0,1)$:

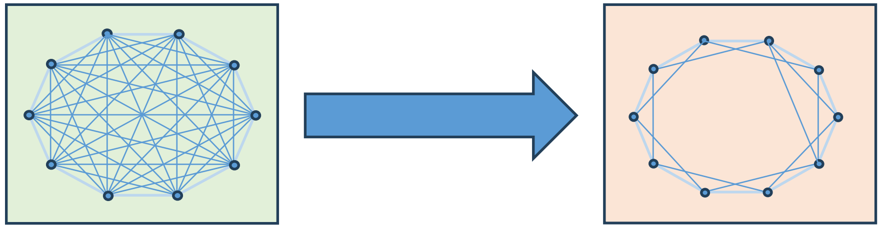
Let $\mathcal{Q}_{\text{ideal}}$ be a circuit with		There is a circuit $\mathcal{Q}_{\text{FT}}$ with
n qubits	**Then**:	$n \cdot O(1)$ qubits
$T(n) = O(\text{poly}(n))$ depth		$T(n) \cdot \exp\left(O(\log^2(\log(n/\varepsilon)))\right)$ depth

such that a noisy implementation of $\mathcal{Q}_{\text{FT}}$ with local stochastic noise of strength $p \leq p_0$ has an output distribution whose L^1-distance to the output distribution of $\mathcal{Q}_{\text{ideal}}$ bounded by ε.

[1] Yamasaki and Koashi, Nat. Phys. no. 20, Feb 2024

Main result: Fault-tolerance with **local operations** in 3D

Can the (ideal) circuit $\mathcal{Q}_{\text{ideal}}$ be simulated using noisy, local operations?



Theorem [1] There is a threshold error strength $p_0 > 0$ such that for large n and $\varepsilon \in (0,1)$:

Let $\mathcal{Q}_{\text{ideal}}$ be a circuit with
- n qubits
- $T(n) = O(\text{poly}(n))$ depth

Then:

There is a **3D-local** circuit $\mathcal{Q}_{\text{FT}}$ with
- $n \cdot O\left(n^{1/2}\log^3 n\right)$ qubits
- $T(n) \cdot \exp(O(\log^2(\log(n/\varepsilon)))\,)$ depth

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[1] RK and Shin Ho Choe, arXiv:2402.13863

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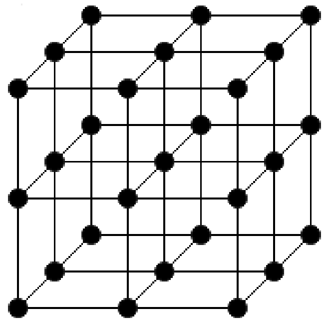

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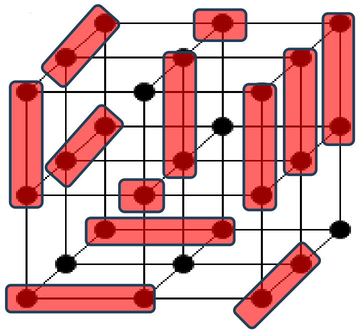

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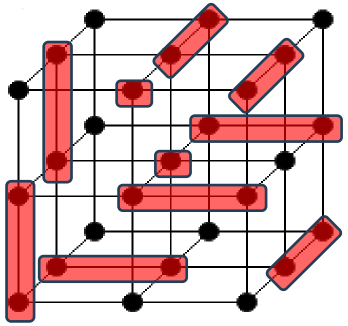

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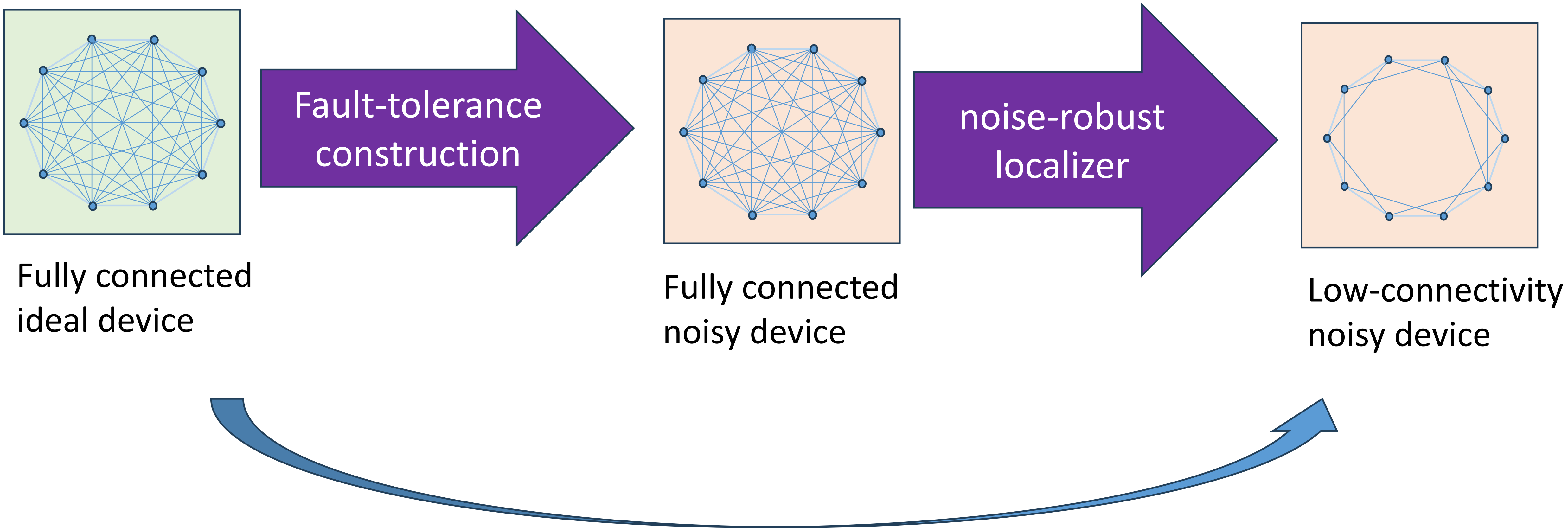
This talk: How to use noisy, local operations

noisy qubits/ operations

ideal qubits/ operations

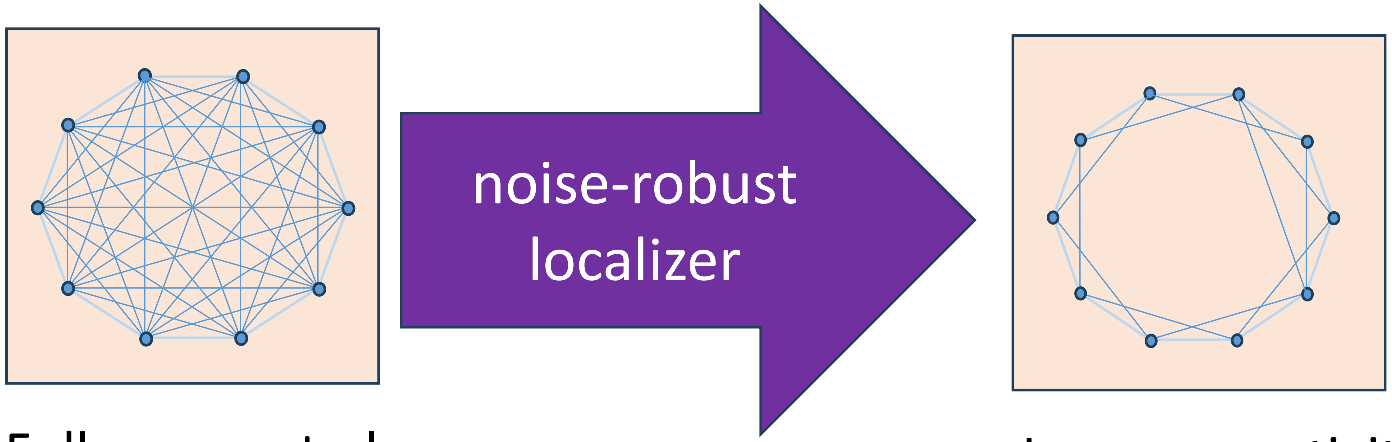


Main consequence:

Overhead-efficient fault-tolerance constructions incorporating locality constraints.

How to use noisy, local operations instead of noisy, (general) operations

noisy qubits/
operations

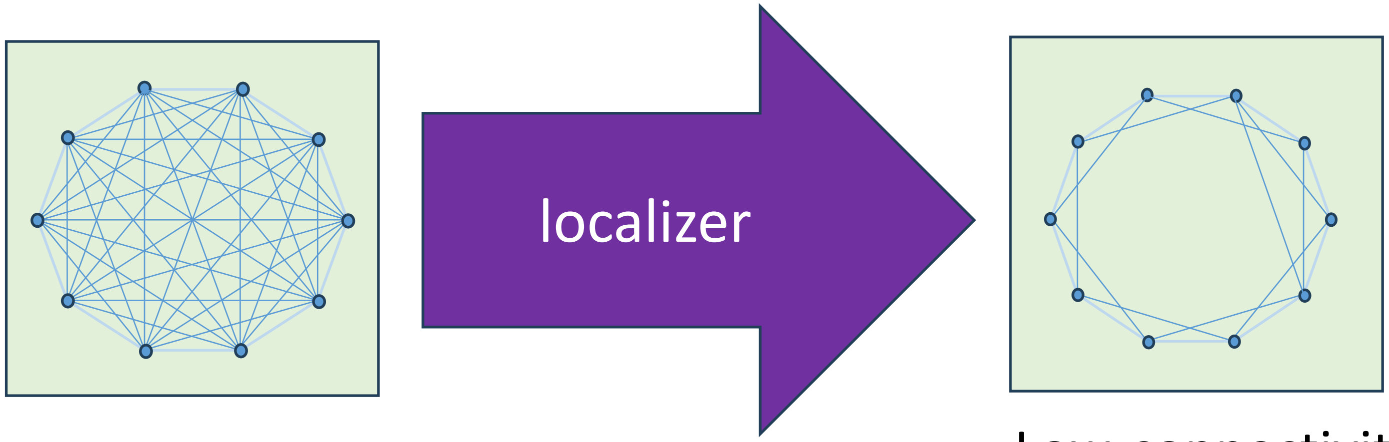


Fully connected
noisy device

Low-connectivity
noisy device

How to use local operations

ideal qubits/
operations

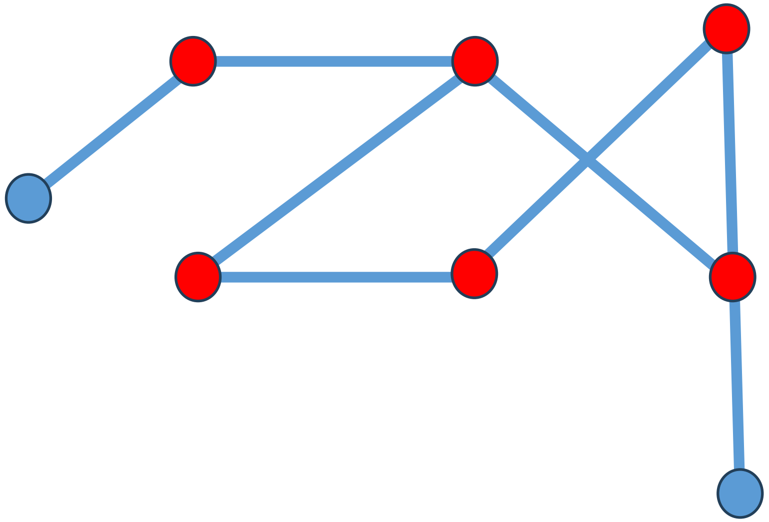


Fully connected
ideal device

Low-connectivity
ideal device

ignoring noise

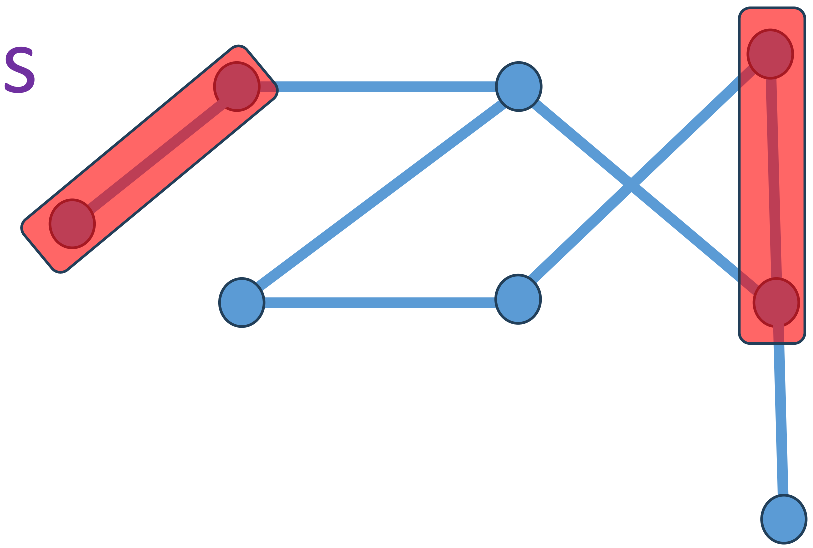
Routing of qubits in graphs



Given:

- A graph $G = (V, E)$ with a qubit Q_v at each vertex v.
- A special subset $S = \{v_1, \ldots, v_k\}$ of vertices.

Routing of qubits in graphs



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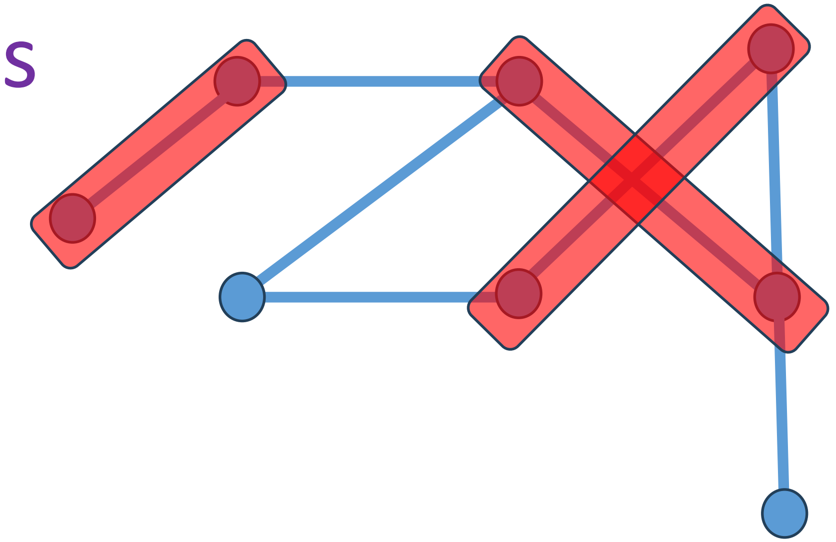

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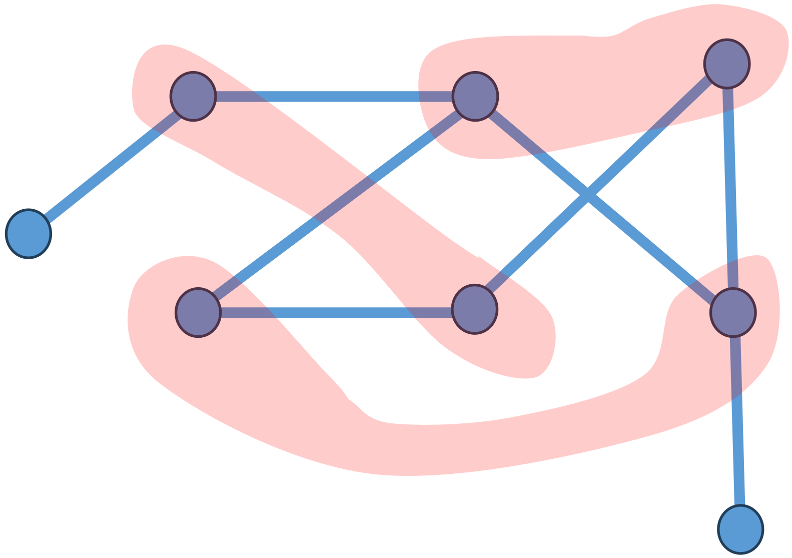

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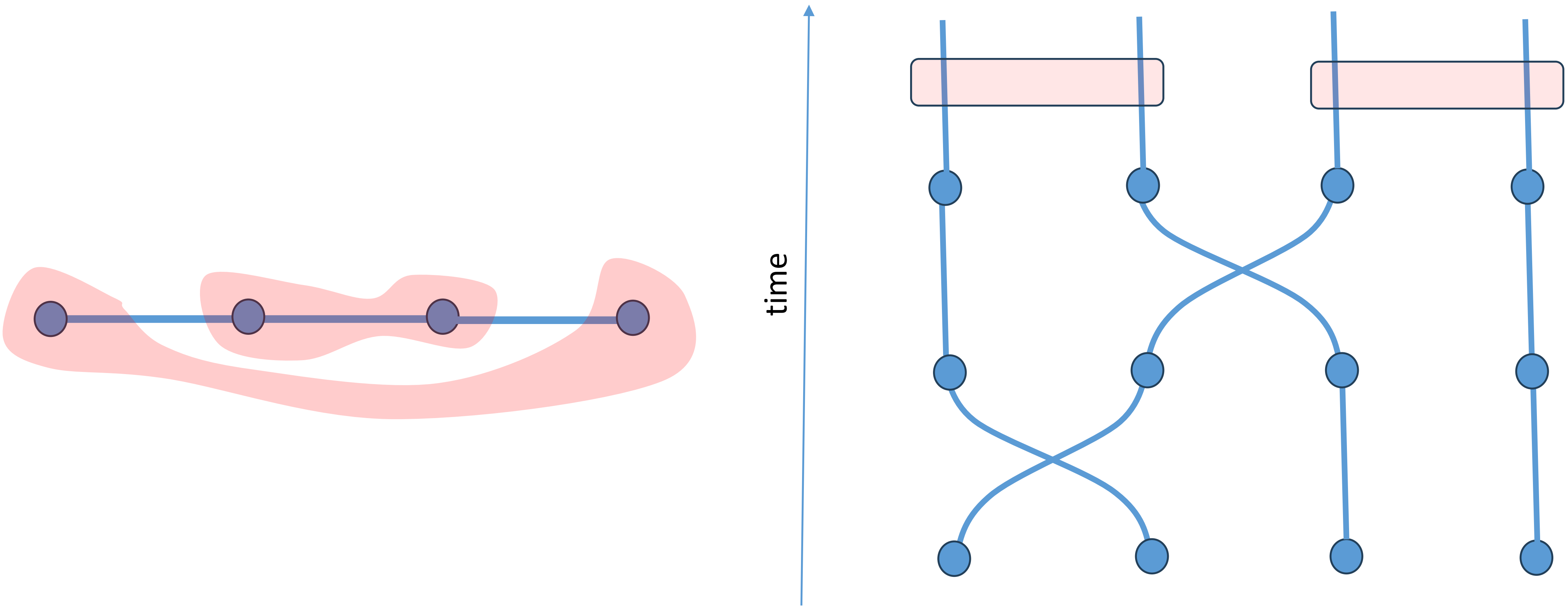
Capabilities: Can apply circuits composed of local and nearest-neighbor operations.

