Dynamical Black Hole Entropy

Robert M. Wald

S. Hollands, R.M. Wald, and V.G. Zhang, arXiv:2402.00818.

Black Hole Entropy

As Iyer and I showed in 1994, in an arbitrary theory of gravity obtained from a diffeomorphism covariant Lagrangian, stationary black holes satisfy a "first law of black hole mechanics," thereby enabling one to identify a quantity representing black hole entropy. However, our derivation required evaluation of the entropy on the bifurcation surface, $\mathcal{B}$, of the black hole, thus restricting the validity of our formula for entropy to stationary black holes and their linear perturbations, evaluated at the "time" represented by $\mathcal{B}$. It is of considerable interest to obtain an expression for the entropy of a non-stationary black hole in a general theory of gravity at a "time" represented by an arbitrary cross-section $\mathcal{C}$, since this would allow one to investigate whether, classically, black hole entropy satisfies a second law (i.e., whether it is non-decreasing with time) and whether, semiclassically, black hole entropy satisfies a generalized second law (i.e., whether the sum of black hole entropy and a matter contribution to entropy is non-decreasing).

Previous Work on Dynamical Black Hole Entropy

Iyer and I proposed a formula for dynamical black hole entropy, but our formula was not field redefinition invariant, and we retracted our proposal in a "note in proof" in the published version of their paper. In 2013, Dong proposed a formula for dynamical black hole entropy in theories whose Lagrangian is an arbitrary function of curvature (but not derivatives of curvature) based on holographic entanglement arguments. In 2015, Wall proposed a general prescription for dynamical black hole entropy for perturbations of a stationary black hole. When he evaluated this in the case of a Lagrangian that is a function of curvature and obtained agreement with Dong's formula.

Our Approach and Results

We apply a new strategy to the definition of dynamical black hole entropy—intended to be applied only at leading nontrivial order in perturbation theory about a stationary black hole—based upon the validity of a local, "physical process version" of the first law of black hole mechanics. In the case of general relativity, we obtain a nontrivial dynamical correction to the Bekenstein-Hawking entropy formula, namely we obtain

$$S_{\mathcal{C}} = \frac{A[\mathcal{C}]}{4} - \frac{1}{4} \int_{\mathcal{C}} V \vartheta$$

where $A[\mathcal{C}]$ is the area of the cross-section $\mathcal{C}$ (so $A[\mathcal{C}]/4$ is the usual Bekenstein-Hawking entropy), V is an affine parameter of the null generators of the horizon (with $V = 0$ corresponding to the bifurcation surface $\mathcal{B}$), and ϑ is the expansion of these generators. For general theories, our formula differs from the Dong-Wall formula, but there is a close relation between them.

Lagrangian Formalism

Lagrangian:

$$\mathbf{L} = \mathbf{L}(g_{ab}, R_{abcd}, \nabla_{a_1} R_{bcde}, \cdots, \nabla_{(a_1} \cdots \nabla_{a_m)} R_{bcde}; \psi)$$

Symplectic potential:

$$\delta \mathbf{L} = \mathbf{E} \delta \phi + d\boldsymbol{\theta}$$

Symplectic current:

$$\boldsymbol{\omega}(\phi, \delta_1 \phi, \delta_2 \phi) = \delta_1 \boldsymbol{\theta}(\phi, \delta_2 \phi) - \delta_2 \boldsymbol{\theta}(\phi, \delta_1 \phi)$$

Noether current associated to vector field χ^a:

$$\mathbf{J}[\chi] = \boldsymbol{\theta}(\phi, \mathcal{L}_\chi \phi) - \chi \cdot \mathbf{L}$$

Lagrangian Formalism (Continued)

Noether Charge:

$$\mathbf{J}[\chi] = d\mathbf{Q}[\chi] + \chi^a \boldsymbol{\mathcal{C}}_a$$

"Fundamental Identity":

$$\boldsymbol{\omega}(\phi, \delta\phi, \mathcal{L}_\chi \phi) = \chi \cdot (\mathbf{E}\delta\phi) + \chi^a \delta\boldsymbol{\mathcal{C}}_a + d[\delta\mathbf{Q} - \chi \cdot \boldsymbol{\theta}(\phi, \delta\phi)]$$

