Symmetry-Protected Topological (SPT) Phases and Duality

“Extreme Universe”
Online Colloquium Part II
(Interdisciplinary Online Seminar)
February 6, 2025

in memory of Ian Affleck

Masaki Oshikawa
ISSP & Kavli IPMU, UTokyo

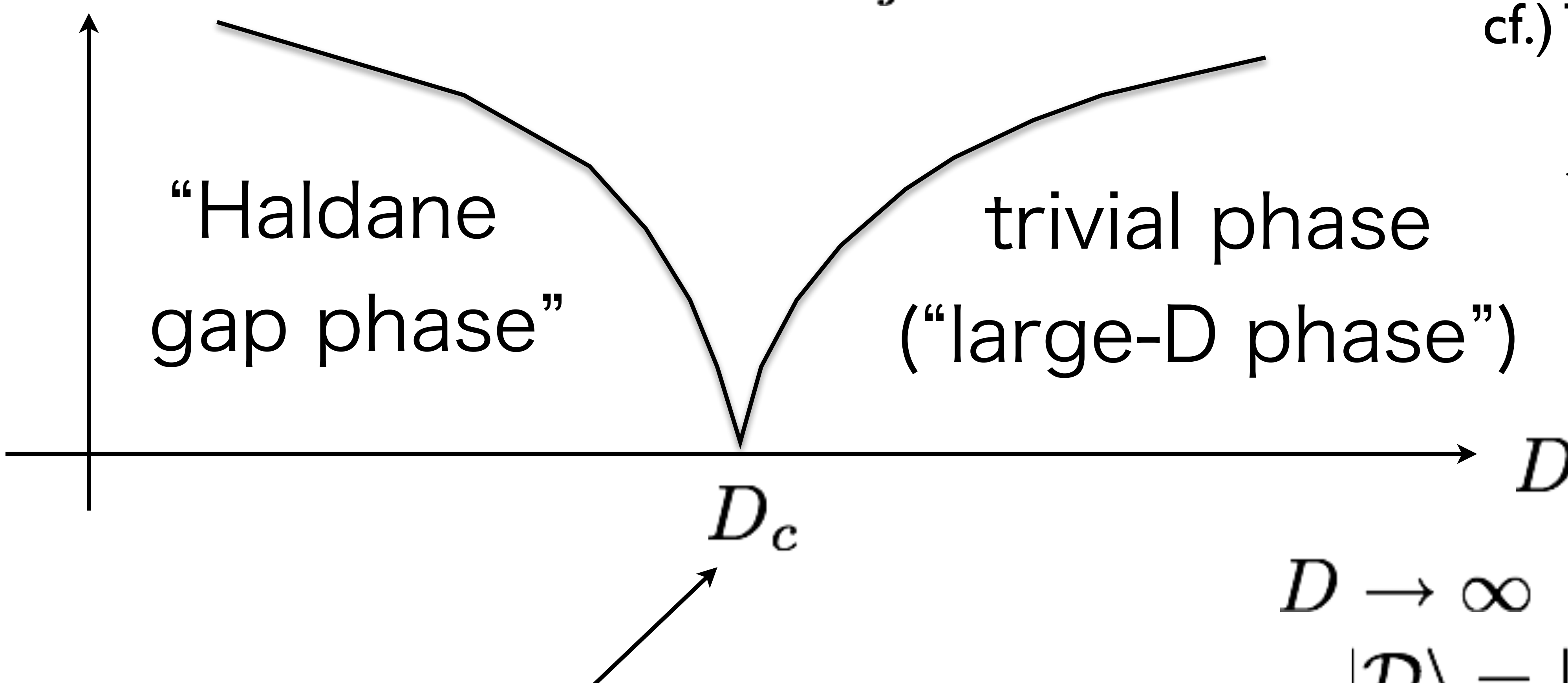
In Memoriam: Ian Affleck (2 July 1952 - 4 Oct 2024)

My memoir on Ian: https://bit.ly/3YfV7Gn

S=1 Haldane Phase and QPT

$$\mathcal{H} = J \sum_j \left(\vec{S}_j \cdot \vec{S}_{j+1} + D (S_j^z)^2 \right).$$

gap



quantum phase transition

cf.) Tasaki's talk

$$D = \frac{1-s}{s}$$

$D \to \infty$

$$|\mathcal{D}\rangle = |000000\ldots\rangle$$

Why there is the transition?

Modern understanding: Haldane phase is a SPT!

History of Topological Phases

1980〜 Integer Quantum Hall states

Thouless-Kohmoto-Nightingale-den Nijs: σ_{xy} =Chern #

1982〜 Fractional Quantum Hall states

Laughlin wavefunction, Chern-Simons field theory, …

2005〜 Topological Insulators (free electrons)

Kane-Mele, …

interacting quantum many-body systems

2009〜 **S**ymmetry-**P**rotected **T**opological Phases

Gu-Wen, Pollmann-Berg-Turner-MO,…

History of Topological Phases

1980〜 Integer Quantum Hall states

Thouless-Kohmoto-Nightingale-den Nijs: σ_{xy} =Chern #

1981〜 "Haldane Conjecture"

AKLT states, Haldane gap phase, …

1982〜 Fractional Quantum Hall states

archetypal example!

Laughlin wavefunction, Chern-Simons field theory, …

2005〜 Topological Insulators (free electrons)

Kane-Mele, …

interacting quantum many-body systems

2009〜 **S**ymmetry-**P**rotected **T**opological Phases

Gu-Wen, Pollmann-Berg-Turner-MO,…

PHYSICAL REVIEW B VOLUME 40, NUMBER 7 1 SEPTEMBER 1989

Preroughening transitions in crystal surfaces and valence-bond phases in quantum spin chains

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(Received 10 April 1989)

We show that disordered flat phases in crystal surfaces are equivalent to valence-bond-type phases in integer and half-integer spin quantum chains. In the quantum spin representation the disordered flat phase represents a fluid-type phase with long-range antiferromagnetic spin order. This order is stabilized dynamically by the hopping of the particles and short-range spin-exchange interactions. The mass of Néel solitons is finite. Numerical finite-size-scaling results confirm this. We identify the order parameter of the valence-bond phase. The Haldane conjecture suggests a fundamental difference between half-integer and integer antiferromagnetic Heisenberg spin chains. We find that disordered flat phases are realized in both cases, have exactly the same type of long-range antiferromagnetic spin order, and are stabilized by exactly the same mechanism. They differ only in the mathematical formulation of broken symmetry in the spin representation. We suggest experimental methods of observing disordered flat phases in crystal surfaces.

It is impossible to define local order parameters that distinguish these two phases. The local order parameters of Sec. II E become nonlocal string operators in the spin-1 formulation (where the surface configuration is characterized by the steps). Recall the Ising-type order parame-

ırameter ψ, Eq. (2.8), which van-
: in the DOF and BCSOS flat

$$\left| i\pi \sum_{m=1}^{n} S_M^z \right| S_n^z \left| 0 \right\rangle , \qquad (4.4)$$

and its square is the limiting value of the correlation function, Eq. (2.7),

$$G_s(n) = \left\langle 0 \left| S_{n_0}^z \exp \left[i\pi \sum_{m=n_0}^{n+n_0} S_m^z \right] S_{n+n_0}^z \right| 0 \right\rangle . \qquad (4.5)$$

Physica Scripta. Vol. T27, 156–159, 1989.