Problem input: A pairing $\left\{\left(v_{i_r}, v_{j_r}\right)\right\}_{r=1}^{k}$ of the vertices of S

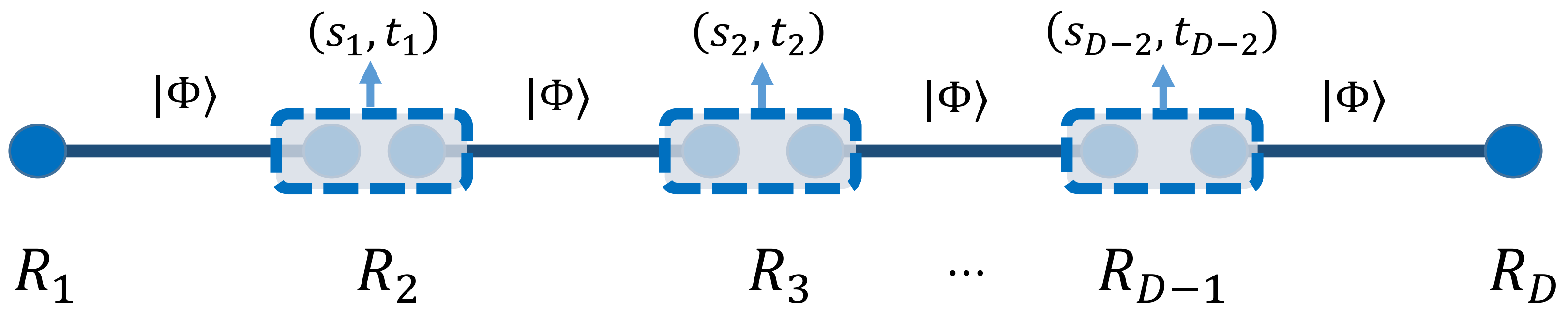
Goal: Apply a tensor product $\otimes_{r=1}^{k} \mathcal{M}^{(r)}_{Q_{v_{i_r}} Q_{v_{j_r}}}$ of two-qubit operations.

Routing of qubits in graphs: SWAP-based protocols



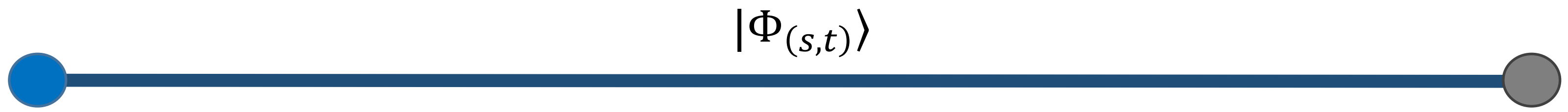
The number of SWAP-gate layers needed may scale linearly (in 1D)!

Entanglement swapping



Start with $D-1$ EPR pairs arranged on a line

Perform Bell measurements between neighboring pairs



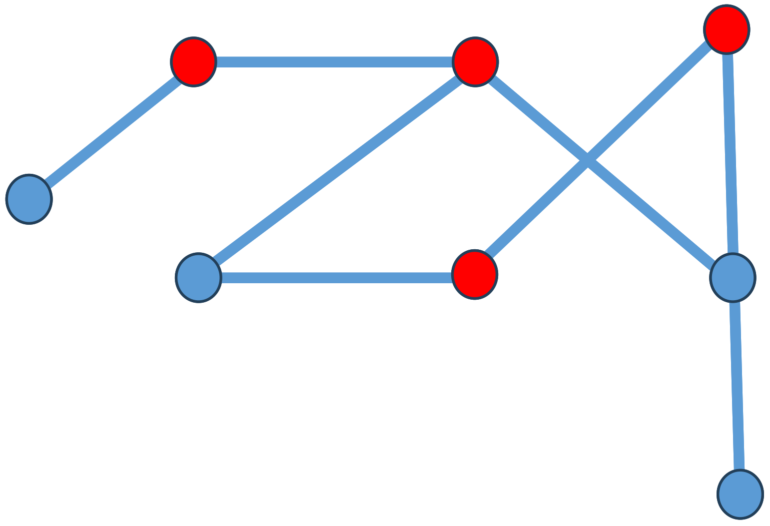
The state after the Bell measurements is equivalent to a "Pauli-corrupted" long-range entangled Bell state

$|\Phi_{(s,t)}\rangle = (I \otimes Z^s X^t)|\Phi\rangle$ with Bell state determined by

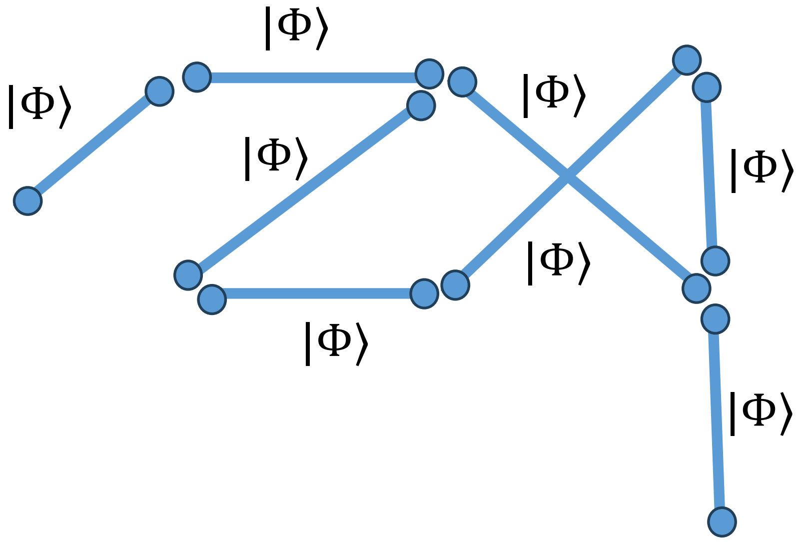
$$s = \sum_{j=1}^{D-2} s_j \pmod 2,$$
$$t = \sum_{j=1}^{D-2} t_j \pmod 2.$$

From qubit routing to parallel routing (using entanglement-swapping)

Original graph



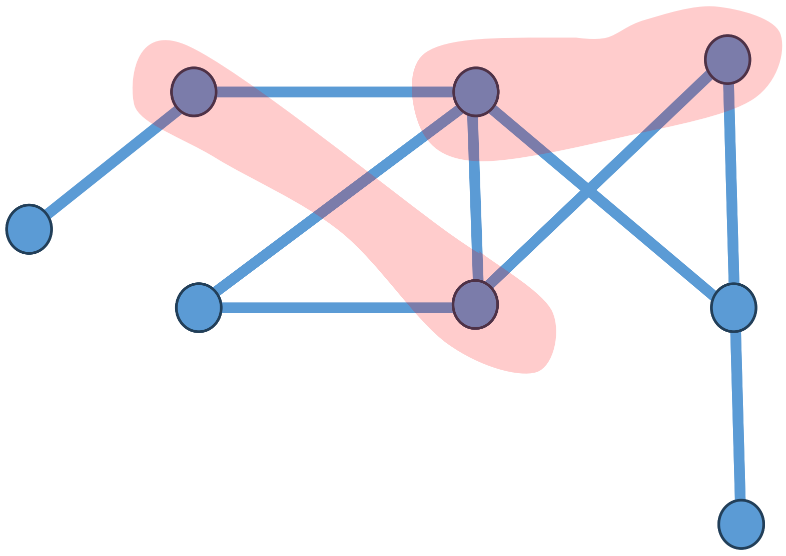
Entanglement structure



Each edge is replaced by a Bell state $|\Phi\rangle$

From qubit routing to parallel routing (using entanglement-swapping)

Original graph pairing $\{(v_{i_r}, v_{j_r})\}_{r=1}^{k}$



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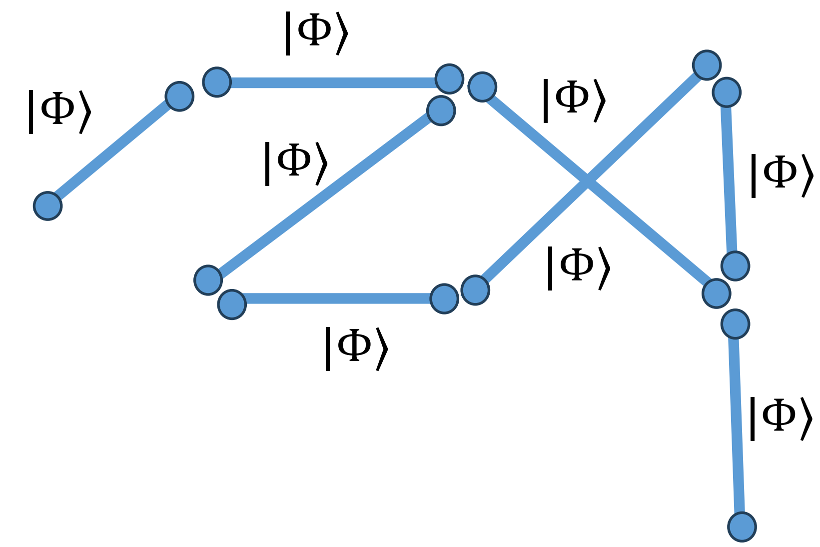

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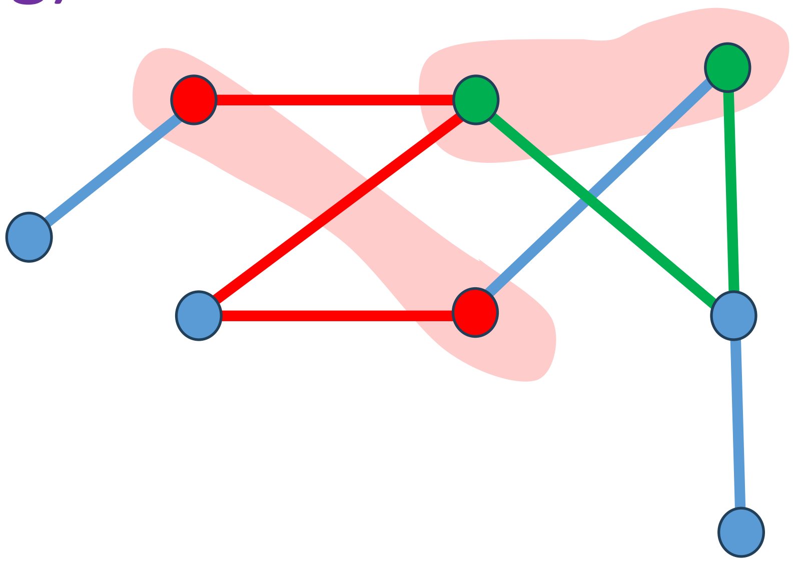
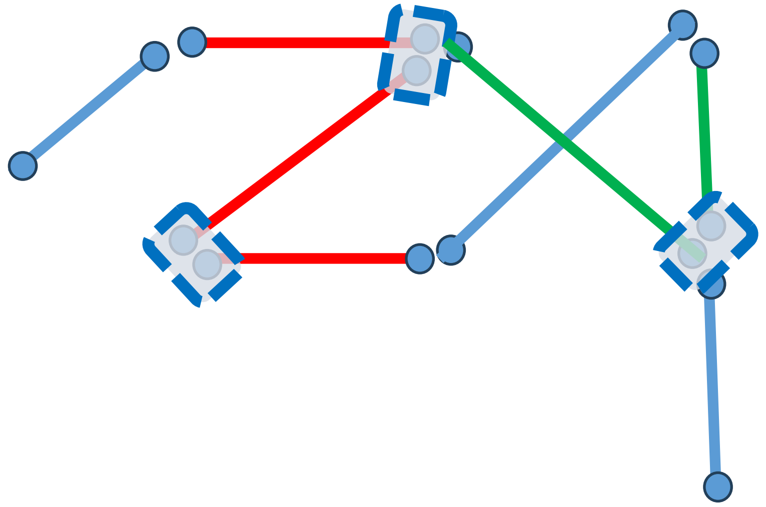


pairing $\{(v_{i_r}, v_{j_r})\}_{r=1}^{k}$

Edge-disjoint family $\{\pi_r\}_{r=1}^{k}$ of paths whose endpoints $\partial\pi_r = \{v_{i_r}, v_{j_r}\}$ correspond to pairing $\{(v_{i_r}, v_{j_r})\}_{r=1}^{k}$

Entanglement structure



Entanglement swapping along each path π_r (executed in parallel)

A combinatorial problem: Parallel routing in a graph

Given: A graph $G = (V, E)$

Definition: A subset $S = \{v_1, \dots, v_{2k}\}$ of vertices is called **parallel-routable** $:\Leftrightarrow$

For any pairing $\{(v_{i_r}, v_{j_r})\}_{r=1}^{k}$ of S, there is collection $\{\pi_r\}_{r=1}^{k}$ of paths such that

- $\{\pi_r\}_{r=1}^{k}$ are pairwise edge-disjoint
- $\partial\pi_r = \{v_{i_r}, v_{j_r}\}$ for each $r = 1, \dots, k$.

Problem: Find a parallel-routable set S of maximal size.

Parallel routing in 2D grid graphs

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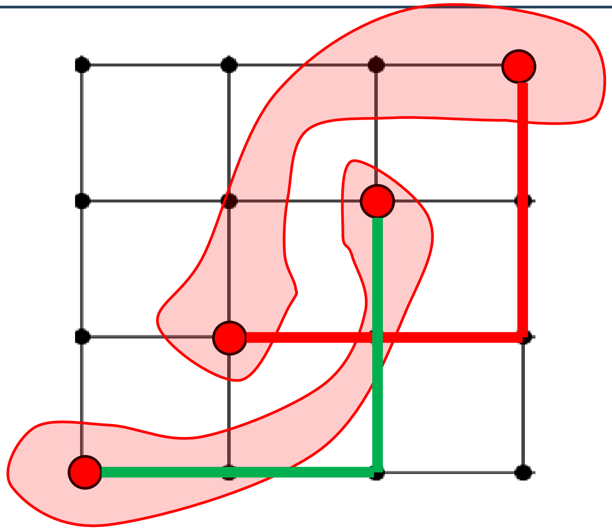
Theorem: The 2D grid graph $P_L \times P_L$ contains a parallel-routable set S of size $|S| = L$.

The length of each path π_r is $\leq 2L$, and the paths can be efficiently computed from the pairing.

Proof: Consider the set $S = \{v_r = (r, r) \mid r = 1, \dots, L\}$. □



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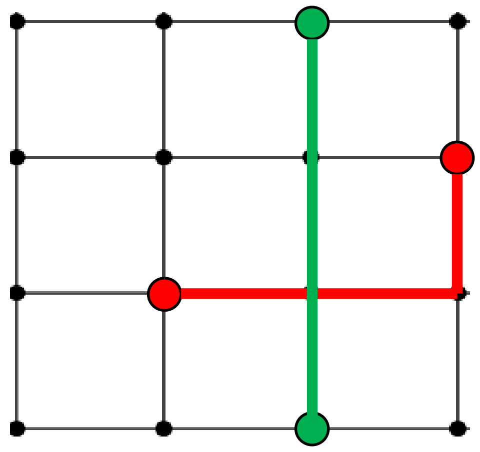

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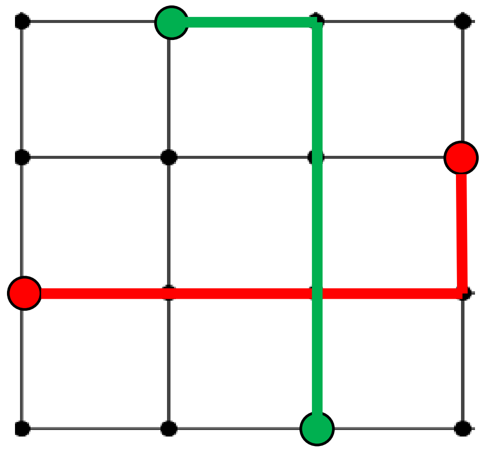

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Lemma A sufficient condition for the existence of $\{\pi_r\}_{r=1}^{k}$ is

(*)

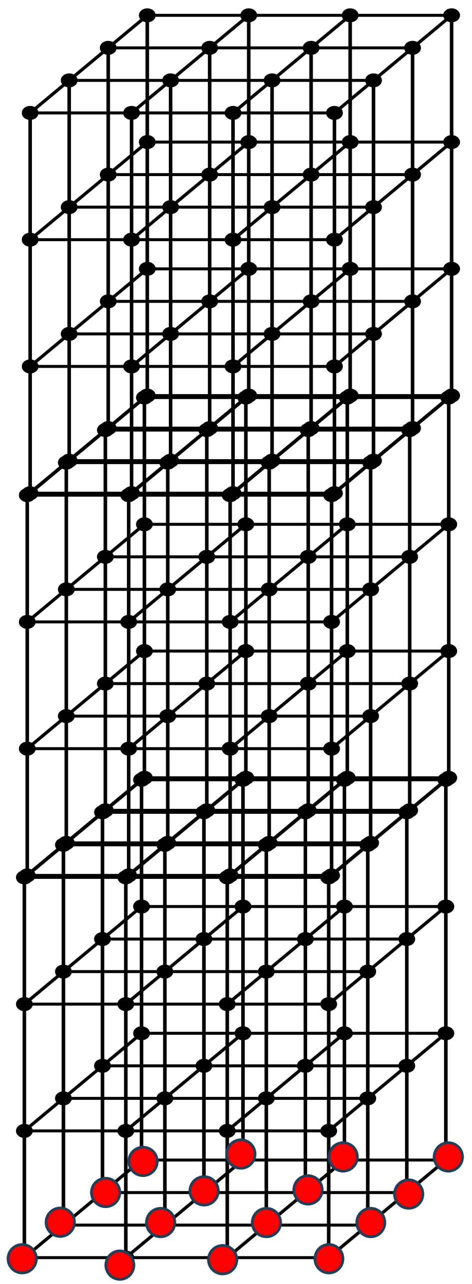
$$\{x(v_{i_r}), x(v_{j_r})\} \cap \{x(v_{i_s}), x(v_{j_s})\} = \emptyset \quad \text{for} \quad r \neq s$$
$$\{y(v_{i_r}), y(v_{j_r})\} \cap \{y(v_{i_s}), y(v_{j_s})\} = \emptyset \quad \text{for} \quad r \neq s$$

"Projections of endpoints onto coordinate axes do not intersect for different pairs."

Parallel routing in 3D grid graphs

Theorem: The 3D grid graph $P_L \times P_L \times P_{4L}$ contains a parallel-routable set S of size $|S| = L^2$. Corresponding paths have length at most 10L.

