The Many Faces of Quantum Entropy: From Divergence Measures to Conditional Independence

Angela Capel
(University of Cambridge)

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Classical and Quantum Entropies

What is Entropy? From Classical to Quantum Information

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- It measures the average information per message.
- It is crucial for data compression, transmission and cryptography.
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Why do we care?

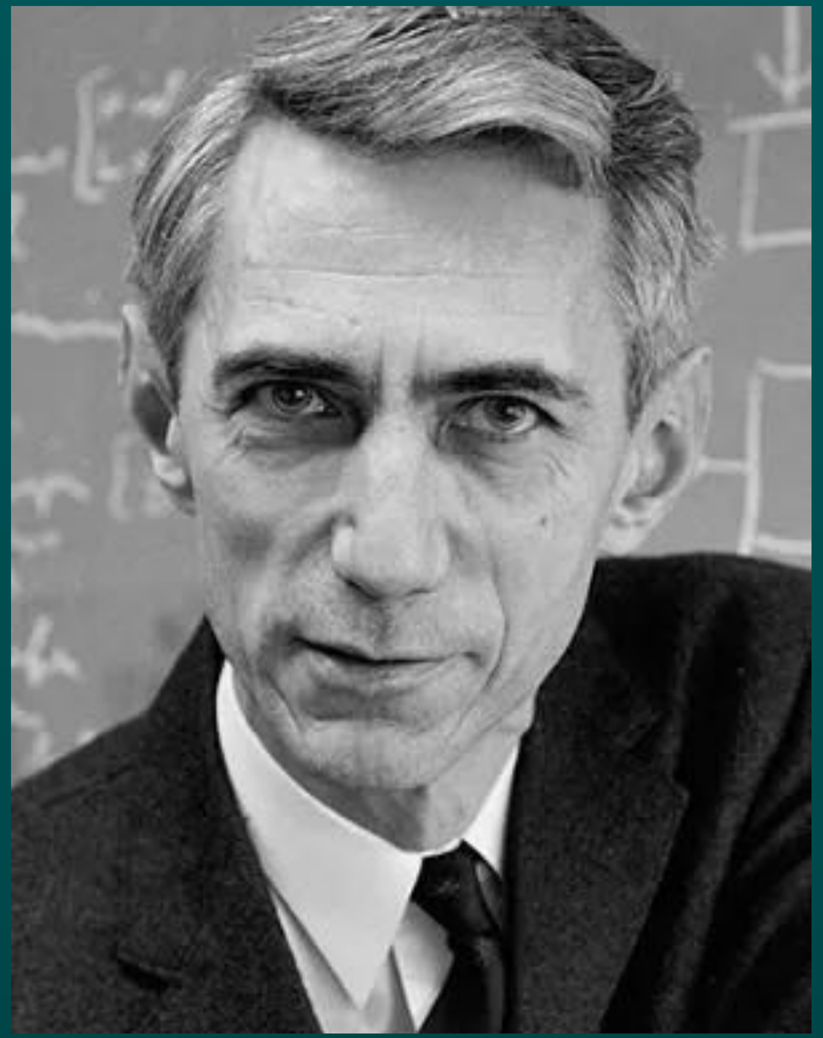
- It sets limits on how efficiently we can store, transmit or hide information.
- It helps us understand correlations, irreversibility and complexity in both classical and quantum worlds.

Classical Entropies

Shannon entropy:

- It is the foundational measure of uncertainty in a discrete random variable.

$$H(X) := -\sum_x p_x \log(p_x)$$



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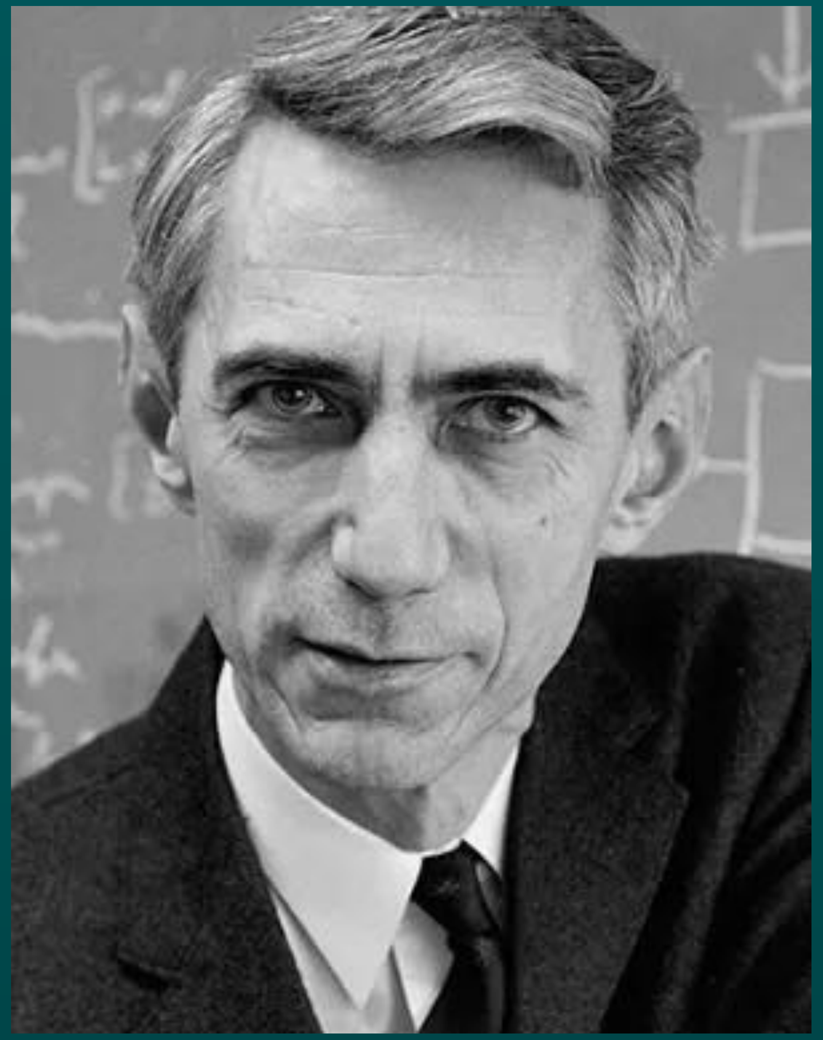

Shannon

Some properties:

- Non-negativity: $H(X) \geq 0$
- Maximum value: $H(X) = \log(n)$ when $p_x = \frac{1}{n} \quad \forall x = 1, \ldots, n$
- Minimum value: $H(X) = 0$ for $p_i = 1$ and $p_j = 0 \quad \forall j \neq i$

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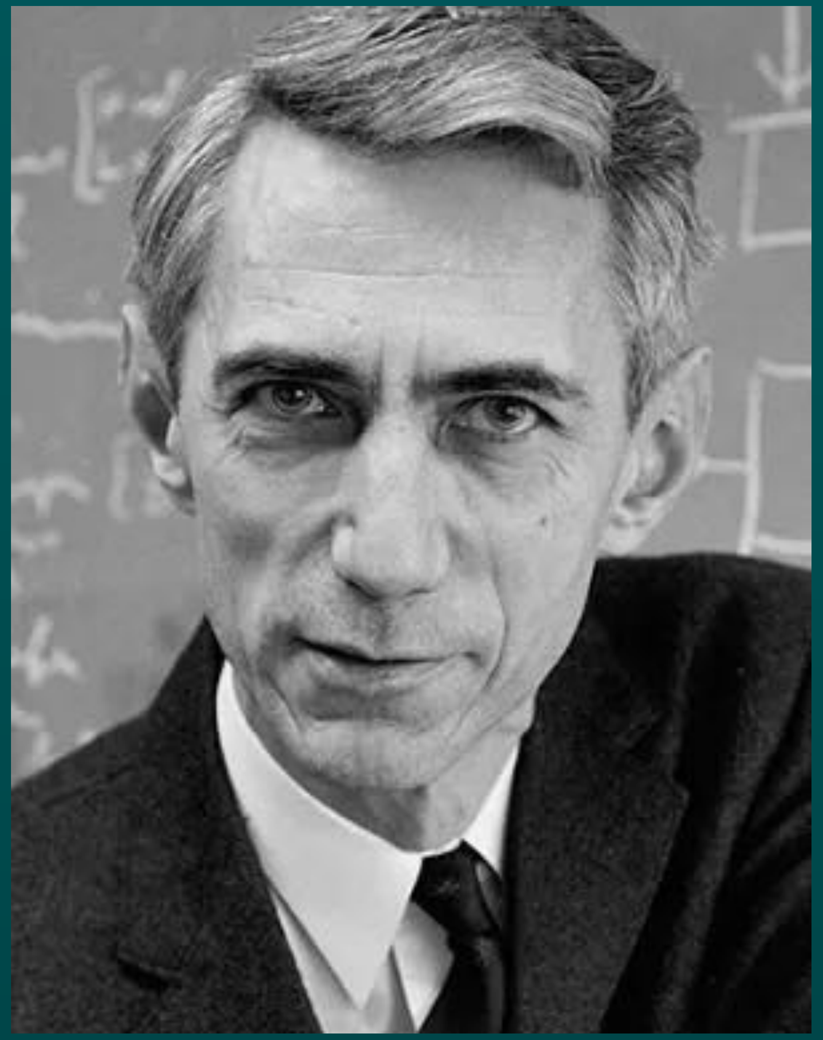

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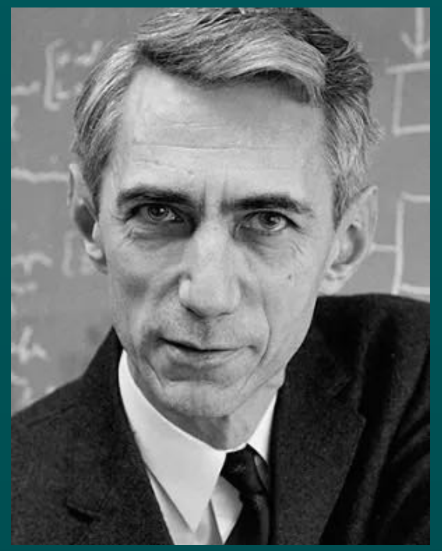
Some interpretations/applications:

- Average number of bits needed to encode outcomes of X
- Optimal rate for lossless compression
- Used in statistical mechanics, machine learning and cryptography

Classical Entropies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Other quantities

Rényi entropies:

- A one-parameter family generalising the Shannon entropy. $H_\alpha(X) = \frac{1}{1-\alpha}\log\left(\sum_{x=1}^{n} p_x^\alpha\right)$

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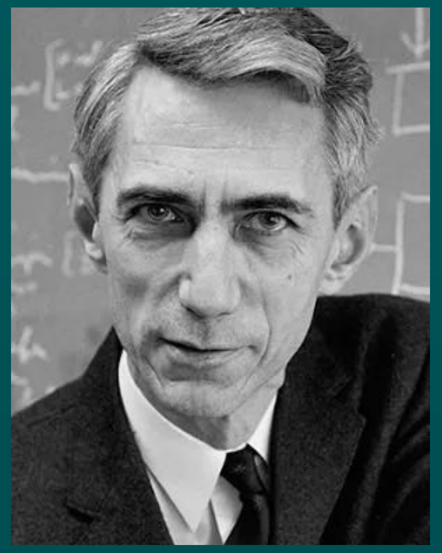
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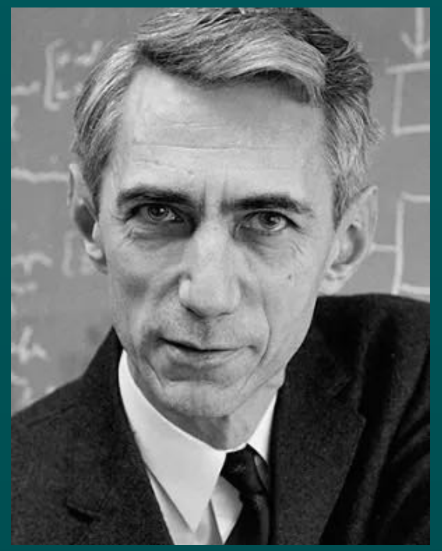
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Mutual information:

- It measures shared information between variables. $I(X:Y) = H(X) + H(Y) - H(X,Y)$

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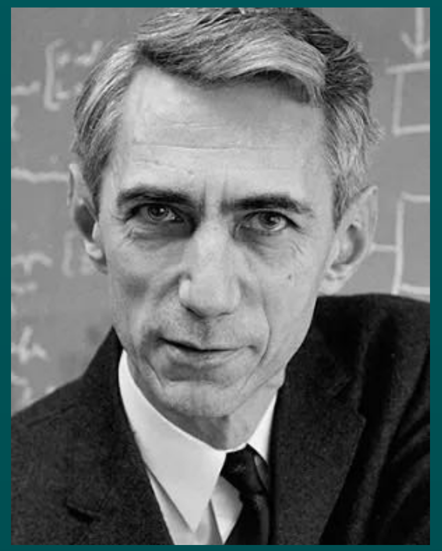
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Rényi divergences:

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How to extend this Quantumly?

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Quantum Information

Von Neumann entropy

$$S(\rho) := -\mathrm{Tr}[\rho\log(\rho)]$$

$$\left(S(\rho) = -\sum_x \lambda_x \log(\lambda_x) \right)$$

Quantum Rényi entropies

$$S_\alpha(\rho) = \frac{1}{1-\alpha}\log\mathrm{Tr}[\rho^\alpha]$$

$$\alpha \in (0,1)\cup(1,\infty)$$

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$$I_\rho(A:B) = -S(\rho_{AB}) + S(\rho_A) + S(\rho_B)$$

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Quantum Information

Many possibilities!!!

Quantum Relative Entropies

$\mathcal{H}, \mathcal{K}$ finite dimensional , $\rho, \sigma \in \mathcal{S}(\mathcal{H})$

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Any more?

Geometric relative entropies and barycentric Rényi divergences

Milán Mosonyi,[1,2,3,*] Gergely Bunth,[1,2,†] and Péter Vrana[1,4,‡]

Idea: Interpolation between both.

Data processing inequality

Relative entropies and Data-processing inequality

$\mathcal{H}, \mathcal{K}$ finite dimensional , $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, $\mathcal{T} : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K})$ CPTP map (quantum channel)

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A quantum channel is a linear map $T : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{K})$ such that

1. Trace preserving, i.e. $\mathrm{Tr}[\mathcal{T}(\rho)] = \mathrm{Tr}[\rho] \;\; \forall \rho \in \mathcal{S}(\mathcal{H})$.
2. Positive: $\rho \geq 0$, then $\mathcal{T}(\rho) \geq 0$.
3. Completely positive: For all $n \in \mathbb{N}_0$, $\mathcal{T} \otimes \mathrm{id}_n$ is positive, with id_n the identity map on $\mathcal{B}(\mathbb{C}^n)$.

i.e. a completely positive trace preserving (CPTP) map.