Hidden Topological Order in Integer Quantum Spin Chains

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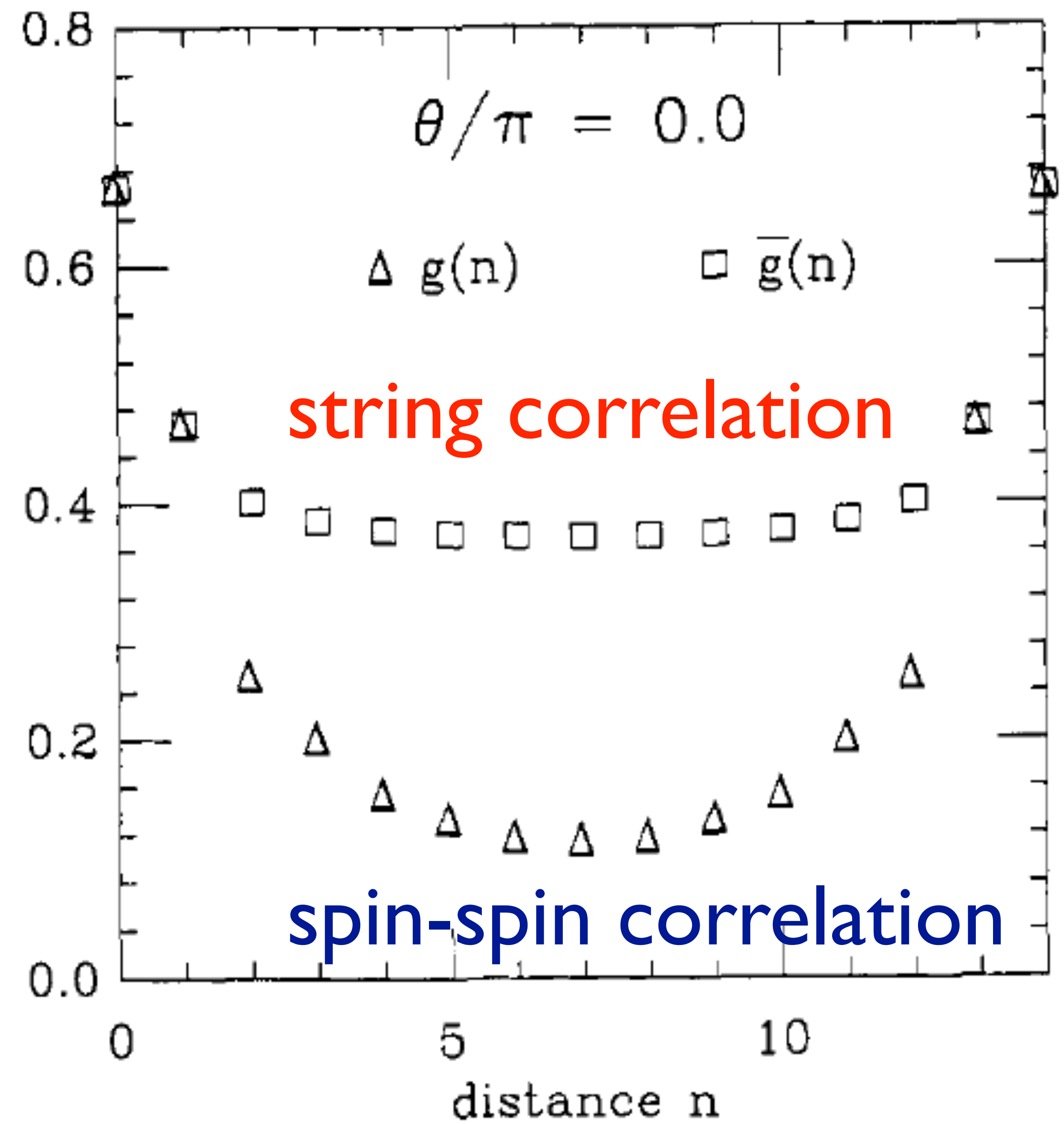
Received June 8, 1988, accepted July 25, 1988

3. Conclusions

We have investigated analogies between integer quantum spin-chains and the fractional quantum Hall effect. Both systems appear to have disordered liquid ground states but because of subtle topological effects, they both have an excitation gap. This topological order is not visible in the ordinary two-point correlation function, but can be detected by defining a special singular-gauge correlation function.



"Haldane gap phase" was already recognized around 1990 as a "topological phase" BUT its meaning was unclear

What We Got Right by 1990s

Haldane gap phase is

a nontrivial phase without any conventional order

→ some sort of "topological phase"

Nontrivial characteristics

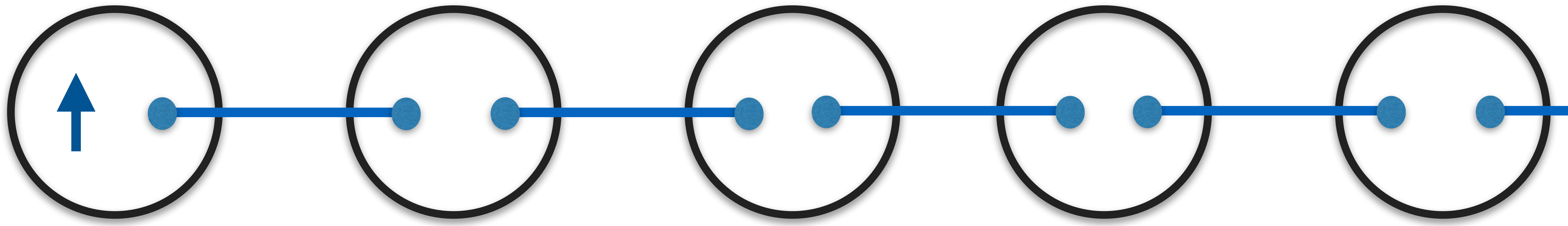
- string order parameter   denNijs-Rommelse 1988
- edge states

Kennedy 1990

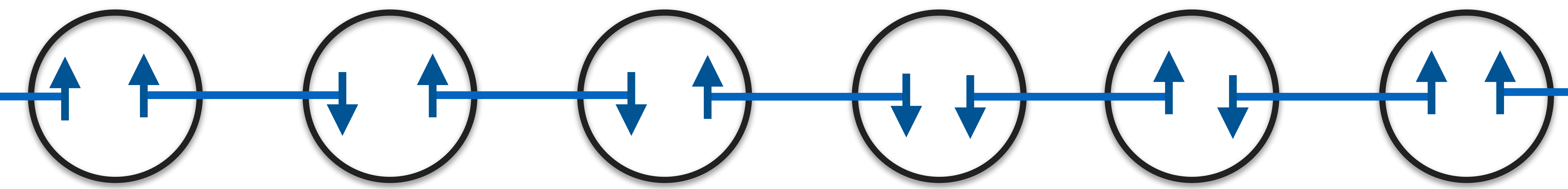
Hidden $Z_2 \times Z_2$ Symmetry Breaking

Kennedy-Tasaki 1992

Nontrivial Features of Haldane Phase



Open boundary condition : "edge state" of $S=1/2$ [Kennedy 1990]



$S^z = +1$ 0 0 -1 0 +1

non-local "string" order

[den Nijs-Rommelse 1989]

$$\mathcal{O}^z_{str} = \lim_{|j-k|\to\infty} \langle S^z_j \exp\left(i\pi \sum_{l=j}^{k-1} S^z_l \right) S^z_k \rangle$$

How the KT transformation works

$$U_{\mathrm{KT}} = \prod_{j<k} \exp\left(i\pi S_j^z S_k^x\right)$$

[simplified expression in M.O. 1992]

$$U_{\mathrm{KT}} S_j^z U_{\mathrm{KT}}^\dagger = \exp\left(i\pi \sum_{l<j} S_l^z\right) S_j^z$$

$$U_{\mathrm{KT}} S_j^x U_{\mathrm{KT}}^\dagger = S_j^x \exp\left(i\pi \sum_{j<l} S_l^x\right)$$

Kennedy-Tasaki transformation is a well-defined **unitary** for a finite chain with the open boundary condition which we assume for the moment (will come back later)

KT transformation of H

$$\mathcal{H} = J\sum_j \vec{S}_j \cdot \vec{S}_{j+1} + D\sum_j \left(S_j^z\right)^2$$

$$\begin{aligned}\tilde{\mathcal{H}} &= U_{\mathrm{KT}}\mathcal{H}U_{\mathrm{KT}}^{\dagger} \\ &= J\sum_j \left(S_j^x e^{i\pi S_{j+1}^x} S_{j+1}^x + S_j^y e^{i\pi(S_j^z + S_{j+1}^x)} S_{j+1}^y + S_j^z e^{i\pi S_j^z} S_{j+1}^z\right) + D\sum_j \left(S_j^z\right)^2\end{aligned}$$

lacks the global SU(2) spin rotation symmetry, but still has the discrete global symmetry (π-rotation about x, y, & z axes)

dihedral group $D_2 = Z_2 \times Z_2$

Consequence of Dual SSB (I)

Suppose that the full global D_2 symmetry of the dual system is spontaneously broken

dual system: $$\langle S_j^z S_k^z \rangle \to \text{const.} \neq 0 \quad (k - j \to \infty)$$

⇕

original system:

$$\langle S_j^z \exp\left(i\pi \sum_{j \le l < k} S_l^z \right) S_k^z \rangle \to \text{const.} \neq 0 \quad (k - j \to \infty)$$

long-range "string order"!!

Consequences of Dual SSB (II)

Full global D2 symmetry of the dual system is spontaneously broken

↓

Dual system has 4-fold (quasi-)degenerate ground states

↓

Original system also has 4-fold (quasi-)degenerate ground states (only) for the open boundary condition

Edge state!