Proof: Consider the set $S = \{(x, y, 1) \mid x, y \in \{1, \dots, L\}\} \quad =: \mathcal{F}_1$



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We use a greedy algorithm that given a pairing $\left\{\left(v_{i_r}, v_{j_r}\right)\right\}_{r=1}^{k}$ constructs a collection $\{\pi_r\}_{r=1}^{k}$ of paths as follows:

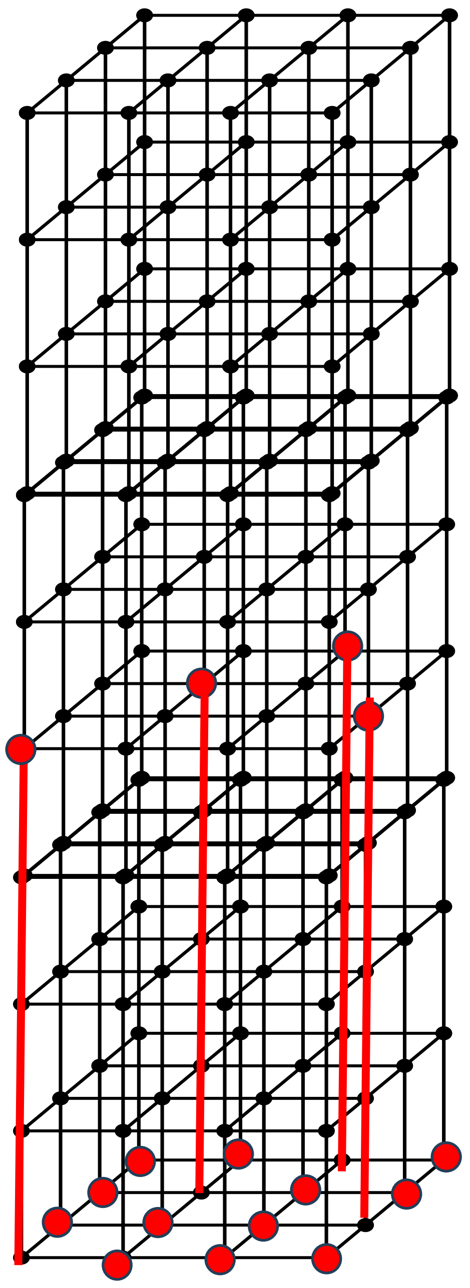
For $r = 1, \dots, k$:

1. Find the minimal floor level z such that adding $\left\{\Pi_z v_{i_r}, \Pi_z v_{j_r}\right\}$ (where $\pi_z(x, y, 1) = (x, y, z)$) to the floor

   $$\mathcal{F}_z := \{(x, y, z) \mid x, y \in \{1, \dots, L\}\}$$

   does not violate (*)

2. Then use "vertical" elevators and the Lemma to construct π_r



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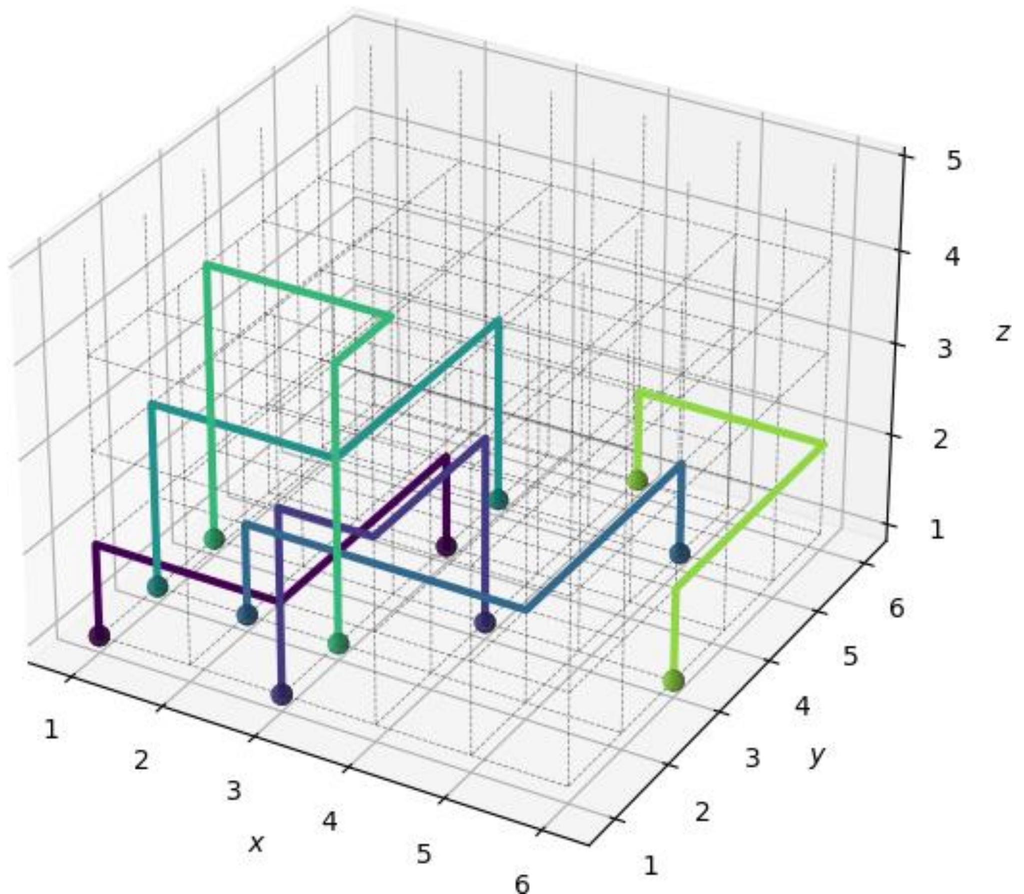
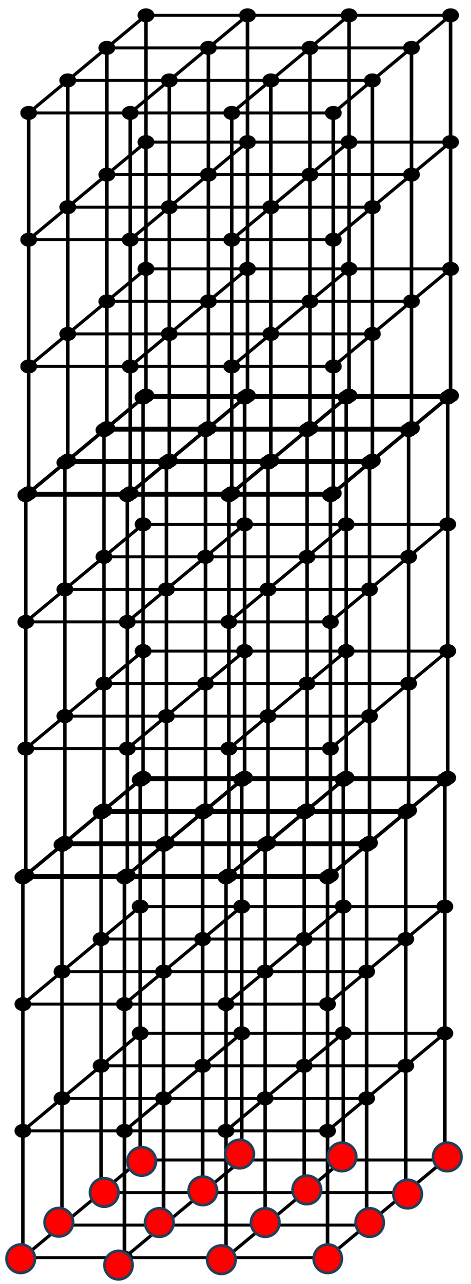
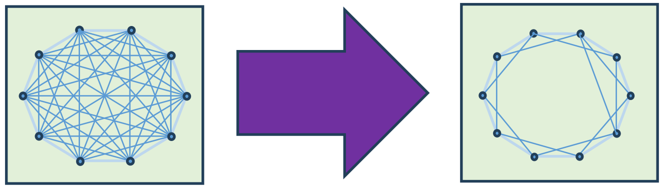
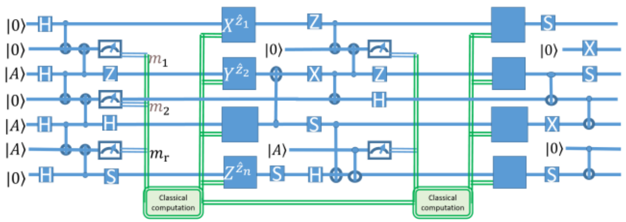




(Ideal) localization





Given: adaptive quantum circuit $\mathcal{Q}$

on $\boldsymbol{n}$ qubits

of depth $\boldsymbol{T}$

involving **non-local** operations

localizer

Theorem:

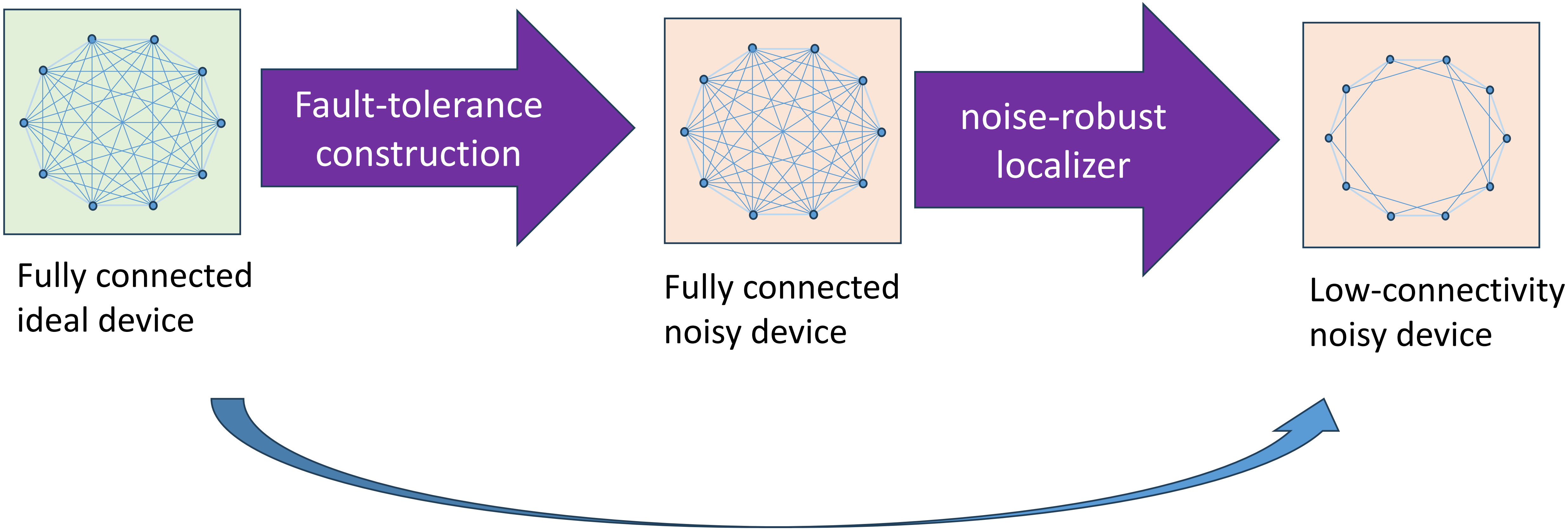
There is an adaptive circuit $\mathcal{Q}'$ with the following properties:

1. $\mathcal{Q}'$ uses $\boldsymbol{n} \cdot \boldsymbol{O}(\boldsymbol{n}^{1/2})$ qubits and is **local** when these are arranged on a **3D grid graph**.
2. $\mathcal{Q}'$ has quantum depth of order $\boldsymbol{O}(\boldsymbol{T})$.
3. $\mathcal{Q}'$ **simulates** $\mathcal{Q}$ exactly

This talk: How to use noisy, local operations

noisy qubits/ operations

ideal qubits/ operations



Main consequence:

Overhead-efficient fault-tolerance constructions incorporating locality constraints.

(Standard) Fault-tolerance constructions

Fully connected ideal device

Fully connected noisy device

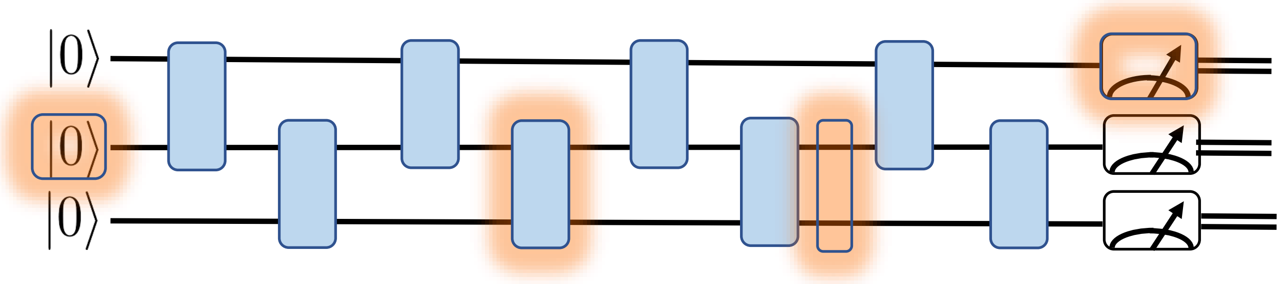
(ignoring locality)

How to realize simulate an (ideal, i.e., noise-free general) circuit by a noisy (general) circuit.

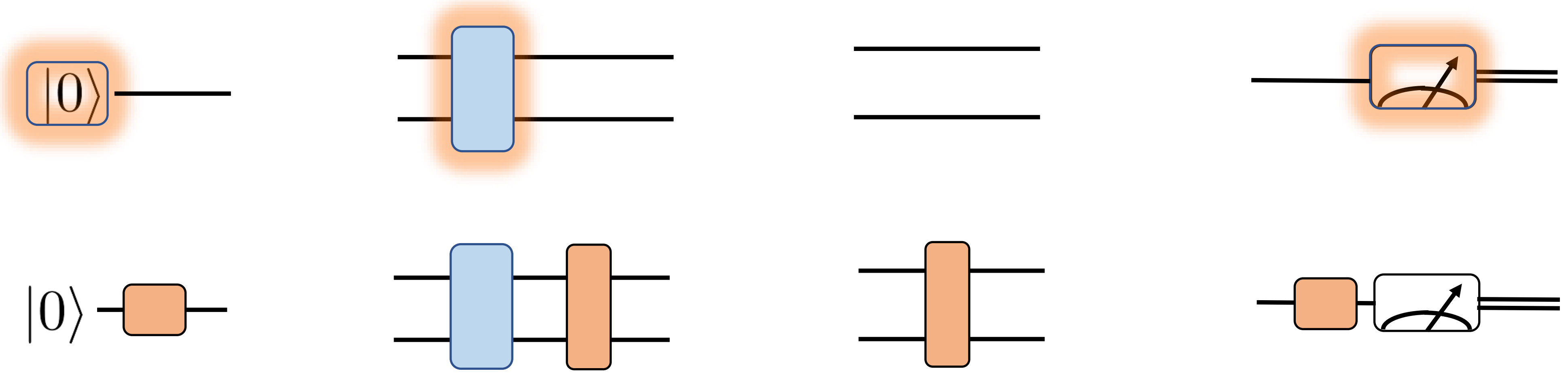
(other researchers' fantastic achievements!)

Noise in quantum circuits: basic properties

Errors can affect all involved operations: preparation, storage, gates and readout.



	preparation	gate	memory/ wait location	measurement
faulty operation				
model:				



Local stochastic noise

Assumption	Justification
Errors are (randomly chosen) **Pauli errors**.	*Probabilistic Pauli noise is no more detrimental than general coherent noise.*

argument based on linearity/operator bases:

Ekert, Macchiavello, Phys. Rev. Lett. **77**, 2585 (1996)

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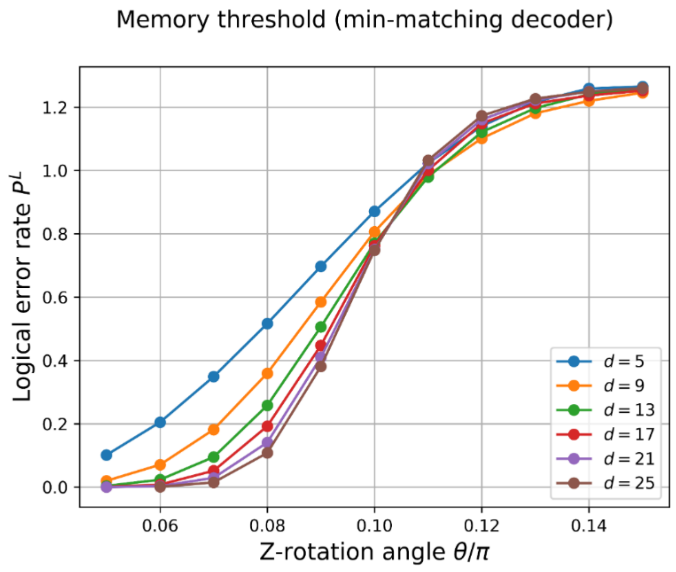

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Errors are (randomly chosen) **Pauli errors**.	*Probabilistic Pauli noise is no more detrimental than general coherent noise.*
High weight errors are exponentially **suppressed** (unlikely).	*Physical processes are typically local/two-body.*

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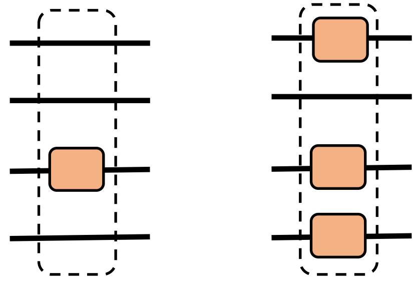
probability of occurence

$\leq O(p)$ $\leq O(p^3)$



Local stochastic noise

Assumption	Justification
Errors are (randomly chosen) **Pauli errors**.	*Probabilistic Pauli noise is no more detrimental than general coherent noise.*
High weight errors are exponentially **suppressed** (unlikely).	*Physical processes are typically local/two-body.*

argument based on linearity/operator bases:

Ekert, Macchiavello, Phys. Rev. Lett. **77**, 2585 (1996)

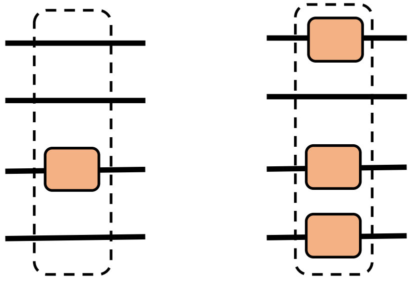
evidence from numerical simulation for surface codes:

Bravyi, Englbrecht, K, Peard npj Quant. Inf., vol. 4, no. 55 (2018)

can be achieved by Pauli twirling

probability of occurence

$\leq O(p)$ $\leq O(p^3)$



Def. A random n-qubit Pauli error E is called **local stochastic noise of strength** $p \in [0,1]$ if

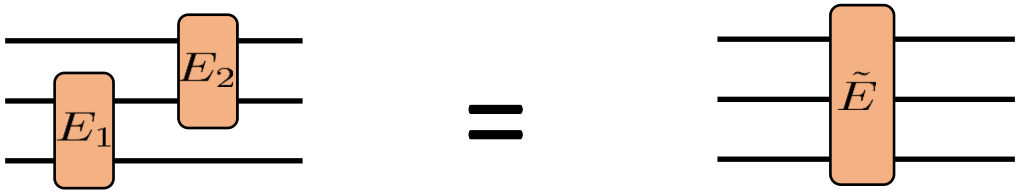
$$\Pr[F \subseteq \mathrm{Supp}(E)] \leq p^{|F|} \text{ for all } F \subseteq \{1, \ldots, n\}$$

Notation: $E \sim \mathcal{N}(p)$.