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Equality conditions

[Petz, '78]

$$D(\rho\|\sigma) = D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$

$$\Updownarrow$$

$$\rho = \sigma^{1/2}\mathcal{T}^*(\mathcal{T}(\sigma)^{-1/2}\mathcal{T}(\rho)\mathcal{T}(\sigma)^{-1/2})\sigma^{1/2}$$

$$\|$$

$$\mathcal{R}^{\sigma}_{\mathcal{T}} \circ \mathcal{T}(\rho)$$

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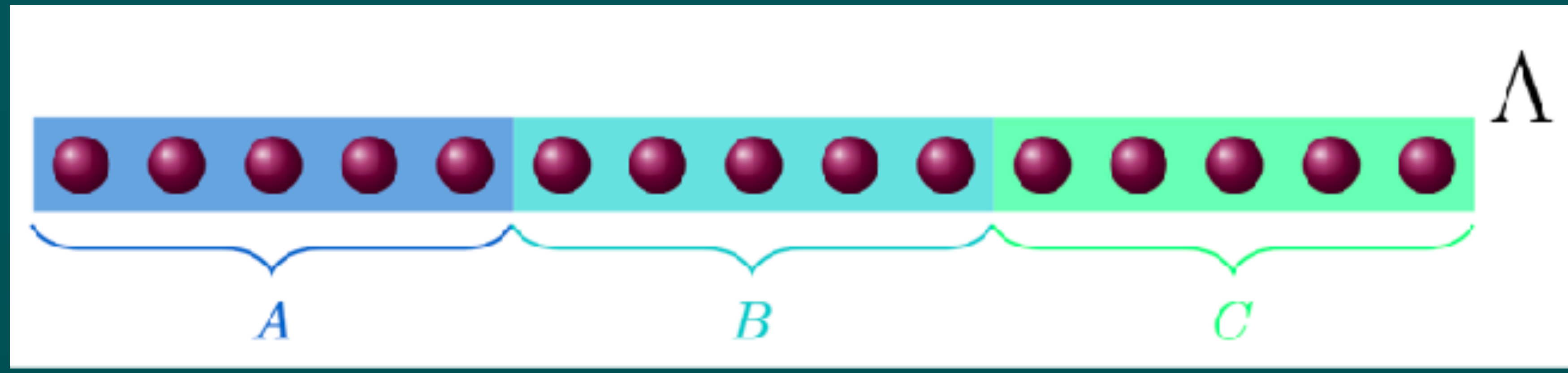
A particular case

$$\mathcal{H}_{ABC},\ \rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC}),\ \sigma_{ABC} = \frac{\mathbf{1}_A}{d_A} \otimes \rho_{BC}$$

$$\mathcal{T} = \mathrm{tr}_C$$



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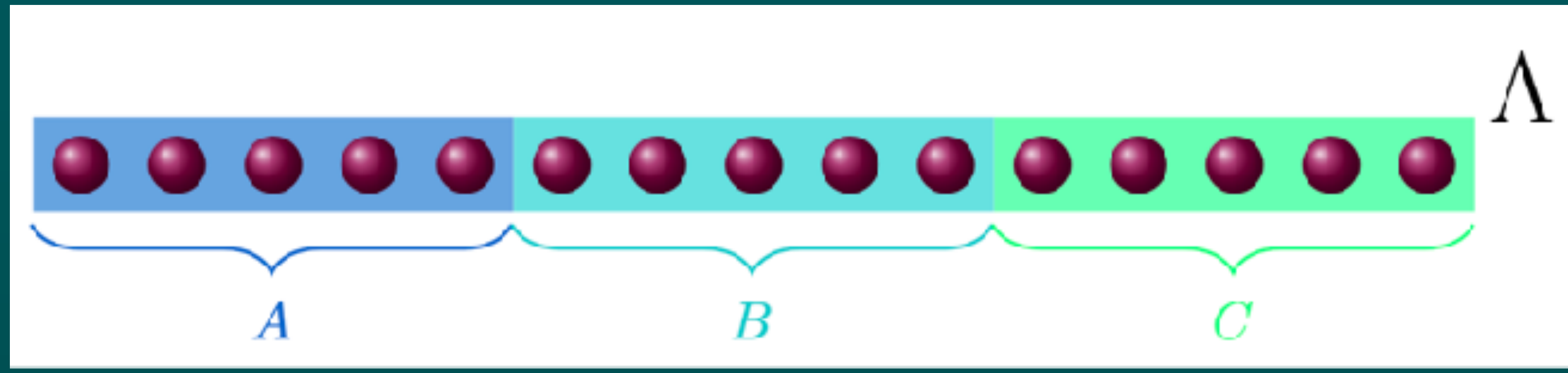

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$$D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$$

$$D(\rho\|\sigma) - D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$

Conditional mutual information

$$\begin{aligned} I_\rho(A:C|B) &= S_\rho(AB) + S_\rho(BC) - S_\rho(ABC) - S_\rho(B) \\ &= D(\rho_{ABC}\|1_A/d_A \otimes \rho_{BC}) - D(\rho_{AB}\|1_A/d_A \otimes \rho_B) \\ &= D(\rho_{ABC}\|\rho_A \otimes \rho_{BC}) - D(\rho_{AB}\|\rho_A \otimes \rho_B) \end{aligned}$$

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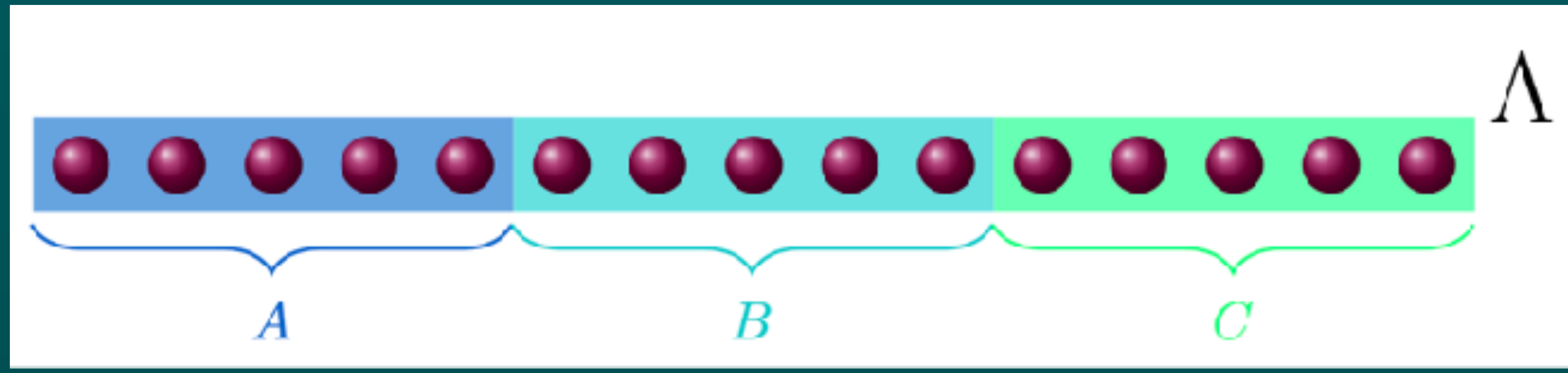

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$$I_\rho(A:C|B) = 0$$

$\Updownarrow$ [Hayden et al., '03]

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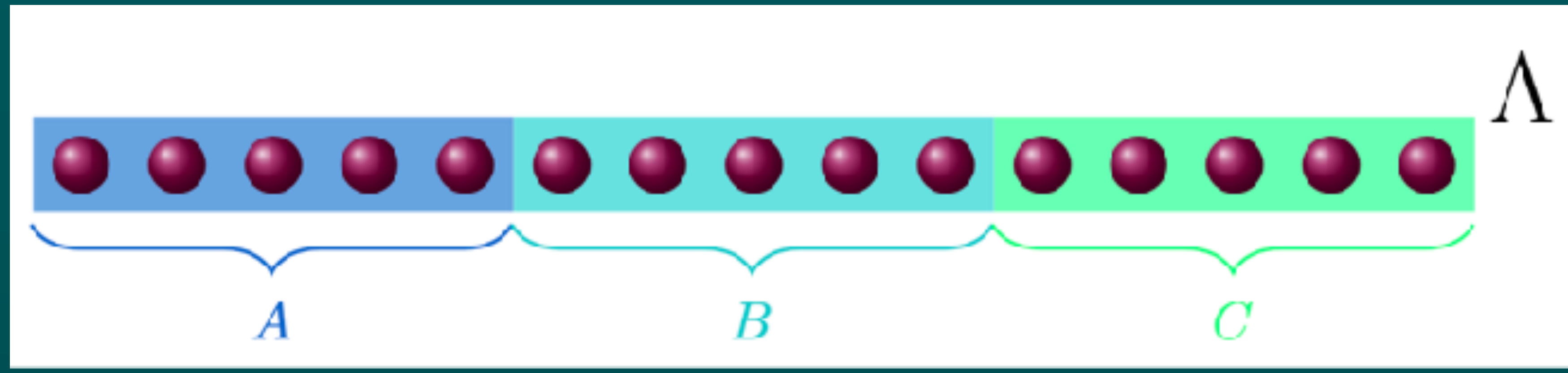

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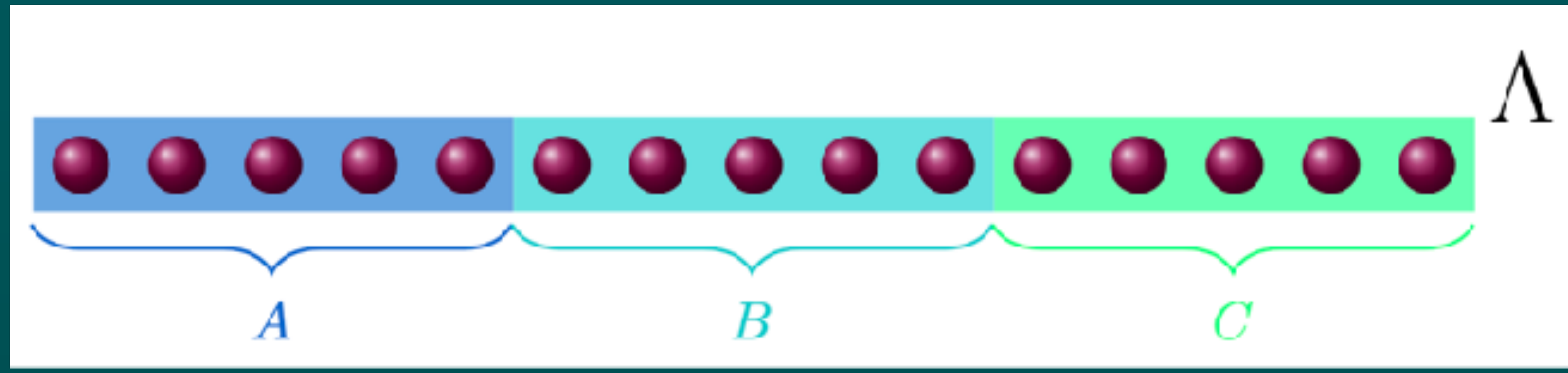

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Conditional mutual information

BS conditional mutual information

$$\widehat{I}_\rho(A:C|B)\ ?$$

$$I_\rho(A:C|B)$$

$$= D(\rho_{ABC}\|1_A/d_A \otimes \rho_{BC}) - D(\rho_{AB}\|1_A/d_A \otimes \rho_B) \longrightarrow \widehat{D}(\rho_{ABC}\|1_A/d_A \otimes \rho_{BC}) - \widehat{D}(\rho_{AB}\|1_A/d_A \otimes \rho_B) = \widehat{I}^{\mathrm{os}}_\rho(A:C|B)$$

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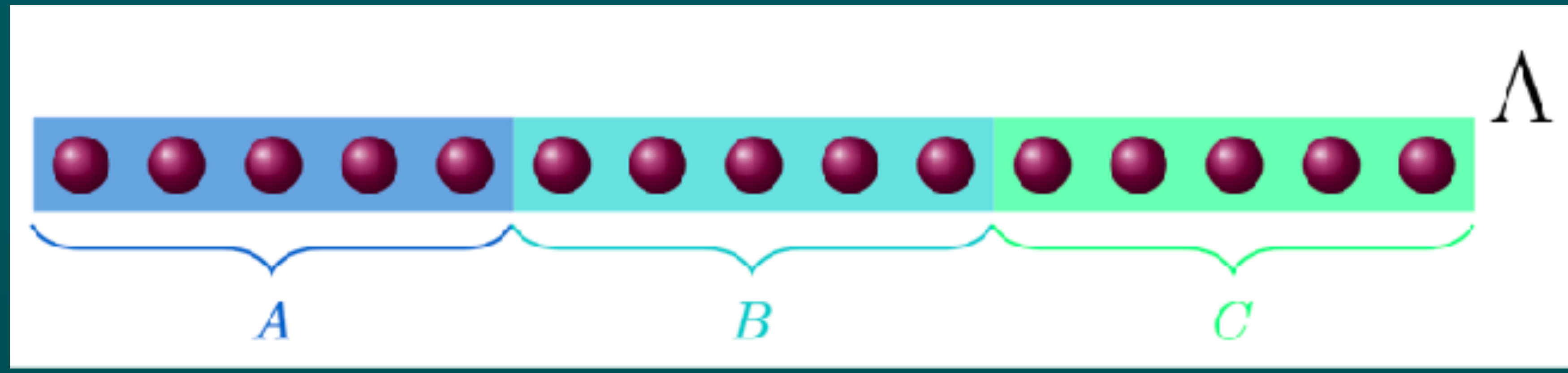

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$$= D(\rho_{ABC}\|\rho_A \otimes \rho_{BC}) - D(\rho_{AB}\|\rho_A \otimes \rho_B) \longrightarrow \widehat{D}(\rho_{ABC}\|\rho_A \otimes \rho_{BC}) - \widehat{D}(\rho_{AB}\|\rho_A \otimes \rho_B) = \widehat{I}^{\mathrm{ts}}_\rho(A:C|B)$$

$$I_\rho(A:C|B) = 0$$

$$\Updownarrow$$

$$\rho_{ABC} = \rho_{BC}^{1/2}\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}\rho_{BC}^{1/2}$$

BS conditional mutual information

$$\widehat{I}_\rho(A:C|B)\ ?$$

$$\widehat{I}^{\mathrm{os,ts}}_\rho(A:C|B) = 0$$

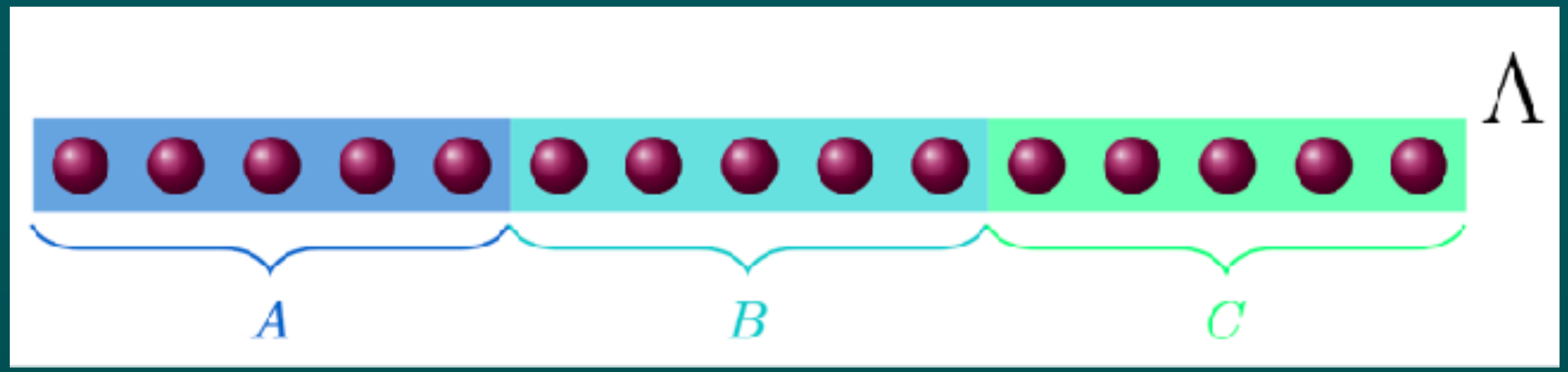
$$\Updownarrow$$

$$\rho_{ABC} = \rho_{BC}\rho_B\rho_{AB}$$

A particular case

$$\mathcal{H}_{ABC},\ \rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC}),\ \sigma_{ABC} = \frac{\mathbf{1}_A}{d_A} \otimes \rho_{BC}$$

$$\mathcal{T} = \mathrm{tr}_C$$



Belavkin–Staszewski relative entropy

$$\widehat{D}(\rho\|\sigma) = \mathrm{Tr}[\rho \log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$$

$$\widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$

BS conditional mutual information

$$\widehat{I}_\rho(A:C|B)\ ?$$

$$\widehat{D}(\rho_{ABC}\|1_A/d_A \otimes \rho_{BC}) - \widehat{D}(\rho_{AB}\|1_A/d_A \otimes \rho_B) = \widehat{I}_\rho^{\mathrm{os}}(A:C|B)$$

$$\widehat{D}(\rho_{ABC}\|\rho_A \otimes \rho_{BC}) - \widehat{D}(\rho_{AB}\|\rho_A \otimes \rho_B) = \widehat{I}_\rho^{\mathrm{ts}}(A:C|B)$$

$$\widehat{I}_\rho^{\mathrm{os,ts}}(A:C|B) = 0$$

$$\Updownarrow$$

$$\rho_{ABC} = \rho_{BC}\rho_B\rho_{AB}$$

$$\widehat{D}(\rho\|\sigma) = \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$

$$\Updownarrow$$

$$\rho = \sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho))$$

$$||$$

$$\mathcal{B}_\mathcal{T}^\sigma \circ \mathcal{T}(\rho)$$

Relative entropies and Recoverability

Umegaki relative entropy

Belavkin-Staszewski relative entropy

Equality conditions

[Petz, '78]