Part I Prof. Hal Tasaki

Gakushuin University

Haldane conjecture, valence-bond picture, SPT phases, and all that in quantum spin chains

Part I **Prof. Hal Tasaki**

Gakushuin University

Haldane conjecture, valence-bond picture, SPT phases, and all that in quantum spin chains

物性研究 58−2 (1992−5)

講義ノート

量子スピン系の理論

Haldane Gap,
Disordered Ground States,
Quantum Spin Liquid
and All That in
Quantum Spin Systems

(1992 年 4 月 6 日受理)

The 35th
Condensed Matter Physics
Summer School
(第35回物性若手夏の学校)

in 1990

Hidden $Z_2 \times Z_2$ symmetry in quantum spin chains with arbitrary integer spin

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Received 9 December 1991, in final form 4 June 1992

$$V = V^{-1} = \prod_{j<k} \exp\left(i\pi S_j^z S_k^x\right)$$

explicitly for several variants of the VBS-type states. In the standard VBS state, the hidden $Z_2 \times Z_2$ symmetry breaks down when S is odd but remains unbroken when S is even. Our results for partially dimerized VBS states suggest that the hidden $Z_2 \times Z_2$

"Hidden Z_2 x Z_2 symmetry breaking" in Haldane gap phase for S=1 ⇒ nontrivial SPT [Gu-Wen 2009]

only if S is odd [Pollmann et al 2009]

What We Missed until 2009

We "almost" got it by 1990s, but
did not quite reach the concept of SPT before 2009
(17 years gap between Kennedy-Tasaki 1992 and Gu-Wen 2009!)

What was missing?

We had not asked the simple (but crucial) question

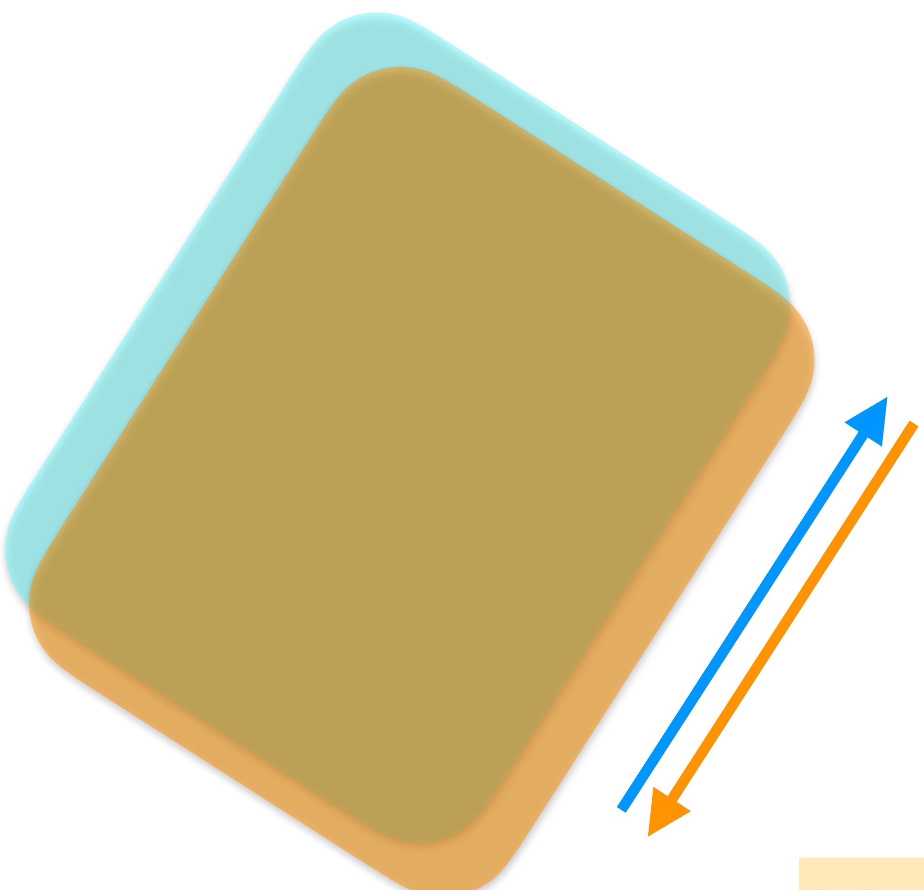
"Under what condition
the Haldane gap phase is distinct from the trivial phase?"

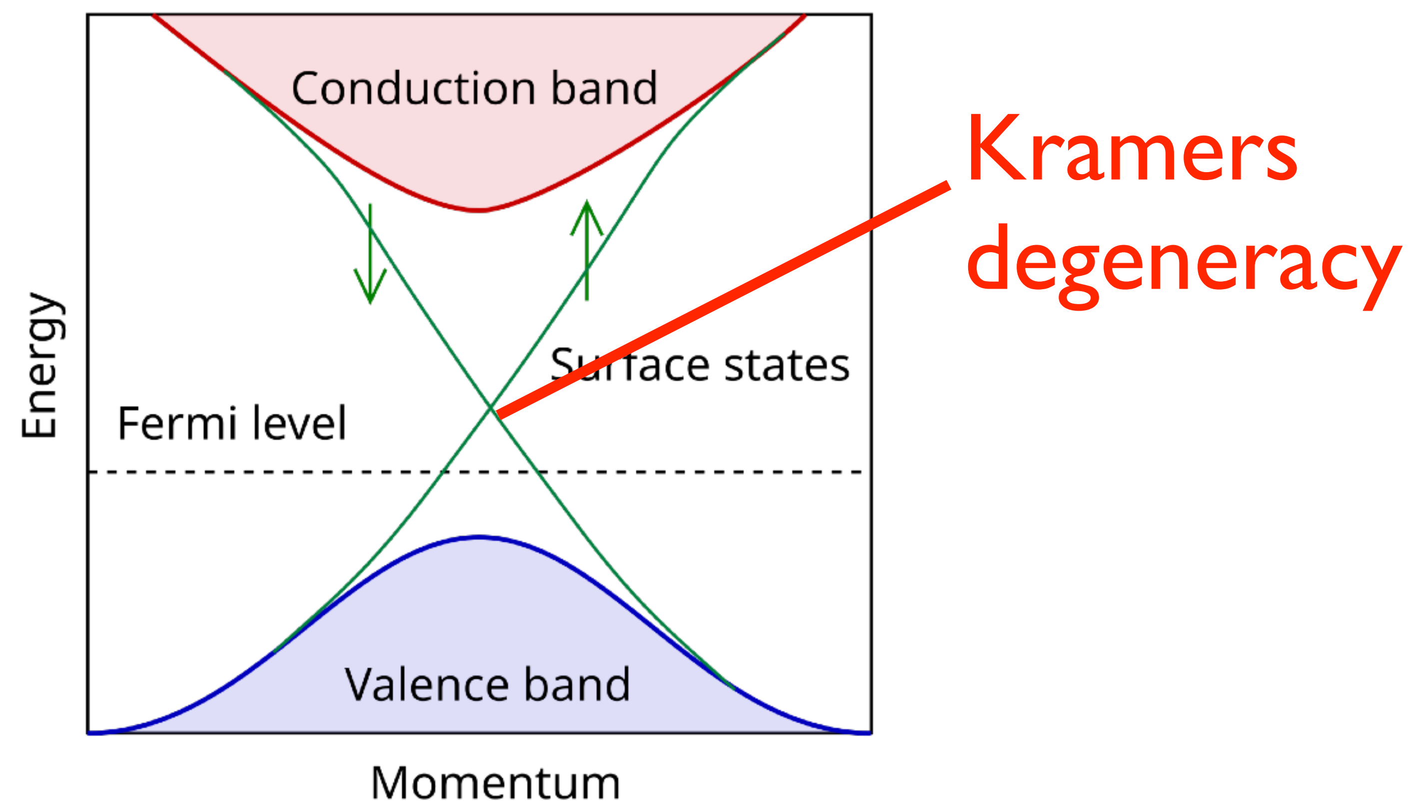
Hint from Topological Insulator

Simple construction of Topological Insulator:

Integer Quantum Hall states for ↑ spin with C=+1

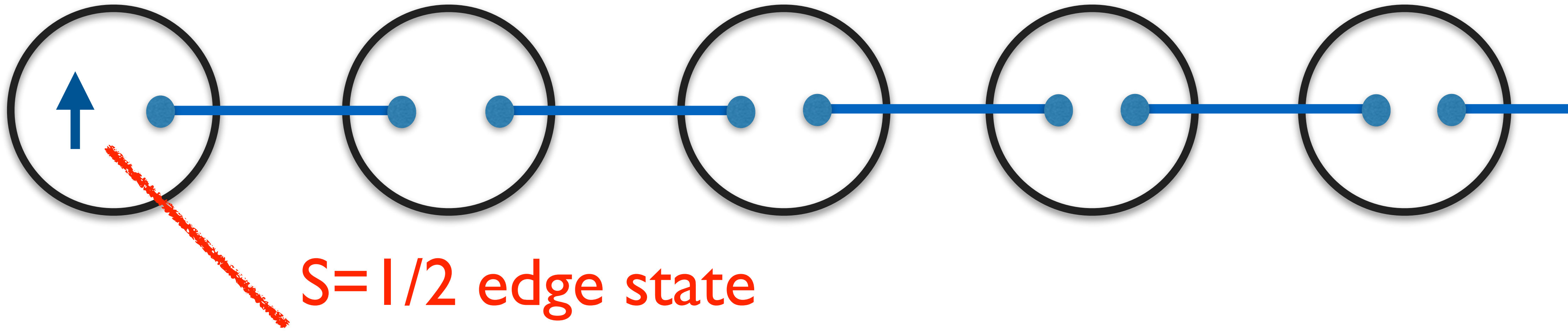
for ↓ spin with C=-1





Helical edge state is protected by Time Reversal through Kramers theorem even when two layers are coupled!