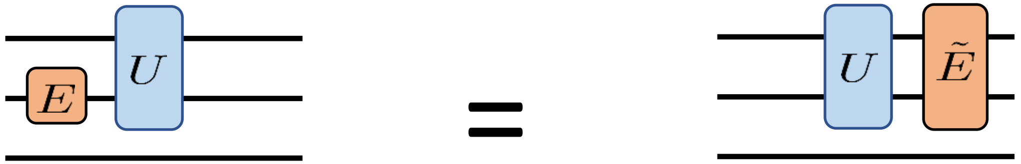
Gottesman, Quant.Info. Comp., vol. 14, no. 15–16, 2014
Fawzi, Grospellier & Leverrier, FOCS 2018

Error transformation rules for probabilistic Pauli noise

Error accumulation:



Error propagation:



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Notation: $E \sim \mathcal{N}(p)$.

Such Pauli errors can be arbitrarily correlated: no locality constraints

Lemma:

- $E \sim \mathcal{N}(p), E' \sim \mathcal{N}(q)$ (possibly dependent)
  $\Longrightarrow$
  $E'E \sim \mathcal{N}(2 \max\{\sqrt{p}, \sqrt{q}\})$
- $E \sim \mathcal{N}(p)$ and C depth-1 Clifford circuit
  $\Longrightarrow$
  $C^{\dagger}EC \sim \mathcal{N}(\sqrt{2p})$

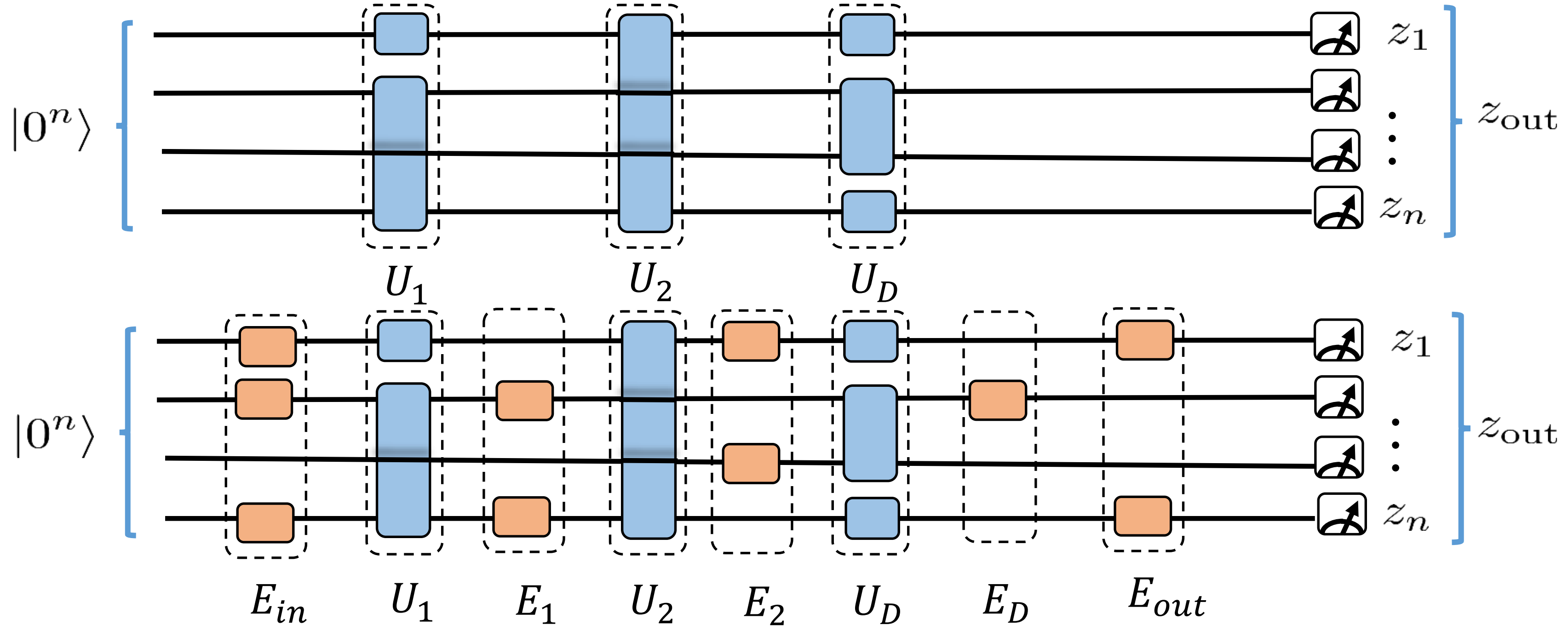
Gottesman, Quant.Info. Comp., vol. 14, no. 15–16, 2014
Fawzi, Grospellier & Leverrier, FOCS 2018

Local stochastic noise in quantum circuits

Let $U = U_D U_{D-1} \cdots U_1$ be a quantum circuit with layers of 1- and 2-qubit gates U_j

ideal circuit

noisy implementation



Def. A **noisy implementation of U with error strength** gives a sample $z_{\text{out}} \in \{0,1\}^n$ from

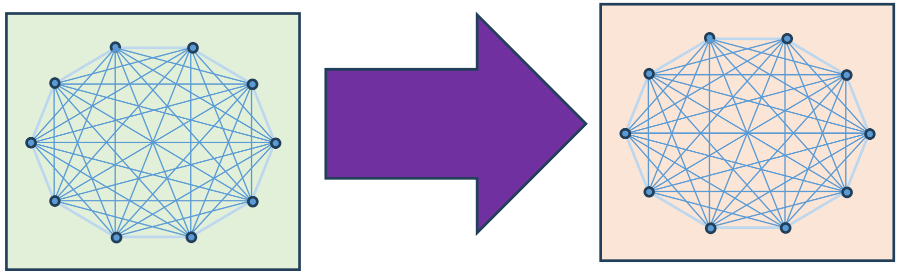
$$P(z_{\text{out}} \mid E_{\text{in}}, E_1, \ldots, E_D, E_{\text{out}}) = \left|\langle z_{\text{out}} | E_{\text{out}} E_D U_D \cdots E_1 U_1 E_{\text{in}} |0^n\rangle\right|^2$$

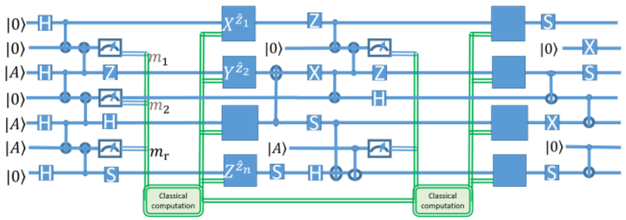
where $E_j \sim \mathcal{N}(p)$ (qubit/gate noise) $E_{in} \sim \mathcal{N}(p)$ (preparation noise)

$E_{out} \sim \mathcal{N}(p)$ (measurement noise)

Universal quantum computation: Fault-tolerance constructions

Can the (ideal) circuit $\mathcal{Q}_{\text{ideal}}$ be simulated using noisy components?





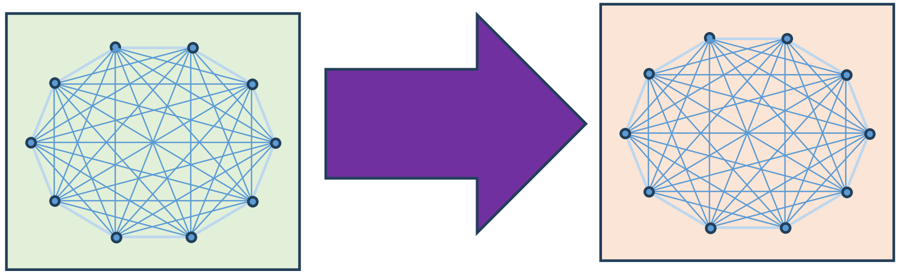
Given: (adaptive) quantum circuit $\mathcal{Q}_{\text{ideal}}$ on n qubits of depth T

Fault-tolerance construction

Larger circuit $\mathcal{Q}_{\text{FT}}$ whose noisy implementation simulates $\mathcal{Q}_{\text{ideal}}$

Fault-tolerance construction of Yamasaki and Koashi

Can the (ideal) circuit $\mathcal{Q}_{\text{ideal}}$ be simulated using noisy components?



Theorem [1] There is a threshold error strength $p_0 > 0$ such that for large n and $\varepsilon \in (0,1)$:

Let $\mathcal{Q}_{\text{ideal}}$ be a circuit with

n qubits

$T(n) = O(\text{poly}(n))$ depth

Then:

There is a circuit $\mathcal{Q}_{\text{FT}}$ with

$n \cdot O(1)$ qubits

$T(n) \cdot \exp\big(O(\log^2(\log(n/\varepsilon)))\big)$ depth

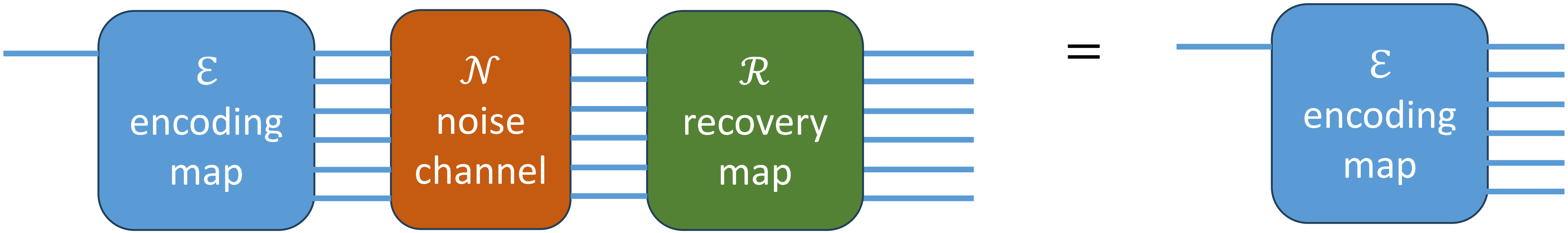
such that a noisy implementation of $\mathcal{Q}_{\text{FT}}$ with local stochastic noise of strength $p \leq p_0$ has an output distribution whose L^1-distance to the output distribution of $\mathcal{Q}_{\text{ideal}}$ bounded by ε.

[1] Yamasaki and Koashi, Nat. Phys. no. 20, Feb 2024

Quantum memories: protecting information against noise

How to protect against noise given by a CPTP map ?

$\mathcal{N}: \mathcal{B}\left((\mathbb{C}^2)^{\otimes n}\right) \rightarrow \mathcal{B}\left((\mathbb{C}^2)^{\otimes n}\right)$ **noise channel**



Encoding map: A CPTP map

$\mathcal{E}: \mathcal{B}\left((\mathbb{C}^2)^{\otimes k}\right) \rightarrow \mathcal{B}\left((\mathbb{C}^2)^{\otimes n}\right)$

Recovery map: A CPTP map

$\mathcal{R}: \mathcal{B}\left((\mathbb{C}^2)^{\otimes n}\right) \rightarrow \mathcal{B}\left((\mathbb{C}^2)^{\otimes n}\right)$

ideally want

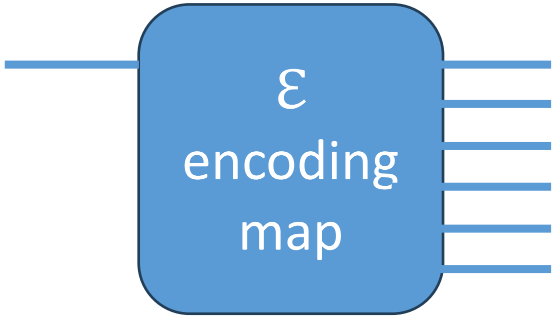
$\mathcal{R} \circ \mathcal{N} \circ \mathcal{E} = \mathcal{E}$

"perfect recovery"

Stabilizer codes

Encoding map: A CPTP map

$$\mathcal{E}: \mathcal{B}\left((\mathbb{C}^2)^{\otimes k}\right) \to \mathcal{B}\left((\mathbb{C}^2)^{\otimes n}\right)$$



The encoded state $\mathcal{E}(\rho)$ has support on a certain subspace $\mathcal{L} \subset (\mathbb{C}^2)^{\otimes n}$, the **code space** of a quantum error-correcting code.

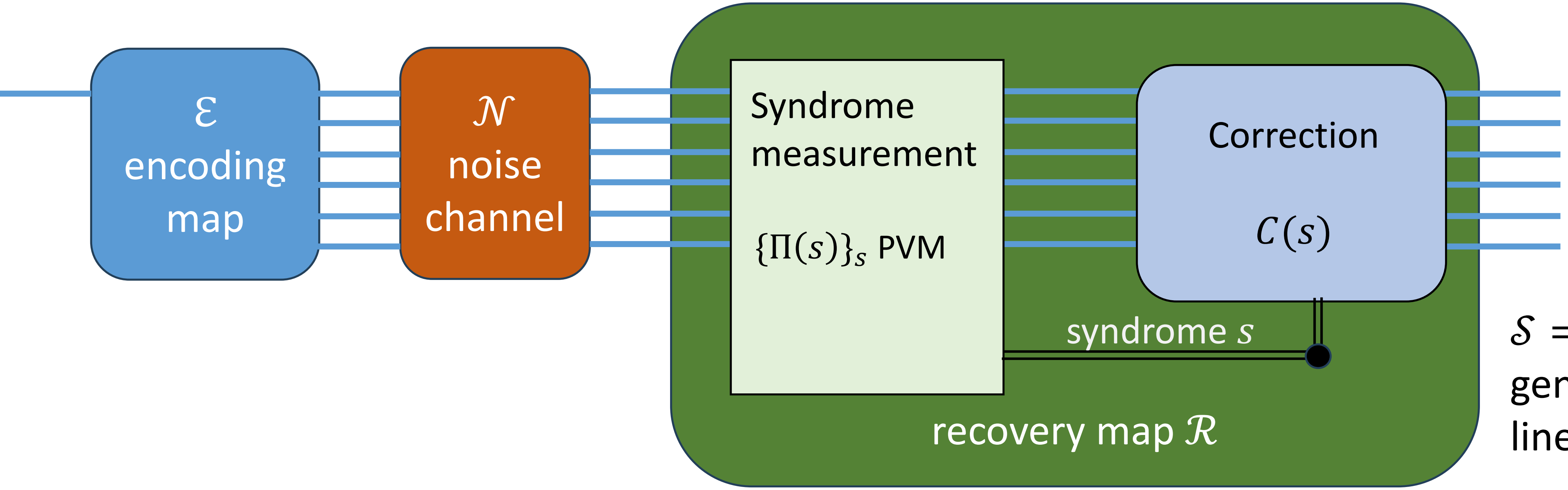
$\mathcal{S}$ **stabilizer group**, i.e., an abelian subgroup of the n-qubit Pauli group $\mathcal{P}_n$ such that $-I \notin \mathcal{S}$.

$\mathcal{S} = \langle S_1, \dots, S_{n-k} \rangle$ generated by $n-k$ linearly independent generators $\{S_j\}_{j=1}^{n-k}$.

$$\mathcal{L}_{\mathcal{S}} = \left\{\Psi \in (\mathbb{C}^2)^{\otimes n} \mid S\Psi = \Psi \text{ for all } S \in \mathcal{S}\right\}$$

code space of stabilizer code

Recovery in a stabilizer code



$\mathcal{S} = \langle S_1, \dots, S_{n-k} \rangle$ generated by $n-k$ linearly independent generators $\{S_j\}_{j=1}^{n-k}$.

Recovery procedure $\mathcal{R}$

1. measure the eigenvalue $(-1)^{s_j}$ of S_j generating a syndrome $s \in \{0,1\}^{n-k}$

   Let $\Pi(s) = \Pi_{j=1}^{n-k} \frac{1}{2}\left(I + (-1)^{s_j} S_j\right)$

   be the corresponding projection.

2. Compute a Pauli correction $C(s) \in \mathcal{P}_n$, i.e., evaluate a function $C : \{0,1\}^{n-k} \to \mathcal{P}_n$.

   Apply $C(s)$.