[Bluhm-C., '20]

$$D(\rho\|\sigma) = D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$
$$\Updownarrow$$
$$\rho = \sigma^{1/2}\mathcal{T}^*(\mathcal{T}(\sigma)^{-1/2}\mathcal{T}(\rho)\mathcal{T}(\sigma)^{-1/2})\sigma^{1/2}$$
$$||$$
$$\mathcal{R}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho)$$

$$\widehat{D}(\rho\|\sigma) = \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$
$$\Updownarrow$$
$$\rho = \sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho))$$
$$||$$
$$\mathcal{B}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho)$$

Relative entropies and Approximate Recoverability

	Umegaki relative entropy	Belavkin-Staszewski relative entropy
Equality conditions [Petz, '78] [Bluhm-C., '20]	$D(\rho\\|\sigma) = D(\mathcal{T}(\rho)\\|\mathcal{T}(\sigma))$ $\Updownarrow$ $\rho = \sigma^{1/2}\mathcal{T}^*(\mathcal{T}(\sigma)^{-1/2}\mathcal{T}(\rho)\mathcal{T}(\sigma)^{-1/2})\sigma^{1/2}$ $\\|$ $\mathcal{R}^\sigma_\mathcal{T} \circ \mathcal{T}(\rho)$	$\widehat{D}(\rho\\|\sigma) = \widehat{D}(\mathcal{T}(\rho)\\|\mathcal{T}(\sigma))$ $\Updownarrow$ $\rho = \sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho))$ $\\|$ $\mathcal{B}^\sigma_\mathcal{T} \circ \mathcal{T}(\rho)$
Approximate version	$D(\rho\\|\sigma) - D(\mathcal{T}(\rho)\\|\mathcal{T}(\sigma)) \leq \varepsilon$ $\Updownarrow$? $\rho \approx \mathcal{R}^\sigma_\mathcal{T} \circ \mathcal{T}(\rho)$	$\widehat{D}(\rho\\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\\|\mathcal{T}(\sigma)) \leq \varepsilon$ $\Updownarrow$? $\rho \approx \mathcal{B}^\sigma_\mathcal{T} \circ \mathcal{T}(\rho)$

Relative entropies and Approximate Recoverability

	Umegaki relative entropy	Belavkin-Staszewski relative entropy
Approximate DPI	$D(\rho\\|\sigma) - D(\mathcal{T}(\rho)\\|\mathcal{T}(\sigma)) \leq \varepsilon$ $\Updownarrow$? $\rho \approx \mathcal{R}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho)$	$\widehat{D}(\rho\\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\\|\mathcal{T}(\sigma)) \leq \varepsilon$ $\Updownarrow$? $\rho \approx \mathcal{B}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho)$

Relative entropies and Strengthened Data-processing inequality

	Umegaki relative entropy	Belavkin-Staszewski relative entropy
Approximate DPI	$D(\rho\|\sigma) - D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \leq \varepsilon$ $\Updownarrow$? $\rho \approx \mathcal{R}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho)$	$\widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \leq \varepsilon$ $\Updownarrow$? $\rho \approx \mathcal{B}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho)$

[Fawzi-Renner, '15]

[Junge et al, '18]

[Carlen-Vershynina, '20]

[Gao-Wilde, '21]

$$\left(\frac{\pi}{8}\right)^4 \|\rho^{-1}\|^{-2} \|\mathcal{T}(\rho)^{-1}\|^{-2} \|\mathcal{R}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho) - \rho\|_1^4 \leq D(\rho\|\sigma) - D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$

Relative entropies and Strengthened Data-processing inequality

	Umegaki relative entropy	Belavkin-Staszewski relative entropy
Approximate DPI	$D(\rho\|\sigma) - D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \leq \varepsilon$ $\Updownarrow$? $\rho \approx \mathcal{R}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho)$	$\widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \leq \varepsilon$ $\Updownarrow$? $\rho \approx \mathcal{B}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho)$
[Fawzi-Renner, '15] [Junge et al, '18] [Carlen-Vershynina, '20] [Gao-Wilde, '21] [Bluhm-C., '20]	$\left(\frac{\pi}{8}\right)^4 \|\rho^{-1}\|^{-2}\|\mathcal{T}(\rho)^{-1}\|^{-2}\|\mathcal{R}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho) - \rho\|_1^4 \leq D(\rho\|\sigma) - D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$	$\left(\frac{\pi}{8}\right)^4 \|\rho^{-1/2}\sigma\rho^{-1/2}\|^{-4}\|\rho^{-1}\|^{-2}\|\mathcal{B}_{\mathcal{T}}^{\rho} \circ \mathcal{T}(\sigma) - \sigma\|_1^4 \leq \widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$

Relative entropies and Approximate Data-processing inequality

	Umegaki relative entropy	Belavkin-Staszewski relative entropy
Approximate DPI	$D(\rho\|\sigma) - D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \leq \varepsilon$ $\Updownarrow$? $\rho \approx \mathcal{R}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho)$	$\widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \leq \varepsilon$ $\Updownarrow$? $\rho \approx \mathcal{B}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho)$

[Fawzi-Renner, '15]

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[Carlen-Vershynina, '20]

[Gao-Wilde, '21]

$$\left(\frac{\pi}{8}\right)^4 \|\rho^{-1}\|^{-2} \|\mathcal{T}(\rho)^{-1}\|^{-2} \|\mathcal{R}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho) - \rho\|_1^4$$

$$\leq D(\rho\|\sigma) - D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$

$$\leq ?$$

Relative entropies and Approximate Data-processing inequality

	Umegaki relative entropy	Belavkin-Staszewski relative entropy
Approximate DPI	$D(\rho\|\sigma) - D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \leq \varepsilon$ $\Updownarrow$? $\rho \approx \mathcal{R}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho)$	$\widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \leq \varepsilon$ $\Updownarrow$? $\rho \approx \mathcal{B}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho)$

[Fawzi-Renner, '15]

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$$\left(\frac{\pi}{8}\right)^4 \|\rho^{-1}\|^{-2} \|\mathcal{T}(\rho)^{-1}\|^{-2} \|\mathcal{R}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho) - \rho\|_1^4 \leq D(\rho\|\sigma) - D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \leq ?$$

$$\left(\frac{\pi}{8}\right)^4 \|\rho^{-1/2}\sigma\rho^{-1/2}\|^{-4} \|\rho^{-1}\|^{-2} \|\mathcal{B}_{\mathcal{T}}^{\rho} \circ \mathcal{T}(\sigma) - \sigma\|_1^4 \leq \widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \leq \text{Here}$$

Reversed DPI for the Belavkin-Staszewski entropy

$\widehat{D}(\rho\|\sigma) = \mathrm{Tr}[\rho \log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$

Theorem (Gondolf, Scalet, Ruiz-de-Alarcón, Alhambra, C. '24) $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, $\mathcal{T} : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K})$

$$\widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \leq \|\rho^{-1/2}\sigma\rho^{-1/2}\|_\infty \|\mathcal{T}(\rho)^{1/2}\|_\infty \|\mathcal{T}(\rho)^{-1/2}\|_\infty$$
$$\cdot \|\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)\|_\infty \|\rho\sigma^{-1}\mathcal{T}^*(\mathcal{T}(\sigma)\mathcal{T}(\rho)^{-1}) - \mathbf{1}\|_\infty$$

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$$\widehat{D}(\rho\|\sigma) = \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$
$$\Updownarrow$$
$$\rho = \sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho))$$
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Simplified: $\mathcal{E}$ conditional expectation

$$\boxed{\widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{E}(\rho)\|\,\mathcal{E}(\sigma)) = 0 \Leftrightarrow \sigma = \mathcal{E}(\sigma)\mathcal{E}(\rho)^{-1}\rho =: \mathcal{B}^\rho_{\mathcal{E}}(\sigma)}$$

$$\boxed{\widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{E}(\rho)\|\,\mathcal{E}(\sigma)) \leq \begin{cases} C(\rho,\sigma,\mathcal{E})\,\|\rho\,(\mathcal{B}^\sigma_{\mathcal{E}}(\rho))^{-1} - \mathbb{1}\,\|_\infty \\ C'(\rho,\sigma,\mathcal{E})\,\|\sigma\,(\mathcal{B}^\rho_{\mathcal{E}}(\sigma))^{-1} - \mathbb{1}\,\|_\infty \end{cases}}$$

Reversed DPI for the Belavkin-Staszewski entropy

$\widehat{D}(\rho\|\sigma) = \mathrm{Tr}[\rho \log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$

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$$\widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \leq \|\rho^{-1/2}\sigma\rho^{-1/2}\|_\infty \|\mathcal{T}(\rho)^{1/2}\|_\infty \|\mathcal{T}(\rho)^{-1/2}\|_\infty \cdot \|\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)\|_\infty \|\rho\sigma^{-1}\mathcal{T}^*(\mathcal{T}(\sigma)\mathcal{T}(\rho)^{-1}) - \mathbf{1}\|_\infty$$

Simplified: $\mathcal{E}$ conditional expectation

$$\widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) = 0 \Leftrightarrow \sigma = \mathcal{E}(\sigma)\mathcal{E}(\rho)^{-1}\rho =: \mathcal{B}^\rho_\mathcal{E}(\sigma)$$

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Proof

$$\log(X) = \int_0^\infty \left(\frac{1}{t+1} - \frac{1}{t+X} \right) dt$$

+

multiple norm inequalities

Reversed DPI for the Belavkin–Staszewski entropy

$\widehat{D}(\rho\|\sigma) = \mathrm{Tr}[\rho \log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$

Theorem (Gondolf, Scalet, Ruiz-de-Alarcón, Alhambra, C. '24) $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, $\mathcal{T} : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K})$

$$\widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma)) \leq \|\rho^{-1/2}\sigma\rho^{-1/2}\|_\infty \|\mathcal{T}(\rho)^{1/2}\|_\infty \|\mathcal{T}(\rho)^{-1/2}\|_\infty \cdot \|\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho)\|_\infty \|\rho\sigma^{-1}\mathcal{T}^*(\mathcal{T}(\sigma)\mathcal{T}(\rho)^{-1}) - \mathbf{1}\|_\infty$$

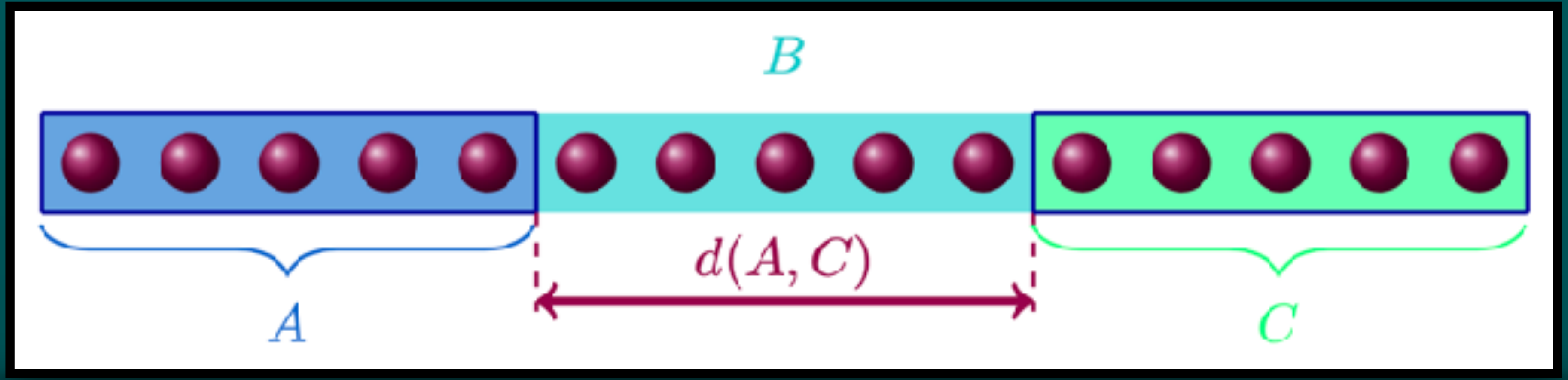
Approximate DPI: $\mathcal{E}$ conditional expectation

$$\left(\frac{\pi}{8}\right)^4 \left\|\rho^{-1/2}\sigma\rho^{-1/2}\right\|_\infty^{-4} \|\mathcal{E}(\rho)^{-1}\|_\infty^{-2} \|\mathcal{B}_\mathcal{E}^\rho(\sigma) - \sigma\|_2^4 \leq \widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{E}(\rho)\|\,\mathcal{E}(\sigma)) \leq \left\|\rho^{-1/2}\sigma\rho^{-1/2}\right\|_\infty \|\mathcal{E}(\rho)^{-1}\|_\infty^{1/2} \|\sigma^{-1}\rho\|_\infty \|\mathcal{B}_\mathcal{E}^\rho(\sigma) - \sigma\|_\infty$$

A particular case

$$\mathcal{H}_{ABC},\ \rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC}),\ \sigma_{ABC} = \frac{\mathbf{1}_A}{d_A} \otimes \rho_{BC}$$

$$\mathcal{T} = \mathrm{tr}_C$$



Umegaki relative entropy

$$D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$$

$$D(\rho\|\sigma) - D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$

Belavkin-Staszewski relative entropy

$$\widehat{D}(\rho\|\sigma) = \mathrm{Tr}[\rho\log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$$

$$\widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$

Conditional mutual information

$$I_\rho(A:C|B)$$

$$= D(\rho_{ABC}\|1_A/d_A \otimes \rho_{BC}) - D(\rho_{AB}\|1_A/d_A \otimes \rho_B) \longrightarrow \widehat{D}(\rho_{ABC}\|1_A/d_A \otimes \rho_{BC}) - \widehat{D}(\rho_{AB}\|1_A/d_A \otimes \rho_B) = \widehat{I}_\rho^{\mathrm{os}}(A:C|B)$$

$$= D(\rho_{ABC}\|\rho_A \otimes \rho_{BC}) - D(\rho_{AB}\|\rho_A \otimes \rho_B) \longrightarrow \widehat{D}(\rho_{ABC}\|\rho_A \otimes \rho_{BC}) - \widehat{D}(\rho_{AB}\|\rho_A \otimes \rho_B) = \widehat{I}_\rho^{\mathrm{ts}}(A:C|B)$$

$$I_\rho(A:C|B) = 0$$

$$\Updownarrow$$

$$\rho_{ABC} = \rho_{BC}^{1/2}\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}\rho_{BC}^{1/2}$$

BS conditional mutual information

$$\widehat{I}_\rho(A:C|B)\ ?$$

$$\widehat{I}_\rho^{\mathrm{os,ts}}(A:C|B) = 0$$

$$\Updownarrow$$

$$\rho_{ABC} = \rho_{BC}\rho_B\rho_{AB}$$

A particular case

Umegaki relative entropy

$$D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$$

Belavkin-Staszewski relative entropy

$$\widehat{D}(\rho\|\sigma) = \mathrm{Tr}[\rho\log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$$

Conditional mutual information

BS conditional mutual information

$$I_\rho(A:C|B) = D(\rho_{ABC}\|1_A/d_A\otimes\rho_{BC}) - D(\rho_{AB}\|1_A/d_A\otimes\rho_B) \longrightarrow \widehat{D}(\rho_{ABC}\|1_A/d_A\otimes\rho_{BC}) - \widehat{D}(\rho_{AB}\|1_A/d_A\otimes\rho_B) = \widehat{I}^{\mathrm{os}}_\rho(A:C|B)$$

$$I_\rho(A:C|B) = 0$$

$$\Updownarrow$$

$$\rho_{ABC} = \rho_{BC}^{1/2}\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}\rho_{BC}^{1/2}$$

$$\widehat{I}^{\mathrm{os,ts}}_\rho(A:C|B) = 0$$

$$\Updownarrow$$

$$\rho_{ABC} = \rho_{BC}\rho_B\rho_{AB}$$

Approximate version?

$$\left(\frac{\pi}{8}\right)^4 \|\rho^{-1}\|^{-2}\|\mathcal{T}(\rho)^{-1}\|^{-2}\|\mathcal{R}^\sigma_{\mathcal{T}}\circ\mathcal{T}(\rho)-\rho\|_1^4$$
$$\leq D(\rho\|\sigma) - D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$
$$\leq ?$$

$$\left(\frac{\pi}{8}\right)^4 \|\rho^{-1/2}\sigma\rho^{-1/2}\|^{-4}\|\rho^{-1}\|^{-2}\|\mathcal{B}^\rho_{\mathcal{T}}\circ\mathcal{T}(\sigma)-\sigma\|_1^4$$
$$\leq \widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$
$$\leq \text{Here}$$

Conditional independence of quantum Gibbs states in 1D

Umegaki relative entropy

$$D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$$

Belavkin-Staszewski relative entropy

$$\widehat{D}(\rho\|\sigma) = \mathrm{Tr}[\rho\log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$$

Conditional mutual information

BS conditional mutual information

$$I_\rho(A:C|B) = D(\rho_{ABC}\|1_A/d_A\otimes\rho_{BC}) - D(\rho_{AB}\|1_A/d_A\otimes\rho_B) \longrightarrow \widehat{D}(\rho_{ABC}\|1_A/d_A\otimes\rho_{BC}) - \widehat{D}(\rho_{AB}\|1_A/d_A\otimes\rho_B) = \widehat{I}^{\mathrm{os}}_\rho(A:C|B)$$

$$I_\rho(A:C|B) = 0$$

$$\Updownarrow$$

$$\rho_{ABC} = \rho_{BC}^{1/2}\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}\rho_{BC}^{1/2}$$

$$\widehat{I}^{\mathrm{os,ts}}_\rho(A:C|B) = 0$$

$$\Updownarrow$$

$$\rho_{ABC} = \rho_{BC}\rho_B\rho_{AB}$$

Approximate version?