Even Simpler in the Haldane Gap Phase!



S=1/2 edge state
degeneracy protected by Kramers Theorem
under Time Reversal!

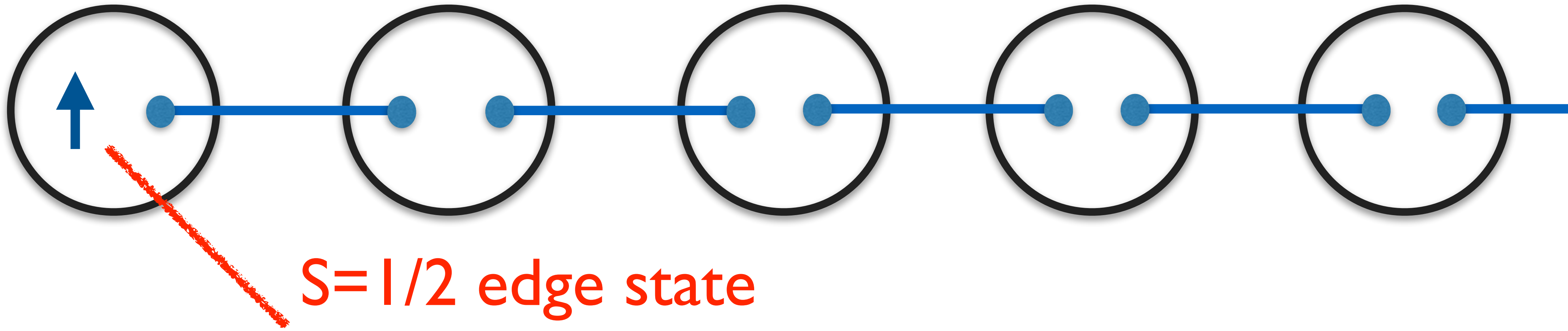
S=1 Haldane gap phase with the edge state is distinct from trivial phase under the Time Reversal, i.e. there must be a phase transition between the Haldane gap phase and a trivial phase under the Time Reversal

S=1 Haldane phase is a Symmetry-Protected Topological phase protected by the Time Reversal

Pollmann-Berg-Turner-MO, arXiv:0909.4059

cf.) Gu-Wen arXiv:0903.1069

Even Simpler in the Haldane Gap Phase!



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Rejected by PRL→PRB2012

PHYSICAL REVIEW B 81, 064439 (2010)

arXiv:0910.1811

Entanglement spectrum of a topological phase in one dimension

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(Received 9 October 2009; revised manuscript received 29 January 2010; published 26 February 2010)

within Matrix Product States

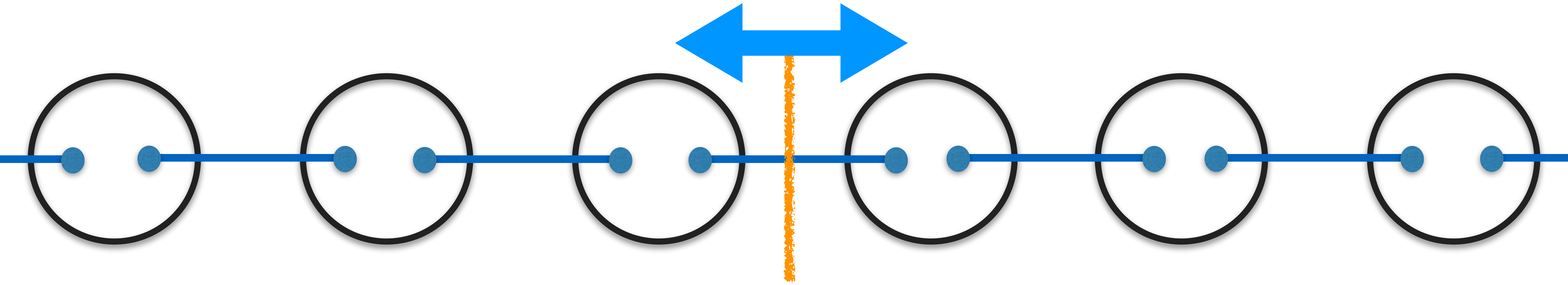
$$|\Psi\rangle = \sum_{\{m_j\}} \mathrm{tr}[\Gamma_{m_1}\Lambda \ldots \Gamma_{m_L}\Lambda]|m_1 \ldots m_L\rangle.$$

S=1 Haldane gap phase is a SPT protected by any one of

- Time Reversal
- Bond-centered Spatial Inversion
- Global $D_2=Z_2\times Z_2$ Spin Rotation

symmetries

Bond-Centered Inversion: Heuristics

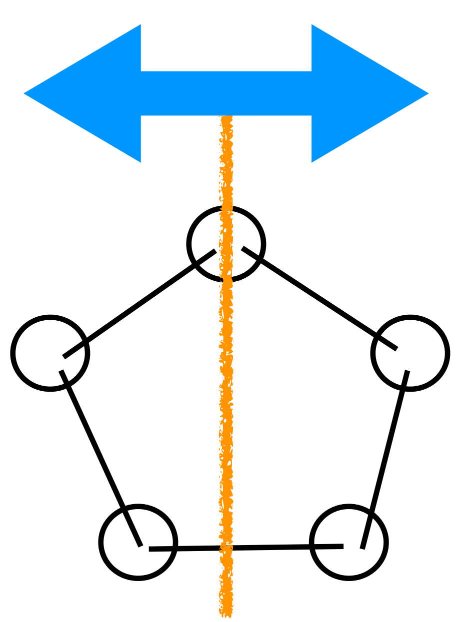


 $= \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$: parity odd

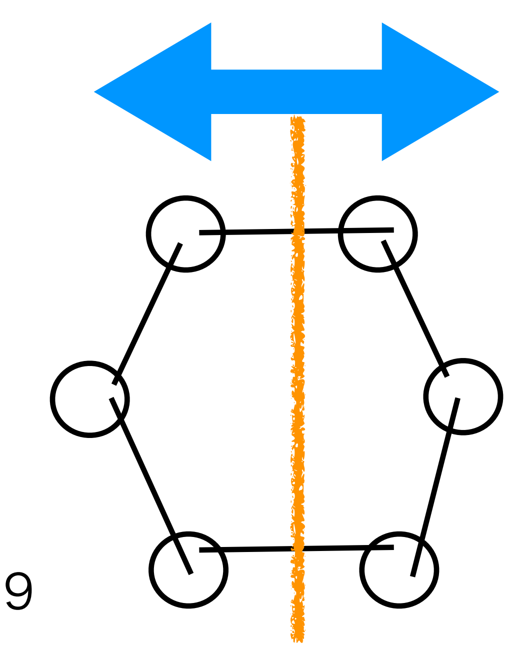
cf.) trivial large-*D* state $|000\ldots000\rangle$: parity even

The S=1 AKLT state is "intrinsically" odd parity under the bond-centered spatial inversion

SPT protected by the inversion!



parity odd



parity even

Modern View of the KT Duality

THE question lacking in 1990s:
When does the hidden $Z_2 \times Z_2$ symmetry breaking argument work?