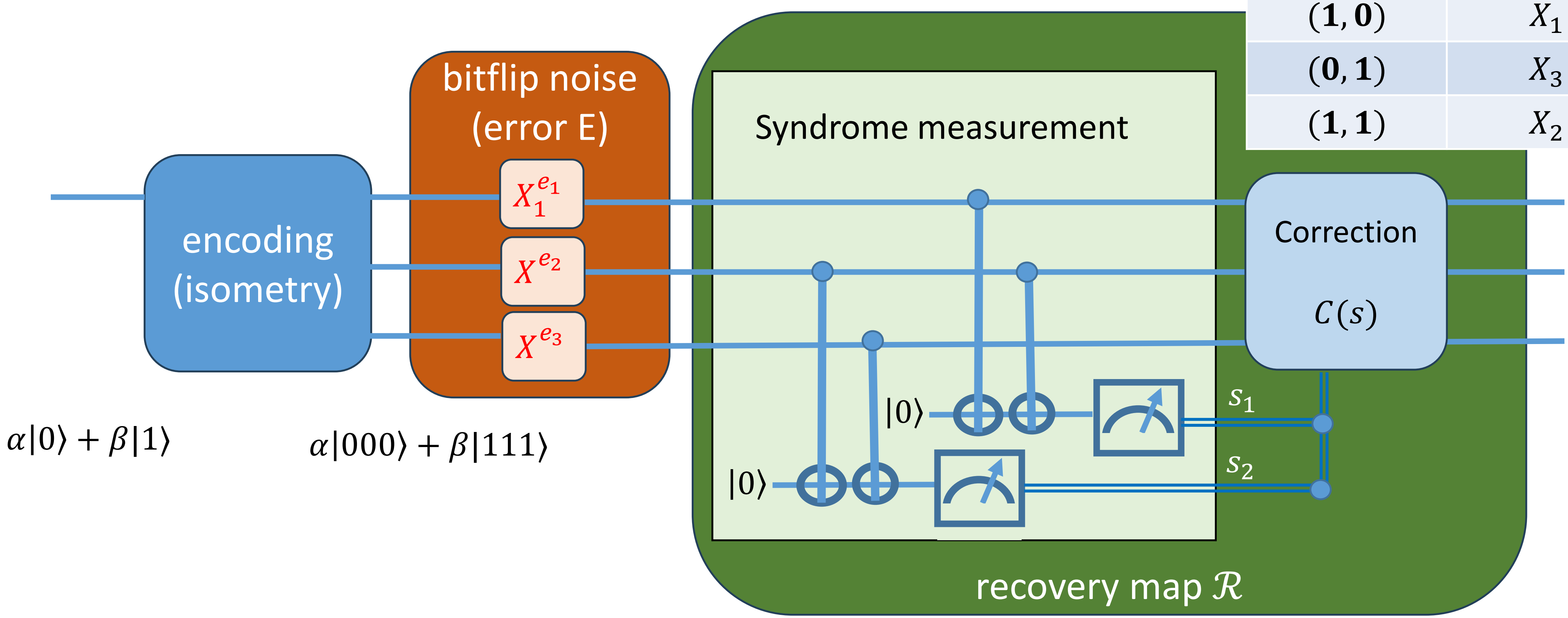
3-qubit repetition code

$\mathcal{L}_{\mathcal{S}} = \text{span}\{|000\rangle, |111\rangle\}$

Example: 3-qubit repetition code $\quad \mathcal{S} = \langle Z_1 Z_2, Z_2 Z_3 \rangle$

Logical operators $\quad \bar{X} = X_1 X_2 X_3, \bar{Z} = Z_1$

$s = (s_1, s_2)$	$C(s)$
$(0,0)$	I
$(1,0)$	X_1
$(0,1)$	X_3
$(1,1)$	X_2



If recovery fails, we must have $(e_1, e_2) = (1,1)$ or $(e_1, e_3) = (1,1)$ or $(e_2, e_3) = (1,1)$

3-qubit repetition code

$\mathcal{L}_{\mathcal{S}} = \text{span}\{\,|000\rangle, |111\rangle\}$

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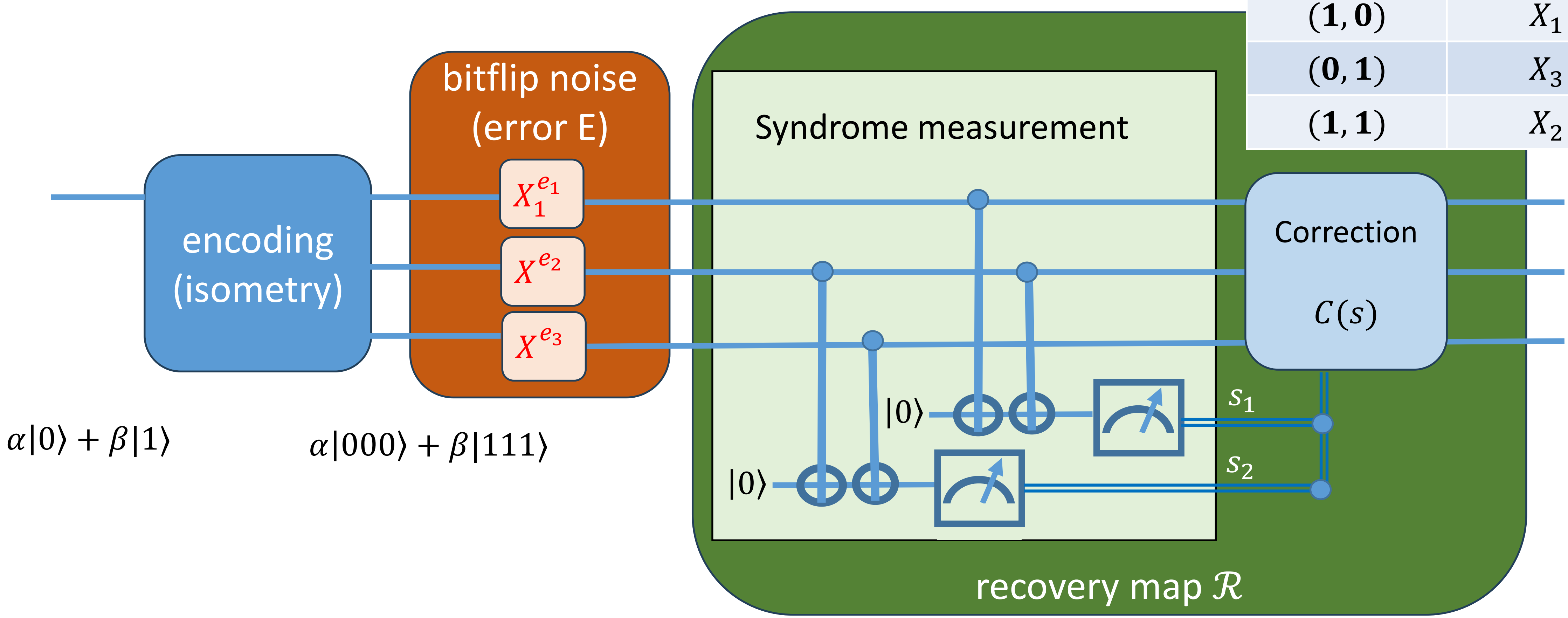
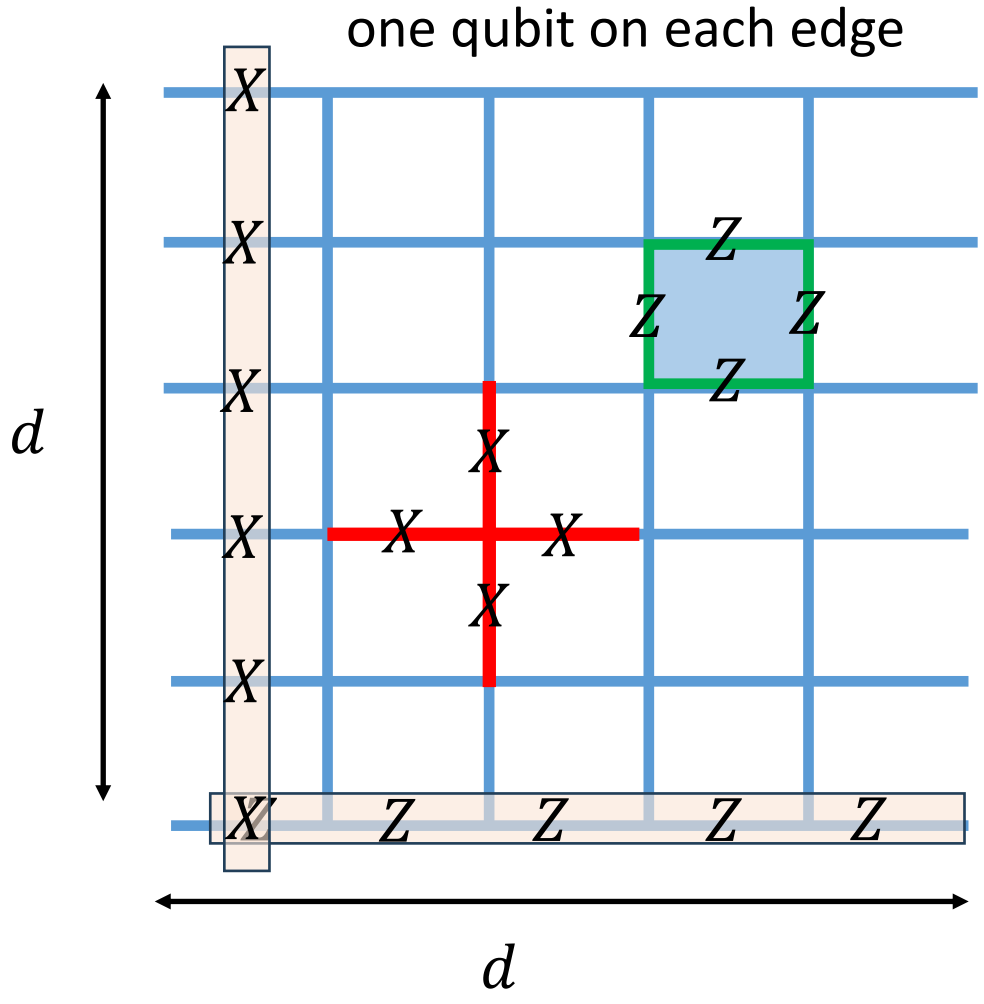


$$\Pr[\text{ recovery fails }] \leq \Pr_E[\{1,2\} \subseteq \text{supp}(E)] + \Pr_E[\{1,3\} \subseteq \text{supp}(E)] + \Pr_E[\{2,3\} \subseteq \text{supp}(E)]$$

$$\leq 3\,p^2 =: f(p)$$

if $E \sim N(p)$ is local stochastic

The surface code: A $[n, 1, \Theta(n^{1/2})]$-code with geometrically local generators



- encodes 1 qubit into $n = d^2 + (d-1)^2$ physical qubits
- Code space is ground space of gapped local Hamiltonians with 4-qubit interactions
- Code has distance d : no operator with support of size O(1) can distinguish ground states

Logical operators

$$\bar{X} = \prod_{e \in P} X_e$$

$$\bar{Z} = \prod_{e \in P'} Z_e$$

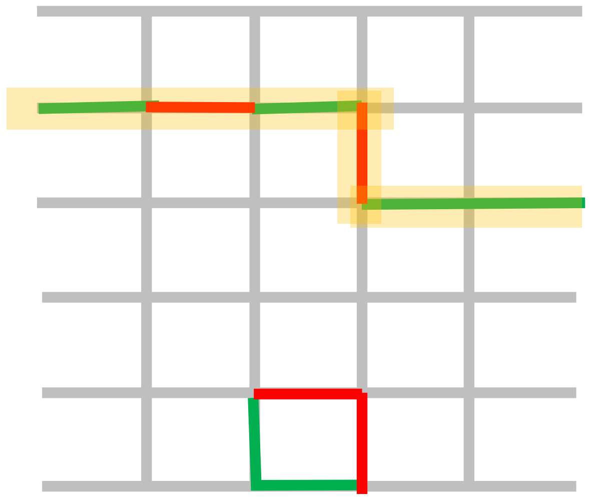
where P (resp. P') connect left/right (resp. top/bottom) boundaries.

Combinatorial upper bounds on failure probability

$\Pr[\text{ recovery fails }] \leq \sum_{m\in\mathcal{M}} \Pr_E[D_m \subseteq \mathrm{supp}(E)]$

$\{D_m\}_{m\in\mathcal{M}}$ is a certain family of subsets of qubits

$f(p) := \sum_{m\in\mathcal{M}} p^{|D_m|}$ for $p \in [0,1]$



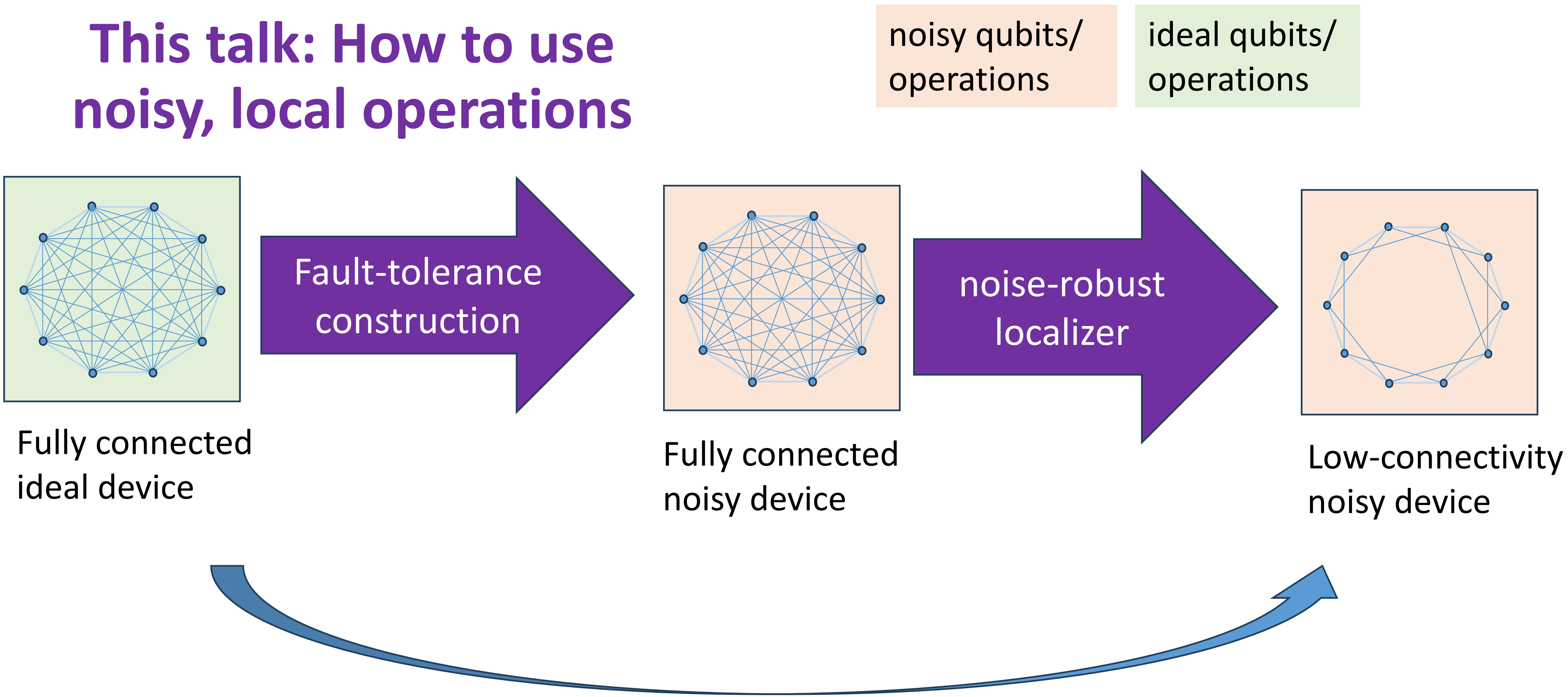
Lemma: (Combinatorics) $f(p) \leq \mathrm{poly}(d) \cdot (p/p_0)^{d/2}$ for all $p \leq p_0 = 1/36 \approx 0.028$

Corollary: Can recover from ***local stochastic error*** E ***of strength*** $\boldsymbol{p \leq p_0}$ for sufficiently large d

Proof: Use the Definition of local stochastic errors!

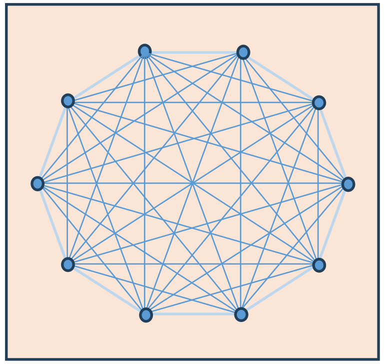
Dennis, Kitaev, Preskill, J. Math. Phys. 43, 4452-4505 (2002).
http://theory.caltech.edu/~preskill/ph219/fault-tolerance-2011.pdf
Fowler Phys. Rev. Lett. 109, 180502 (2012)

This talk: How to use noisy, local operations

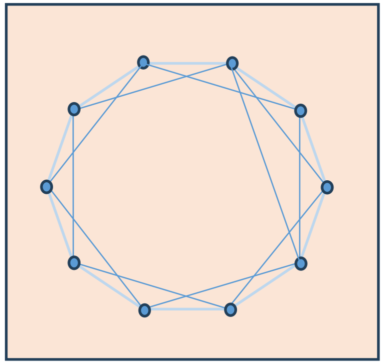


Main consequence:
Overhead-efficient fault-tolerance constructions incorporating locality constraints.

Noise-robust localization

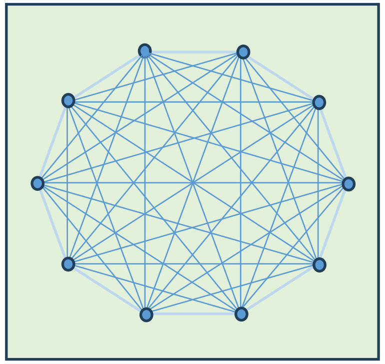


noise-robust localizer

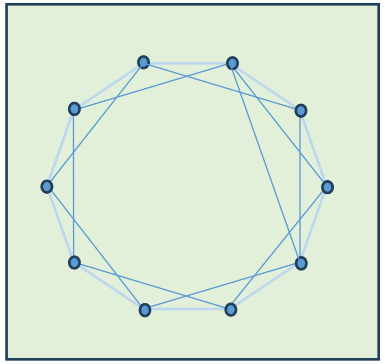


Fully connected noisy device

Low-connectivity noisy device



localizer



Fully connected ideal device

Low-connectivity ideal device

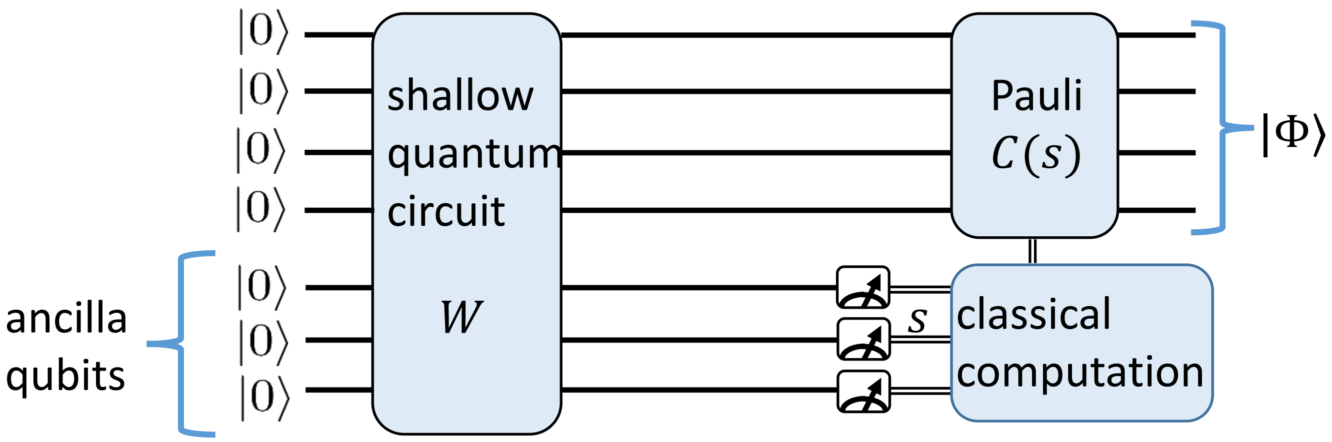
Basic building blocks:

- Bell (resource) states
- Entanglement-Swapping

Single-shot (stabilizer) state preparation protocols

Prepare a (stabilizer) state $\boldsymbol{\Phi} \in \left(\mathbb{C}^2\right)^{\otimes r}$ as follows:

1. Apply a constant-depth Clifford circuit W to $|0^N\rangle$.
2. Apply single-qubit measurements to $N - r$ qubits, resulting in a measurement results $s \in \{0,1\}^{N-r}$.
3. Apply a Pauli correction $C(s) \in \mathcal{P}_r$ to the remaining qubits.