If χ^a is a Killing field and the equations of motion are hold in the background spacetime ($\mathbf{E} = 0$), then the fundamental identity reduces to

$$d[\delta\mathbf{Q} - \chi \cdot \boldsymbol{\theta}(\phi, \delta\phi)] = -\chi^a \delta\boldsymbol{\mathcal{C}}_a$$

First Law of Black Hole Mechanics

Consider a source-free perturbation $\delta \boldsymbol{\mathcal{C}}_a = 0$ of a stationary, non-extremal black hole with horizon Killing field ξ^a. Integrate

$$d[\delta \mathbf{Q}[\xi] - \xi \cdot \boldsymbol{\theta}(\phi, \delta\phi)] = 0$$

over a hypersurface extending from the bifurcation surface $\mathcal{B}$ to spatial infinity. Obtain

$$\frac{\kappa}{2\pi} \delta S = \delta M - \Omega_H \delta J$$

where

$$S \equiv \frac{2\pi}{\kappa} \int_{\mathcal{B}} \mathbf{Q}[\xi]$$

Evaluation at $\mathcal{B}$ (where $\xi^a = 0$) was needed to get rid of the $\xi \cdot \boldsymbol{\theta}$ that otherwise would have appeared in the horizon boundary term. This restricts the validity of the formula to "time" $\mathcal{B}$.

Entropy on an Arbitrary Cross-Section $\mathcal{C}$

We would like to define $S_{\mathcal{C}}$ on a cross-section $\mathcal{C}$ of the event horizon $\mathcal{H}$ so that

$$\delta S_{\mathcal{C}} = \frac{2\pi}{\kappa} \int_{\mathcal{C}} [\delta \mathbf{Q}[\xi] - \xi \cdot \boldsymbol{\theta}(\phi, \delta\phi)]$$

If, on $\mathcal{H}$, the symplectic potential $\boldsymbol{\theta}$ were of the form $\boldsymbol{\theta} = \delta \boldsymbol{B}_{\mathcal{H}}$ for some quantity $\boldsymbol{B}_{\mathcal{H}}$ defined on $\mathcal{H}$, then the desired quantity $S_{\mathcal{C}}$ could be defined by

$$S_{\mathcal{C}} = \frac{2\pi}{\kappa} \int_{\mathcal{C}} [\mathbf{Q}[\xi] - \xi \cdot \boldsymbol{B}_{\mathcal{H}}]$$

However, it is not possible for $\boldsymbol{\theta}$ to be of the form of a total variation $\delta \boldsymbol{B}_{\mathcal{H}}$ on $\mathcal{H}$ in general because such a form would imply the vanishing of the symplectic current flux through the horizon (since the symplectic current $\boldsymbol{\omega}$ is the antisymmetrized second variation of $\boldsymbol{\theta}$).

Analogous Example: ADM Mass

Want to define ADM mass, M, at spatial infinity with respect to an asymptotic time translation t^a so that it is the Hamiltonian conjugate to t^a, i.e., so that it generates asymptotic time translations on phase space. This requires that

$$\delta M = \int_\infty [\delta \mathbf{Q}[t] - t \cdot \boldsymbol{\theta}(\phi, \delta\phi)]$$

However, the symplectic current goes to zero sufficiently rapidly that the symplectic current flux vanishes at spatial infinity, so there is no obstruction to finding a quantity $\boldsymbol{B}_\infty$ such that to leading asymptotic order

$$\boldsymbol{\theta} = \delta \boldsymbol{B}_\infty$$

One may then define the ADM mass as

$$M = \int_\infty [\mathbf{Q}[t] - t \cdot \boldsymbol{B}_\infty]$$

and there is no difficulty in obtaining an M that satisfies the desired relation.