[Bluhm et al., '23] (follows from [Winter '16] and [Sutter-Renner '17])

$$\begin{aligned}\left(\frac{\pi}{8}\right)^4 \|\rho_B^{-1}\|_\infty^{-2}\,\|\rho_{ABC}^{-1}\|_\infty^{-2}\,\left\|\rho_{ABC} - \rho_{AB}^{1/2}\rho_B^{-1/2}\rho_{BC}\rho_B^{-1/2}\rho_{AB}^{1/2}\right\|_1^4 &\\ \leq I_\rho(A:C|B) &\\ \leq 2\left(\log\min\{d_A, d_C\}+1\right)\left\|\rho_{ABC} - \rho_{AB}^{1/2}\rho_B^{-1/2}\rho_{BC}\rho_B^{-1/2}\rho_{AB}^{1/2}\right\|_1^{1/2}\end{aligned}$$

$$\left(\frac{\pi}{8}\right)^4 \left\|\rho_{ABC}^{-1/2}\rho_{BC}\rho_{ABC}^{-1/2}\right\|_\infty^{-4}\|\rho_{AB}^{-1}\|_\infty^{-2}\|\rho_B\rho_{AB}^{-1}\rho_{ABC} - \rho_{BC}\|_\infty$$

$$\leq \widehat{I}^{\mathrm{os}}_\rho(A:C|B) \leq$$

$$\left\|\rho_{ABC}^{-1/2}\rho_{BC}\rho_{ABC}^{-1/2}\right\|_\infty(\|\rho_{AB}^{-1}\|_\infty\|\rho_{AB}\|_\infty)^{1/2}\|\rho_B^{-1}\rho_{AB}\|_\infty\|\rho_{ABC}\rho_{BC}^{-1}\rho_{AB}\rho_B^{-1} - \mathbb{1}\|_\infty$$

Applications of Approximate DPIs in Mathematical Physics

Bridging Quantum Information and Mathematical Physics

Quantum Information

Mathematical Physics

Bridging Quantum Information and Mathematical Physics

Quantum Information

Mathematical Physics

Bridging Quantum Information and Mathematical Physics

Quantum Information

Entropic inequalities
(reversed and
strengthened DPI)

Mathematical Physics

Conditional independence
(of 1D Gibbs states)

Bridging Quantum Information and Mathematical Physics

Quantum Information

Entropic inequalities
(reversed and
strengthened DPI)

Mathematical Physics

Conditional independence
(of 1D Gibbs states)

Algorithms

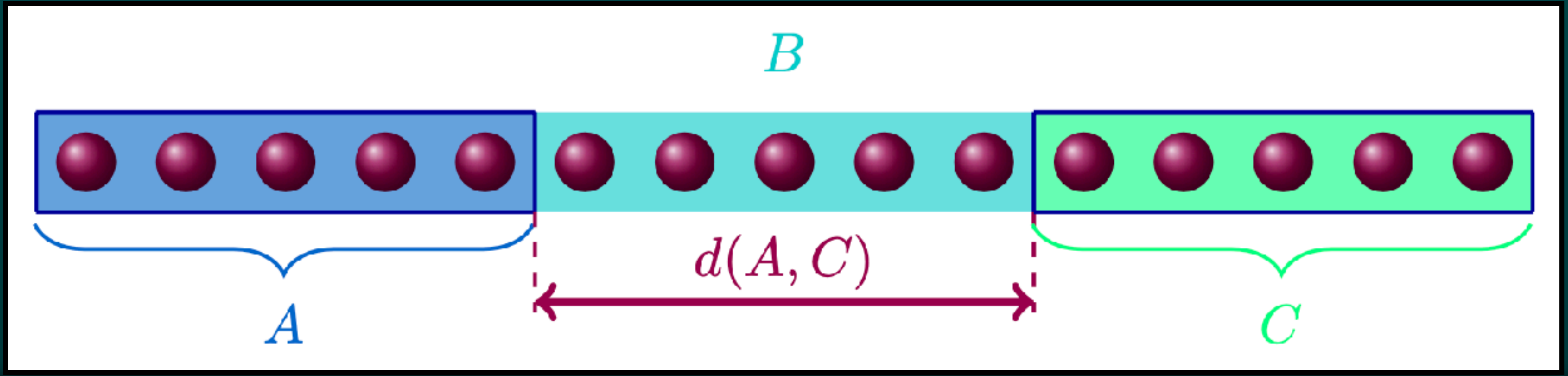
Efficient MPO approximation
of 1D Gibbs states

Step I:

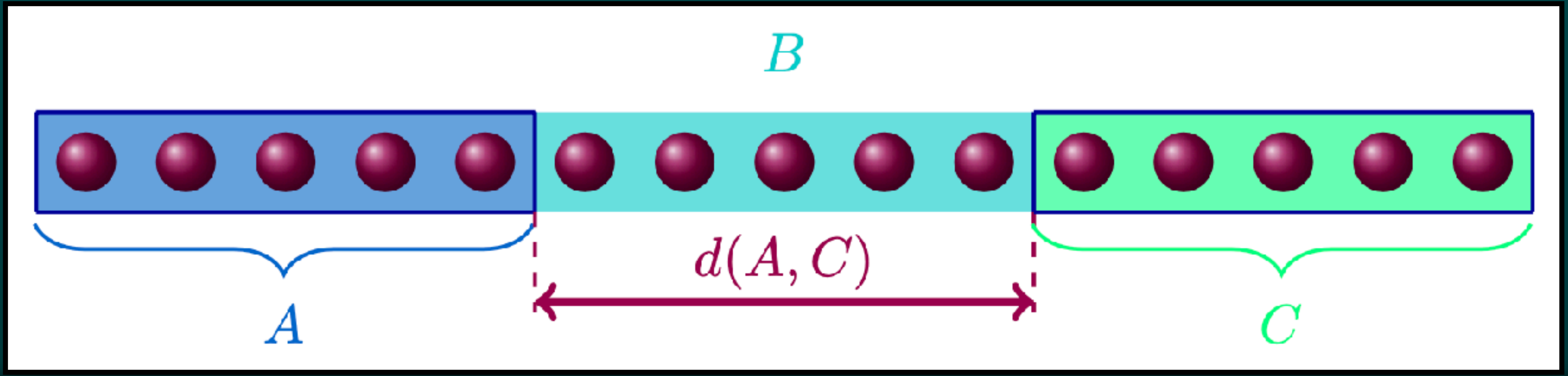
Reversed DPI for the Belavkin-Staszewski relative entropy

$$\left(\frac{\pi}{8}\right)^4 \left\|\rho^{-1/2}\sigma\rho^{-1/2}\right\|_\infty^{-4} \|\mathcal{E}(\rho)^{-1}\|_\infty^{-2} \|\mathcal{B}^\rho_\mathcal{E}(\sigma) - \sigma\|_2^4$$
$$\leq \widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{E}(\rho)\|\,\mathcal{E}(\sigma)) \leq$$
$$\left\|\rho^{-1/2}\sigma\rho^{-1/2}\right\|_\infty \|\mathcal{E}(\rho)^{-1}\|_\infty^{1/2} \|\sigma^{-1}\rho\|_\infty \|\mathcal{B}^\rho_\mathcal{E}(\sigma) - \sigma\|_\infty$$

STEP II:
CONDITIONAL INDEPENDENCE OF QUANTUM GIBBS STATES IN 1D

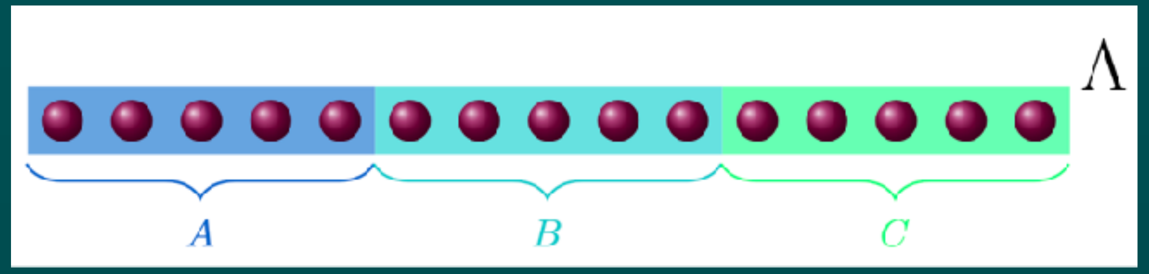


Quantum Gibbs states in 1D



Introduction to the setting

Study of quantum Gibbs states in 1D

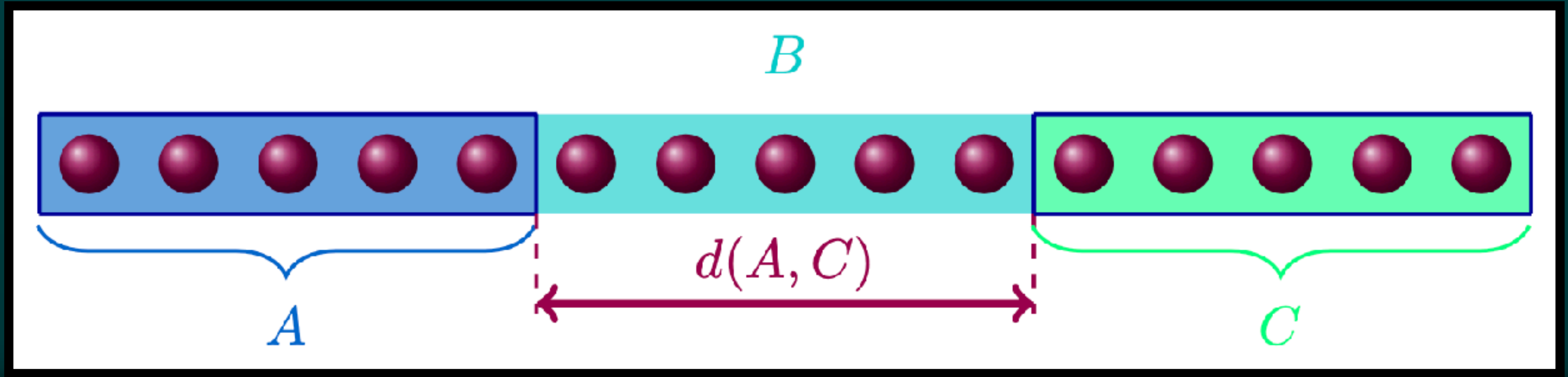


- Spin chain: $\Lambda \subset\subset \mathbb{Z}$
- Hamiltonian: $H_\Lambda = \sum_{X \subset \Lambda} H_X$
  - Finite-range (k-local interactions):
    $H_X = 0 \text{ for } \mathrm{diam}(X) > k \quad \text{and} \quad \|H_X\| < J \ \ \forall X \subset \Lambda$
  - Short-range (exponentially-decaying interactions):
    $\|H_\Lambda\|_\lambda := \sup_{x \in V} \sum_{X \ni x} \|H_X\| e^{\lambda |X|} < \infty$
- Gibbs state (at inverse temperature $\beta > 0$): $\rho^\Lambda := \frac{e^{-\beta H_\Lambda}}{\mathrm{Tr}[e^{-\beta H_\Lambda}]}$

Objectives of this work

Part II: Conditional independence of quantum Gibbs states in 1D

- Spin chain: $\Lambda \subset\subset \mathbb{Z}$
- Hamiltonian: $H_\Lambda = \sum_{X \subset \Lambda} H_X$
- Gibbs state (at $\beta > 0$): $\rho^\Lambda := \dfrac{e^{-\beta H_\Lambda}}{\mathrm{Tr}[e^{-\beta H_\Lambda}]}$



Decay of correlations

Covariance

$$\mathrm{Cov}_{\rho^\Lambda}(A,C) = \sup_{\|O_A\|=\|O_C\|=1} \left|\mathrm{Tr}[\rho^\Lambda O_A O_C] - \mathrm{Tr}[\rho^\Lambda O_A]\mathrm{Tr}[\rho^\Lambda O_C]\right|$$

Mutual information $\rho^\Lambda \equiv \rho$

$$I_\rho(A:C) = D(\rho_{AC}\|\rho_A \otimes \rho_C) = \mathrm{Tr}[\rho_{AC}(\log\rho_{AC} - \log\rho_A \otimes \rho_C)]$$

Conditional independence

Conditional mutual information

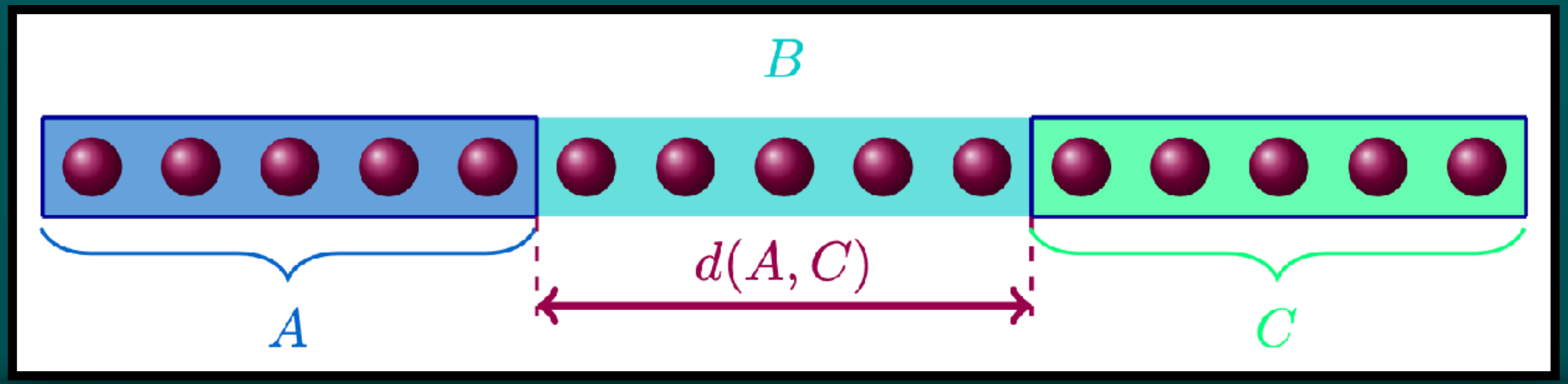
$$I_\rho(A:C|B) = S_\rho(AB) + S_\rho(BC) - S_\rho(ABC) - S_\rho(B)$$

Here different quantity!