$(C: \{0,1\}^{N-r} \rightarrow \mathcal{P}_r$ should be efficiently computable.)

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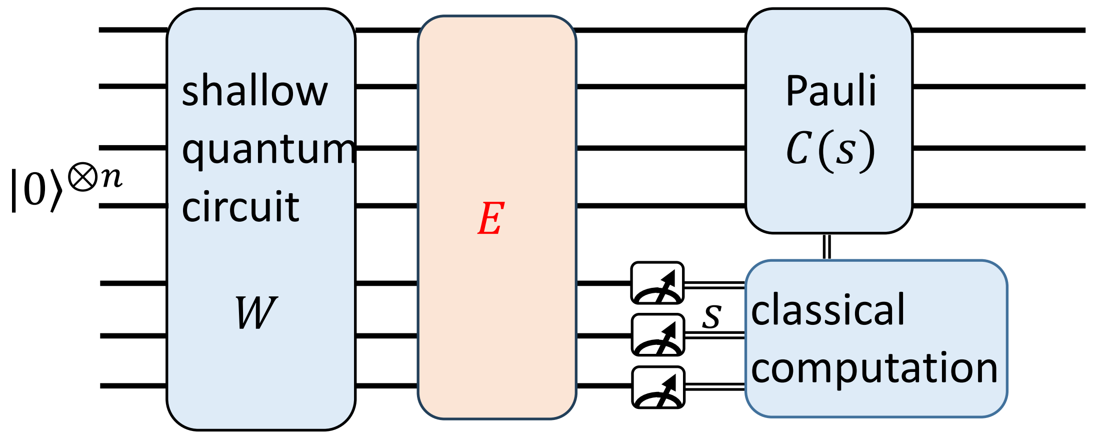
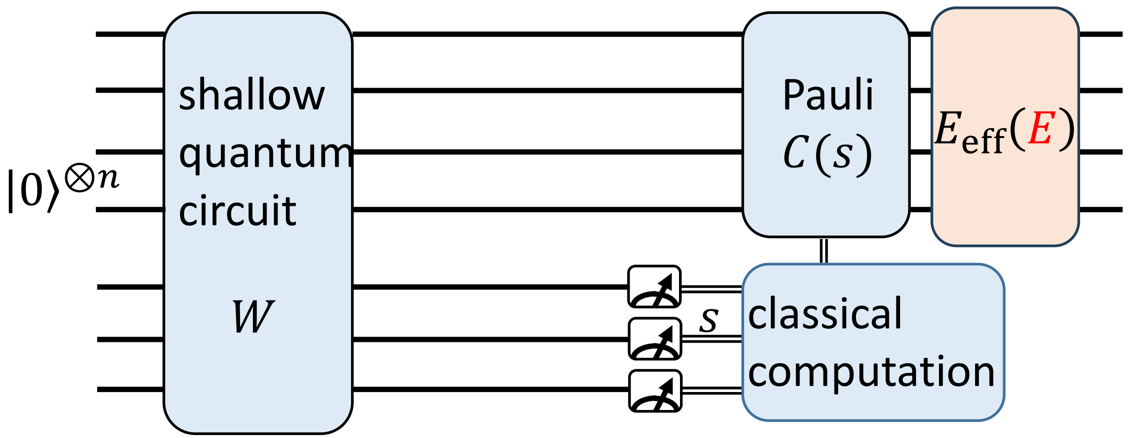


$$E_{\text{eff}}(E)|\Phi\rangle \propto C(s)(I \otimes \langle s|)E|0^N\rangle$$



State preparation: Robustness

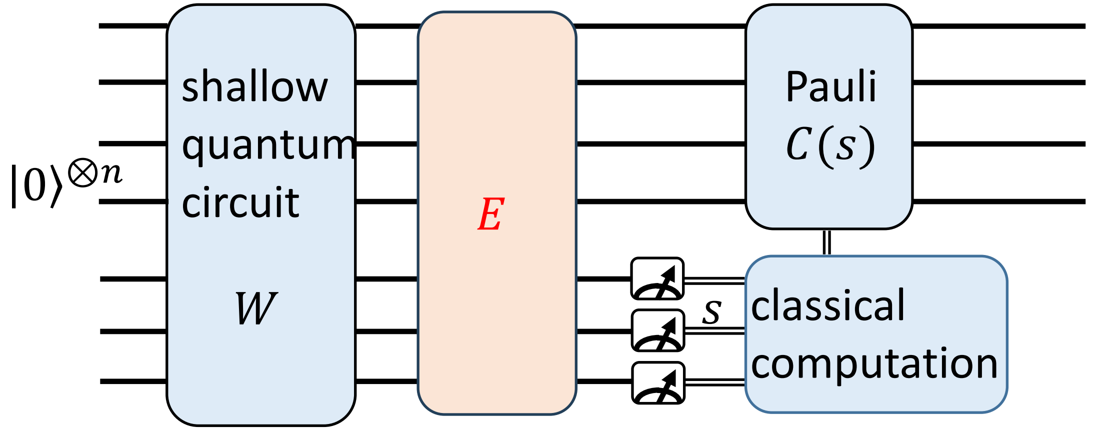
$$\text{FAIL} \coloneqq \{E \in \mathcal{P}_n \mid E_{\text{eff}}(E)|\Phi\rangle \neq |\Phi\rangle\}$$

Definition: Let $f: [0,1] \to [0,1]$. The protocol is called ***f*-robust** if there is a family $\{D_m\}_{m\in\mathcal{M}} \subset 2^{[n]}$ such that

(a) For every $E \in$ FAIL:

$$\exists\, m \in \mathcal{M} \text{ such that } D_m \subseteq \text{supp}\,(E)$$

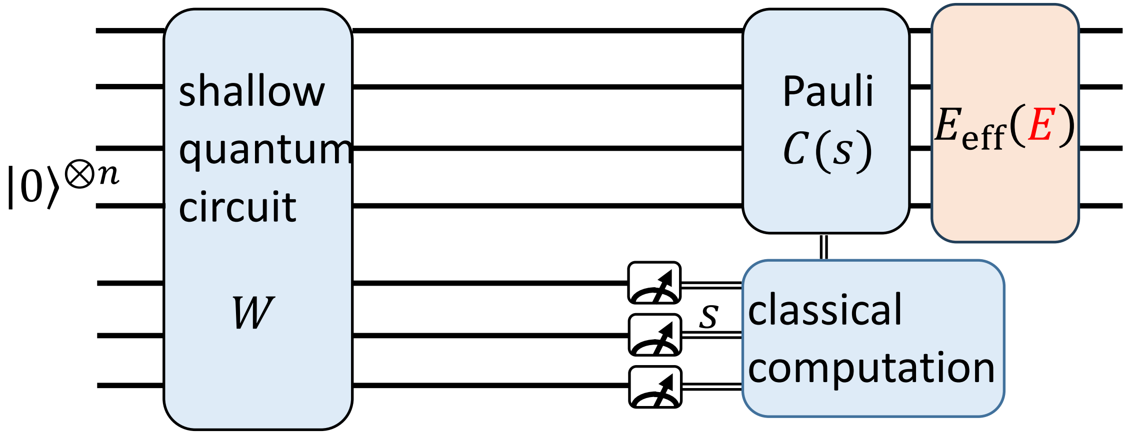
(b) $\sum_{m\in\mathcal{M}} p^{|D_m|} \leq f(p)$ for all $p \in [0,1]$.



$$E_{\text{eff}}(E)|\Phi\rangle \propto C(s)(I \otimes \langle s|)E|0^N\rangle$$

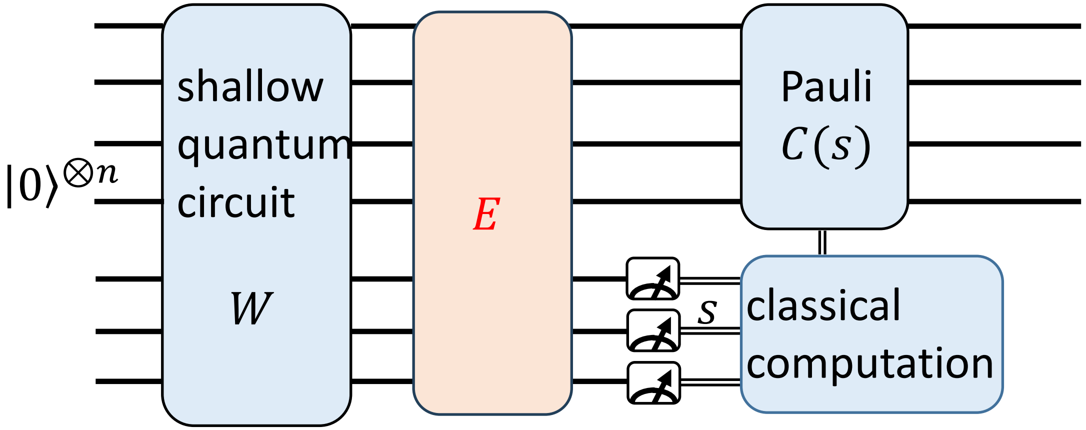
We are often only interested in the probability

$$\Pr\,[\text{protocol fails to prepare } |\Phi\rangle] = \Pr[\text{E} \in \text{FAIL}]$$



State preparation: Robustness and failure probability

$$\mathrm{FAIL} := \{E \in \mathcal{P}_n \mid E_{\mathrm{eff}}(E)|\Phi\rangle \neq |\Phi\rangle\}$$



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(b) $\sum_{m\in\mathcal{M}} p^{|D_m|} \leq f(p)$ for all $p \in [0,1]$.

Lemma An f-robust protocol π fault-tolerantly prepares the state Φ under local stochastic noise $E \sim \mathcal{N}(p)$:

$$\Pr\,[\pi \text{ fails}\,] \leq f(p) \text{ for any } p \in [0,1].$$

Proof. Consider local stochastic noise $E \sim \mathcal{N}(p)$ with strength $p \leq p_0$. Then

$$\Pr[E \in \mathrm{FAIL}] \leq \sum_{m\in\mathcal{M}} \Pr[D_m \subseteq \mathrm{supp}(E)]$$ by the union bound and (a)

$$\leq \sum_{m\in\mathcal{M}} p^{|D_m|}$$ by the assumption $E \sim \mathcal{N}(p)$

$$\leq f(p)$$ by (b)

Raussendorf, Bravyi, Harrington, PRA (2005)

Robustness of a 3D-local state preparation procedure

Theorem (Quantum bus architecture).

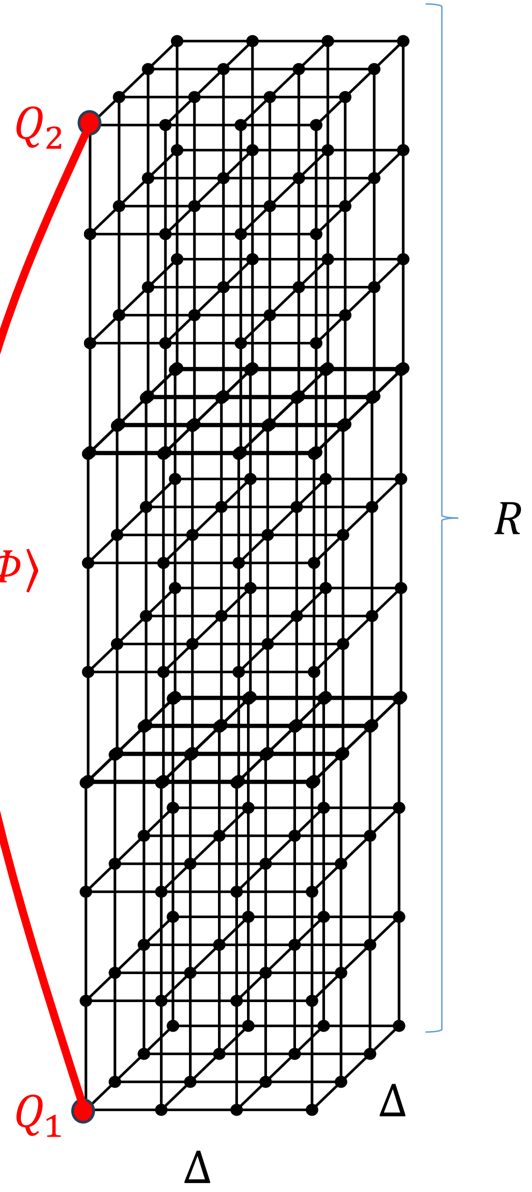
For any $R \geq 2$, $\Delta \in \mathbb{N}$ with $R \leq e^{\frac{\Delta}{8}}$ there is a circuit π such that:

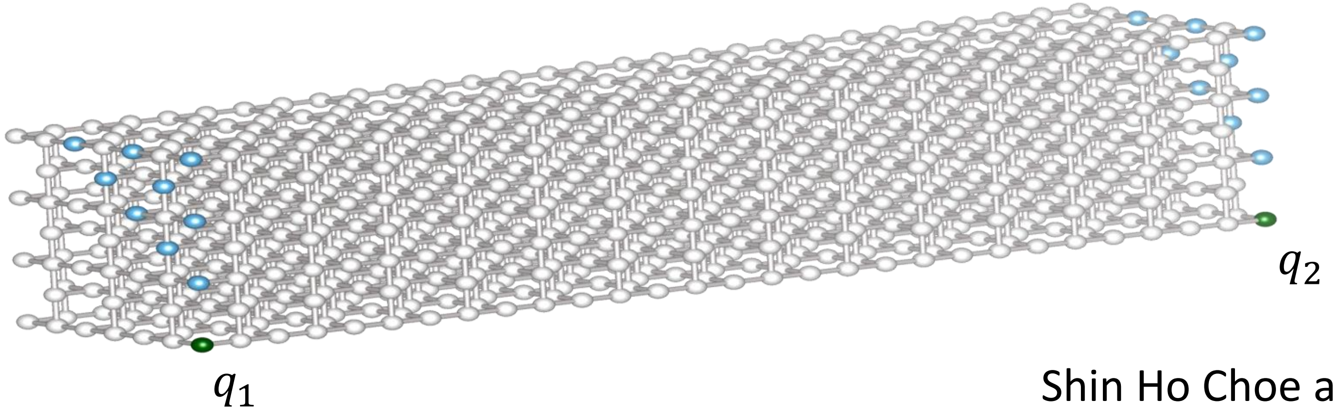
(i) π is local and constant-depth on the grid graph $P_\Delta \times P_\Delta \times P_R$.

(ii) π *prepares the state* $|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

(iii) **π is f-robust for a function $f: [0,1] \to [0,1]$ that satisfies**

$$f(p) = \frac{p}{p_0} \quad \text{for any } p \leq p_0 := 1/5004.$$





Shin Ho Choe and RK, arXiv: 2209.09774

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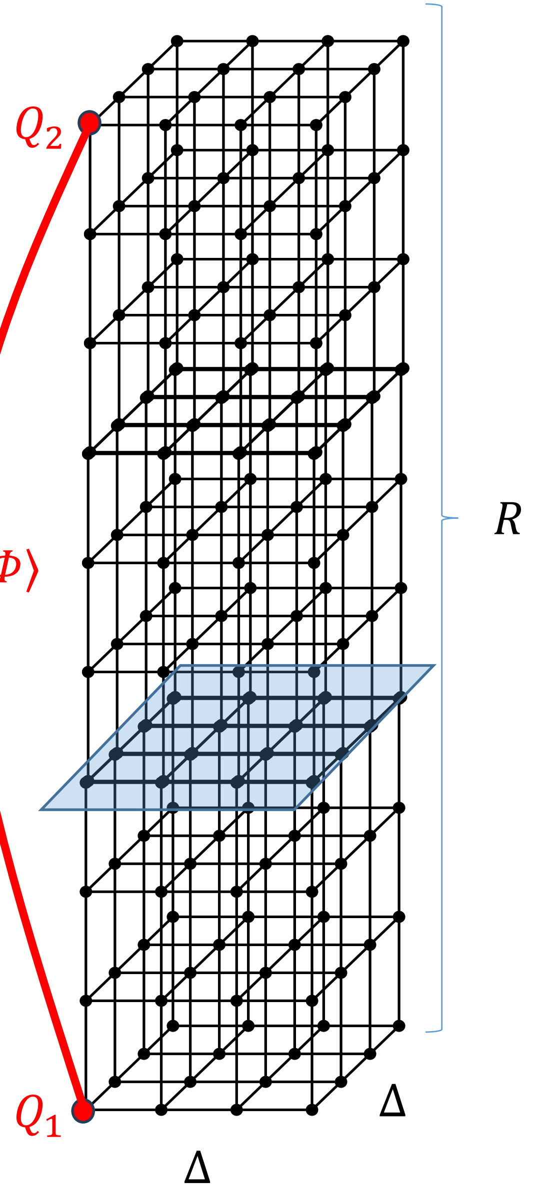
$$\boldsymbol{f(p) = \frac{p}{p_0}} \quad \textbf{for any } \boldsymbol{p \leq p_0 := 1/5004.}$$

Creates a constant-fidelity Bell pair at a distance exponential in $\sqrt{N_{repeater}}$ where the number of qubits per slice is $N_{repeater} = \Delta^2$.

(We know that the maximal distance for any protocol is $O(N_{repeater})$.



State preparation up to local stochastic noise

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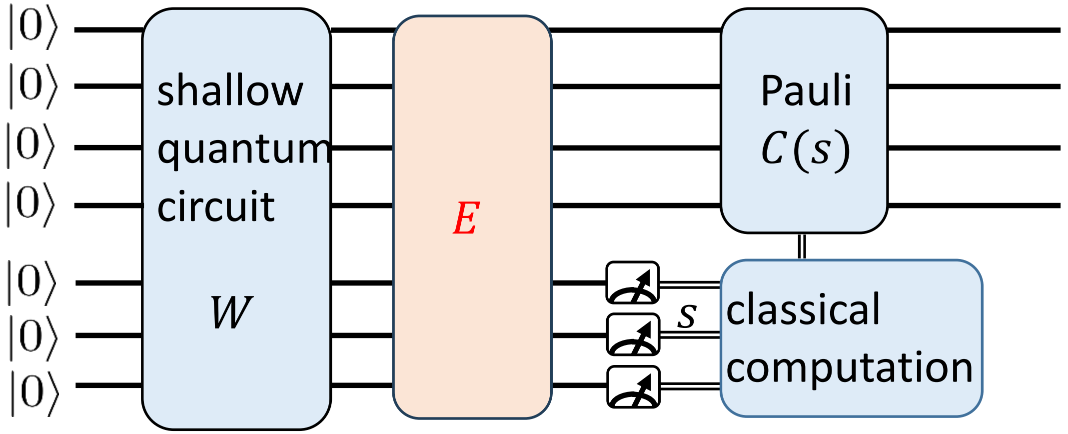


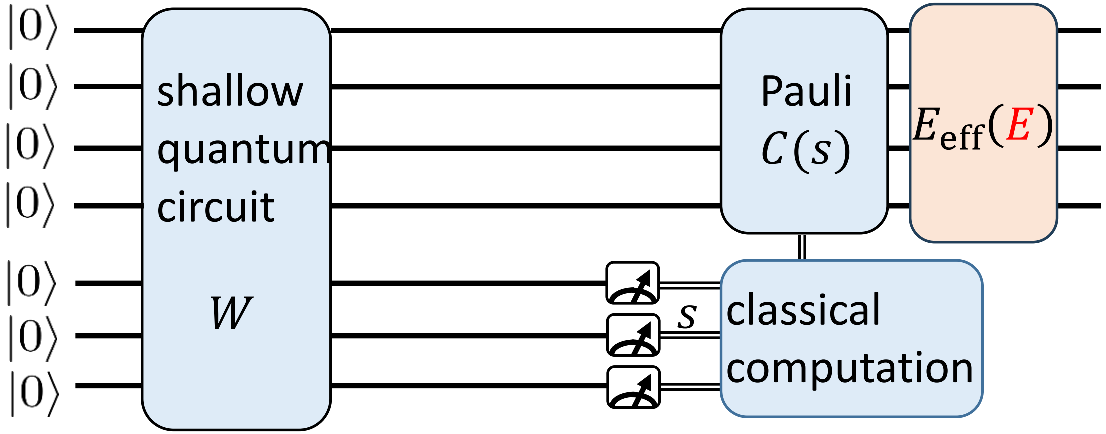
$$E_{\text{eff}}(E)|\Phi\rangle \propto C(s)(I \otimes \langle s|)E|0^N\rangle$$

Definition: The protocol prepares $|\Phi\rangle$

up to local stochastic noise of strength q

under local stochastic noise of strength p

if $\boldsymbol{E \sim \mathcal{N}(p)}$ implies that $\boldsymbol{E_{\text{eff}}(E) \sim \mathcal{N}(q)}$.