Hidden $Z_2 x Z_2$ symmetry breaking is useful
iff the dual Hamiltonian is local (short-range int.)
⇔ the original Hamiltonian has global $Z_2 \times Z_2$ symmetry

$$U_{\mathrm{KT}} S_j^z U_{\mathrm{KT}}^\dagger = \exp\left(i\pi \sum_{l<j} S_l^z \right) S_j^z$$

$$U_{\mathrm{KT}} S_j^x U_{\mathrm{KT}}^\dagger = S_j^x \exp\left(i\pi \sum_{j<l} S_l^x \right)$$

If the Hamiltonian has the global $Z_2 \times Z_2$ symmetry,
the phase with the SSB of the hidden $Z_2 \times Z_2$ symmetry
is well-defined and separated from the trivial phase by
a quantum phase transition **= $Z_2 \times Z_2$ protected SPT!!**

PBTO arXiv:0909.4059

Hidden $Z_2 \times Z_2$ symmetry in quantum spin chains with arbitrary integer spin

Masaki Oshikawa†‡
Institute of Physics, University of Tokyo at Komaba, Komaba, Meguro-ku, Tokyo 153, Japan

Received 9 December 1991, in final form 4 June 1992

explicitly for several variants of the VBS-type states. In the standard VBS state, the hidden $Z_2 \times Z_2$ symmetry breaks down when S is odd but remains unbroken when S is even. Our results for partially dimerized VBS states suggest that the hidden $Z_2 \times Z_2$

Even *S*: edge spin (S/2) is integer; no Kramers theorem protection

even number of S=1/2 singlets per bond;

even parity under bond-centered spatial inversion

(same as the trivial state!)

My 1992 finding had captured that

"Haldane gap phase" is essentially trivial for even *S*

SPT only for odd *S* !

(although we did not appreciate this until 2009)

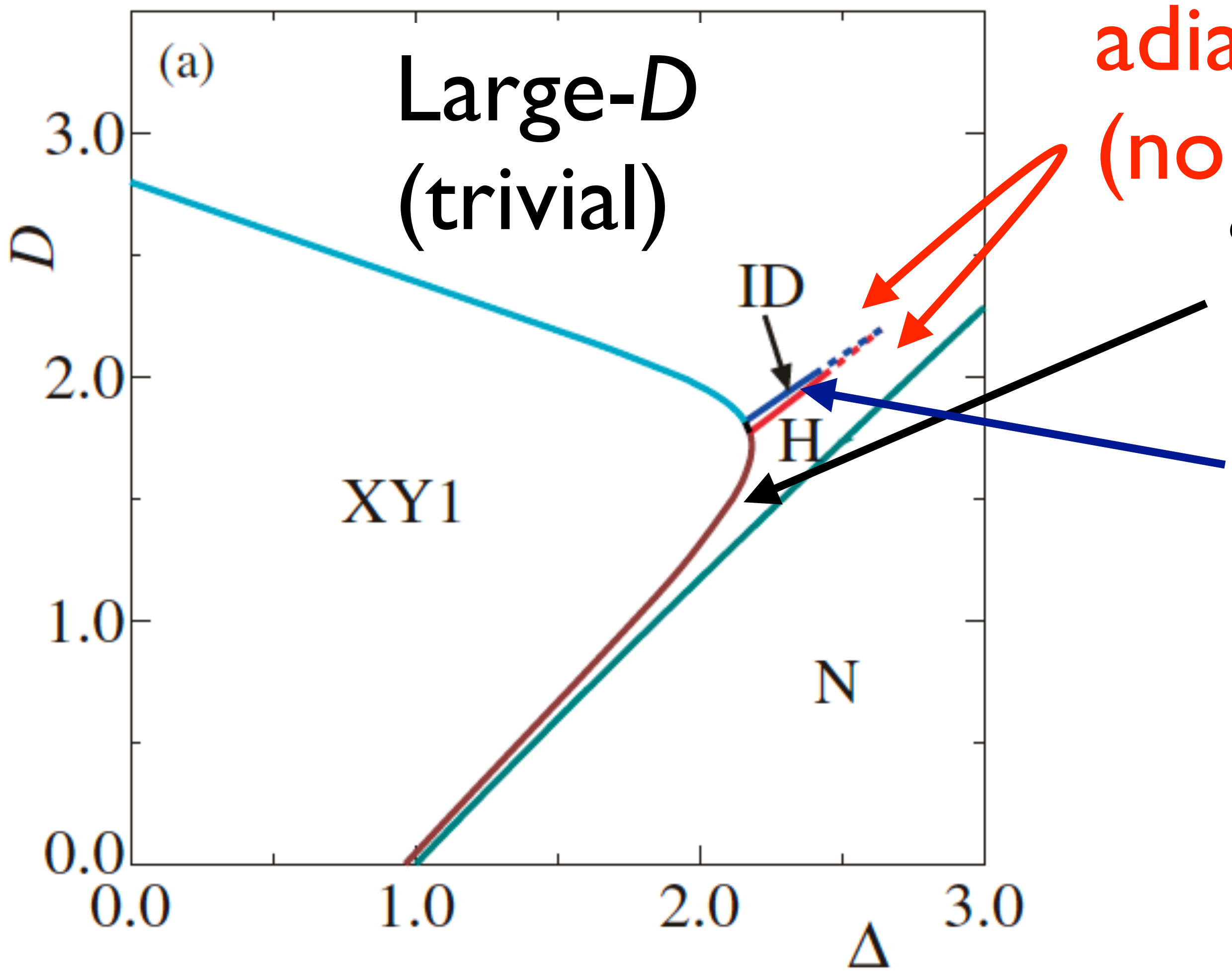
PBTO arXiv:0909.4059

Journal of the Physical Society of Japan **80** (2011) 043001 LETTERS
DOI: 10.1143/JPSJ.80.043001

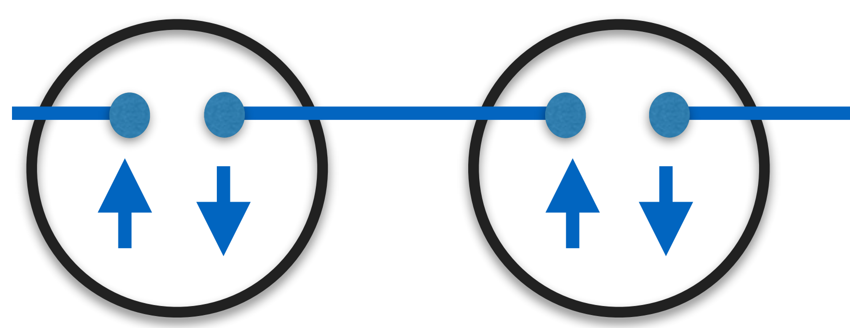
Haldane, Large-*D*, and Intermediate-*D* States in an $S = 2$ Quantum Spin Chain with On-Site and XXZ Anisotropies

Takashi TONEGAWA*, Kiyomi OKAMOTO[1], Hiroki NAKANO[2], Tôru SAKAI[2,3,4], Kiyohide NOMURA[5], and Makoto KABURAGI

$$\mathcal{H} = \sum_j (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z) + D \sum_j (S_j^z)^2,$$



[M.O. 1992]



Kennedy-Tasaki Duality in 21st Century

Many of the nontrivial features of the "Haldane gap phase" were recognized and unified as a consequence of "hidden symmetry breaking" in 1990s although the concept of SPT was (just) missing

Great progress in understanding SPT phases since the discovery/proposal in 2009

Revisit the Kennedy-Tasaki duality with the modern understanding
- reformulation of the Kennedy-Tasaki duality
- applications, especially to construction of gapless SPTs

←Linhao LI

Yunqin→ ZHENG

Linhao Li, Yunqin Zheng, & MO
arXiv:2301.07899, 2307.04788

KT transformation on a ring?

$$U_{\mathrm{KT}} S_j^z U_{\mathrm{KT}}^\dagger = \exp\left(i\pi \sum_{l<j} S_l^z\right) S_j^z$$

$$U_{\mathrm{KT}} S_{L+1}^z U_{\mathrm{KT}}^\dagger = \exp\left(i\pi \sum_{l=1}^{L} S_l^z\right) S_{L+1}^z$$

generator of global Z2xZ2 symmetry!

$$R^x = \exp\left(i\pi \sum_{l=1}^{L} S_l^z\right) = (-1)^{u_x} \qquad S_{L+1}^z = (-1)^{t_z} S_1^z$$

Dual spins obey:

$$u'_z = u_z \mod 2$$
$$t'_z = t_z + u_x \mod 2,$$

Two Interpretations

1) boundary conditions for the original & dual spins are given
   → only the "right" symmetry sector survives
   → **KT transformation is non-invertible**/non-unitary

2) boundary condition (periodic/twisted) is an auxiliary degree of freedom
   → separate Hilbert spaces for periodic/twisted b.c.
   → KT transformation is unitary on the extended Hilbert space

cf.) similar phenomena in Kramers-Wannier duality
only the "right" symmetry sector survives on a ring