$S_\rho(X) = -\mathrm{Tr}[\rho_X \log \rho_X]$

Conditional independence of quantum Gibbs states in 1D

$\widehat{D}(\rho\|\sigma) = \mathrm{Tr}[\rho \log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$



BS conditional mutual informations

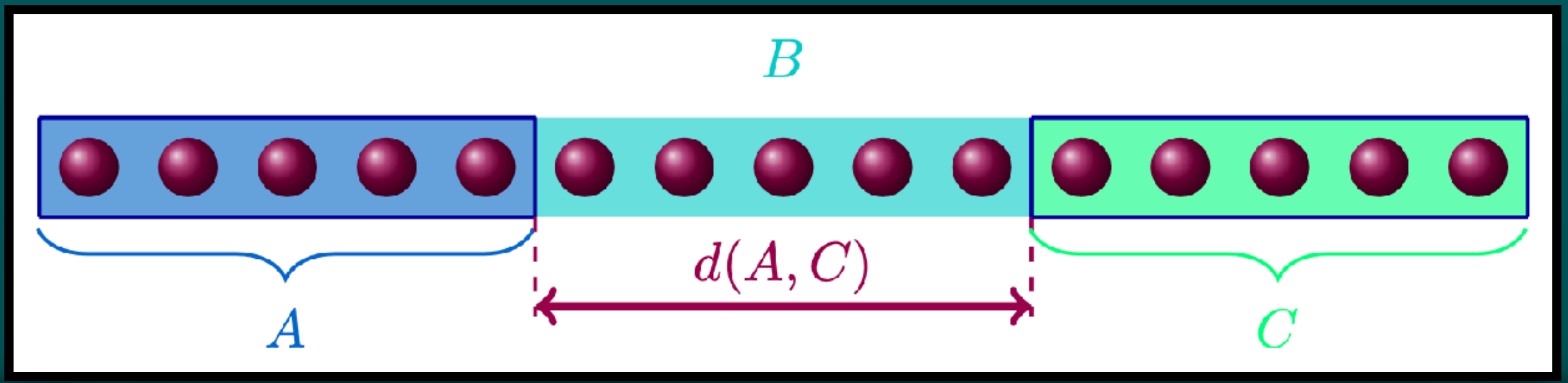
One-sided: $\widehat{I}^{\mathrm{os}}_\rho(A:C|B) = \widehat{D}(\rho_{ABC}\|1_A/d_A \otimes \rho_{BC}) - \widehat{D}(\rho_{AB}\|1_A/d_A \otimes \rho_B)$

Two-sided: $\widehat{I}^{\mathrm{ts}}_\rho(A:C|B) = \widehat{D}(\rho_{ABC}\|\rho_A \otimes \rho_{BC}) - \widehat{D}(\rho_{AB}\|\rho_A \otimes \rho_B)$

Reversed: $\widehat{I}^{\mathrm{rev}}_\rho(A:C|B) = \widehat{D}(\rho_A \otimes \rho_{BC}\|\rho_{ABC}) - \widehat{D}(\rho_A \otimes \rho_B\|\rho_{AB})$

Conditional independence of quantum Gibbs states in 1D

$\widehat{D}(\rho\|\sigma) = \mathrm{Tr}[\rho \log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$



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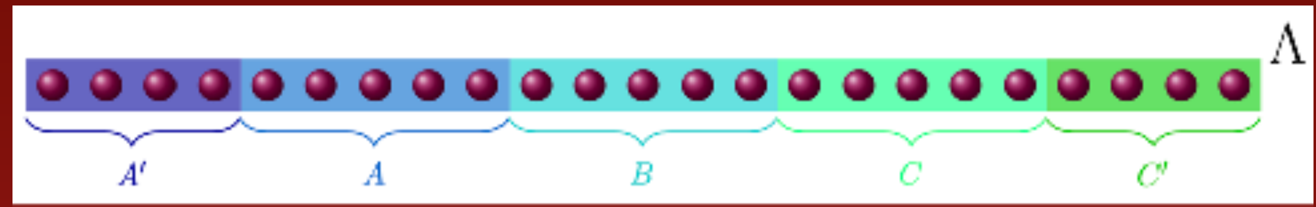
Reversed: $\widehat{I}^{\mathrm{rev}}_\rho(A:C|B) = \widehat{D}(\rho_A \otimes \rho_{BC}\|\rho_{ABC}) - \widehat{D}(\rho_A \otimes \rho_B\|\rho_{AB})$

Theorem (Gondolf, Scalet, Ruiz-de-Alarcón, Alhambra, C. '24)



$\rho = \dfrac{e^{-\beta H_\Lambda}}{\mathrm{Tr}[e^{-\beta H_\Lambda}]}$

$$\widehat{I}^{\mathrm{x}}_\rho(A:C|B) \leq c\, e^{\alpha(|A|+|B|)}\, \delta(|B|)$$

$\mathrm{x} \in \{\mathrm{os}, \mathrm{ts}, \mathrm{rev}\}$ $\quad \ell \mapsto \delta(\ell)$ superexponentially decaying

Conditional independence of quantum Gibbs states in 1D

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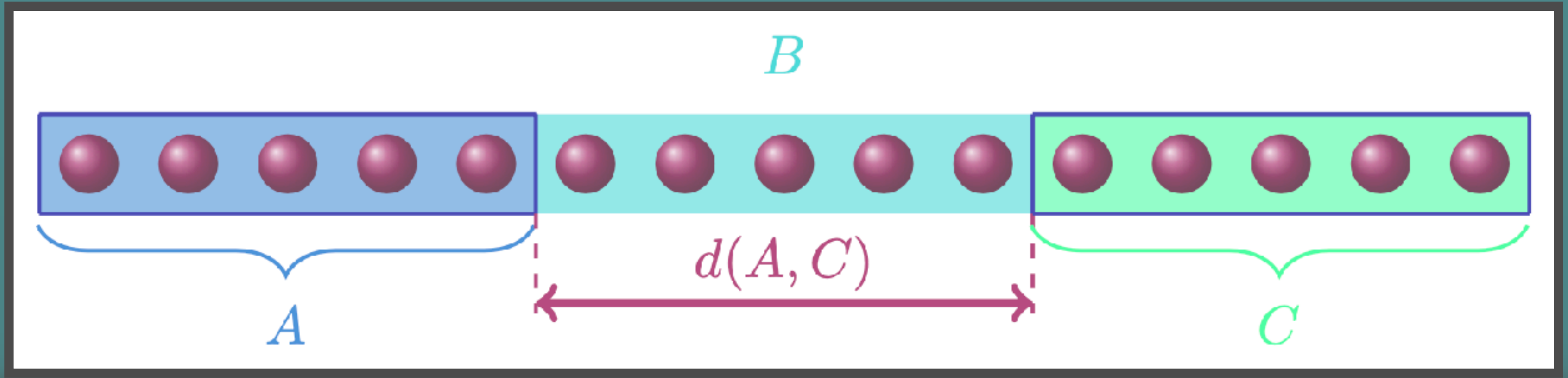

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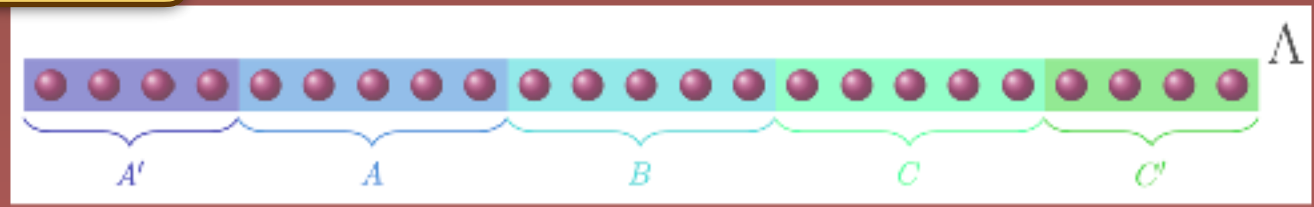
Reversed: $\widehat{I}^{\mathrm{rev}}_\rho(A:C|B) = \widehat{D}(\rho_A \otimes \rho_{BC}\|\rho_{ABC}) -$

[Kuwahara, '24]

$$I_\rho(A:C|B) \le c\,\mathrm{e}^{-\alpha|B|}$$

The BS-CMI decays faster than the CMI!

Theorem (Gondolf, Scalet, Ruiz-de-Alarcón, Alhambra, C. '24)



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Reversed data processing inequality

[Bluhm-C.-Pérez Hernández, '22]

$$\widehat{I}^{\mathrm{x}}_\rho(A:C|B) \le c\,\mathrm{e}^{\alpha(|A|+|B|)}\,\|\rho_{ABC}\rho_{BC}^{-1}\rho_B\rho_{AB}^{-1} - 1_{ABC}\|_\infty \le c\,\mathrm{e}^{\alpha(|A|+|B|)}\,\delta(|B|)$$

Conditional independence of quantum Gibbs states in 1D

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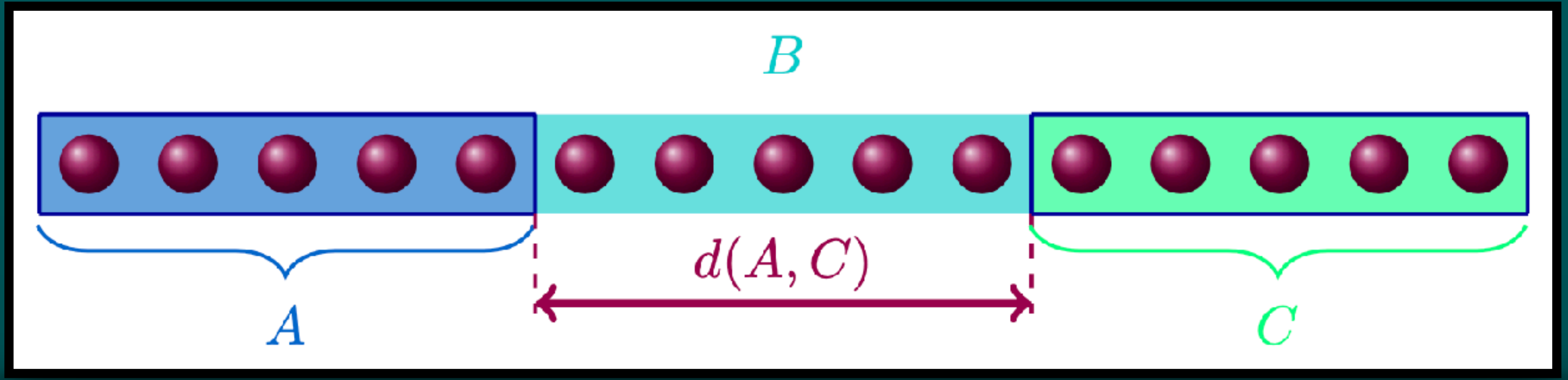

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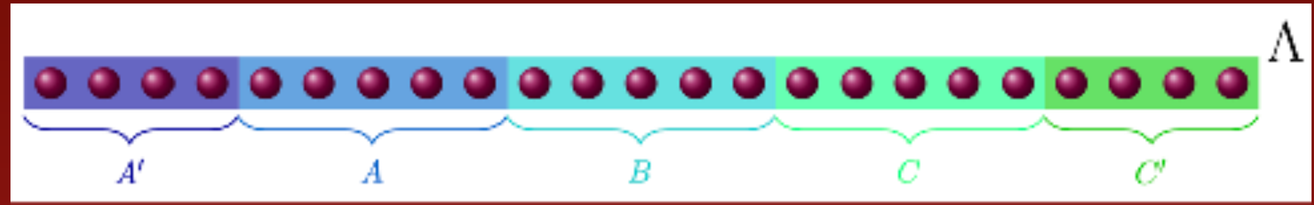

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Conditional independence of quantum Gibbs states in 1D

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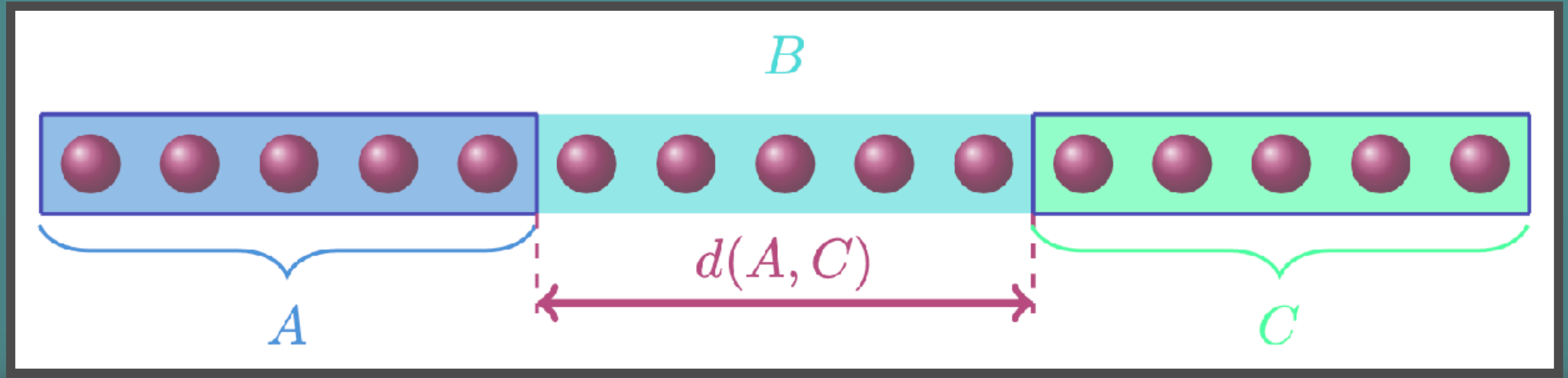

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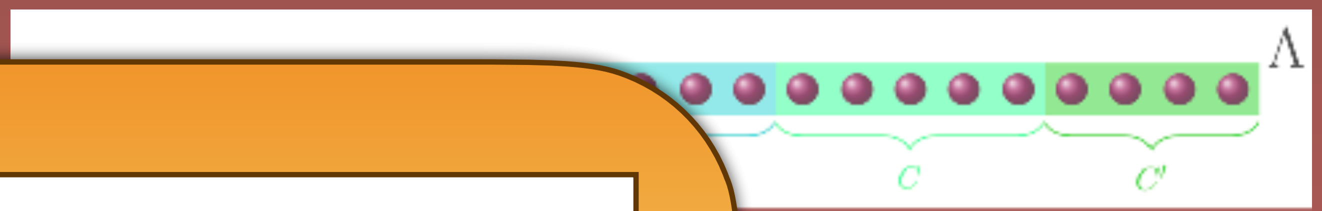

$$\rho = \frac{e^{-\beta H_\Lambda}}{\mathrm{Tr}[e^{-\beta H_\Lambda}]}$$

$$\widehat{I}^{\mathrm{os}}_\rho(A:C|B) \leq \left\|\rho_{ABC}^{-1/2}\rho_{BC}\rho_{ABC}^{-1/2}\right\|_\infty \|\rho_{ABC}\|_1 \left\|\rho_{AB}^{1/2}\right\|_\infty \left\|\rho_{AB}^{-1/2}\right\|_\infty \left\|\rho_B^{-1}\rho_{AB}\right\|_\infty \left\|\rho_{ABC}\rho_{BC}^{-1}\rho_B\rho_{AB}^{-1} - 1\right\|_\infty \mapsto \delta(\ell)$$

superexponentially decaying

Reversed data processing inequality

[Bluhm-C.-Pérez Hernández, '22]

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Conditional independence of quantum Gibbs states in 1D