A parallel repetition theorem

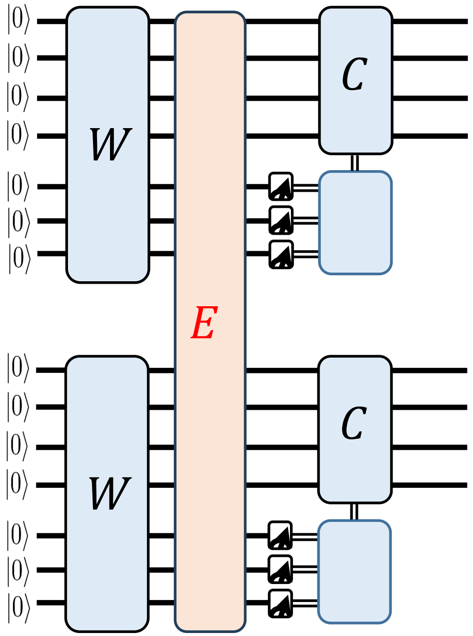
Theorem (Parallel repetition)

Let π be a f-robust protocol preparing $|\Phi\rangle \in (\mathbb{C}^2)^{\otimes r}$.

Then: $\pi \times \pi$ prepares $|\Phi\rangle \otimes |\Phi\rangle$

up to local stochastic noise of strength $f(p)^{1/r}$.

under local stochastic noise of strength $\boldsymbol{p}$



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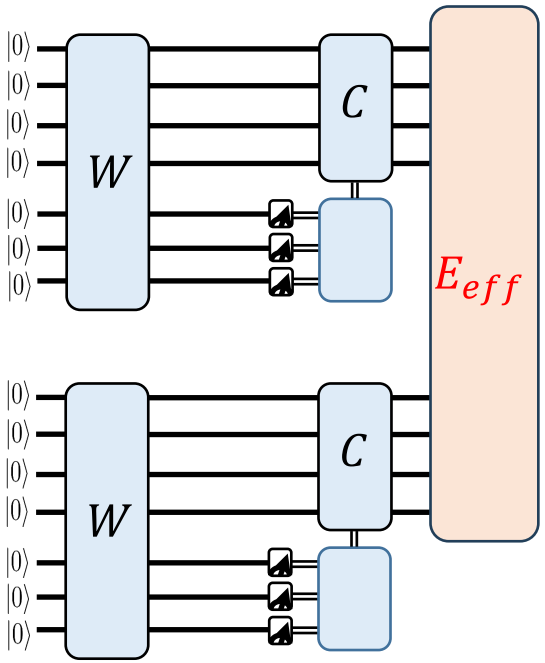
up to local stochastic noise of strength $\boldsymbol{f(p)^{1/r}}$**.**

under local stochastic noise of strength p



A parallel repetition theorem

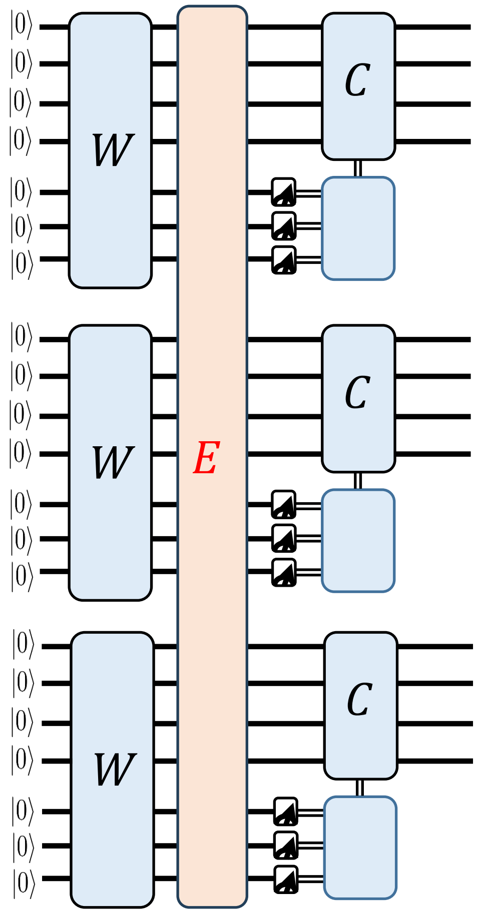
Theorem (Parallel repetition)

Let π be a f-robust protocol preparing $|\Phi\rangle \in (\mathbb{C}^2)^{\otimes r}$.

Then: $\pi^{\times k}$ prepares $|\Phi\rangle^{\otimes k}$

up to local stochastic noise of strength $f(p)^{1/r}$.

under local stochastic noise of strength p



Effective output noise of strength
independent of the number of parallel repetitions.

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Effective output noise of strength
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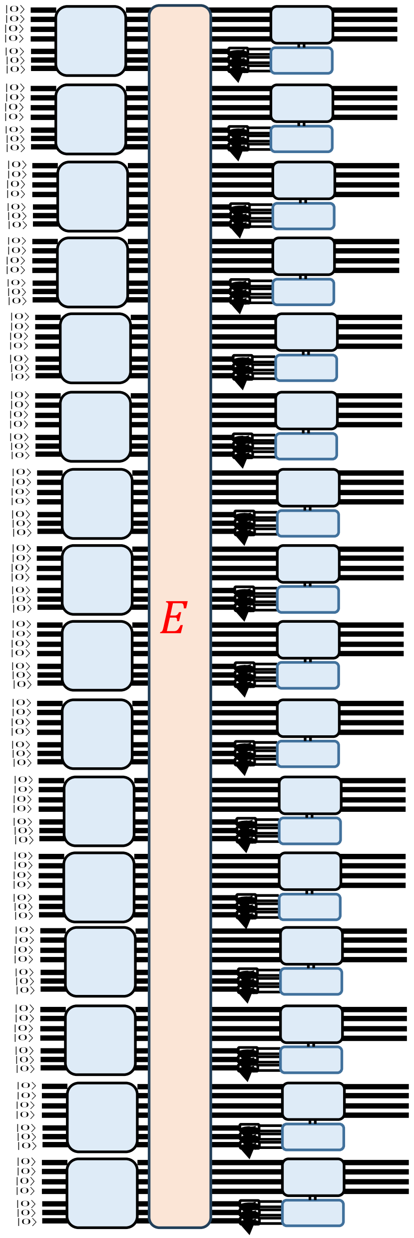
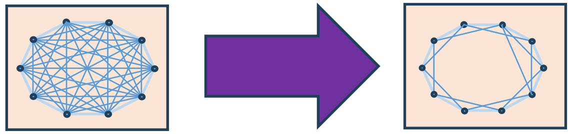
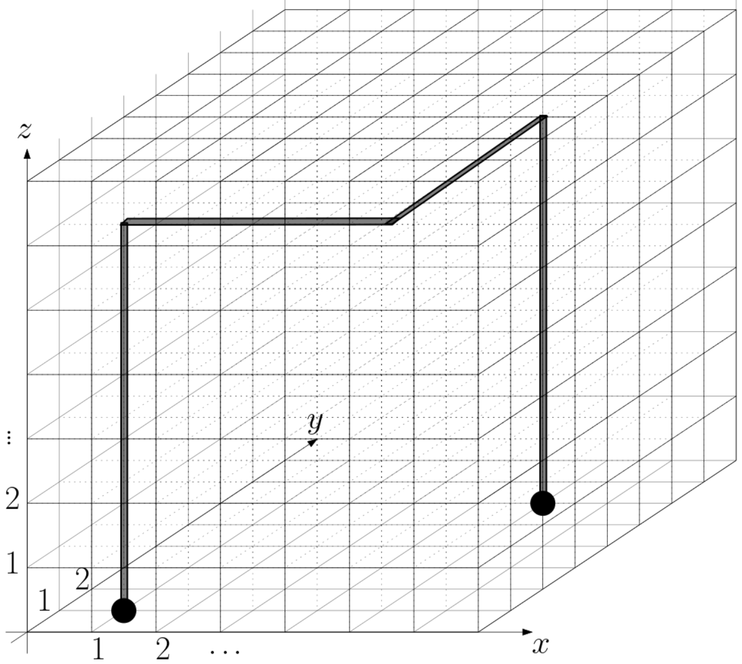
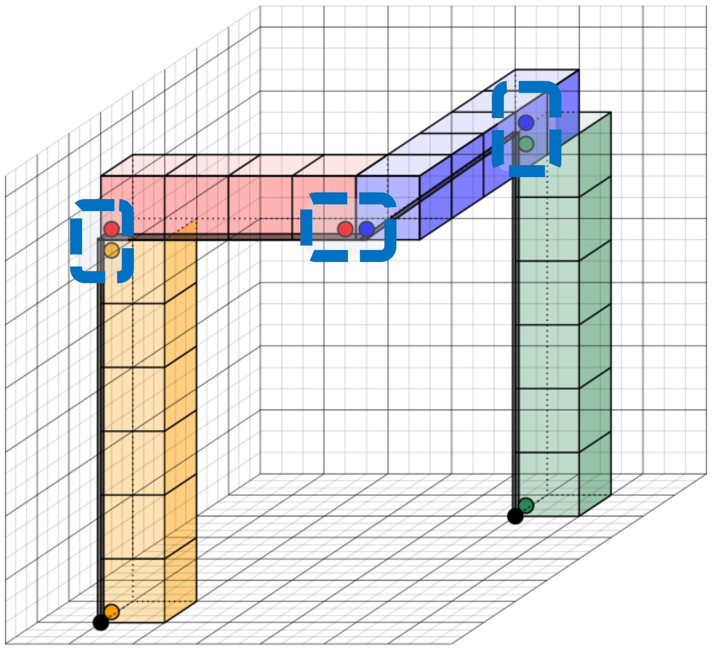


Putting it all together: basic idea



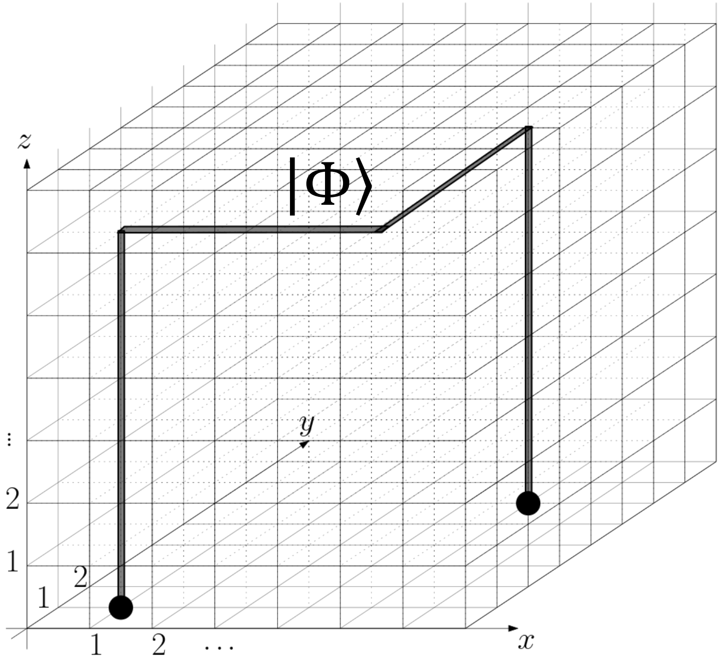


Path P used in pairing



4 buses applied in parallel

fault-tolerant bus &
parallel repetition theorem



3 entanglement swapping measurements

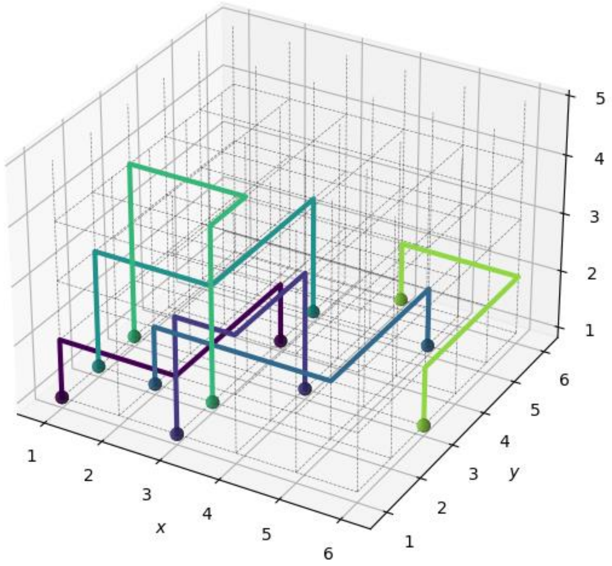
circuit level analysis

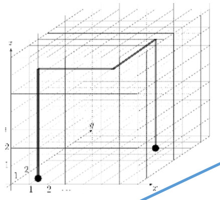
Result:

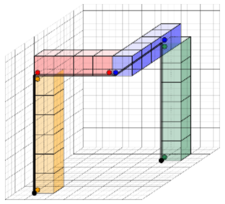
$|\Phi\rangle^{\otimes 4}$ with local stochastic noise

$|\Phi\rangle$ constant-fidelity Bell pair

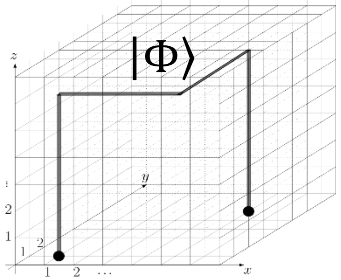
Putting it all together







⋮



⋮

$k = L^2/2$ paths $P_1, \ldots, P_k$

$4\,k$ buses applied in parallel

$3\,k$ entanglement swapping measurements

fault-tolerant bus &
parallel repetition theorem

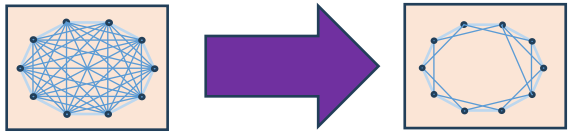
parallel repetition theorem
for entanglement swapping

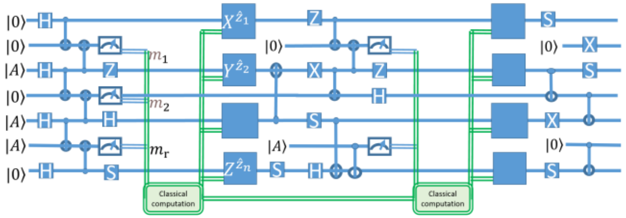
Result:

$|\Phi\rangle^{\otimes 4k}$ with local stochastic noise

$|\Phi\rangle^{\otimes k}$ up to local stochastic noise

Main result: noise-robust localization





Given: adaptive quantum circuit $\mathcal{Q}$

on $\boldsymbol{n}$ qubits

of depth $\boldsymbol{T}$

involving **non-local** operations

noise-robust localizer

Theorem:

There is an adaptive circuit $\mathcal{Q}'$ with

the following properties:

1. $\mathcal{Q}'$ uses $\boldsymbol{n \cdot O\left(n^{1/2} log^3 n\right)}$ qubits

and is **local** when these are arranged

on a **3D grid graph**.

2. $\mathcal{Q}'$ has quantum depth of order $\boldsymbol{O(T)}$.

3. $\mathcal{Q}'$ **simulates** $\mathcal{Q}$ exactly

4. A noisy implementation of $\mathcal{Q}'$ with

noise of strength p is equivalent to a

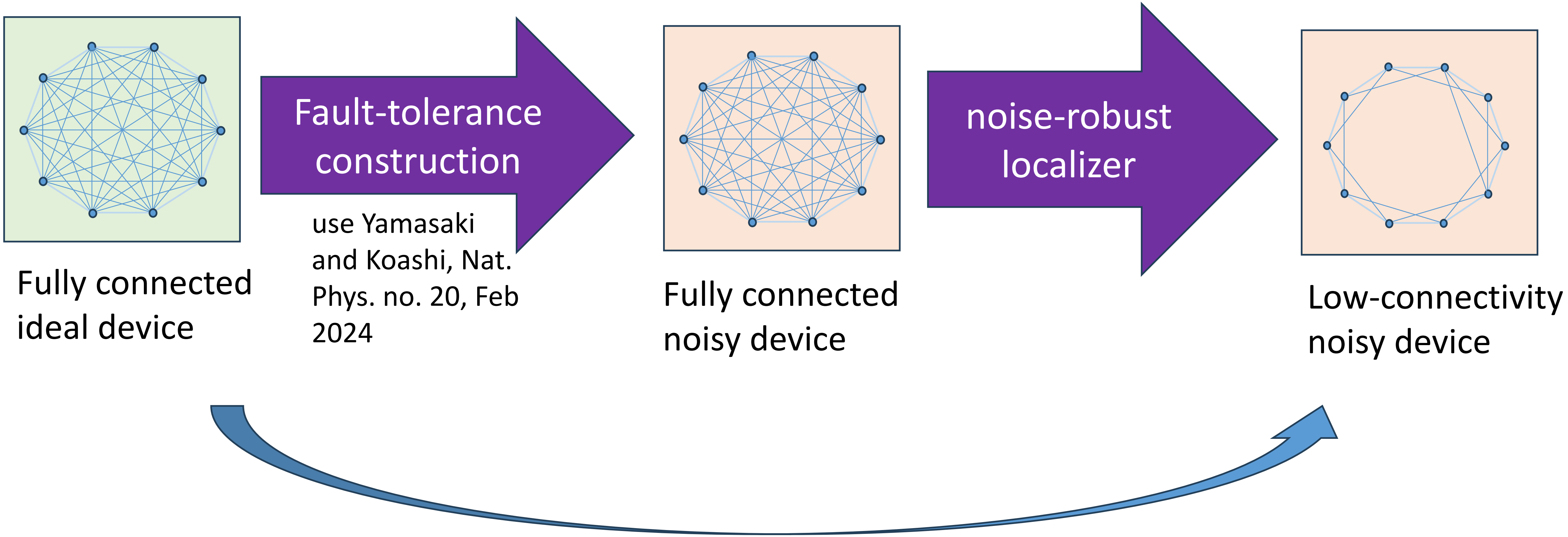
noisy implementation of $\mathcal{Q}$

with noise of strength Cp^c

Application I of localizers: Fault-tolerant computation

noisy qubits/ operations

ideal qubits/ operations



Main consequence I:

Overhead-efficient fault-tolerance constructions incorporating locality constraints.