Kramers-Wannier duality can be defined as
a unitary transformation on an open chain

Field-Theory Formulation

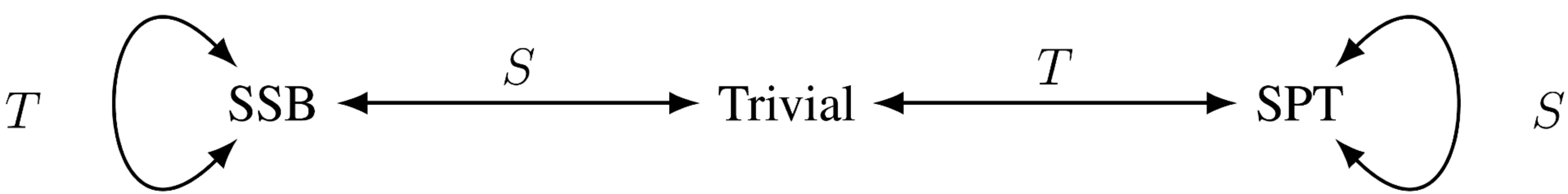
topological manipulations

S: gauging Z2xZ2

$$S : Z_{S_{12}\mathcal{X}}[A_1, A_2] := \frac{1}{|H^0(X_2, \mathbb{Z}_2)|^2} \sum_{a_1, a_2 \in H^1(X_2, \mathbb{Z}_2)} Z_{\mathcal{X}}[a_1, a_2](-1)^{\int_{X_2} a_1 A_2 + a_2 A_1}$$

T: stacking a Z2xZ2 SPT

$$T : Z_{T_{12}\mathcal{X}}[A_1, A_2] := Z_{\mathcal{X}}[A_1, A_2](-1)^{\int_{X_2} A_1 A_2}.$$



Kennedy-Tasaki = *STS*

how to implement this on lattice?

KT transformation for S=1/2

consider a system of two species of S=1/2: σ and τ

Z2xZ2 symmetry generated by $$U_\sigma = \prod_{i=1}^{L} \sigma_i^x, \qquad U_\tau = \prod_{i=1}^{L} \tau_{i-\frac{1}{2}}^x.$$

S: gauging by Z2xZ2 ⇔ Kramers-Wannier for σ, τ

$$\mathcal{N}\,|\{s_i^\sigma, s_{i-\frac{1}{2}}^\tau\}\rangle = \frac{1}{2^L}\sum_{\{\widehat{s}^\sigma_{j-\frac{1}{2}}, \widehat{s}^\tau_j\}} (-1)^{\sum_{j=1}^{L} s_j^\sigma(\widehat{s}^\sigma_{j-\frac{1}{2}}+\widehat{s}^\sigma_{j+\frac{1}{2}})+t_\sigma \widehat{s}^\sigma_{\frac{1}{2}}+\widehat{s}^\tau_j(s^\tau_{j-\frac{1}{2}}+s^\tau_{j+\frac{1}{2}})+\widehat{t}_\tau s^\tau_{\frac{1}{2}}}\,|\{\widehat{s}^\sigma_{j-\frac{1}{2}}, \widehat{s}^\tau_j\}\rangle$$

T: stacking with Z2xZ2 ⇔ "Domain wall decoration"

$$U_{\mathrm{DW}}\,|\{\widehat{s}^\sigma_{i-\frac{1}{2}}, \widehat{s}^\tau_i\}\rangle = (-1)^{\sum_{j=1}^{L} \widehat{s}^\tau_j(\widehat{s}^\sigma_{j-\frac{1}{2}}+\widehat{s}^\sigma_{j+\frac{1}{2}})+\widehat{t}_\tau \widehat{s}^\sigma_{\frac{1}{2}}}\,|\{\widehat{s}^\sigma_{i-\frac{1}{2}}, \widehat{s}^\tau_i\}\rangle\,.$$

$$\mathcal{N}_{\mathrm{KT}} = \mathcal{N}\,U_{\mathrm{DW}}\mathcal{N}.$$

Symmetry/Twist Sectors

Symmetry sectors for σ, τ

$u_{\sigma,\tau}$ = 0, 1 (even/odd under spin flip)

Twist sectors for σ, τ

$t_{\sigma,\tau}$ = 0, 1 (periodic/antiperiodic boundary condition on ring)

dual spin original spin

$$(u'_\sigma, u'_\tau, t'_\sigma, t'_\tau) = (u_\sigma, u_\tau, u_\tau + t_\sigma, u_\sigma + t_\tau).$$

Similar to the original KT for S=1 $t'_z = t_z + u_x \mod 2,$

(in fact we have shown the equivalence between the KTs)

Construction of SPT

Two decoupled Ising chains in the ordered phase

$$H_{\mathrm{SSB}} = -\sum_{i=1}^{L} \left(\sigma^z_{i-1}\sigma^z_i + \tau^z_{i-\frac{1}{2}}\tau^z_{i+\frac{1}{2}} \right)$$

Z2xZ2 fully broken spontaneously

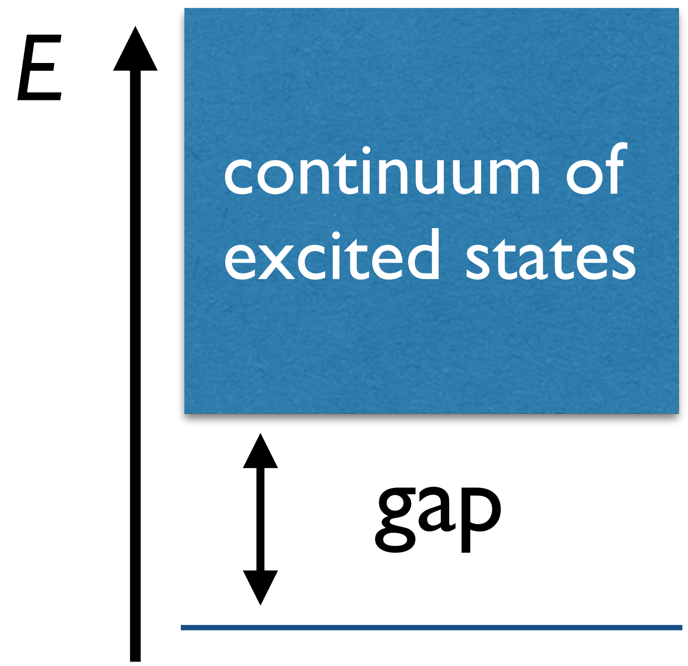
Kennedy-Tasaki duality mapping

$\mathcal{N}_{\mathrm{KT}}$

$$H_{\mathrm{SPT}} = -\sum_{j=1}^{L} \left(\sigma^z_{j-1}\tau^x_{j-\frac{1}{2}}\sigma^z_j + \tau^z_{j-\frac{1}{2}}\sigma^x_j\tau^z_{j+\frac{1}{2}} \right) ,$$

1D “cluster model”: Z2xZ2 SPT

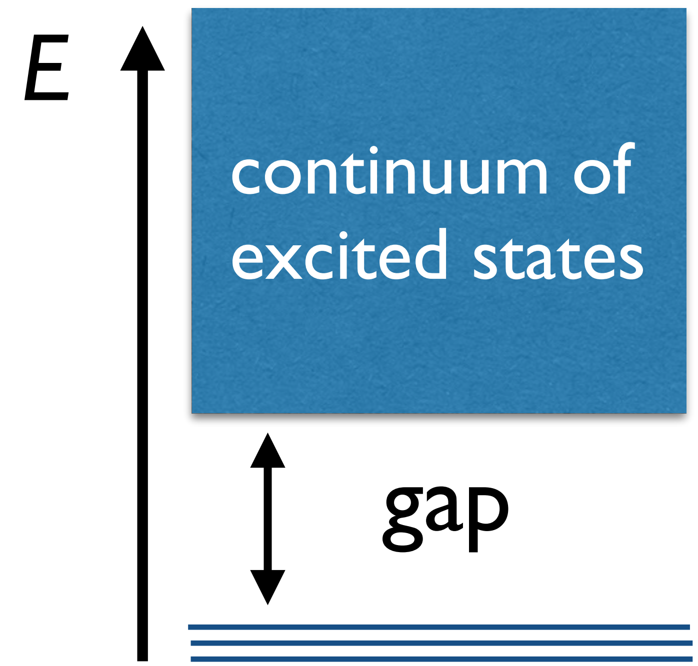
Low-Energy Spectra



unique groundstate
(without edge)