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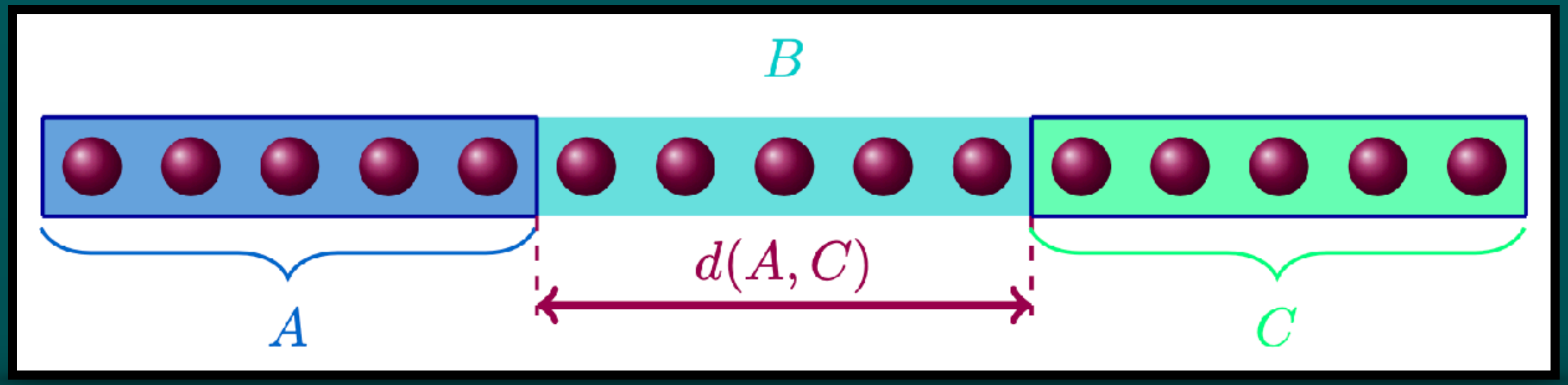

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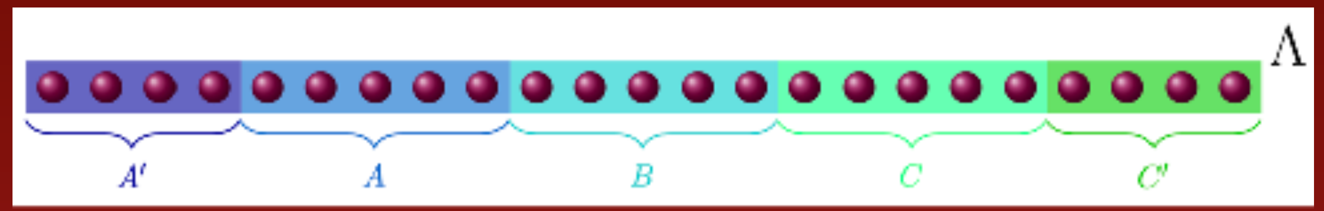

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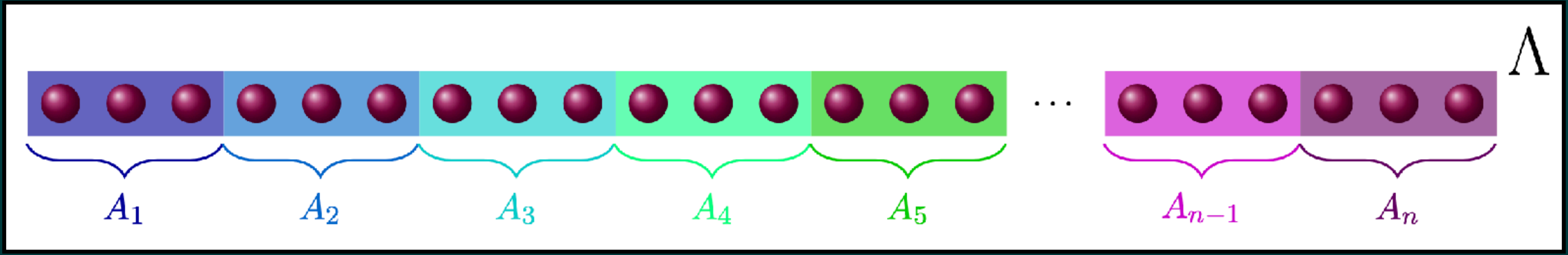
Step III:
Learning of quantum Gibbs states in 1D



Objectives of this work

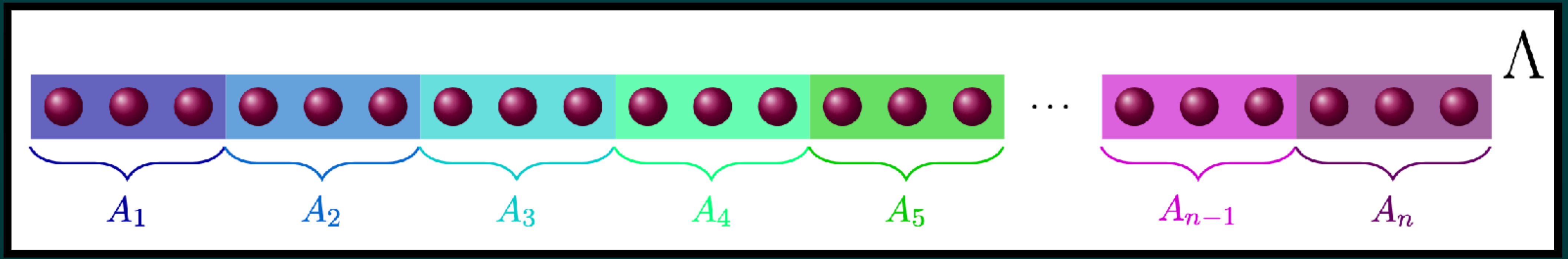
Part III: Learning of quantum Gibbs states in 1D

(via Matrix Product Operators approximation)

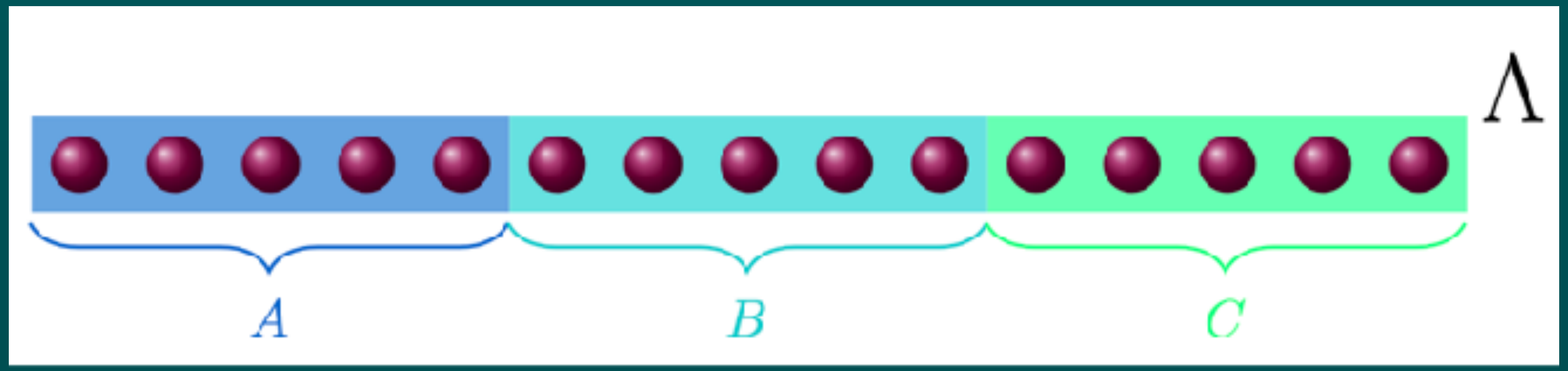
- Spin chain: $\Lambda \subset\subset \mathbb{Z}$
- Hamiltonian: $H_\Lambda = \sum_{X \subset \Lambda} H_X$
- Gibbs state (at $\beta > 0$): $\rho^\Lambda := \frac{e^{-\beta H_\Lambda}}{\mathrm{Tr}[e^{-\beta H_\Lambda}]}$

For ρ^Λ, we construct a MPO $\mathcal{M}$ from its marginals such that

$$\|\rho^\Lambda - \mathcal{M}\|_1 \leq \varepsilon$$



LEARNING OF QUANTUM GIBBS STATES IN 1D

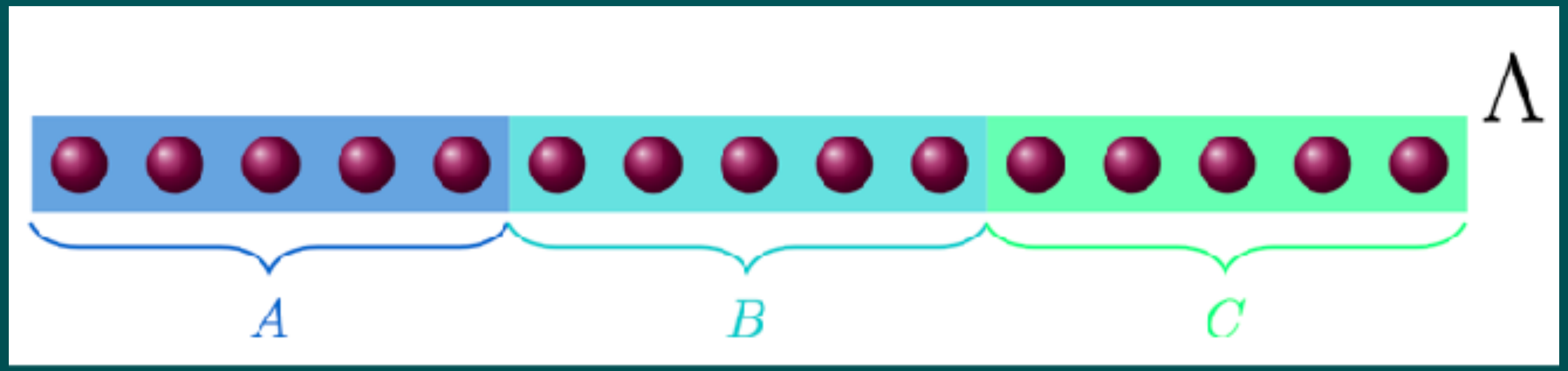


$$\rho = \frac{e^{-\beta H_\Lambda}}{\mathrm{Tr}[e^{-\beta H_\Lambda}]}$$

Recovery condition:

$$\Phi(X) = \rho_B^{1/2}(\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2})^{1/2}\rho_B^{-1/2}X\rho_B^{-1/2}(\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2})^{1/2}\rho_B^{1/2}$$

Learning of quantum Gibbs states in 1D



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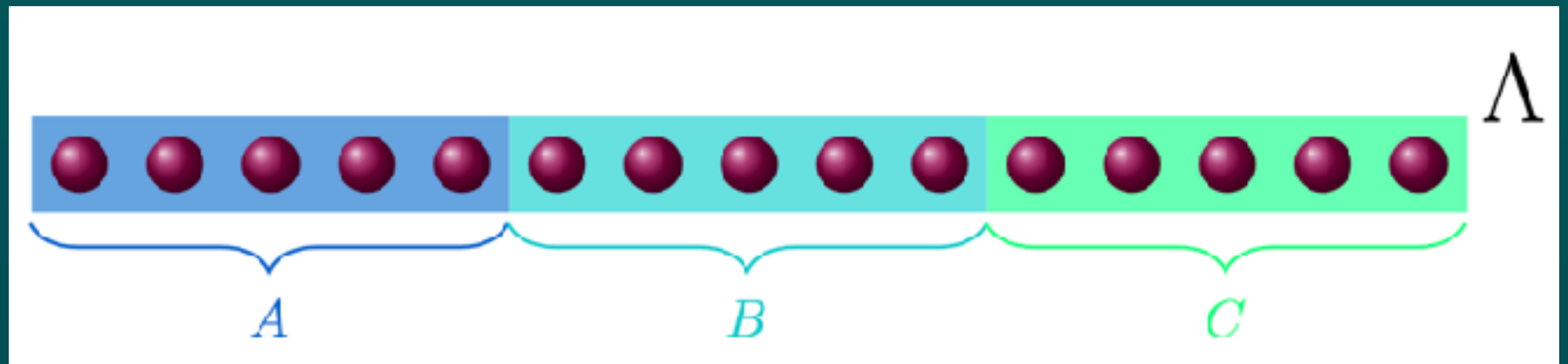
Another strenghthened DPI:

$$\widehat{I}_\rho^{rev}(A:C|B) \geq \left(\frac{\pi}{8}\right)^4 \|\rho_{BC}^{-1/2}\rho_{ABC}\rho_{BC}^{-1/2}\|_\infty^{-2} \|\Phi(\rho_{BC}) - \rho_{ABC}\|_1^4$$

[Bluhm-C., '20]

[Carlen-Vershynina, '20]

Learning of quantum Gibbs states in 1D



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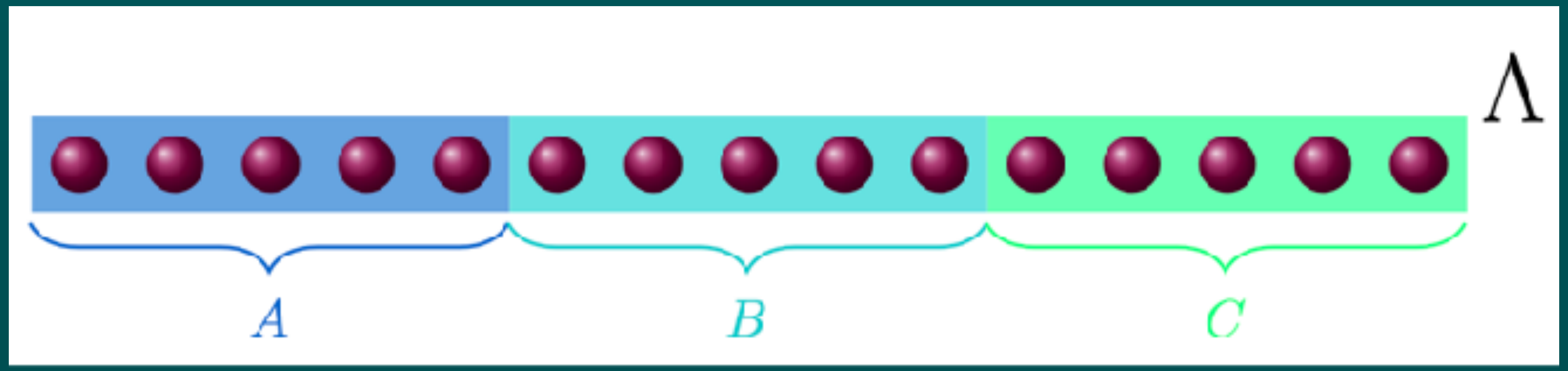
[Bluhm-C., '20]

[Carlen-Vershynina, '20]

Consequence:

$$\widehat{I}_\rho^{rev}(A:C|B) = 0 \;\Rightarrow\; \Phi(\rho_{BC}) = \rho_{ABC}$$

Learning of quantum Gibbs states in 1D



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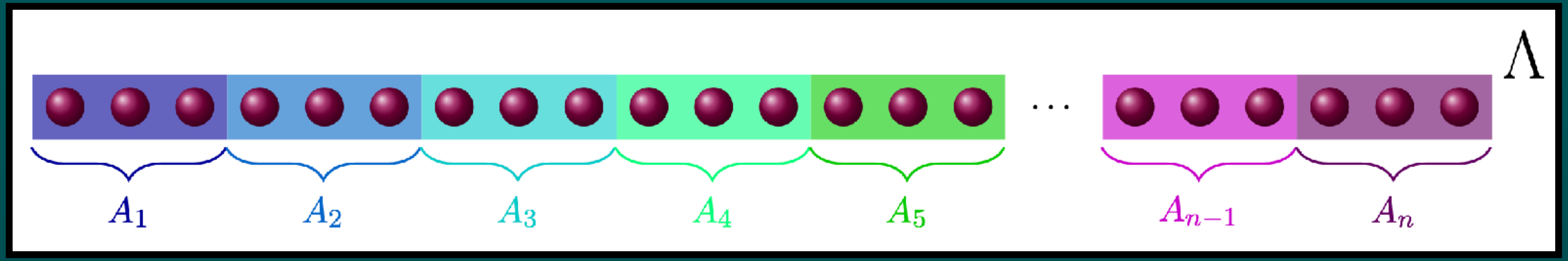
Consequence:

$$\widehat{I}_\rho^{rev}(A:C|B) = 0 \;\Rightarrow\; \Phi(\rho_{BC}) = \rho_{ABC}$$

The converse is also true!