Closely Analogous Example: Bondi Mass

Would like to define Bondi mass, M_B, so that on a cross-section $\mathcal{C}$ of null infinity we have

$$\delta M_B = \int_{\mathcal{C}} [\delta \mathbf{Q}[t] - t \cdot \boldsymbol{\theta}(\phi, \delta\phi)]$$

However, this is not possible, since the symplectic flux does not vanish at null infinity. Nevertheless, Zoupas and I found that one could define a symplectic potential $\boldsymbol{\theta}'(\phi, \delta\phi)$ at $\mathscr{I}^+$ such that $\boldsymbol{\theta}' = 0$ in a stationary background ϕ. We then defined $\boldsymbol{B}_{\mathscr{I}^+}$ by

$$\boldsymbol{\theta} - \boldsymbol{\theta}' = \delta \boldsymbol{B}_{\mathscr{I}^+}$$

and we defined the Bondi mass relative to an asymptotic time translation t^a on a cross-section, $\mathcal{C}$, of $\mathscr{I}^+$ by

$$M_B = \int_{\mathcal{C}} [\mathbf{Q}[t] - t \cdot \boldsymbol{B}_{\mathscr{I}^+}]$$

Dynamical Black Hole Entropy

We follow the same strategy that Zoupas and I used at null infinity. We prove that for first order perturbations of a stationary black hole, there exists a quantity $\boldsymbol{B}_{\mathcal{H}}$ defined on the black hole horizon $\mathcal{H}$ that satisfies

$$\boldsymbol{\theta} = \delta \boldsymbol{B}_{\mathcal{H}}$$

(This is equivalent to the requirement of the existence of an alternative symplectic potential $\boldsymbol{\theta}'$ that vanishes for a stationary background, since for a non-stationary background, one could define $\boldsymbol{\theta}' = \boldsymbol{\theta} - \delta \boldsymbol{B}_{\mathcal{H}}$.) We then define the entropy on an arbitrary cross-section $\mathcal{C}$ of $\mathcal{H}$ by

$$S_{\mathcal{C}} = \frac{2\pi}{\kappa} \int_{\mathcal{C}} [\mathbf{Q}[\xi] - \xi \cdot \boldsymbol{B}_{\mathcal{H}}]$$

Physical Process Version of the First Law

For vacuum perturbations of a stationary black hole, $\delta S_{\mathcal{C}} = \delta S$, i.e., the entropy is "time independent" to first order. In order to obtain nontrivial time variation of black hole entropy, we must either work to second order in perturbation theory in the vacuum case or allow an external stress-energy, δT_{ab}, to be present in the first order perturbation. In the presence of δT_{ab}, the fundamental identity on the black hole horizon becomes

$$d[\delta \mathbf{Q}[\xi] - \xi \cdot \boldsymbol{\theta}(\phi, \delta\phi)] = -\xi^a \delta \mathbf{C}_a$$

Integration over the region of the horizon bounded by cross-sections $\mathcal{C}_1$ and $\mathcal{C}_2$ yields

$$\frac{\kappa}{2\pi}\left[\delta S_{\mathcal{C}_2} - \delta S_{\mathcal{C}_1}\right] = -\int \xi^a \delta \mathbf{C}_a = \int \delta T_{ab} \xi^a k^b \sqrt{h} dV d^{n-2}x$$

where k^a is the tangent to the affinely parametrized generators of the horizon. This has the form of the "physical process version" of the first law

$$\frac{\kappa}{2\pi} \Delta \delta S = \Delta \delta \mathcal{E}$$

Second Law

For first order perturbations with an external stress-energy δT_{ab}, that satisfies the null energy condition, we have $\Delta \delta \mathcal{E} \geq 0$. Thus, the physical process version of the first law then immediately implies that $\Delta \delta S \geq 0$, i.e., the second law holds to first order for matter satisfying the null energy condition. For vacuum perturbations, we have $\Delta \delta S = 0$, so we have to work to second order to see the change in entropy. We obtain,

$$\frac{\kappa}{2\pi} \Delta \delta S = \int \boldsymbol{\omega}(\delta g, \mathcal{L}_\xi \delta g) - d[\xi \cdot (\delta \boldsymbol{\theta} - \delta^2 \mathbf{B}_{\mathcal{H}})]$$

The integrand on the right side is what Hollands and I called the "modified canonical energy" so the second law holds at leading order for vacuum perturbations if and only if the modified canonical energy is positive. This holds in general relativity but would not be expected to hold in more general theories of gravity.