- trivial
- SPT

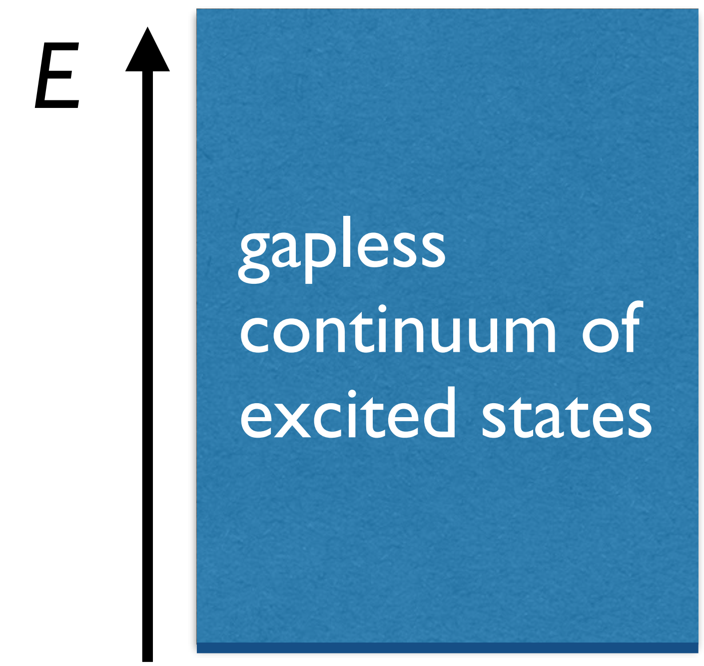
etc.



degenerate
ground states

- SSB
- topological order

etc.



ground state

- metal
- semimetal
- gapless SPT

etc.

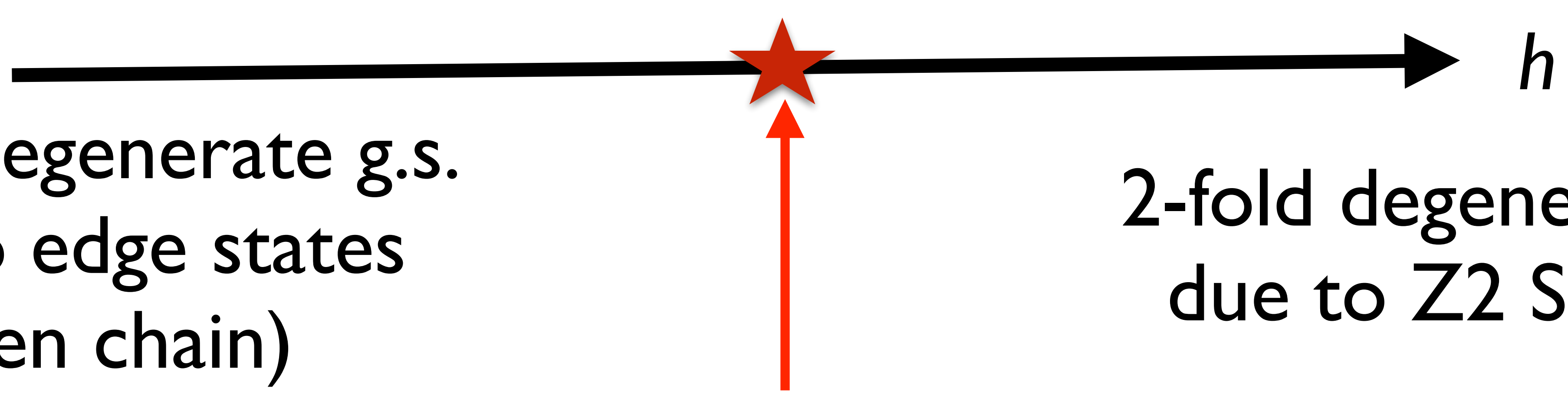
SPT-SSB Phase Transition

$$H = -\sum_i \left(\tau^z_{i-\frac{1}{2}} \sigma^x_i \tau^z_{i+\frac{1}{2}} + \sigma^z_{i-1} \tau^x_{i-\frac{1}{2}} \sigma^z_i + h\sigma^x_i \right)$$

Z2xZ2 SPT

critical point

Z2 SSB (of τ)



4-fold degenerate g.s.
due to edge states
(on open chain)

2-fold degenerate g.s.
due to Z2 SSB

2-fold (exponentially) degenerate g.s.
due to edge states — "gapless SPT"

[Scaffidi, Parker, Vasseur 2017]

Duality Viewpoint

$$H = -\sum_i \left(\tau^z_{i-\frac{1}{2}} \sigma^x_i \tau^z_{i+\frac{1}{2}} + \sigma^z_{i-1} \tau^x_{i-\frac{1}{2}} \sigma^z_i + h\sigma^x_i \right)$$

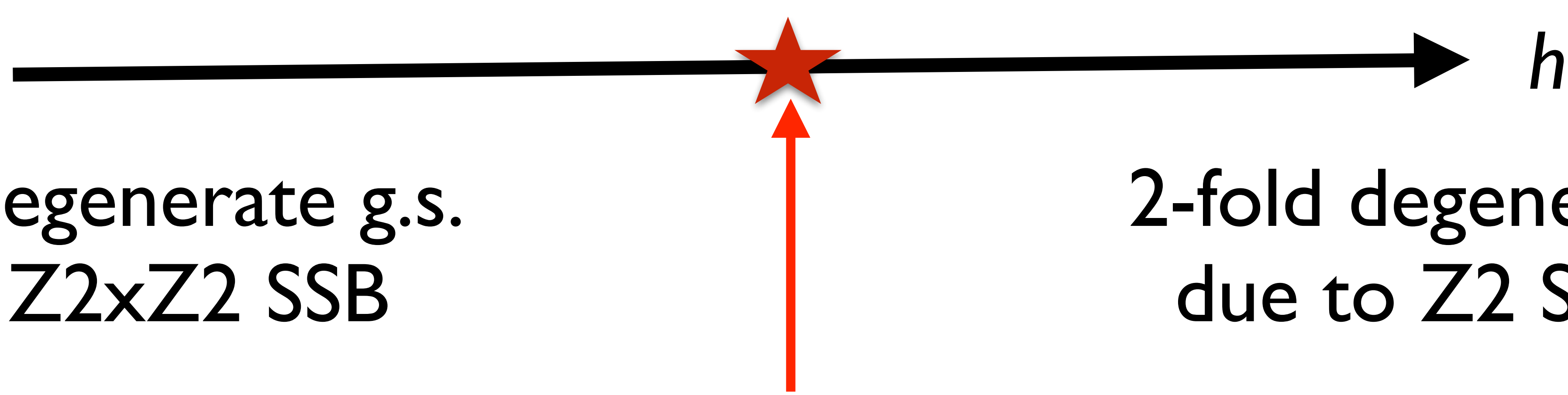
$\Updownarrow \ \mathcal{N}_{\mathrm{KT}}$

$$H_{\mathrm{dual}} = -\sum_i \left(\tau^z_{i-\frac{1}{2}} \tau^z_{i+\frac{1}{2}} + \sigma^z_{i-1} \sigma^z_i + h\sigma^x_i \right)$$

Z2xZ2 SSB

critical point

Z2 SSB (of τ)



4-fold degenerate g.s.
due to Z2xZ2 SSB

2-fold degenerate g.s.
due to Z2 SSB

2-fold (exponentially) degenerate g.s.
remaining due to “spectator” SSB of τ

Intrinsically Gapless SPT

Verresen, Thorngren, Jones, Pollmann, 2019
Thorngren, Vishwanath, Verresen 2020
Li-MO-Zheng 2022, Wen-Potter 2022 etc.