Next part of the talk
[Bluhm, Costa, Jencova]

Learning of quantum Gibbs states in 1D



$$\rho = \frac{e^{-\beta H_\Lambda}}{\mathrm{Tr}[e^{-\beta H_\Lambda}]}$$

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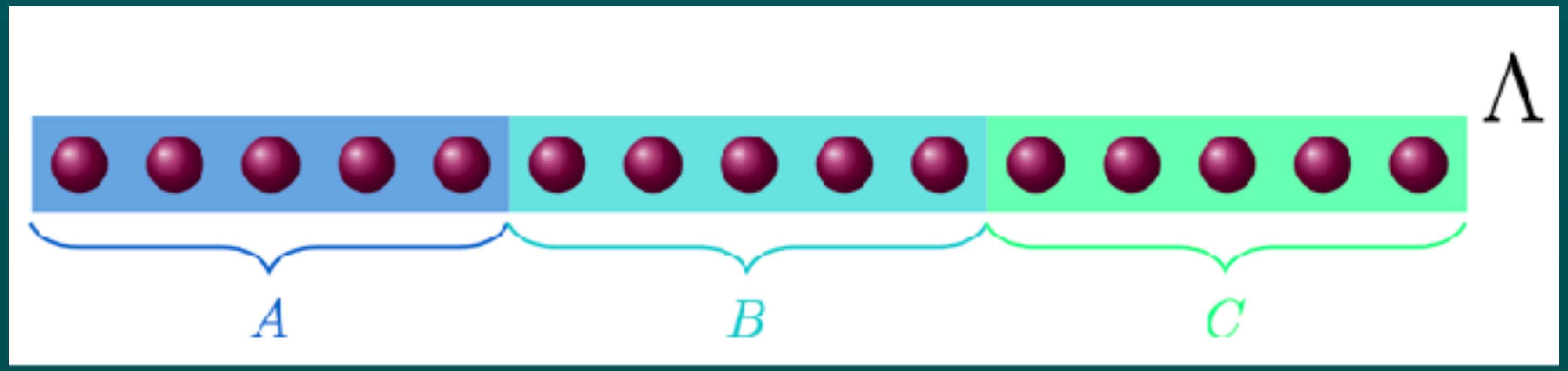
$$\Phi_i(X) = \rho_i^{1/2}(\rho_i^{-1/2}\rho_{i:i+1}\rho_i^{-1/2})^{1/2}\rho_i^{-1/2} X \rho_i^{-1/2}(\rho_i^{-1/2}\rho_{i:i+1}\rho_i^{-1/2})^{1/2}\rho_i^{1/2}$$

$$\left\| \left(\bigcirc_{i=j}^{N-1} \Phi_i \right)(X) \right\|_1 \leq \|\rho_j^{-1}\|_\infty \|X\|_1$$

Learning of quantum Gibbs states in 1D



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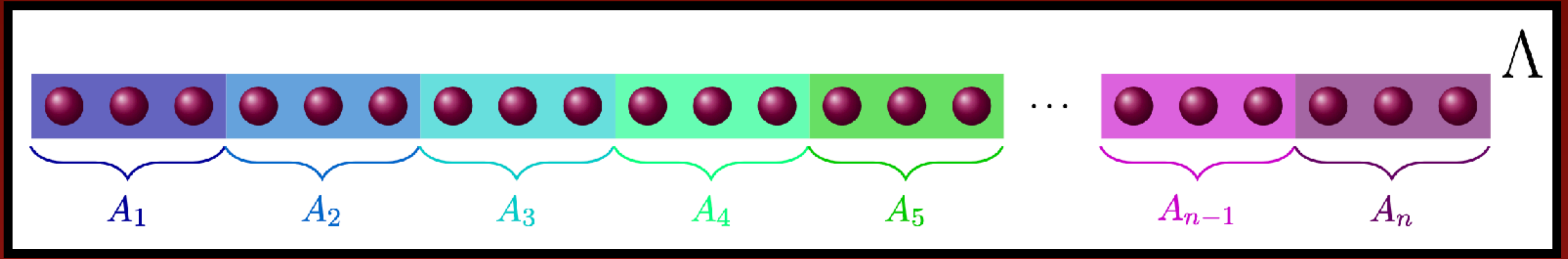
Theorem (Gondolf, Scalet, Ruiz-de-Alarcón, Alhambra, C. '24)



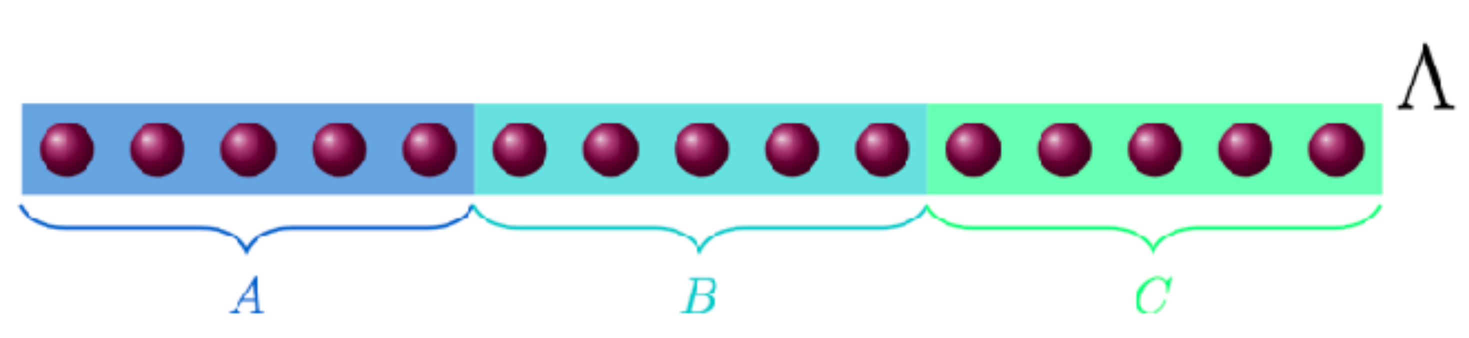
$$\left\|\left(\bigcirc_{i=1}^{N-1}\Phi_i\right)(\rho_1) - \rho_{1:N}\right\|_1 \le \frac{16(N-1)}{\pi} \sup_i D_\infty(1\|\rho_i)\exp(I_\infty(A_1\ldots A_{i-1}:A_i)/2)\ \widehat{I}_\rho^{\mathrm{rev}}(A_i:A_1\ldots A_{i-2}|A_{i-1})^{1/4}$$

MPO of bond dimension $D = \dim(A_i)^3$

$$D_\infty(\rho\|\sigma) = \log\|\sigma^{-1/2}\rho\sigma^{-1/2}\|_\infty$$

Follows from [Bluhm-C., '20] and [Carlen-Vershynina, '19]

LEARNING OF QUANTUM GIBBS STATES IN 1D



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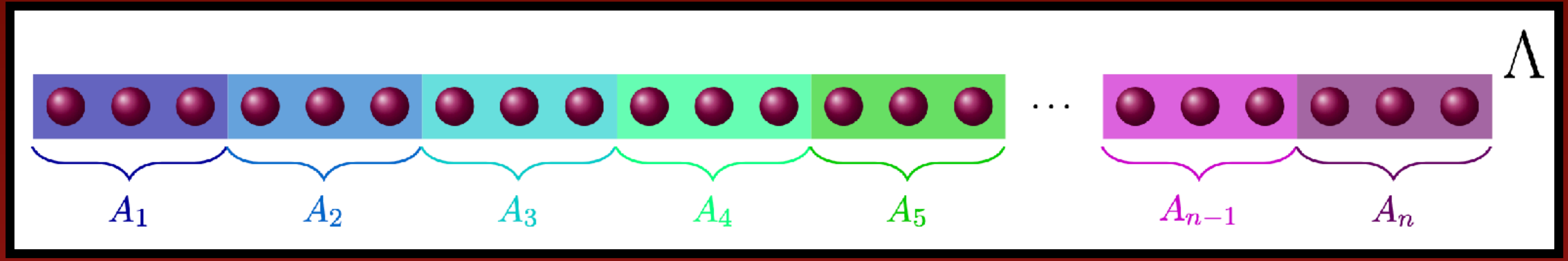


$$\left\|\left(\bigcirc_{i=1}^{N-1}\Phi_i\right)(\rho_1) - \rho_{1:N}\right\|_1 \le \frac{16(N-1)}{\pi}\sup_i D_\infty(1\|\rho_i)\exp(I_\infty(A_1\dots A_{i-1}:A_i)/2)\,\widehat{I}^{\mathrm{rev}}_\rho(A_i:A_1\dots A_{i-2}|A_{i-1})^{1/4}$$

Corollary (Gondolf, Scalet, Ruiz-de-Alarcón, Alhambra, C. '24)

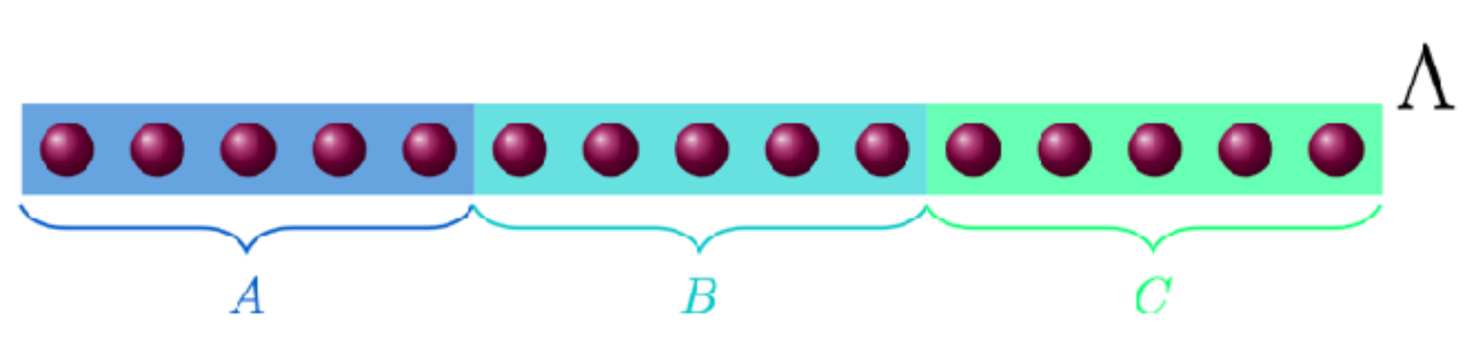
ρ n-site marginal of Gibbs state in TI-1D, ε accuracy. There is an MPO representation of bond dimension

$$D = \exp\left(2\log(d)C_1\frac{\log(n/\varepsilon)+C_2}{\log(\log(n/\varepsilon))}\right)$$

as long as

$$|A_i| = l \ge C_1\frac{\log(n/\varepsilon)+C_2}{\log(\log(n/\varepsilon))}$$

Learning of quantum Gibbs states in 1D



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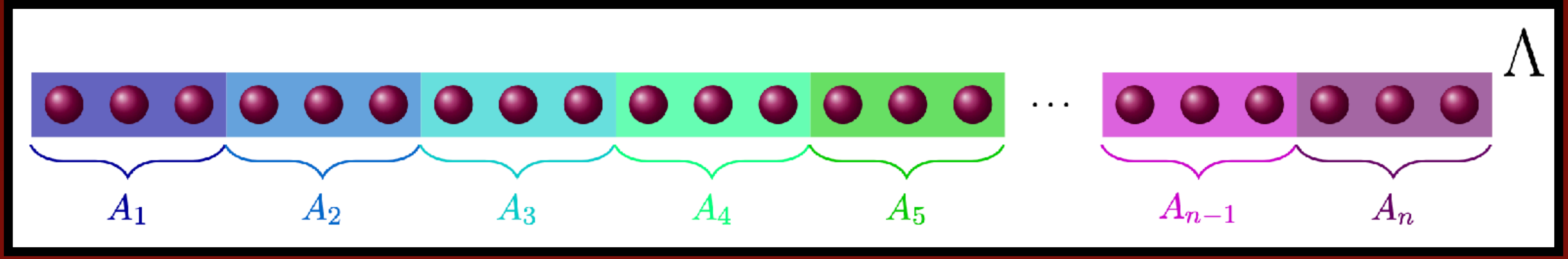


For $\|\rho_i - \hat{\rho}_i\|_1 \leq \delta$ and

$$\hat{\Phi}_i(X) = (\hat{\rho}_i)^{1/2}((\hat{\rho}_i)^{-1/2}\hat{\rho}_{i:i+1}(\hat{\rho}_i)^{-1/2})^{1/2}(\hat{\rho}_i)^{-1/2}X(\hat{\rho}_i)^{-1/2}((\hat{\rho}_i)^{-1/2}\hat{\rho}_{i:i+1}(\hat{\rho}_i)^{-1/2})^{1/2}(\hat{\rho}_i)^{1/2}$$

MPO approximation such that, with probability $\geq 1 - c$

$$\left\| \left(\bigcirc_{i=1}^{N-1} \hat{\Phi}_i \right) (\hat{\rho}_1) - \rho_{1:N} \right\|_1 \leq \varepsilon\,.$$

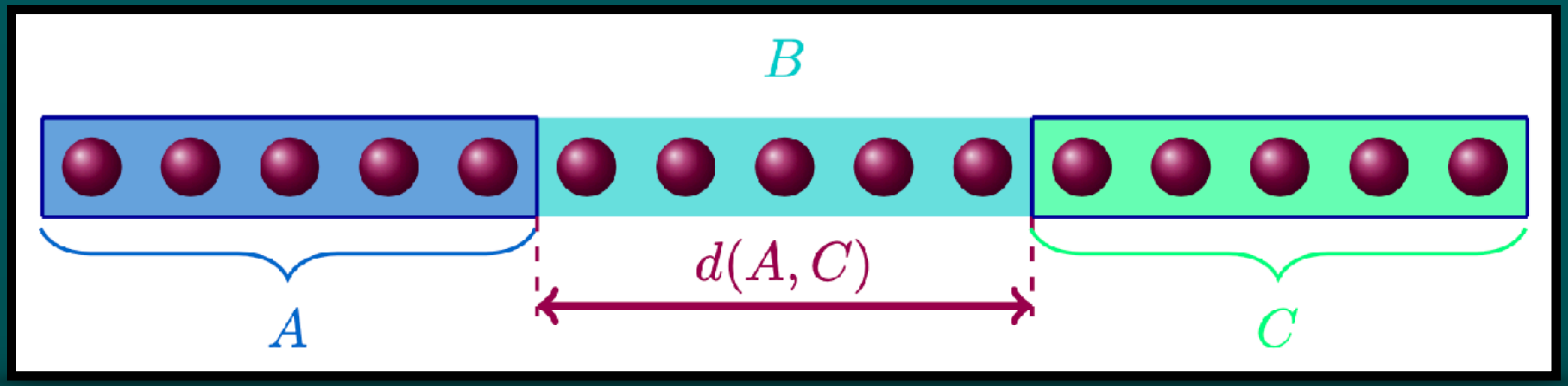
with sample complexity and classical post-processing time $\mathrm{poly}\,(n/\varepsilon)\log(1/c)$

Quantum Markov Chains

A particular case

$$\mathcal{H}_{ABC},\ \rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC}),\ \sigma_{ABC} = \frac{\mathbf{1}_A}{d_A} \otimes \rho_{BC}$$

$$\mathcal{T} = \mathrm{tr}_C$$



Umegaki relative entropy

$$D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$$

$$D(\rho\|\sigma) - D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$

Conditional mutual information

$$\begin{aligned} I_\rho(A:C|B) &\\ &= D(\rho_{ABC}\|1_A/d_A \otimes \rho_{BC}) - D(\rho_{AB}\|1_A/d_A \otimes \rho_B) \\ &= D(\rho_{ABC}\|\rho_A \otimes \rho_{BC}) - D(\rho_{AB}\|\rho_A \otimes \rho_B) \end{aligned}$$

$$I_\rho(A:C|B) = 0$$

$$\Updownarrow$$

$$\rho_{ABC} = \rho_{BC}^{1/2}\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}\rho_{BC}^{1/2}$$