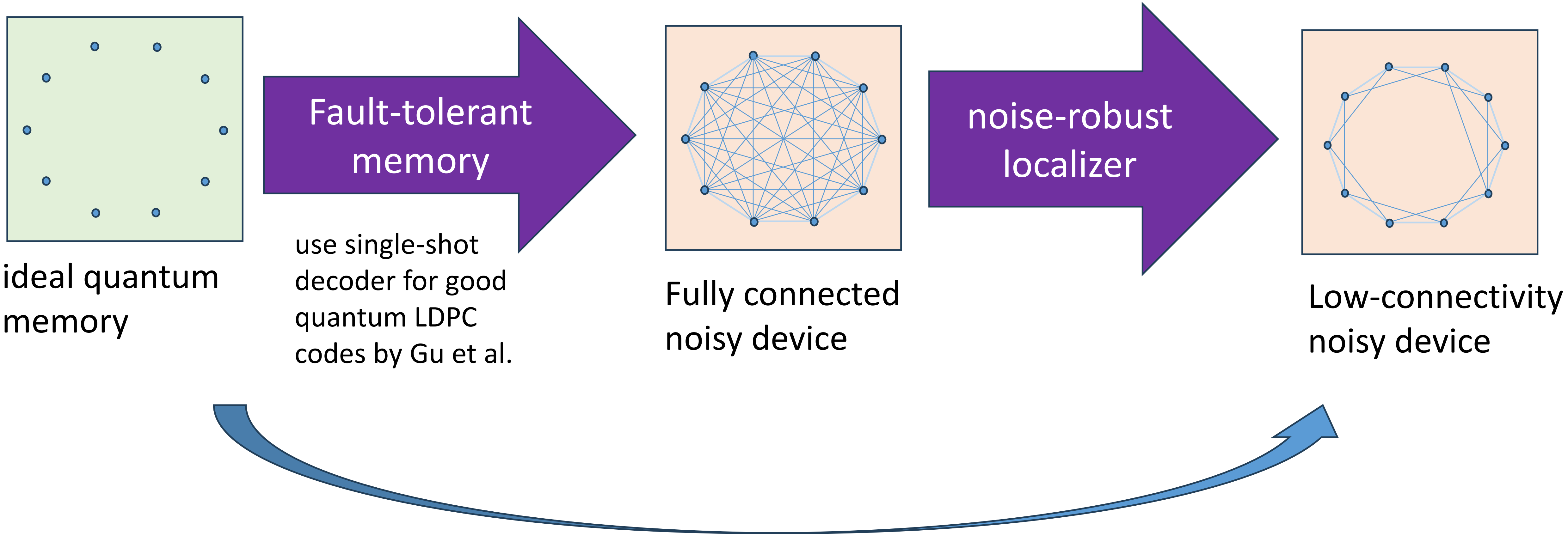
Reference	geometry/locality	physical qubit overhead	quantum depth overhead
[1-4]	1D, 2D & 3D	$\text{poly}(n)$	$\text{poly}(n)$
[5-8]	2D & 3D	$\text{poly}(\text{n})$	$\text{poly}(n)$
our work	quasi-2D-local	$\text{O}(n \log^3 n)$	$\exp O((\log^2 (\log n))$
our work	$3D$-local	$\text{O}(n^{1/2} \log^3 n)$	$\exp O((\log^2 (\log n))$
Fawzi et al. & Gottesman	non-local	$\text{O}(1)$	$\text{O}(n)$
Yamasaki et al.	non-local	$\text{O}(1)$	$\exp O((\log^2 (\log n))$

[1] Aharonov and Ben-Or, SIAM J. Comp. 38, 1207-1282 (2008)
[2] Gottesman, J. Mod. Opt 47, 333-345 (2000)
[3] Svore, Terhal, DiVincenzo, Phys. Rev. A72, 002317 (2005)
[4] Svore, DiVincenzo, Terhal, Quant. Inf. Comp. Vol. 7, No. 4, pp. 297-318 (2007)
[5] Raussendorf, Harrington, Phys. Rev. Lett. 98, 190504 (2007)
[6] Bombin, arXiv:1810.0957
[7] Hormsan, Fowler, Devitt, Meter, NJP, 2012
[8] Litinski, Quantum 2019.
[9] Yamasaki and Koashi, Nat. Phys. no. 20, Feb 2024
[10] Fawzi, Grospellier & Leverrier, FOCS 2018& D. Gottesman, Quant. Inf. Comp. 2014

Application II of localizers: quantum memories

noisy qubits/ operations

ideal qubits/ operations



Main consequence II:

Overhead-efficient fault-tolerant constructions incorporating locality constraints.

Another application: building quantum memories

construction	physical qubit overhead	quantum depth overhead	locality
toric code/surface codes [1]	polynomial	polynomial	2D
hierarchical code [2] by Pattison et al.	polylogarithmic	polynomial	2D
our work together with [3]	polynomial	constant	3D (or quasi-2D)

[1] E. Dennis, A. Kitaev, A. Landahl, J. Preskill, J. Math. Phys. 43, 4452-4505 (2002).

[2] Pattison, Krishna and Preskill, arXiv:2303.04798

[3] Gu, Tang, Caha, Choe, He, Kubica, Commun. Math. Phys. 405, 85 (2024)

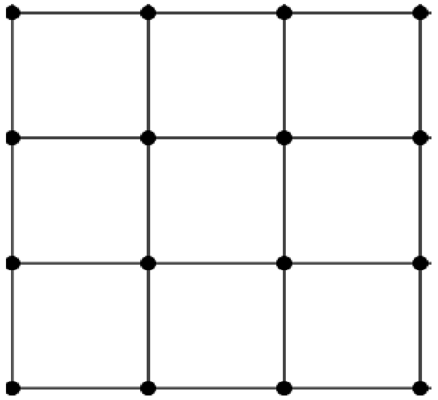
Construction	Circuit depth T	Total number of qubits	Syndrome extraction (Delfosse et al.) bound applies	Parameters saturate bound	Recovery (Baspin et al) bound applies	Parameter saturate bound	Fault-tolerant	Reference
LDPC code implemented with qubit routing	$O(1)$	$O(n^{3/2})$	Yes	Yes	Yes	No	No	Our work (routing)
LPDC code with fault-tolerant routing	$O(1)$ up to polylog factors	$\Theta(n^{3/2}\log^3 n)$	No (controls are not parities)	Yes	Yes	No	Yes	Our work
LDPC code concatenated +O(1)-surface code	$\Theta(\sqrt{n})$	$\Theta(n)$	Yes	Yes	Yes	Yes	No	Pattison et al.
LDPC code concatenated +O(log n)-surface code	$\Theta(\sqrt{n}\log n)$	$\Theta(n\log^2 n)$	No (not LPDC)	Yes	Yes	Yes	Yes	Pattison et al.

[1] N. Delfosse, M. E. Beverland, and M. A. Tremblay, Bounds on stabilizer measurement circuits and obstructions to local implementations of quantum LDPC codes, Sep. 2021. arXiv: 2109.14599.

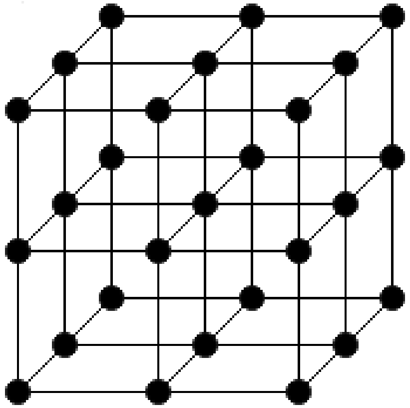
[2] N. Baspin, O. Fawzi, and A. Shayeghi, A lower bound on the overhead of quantum error correction in low dimensions, Feb. 2023. arXiv: 2302.04317.

Local codes in $\mathbb{R}^D$

Bound	Reference
$d \in O(n^{\frac{D-1}{D}})$	[1]
$kd^{\frac{2}{D-1}} \in O(n)$	[2]
$k \in O(n^{\frac{D-2}{D}})$	[3]



$D = 2$



$D = 3$

d code distance
k number of encoded (logical) qubits
n number of physical qubits

No-go theorems (trade-off bounds)

[1] S. Bravyi and B. Terhal, "A no-go theorem for a two-dimensional self-correcting quantum memory based on stabilizer codes," New Journal of Physics, vol. 11, no. 4, p. 043 029, Apr. 2009. DOI: 10.1088/1367 − 2630/11/4/043029.
[2] S. Bravyi, D. Poulin, and B. Terhal, "Tradeoffs for reliable quantum information storage in 2d systems," Phys. Rev. Lett., vol. 104, p. 050503,5 Feb. 2010. DOI: 10.1103/ PhysRevLett. 104.050503.
[3] J. Haah, "A degeneracy bound for homogeneous topological order," SciPost Phys., vol. 10, p. 011, 2021. DOI: 10.21468/SciPostPhys.10.1.011.

Explicit constructions

[1] Elia Portnoy, Local Quantum Codes from Subdivided Manifolds, arXiv:2303.06755
[2] Dominic J. Williamson and Nouédyn Baspin, Layer codes, arXiv:2309.16503
[3] Ting-Chun Lin, Adam Wills, Min-Hsiu Hsieh, Geometrically Local Quantum and Classical Codes from Subdivision, arXiv:2309.16104

The layer code associated with the Shor code

Shor's code $\mathcal{S} = \langle Z_1Z_2, Z_2Z_3, Z_4\,Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9, X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9 \rangle$

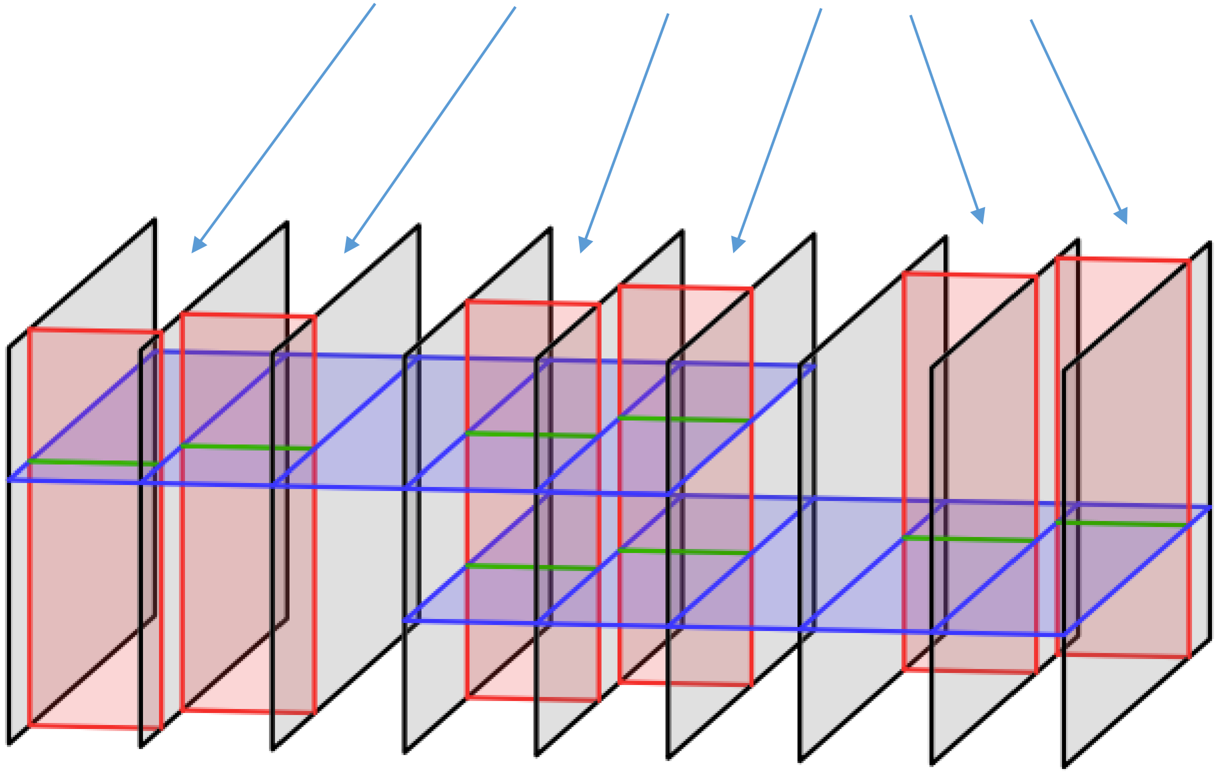


Figure from [1]

[1] Dominic J. Williamson and Nouédyn Baspin, Layer codes, arXiv:2309.16503

The layer code associated with the Shor code

Shor's code $\mathcal{S} = \langle Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9, X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9 \rangle$

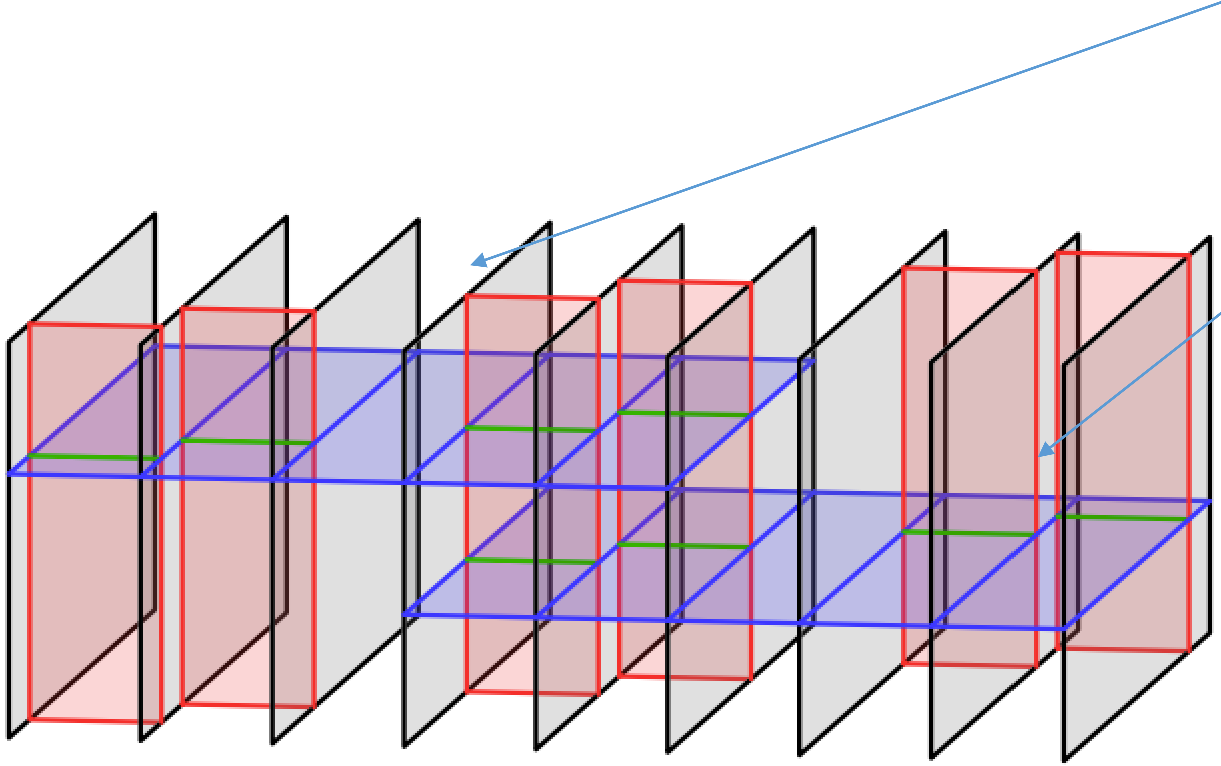
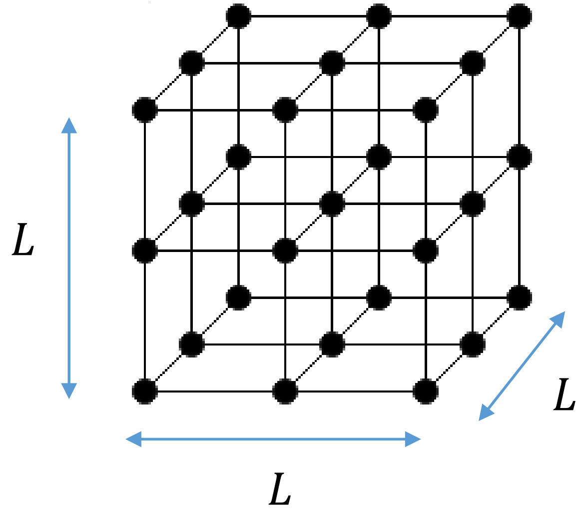


Figure from [1]

[1] Dominic J. Williamson and Nouédyn Baspin, Layer codes, arXiv:2309.16503

The layer code associated with a good quantum LDPC code

Bound	Reference
$d \in O(n^{\frac{D-1}{D}})$	[1]
$kd^{\frac{2}{D-1}} \in O(n)$	[2]
$k \in O(n^{\frac{D-2}{D}})$	[3]



d code distance
k number of encoded (logical) qubits
n number of physical qubits

Applying the layer code construction to a good quantum CSS-code gives a code with parameters

$$[n, k, d] = [\Theta(L^3), \Theta(L), \Theta(L^2)]$$

[1] Dominic J. Williamson and Nouédyn Baspin, Layer codes, arXiv:2309.16503
[2] A. Leverrier, G. Zémor, Quantum Tanner codes, FOCS 2022

Conclusions

- New, geometrically local fault-tolerance constructions with low depth-overhead

- Systematic separation of locality considerations and fault-tolerance design (codes/decoders/gates)

- Many open question:
  - Optimality?
  - 2D-locality?

- Exciting recent new developments in our field!

  Quantum LDPC codes, No low-energy trivial states conjecture, optimal local codes…..

Shin Ho Choe, RK, How to fault-tolerantly realize any quantum circuit with local operations, arXiv:2402.13863

Shin Ho Choe, RK, Long-range data transmission in a fault-tolerant quantum bus architecture, arXiv:2209.09774

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