"topological" features of the gapless SPT phase has no counterpart in a gapped SPT

Entire global symmetry G: non-anomalous
subgroup G_{low} of G acts on low-energy sector anomalously
(cancelled by anomaly in the gapped sector)

Intrinsically Gapless SPT

$$H_{\mathrm{SSB+XX}} = -\sum_{i=1}^{L}\left(\tau^z_{i-\frac{1}{2}}\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\tau^y_{i+\frac{1}{2}} + \sigma^z_{i-1}\sigma^z_i\right).$$

$$\Updownarrow \quad \mathcal{N}_{\mathrm{KT}}$$

$$H_{\mathrm{igSPT}} = -\sum_{i=1}^{L}\left(\tau^z_{i-\frac{1}{2}}\sigma^x_i\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\sigma^x_i\tau^y_{i+\frac{1}{2}} + \sigma^z_{i-1}\tau^x_{i-\frac{1}{2}}\sigma^z_i\right)$$

"intrinsically gapless SPT" protected by Z_4 symmetry
generated by $U_\sigma V_\tau$

$$U_\sigma = \prod_j \sigma^x_j \qquad V_\tau = \prod_{i=1}^{L} e^{\frac{i\pi}{4}\left(1-\tau^x_{i-\frac{1}{2}}\right)}$$

igSPT + Z_4 symmetric perturbation

$$H_{\text{igSPT+pert}} = -\sum_{i=1}^{L}\left(\tau^z_{i-\frac{1}{2}}\sigma^x_i\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\sigma^x_i\tau^y_{i+\frac{1}{2}} + \sigma^z_{i-1}\tau^x_{i-\frac{1}{2}}\sigma^z_i + h\sigma^x_i + h\tau^x_{i-\frac{1}{2}}\right)$$

h respects the Z4 symmetry

the system is trivial in the limit $h \to \infty$

is the igSPT phase stable against a small h? phase diagram?

$\updownarrow$ $\mathcal{N}_{\mathrm{KT}}$

$$H_{\text{XX+pert}} = -\sum_{i=1}^{L}\left(\tau^z_{i-\frac{1}{2}}\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\tau^y_{i+\frac{1}{2}} + h\tau^x_{i-\frac{1}{2}}\right)$$

XY chain in a field

$$H_{\text{SSB+pert}} = -\sum_{i=1}^{L}\left(\sigma^z_{i-1}\sigma^z_i + h\sigma^x_i\right).$$

Transverse Ising chain

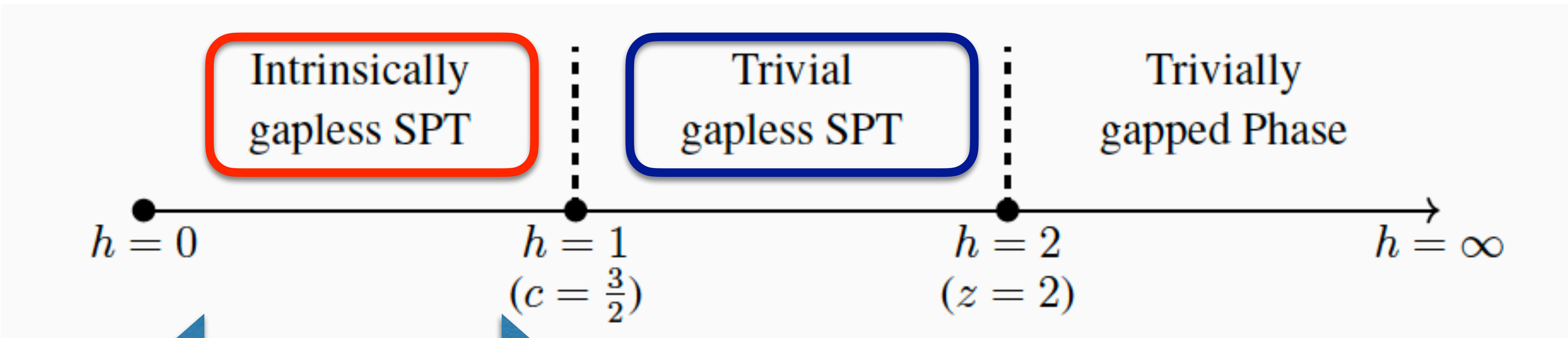
Both exactly solvable!

Phase Diagram

Ising SSB
XY critical (TLL)

Ising trivial
XY critical (TLL)

Ising trivial
XY trivially gapped



$$\langle \sigma_i^z \left(\prod_{k=i}^{j-1} \tau_{k+1/2}^x \right) \sigma_j^z \rangle \sim O(1)$$

$$\langle \tau_{i-1/2}^z \left(\prod_{k=i}^{j-1} \sigma_k^x \right) \tau_{j-1/2}^z \rangle \sim \frac{1}{|i-j|^{2\Delta}}$$

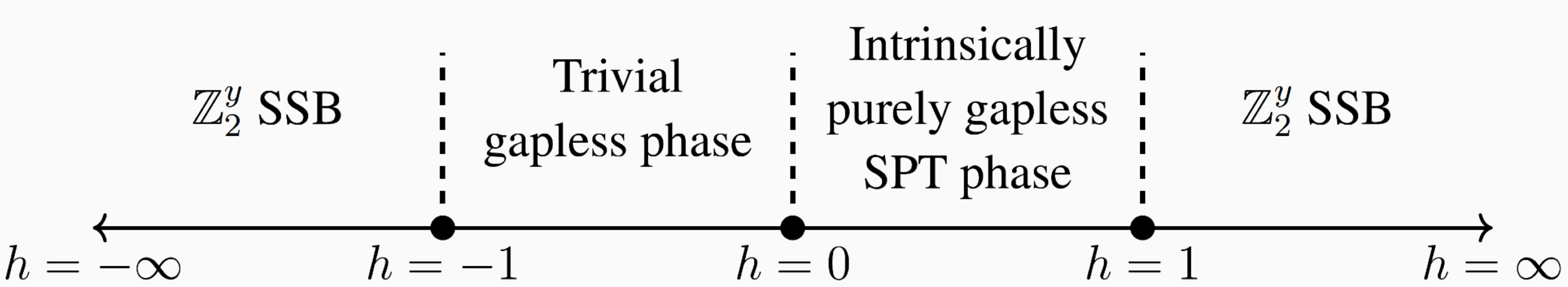
Getting Rid of Gapped Sector?

Replace the gapped SSB in the dual system with a gapless system

$$H^h_{\mathrm{XXZ+XXZ}} = -\sum_{i=1}^{L}\left(\sigma^z_i\sigma^z_{i+1} + \sigma^y_i\sigma^y_{i+1} + h\sigma^x_i\sigma^x_{i+1} + \tau^z_{i-\frac{1}{2}}\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\tau^y_{i+\frac{1}{2}} + h\tau^x_{i-\frac{1}{2}}\tau^x_{i+\frac{1}{2}}\right).$$

$\Updownarrow$ $\mathcal{N}_{\mathrm{KT}}$

$$H^h_{\mathrm{ipgSPTpert}} = -\sum_{i=1}^{L}\left(\sigma^z_i\tau^x_{i+\frac{1}{2}}\sigma^z_{i+1} + \sigma^y_i\tau^x_{i+\frac{1}{2}}\sigma^y_{i+1} + h\sigma^x_i\sigma^x_{i+1} + \tau^z_{i-\frac{1}{2}}\sigma^x_i\tau^z_{i+\frac{1}{2}} + \tau^y_{i-\frac{1}{2}}\sigma^x_i\tau^y_{i+\frac{1}{2}} + h\tau^x_{i-\frac{1}{2}}\tau^x_{i+\frac{1}{2}}\right).$$



(ipgSPT characterized by
symmetry charges in twisted sectors)

Summary

	$\mathcal{N}_{\mathrm{KT}}$	
trivial + trivial	⟷	trivial
Z2 SSB + trivial	⟷	Z2 SSB + trivial
Z2 SSB + Z2 Ising CFT	⟷	Z2xZ2 gapless SPT
Z2 SSB + Z2 SSB	⟷	Z2xZ2 SPT
Z2 SSB + Z4 free boson CFT	⟷	Z4 intrinsically gapless SPT
Z2 free boson CFT + Z2 free boson CFT	⟷	Z2xZ2 purely gapless SPT
Z2 free boson CFT + Z4 free boson CFT	⟷	Z4 intrinsically purely gapless SPT

Recent Developments

arXiv:1803.02369, arXiv:2402.09520
duality for subsystem symmetries

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Subsystem symmetry protected topological order

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Kennedy-Tasaki transformation and non-invertible symmetry in lattice models beyond one dimension

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(Dated: February 16, 2024)

arXiv:2403.00905 duality for fusion category symmetries

Hasse Diagrams for Gapless SPT and SSB Phases with Non-Invertible Symmetries

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arXiv:2405.09754 fermionic Kennedy-Tasaki dualities

Fermionic Non-Invertible Symmetries in (1+1)d: Gapped and Gapless Phases, Transitions, and Symmetry TFTs

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Conclusions

Symmetry-Protected Topological (SPT) phases:
distinct from trivial phase in the presence of symmetry
- short-range entanglement
- edge states
- nonlocal order parameter

Many of the “SPT features” were recognized
in 1980s〜1990s for the Haldane gap phase

Kennedy-Tasaki duality relating SPT to SSB phases
recent resurgence in “generalized Landau paradigm”

Landau was even more right than we thought.
This seems to be a fruitful principle.

(John McGreevy)