Entropy of a Dynamical Black Hole in General Relativity

For the case of general relativity, the additional term $-\xi \cdot \boldsymbol{B}_{\mathcal{H}}$ in our entropy formula gives rise to a"dynamical correction term" to the Bekenstein-Hawking entropy, namely,

$$S_{\mathcal{C}} = \frac{A[\mathcal{C}]}{4} - \frac{1}{4}\int_{\mathcal{C}} V\vartheta$$

Consequently, $S_{\mathcal{C}}$ is smaller than the Bekenstein-Hawking entropy. From the local form of the physical process first law, it follows that, to first order, our entropy increases only when matter crosses the horizon. This contrasts with the Bekenstein-Hawking entropy, which increases before matter is thrown into the black, since the event horizon moves outward in anticipation of matter being thrown in at a later time. We have shown that, to first order, our entropy is, in fact, the area of the apparent horizon corresponding to "time" $\mathcal{C}$, and thus can be determined locally, without knowledge of the future behavior of the spacetime.

Relationship to the Dong-Wall Entropy

Consider the entropy $S[V]$ evaluated on a cross-section of constant affine time V. Then we have

$$S[V] = S_{\mathrm{DW}}[V] - V\frac{\partial}{\partial V}S_{\mathrm{DW}}[V]$$

In particular, we have

$$\frac{\partial}{\partial V}S = -V\frac{\partial^2}{\partial V^2}S_{\mathrm{DW}}$$

Generalized Second Law

Quantum null energy condition (QNEC) applied to Killing horizon:

$$\frac{\partial^2 S_{\rm vN}}{\partial V^2} \leq 2\pi \int_{V=\rm const} \langle T_{ab}\rangle k^a k^b \sqrt{h} d^{n-2}x$$

where

$$S_{\rm vN} = -{\rm tr} \rho \ln \rho$$

is the von Neuman entropy of the matter outside the black hole. Since

$$k^a = \frac{1}{\kappa V}\xi^a$$

this is equivalent to

$$V\frac{\partial^2 S_{\rm vN}}{\partial V^2} \leq \frac{2\pi}{\kappa} \int_{V=\rm const} \langle T_{ab}\rangle \xi^a k^b \sqrt{h} d^{n-2}x = \frac{\partial S}{\partial V}$$

Generalized Second Law (continued)

where the semiclassical version of the local, physical process first law was used. Define

$$S_{\text{dyn matter}} = S_{\text{vN}} - V\frac{\partial S_{\text{vN}}}{\partial V}$$

Then

$$\frac{\partial}{\partial V}S_{\text{dyn matter}} = -V\frac{\partial^2}{\partial V^2}S_{\text{vN}}$$

so QNEC takes the form of the generalized second law

$$\frac{\partial}{\partial V}[S + S_{\text{dyn matter}}] \geq 0$$

Generalized Second Law (continued)

Since we have

$$\frac{\partial S}{\partial V} = -V\frac{\partial^2 S_{\rm DW}}{\partial V^2}$$

we may also express QNEC as

$$V\frac{\partial^2 S_{\rm vN}}{\partial V^2} \le -V\frac{\partial^2 S_{\rm DW}}{\partial V^2}$$

Dividing by V, integrating from V to ∞, and setting $(\partial S_{\rm vN}/\partial V)|_\infty = (\partial S_{\rm DW}/\partial V)|_\infty = 0$, we obtain

$$\frac{\partial}{\partial V}[S_{\rm DW} + S_{\rm vN}] \ge 0$$

Thus, the generalized second law using S and $S_{\rm dyn\,matter}$ is equivalent to QNEC, whereas the generalized second law using $S_{\rm DW}$ and $S_{\rm vN}$ is equivalent to an integrated form of QNEC.

Conclusions

We have proposed a new definition of the entropy of dynamical black holes, intended to be valid to leading nontrivial order for perturbations of stationary black holes.

- For general relativity, it gives a correction to the Bekenstein-Hawking entropy that yields the area of the apparent horizon rather than the area of the event horizon.
- The general definition satisfies a local version of the physical process version of the first law.
- It satisfies the classical second law for first order for perturbations with matter satisfying the null energy condition, and it satisfies the second law for vacuum perturbations if and only if the modified canonical energy flux is positive.
- It satisfies a version of the generalized second law that is equivalent to QNEC.

This should provide plenty of food for thought!