A particular case

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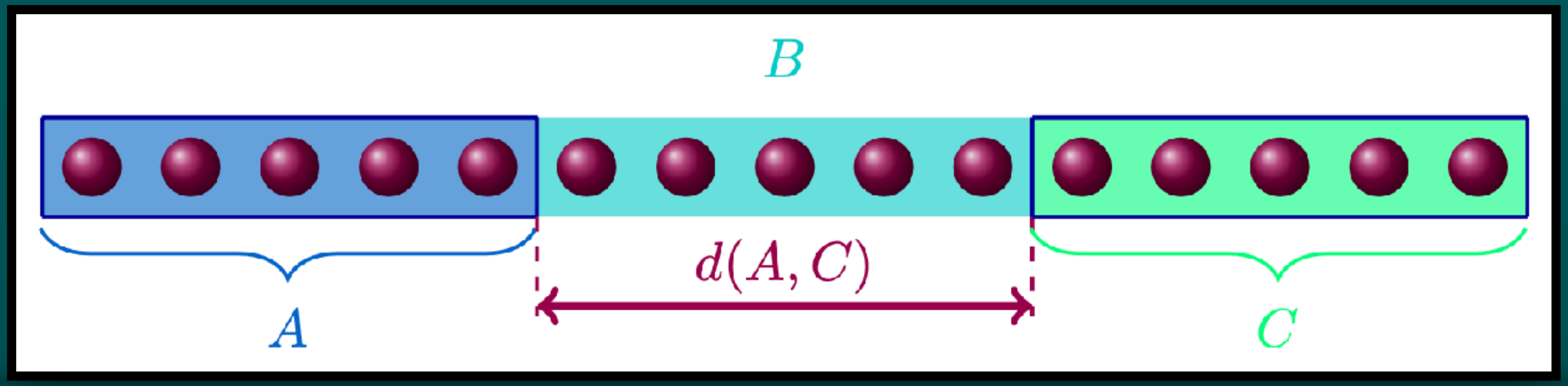

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$$D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$$

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Conditional mutual information

$$\begin{aligned} I_\rho(A:C|B) &= D(\rho_{ABC}\|1_A/d_A \otimes \rho_{BC}) - D(\rho_{AB}\|1_A/d_A \otimes \rho_B) \\ &= D(\rho_{ABC}\|\rho_A \otimes \rho_{BC}) - D(\rho_{AB}\|\rho_A \otimes \rho_B) \end{aligned}$$

$$I_\rho(A:C|B) = 0$$

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$$\rho_{ABC} = \rho_{BC}^{1/2}\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}\rho_{BC}^{1/2}$$

Such a state is called a Quantum Markov Chain

A particular case

$$\mathcal{H}_{ABC},\ \rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC}),\ \sigma_{ABC} = \frac{\mathbf{1}_A}{d_A} \otimes \rho_{BC}$$

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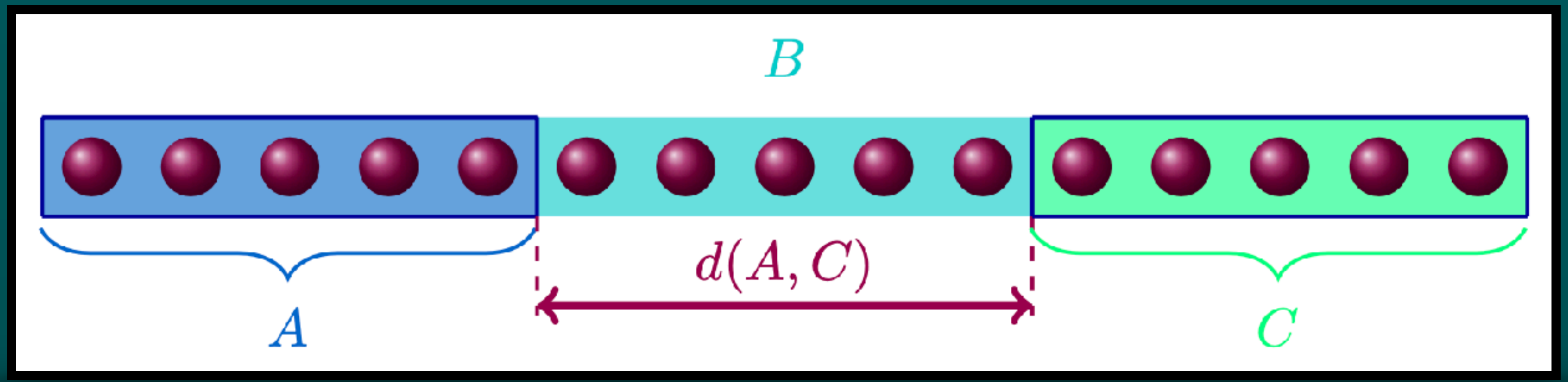

Umegaki relative entropy

$$D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$$

$$D(\rho\|\sigma) - D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$

Belavkin-Staszewski relative entropy

$$\widehat{D}(\rho\|\sigma) = \mathrm{Tr}[\rho\log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$$

$$\widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$

Conditional mutual information

BS conditional mutual information

$$I_\rho(A:C|B)$$

$$\widehat{I}_\rho(A:C|B)\ ?$$

$$= D(\rho_{ABC}\|1_A/d_A \otimes \rho_{BC}) - D(\rho_{AB}\|1_A/d_A \otimes \rho_B) \longrightarrow \widehat{D}(\rho_{ABC}\|1_A/d_A \otimes \rho_{BC}) - \widehat{D}(\rho_{AB}\|1_A/d_A \otimes \rho_B) = \widehat{I}^{\mathrm{os}}_\rho(A:C|B)$$

$$= D(\rho_{ABC}\|\rho_A \otimes \rho_{BC}) - D(\rho_{AB}\|\rho_A \otimes \rho_B) \longrightarrow \widehat{D}(\rho_{ABC}\|\rho_A \otimes \rho_{BC}) - \widehat{D}(\rho_{AB}\|\rho_A \otimes \rho_B) = \widehat{I}^{\mathrm{ts}}_\rho(A:C|B)$$

$$I_\rho(A:C|B) = 0$$

$$\Updownarrow$$

$$\rho_{ABC} = \rho_{BC}^{1/2}\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}\rho_{BC}^{1/2}$$

$$\widehat{I}^{\mathrm{os,ts}}_\rho(A:C|B) = 0$$

$$\Updownarrow$$

$$\rho_{ABC} = \rho_{BC}\rho_B\rho_{AB}$$

A particular case

$$\mathcal{H}_{ABC},\ \rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC}),\ \sigma_{ABC} = \frac{\mathbf{1}_A}{d_A} \otimes \rho_{BC}$$

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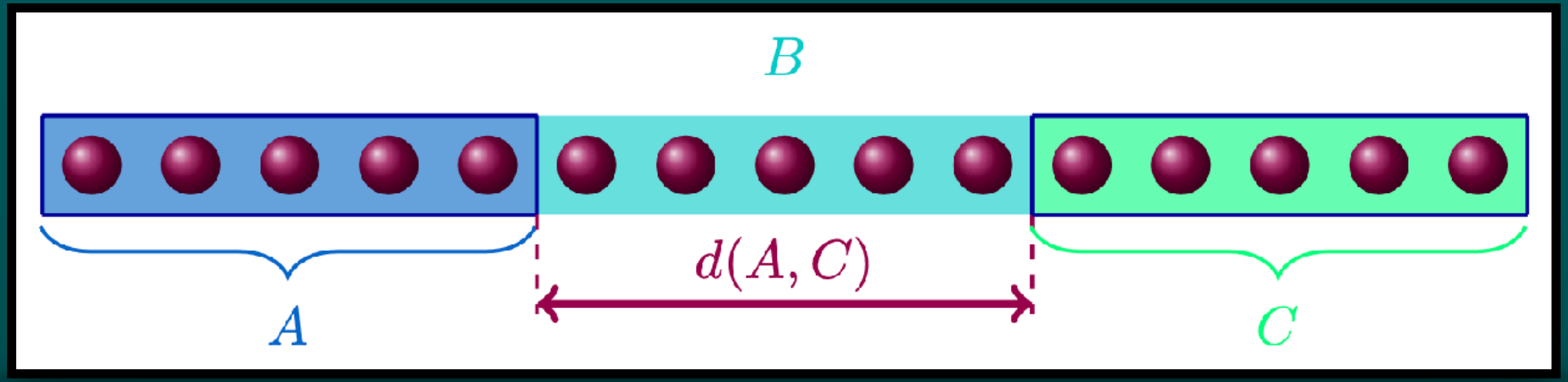

Belavkin–Staszewski relative entropy

$$\widehat{D}(\rho\|\sigma) = \operatorname{Tr}[\rho \log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$$

$$\widehat{D}(\rho\|\sigma) - \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$

BS conditional mutual information

$$\widehat{I}_\rho(A:C|B)\ ?$$

$$\widehat{D}(\rho_{ABC}\|1_A/d_A \otimes \rho_{BC}) - \widehat{D}(\rho_{AB}\|1_A/d_A \otimes \rho_B) = \widehat{I}_\rho^{\mathrm{os}}(A:C|B)$$

$$\widehat{D}(\rho_{ABC}\|\rho_A \otimes \rho_{BC}) - \widehat{D}(\rho_{AB}\|\rho_A \otimes \rho_B) = \widehat{I}_\rho^{\mathrm{ts}}(A:C|B)$$

$$\widehat{I}_\rho^{\mathrm{os,ts}}(A:C|B) = 0$$

$$\Updownarrow$$

$$\rho_{ABC} = \rho_{BC}\rho_B\rho_{AB}$$

We call this state a

Belavkin-Staszewski Quantum Markov Chain

[Bluhm, C., Costa Rico, Jencova, ’25]

How do these Quantum Markov Chains compare?

Quantum Markov Chains

$$I_\rho(A:C|B)=0$$
$$\Updownarrow$$
$$\rho_{ABC}=\rho_{BC}^{1/2}\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}\rho_{BC}^{1/2}$$

Belavkin-Staszewski Quantum Markov Chains

$$\widehat{I}_\rho^{\mathrm{os,ts}}(A:C|B)=0$$
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Belavkin-Staszewski Quantum Markov Chains

$$\widehat{I}_\rho^{\mathrm{os,ts}}(A:C|B) = 0$$
$$\Updownarrow$$
$$\rho_{ABC} = \rho_{BC}\rho_B\rho_{AB}$$

In general:

$$D(\rho\|\sigma) = D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$
$$\Updownarrow$$
$$\rho = \sigma^{1/2}\mathcal{T}^*(\mathcal{T}(\sigma)^{-1/2}\mathcal{T}(\rho)\mathcal{T}(\sigma)^{-1/2})\sigma^{1/2}$$

?

$$\widehat{D}(\rho\|\sigma) = \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$
$$\Updownarrow$$
$$\rho = \sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho))$$

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Belavkin-Staszewski Quantum Markov Chains

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In general:

$$D(\rho\|\sigma) = D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$
$$\Updownarrow$$
$$\rho = \sigma^{1/2}\mathcal{T}^*(\mathcal{T}(\sigma)^{-1/2}\mathcal{T}(\rho)\mathcal{T}(\sigma)^{-1/2})\sigma^{1/2}$$

$$\Rightarrow \qquad \not\Leftarrow$$

$$\widehat{D}(\rho\|\sigma) = \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$
$$\Updownarrow$$
$$\rho = \sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho))$$

[Jencova, Petz, Pitrick, '09], [Hiai, Mosonyi, '17]

How do these Quantum Markov Chains compare?

Quantum Markov Chains

Belavkin-Staszewski Quantum Markov Chains

$$I_\rho(A:C|B) = 0$$
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$$\rho_{ABC} = \rho_{BC}^{1/2}\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}\rho_{BC}^{1/2}$$

$$\widehat{I}_\rho^{\mathrm{os,ts}}(A:C|B) = 0$$
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$$\rho_{ABC} = \rho_{BC}\rho_B\rho_{AB}$$

[Bluhm, C., Costa Rico, Jencova, '25]

In general:

$$D(\rho\|\sigma) = D(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$
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$$\rho = \sigma^{1/2}\mathcal{T}^*(\mathcal{T}(\sigma)^{-1/2}\mathcal{T}(\rho)\mathcal{T}(\sigma)^{-1/2})\sigma^{1/2}$$

$$\widehat{D}(\rho\|\sigma) = \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$$
$$\Updownarrow$$
$$\rho = \sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}\mathcal{T}(\rho))$$

[Jencova, Petz, Pitrick, '09], [Hiai, Mosonyi, '17]

How do these Quantum Markov Chains Relate?

Theorem (Bluhm, C., Costa Rico, Jencova, '25)

Assume that $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$ is such that ρ_B is invertible. Define the state

$$\eta_{ABC} := \frac{1}{d_B} \rho_B^{-1/2} \rho_{ABC} \rho_B^{-1/2}.$$

Then, the following are equivalent:

(i) ρ_{ABC} is a BS-QMC.

(ii) $\rho_{ABC} = \rho_{AB} \rho_B^{-1} \rho_{BC}$.

(iii) The marginals η_{AB} and η_{BC} commute, and we have $\rho_{ABC} = d_B^2 \rho_B^{1/2} \eta_{AB} \eta_{BC} \rho_B^{1/2}$.

(iv) η_{ABC} is a QMC.

(v) There are Hilbert spaces $\mathcal{H}_{B_n^L}$, $\mathcal{H}_{B_n^R}$ and a unitary $U_B : \mathcal{H}_B \to \bigoplus_{n=1}^{N} (\mathcal{H}_{B_n^L} \otimes \mathcal{H}_{B_n^R})$, such that

$$\rho_{ABC} = \rho_B^{1/2} U_B^* \left(\bigoplus_n d_B \, p_n \, \tilde{\eta}_{AB_n^L} \otimes \tilde{\eta}_{B_n^R C} \right) U_B \rho_B^{1/2}$$

for some states $\tilde{\eta}_{AB_n^L}$ on $\mathcal{H}_{AB_n^L}$ and $\tilde{\eta}_{B_n^R C}$ on $\mathcal{H}_{B_n^R C}$ and a probability distribution $\{p_n\}$.

How do these Quantum Markov Chains Relate?

Theorem (Bluhm, C., Costa Rico, Jencova, '25)

Let $\mathcal{T} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a channel and let $U : \mathcal{H} \to \mathcal{K} \otimes \mathcal{H}_E$ be an isometry such that $\mathcal{T} = \mathrm{Tr}_E[U \cdot U^*]$. Let $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ be states such that $\mathrm{supp}(\rho) \leq \mathrm{supp}(\sigma)$. The following conditions are equivalent.

(i) $\widehat{D}(\rho\|\sigma) = \widehat{D}(\mathcal{T}(\rho)\|\mathcal{T}(\sigma))$.

(ii) $\mathcal{T}_\sigma([\rho/\sigma]^2) = \mathcal{T}_\sigma([\rho/\sigma])^2$.

(iii) There is a decomposition and a unitary $S : \mathcal{K} \to \bigoplus_n \mathcal{K}_n^L \otimes \mathcal{K}_n^R$, such that for

$$\rho = U^* \rho_0 U, \quad \sigma = U^* \sigma_0 U,$$

where $\rho_0, \sigma_0 \in \mathcal{B}(\mathcal{K} \otimes \mathcal{H}_E)^+$ are positive and such that $\mathrm{supp}(\rho_0), \mathrm{supp}(\sigma_0) \leq UU^*$, we have

$$\rho_0 = (\mathcal{T}(\sigma)^{1/2} S^* \otimes I_E) \bigoplus_n (\eta_n^L \otimes \eta_n^R)(S\mathcal{T}(\sigma)^{1/2} \otimes I_E)$$
$$\sigma_0 = (\mathcal{T}(\sigma)^{1/2} S^* \otimes I_E) \bigoplus_n (I_{\mathcal{K}_n^L} \otimes \eta_n^R)(S\mathcal{T}(\sigma)^{1/2} \otimes I_E)$$

for some $\eta_n^L \in \mathcal{B}(\mathcal{K}_n^L)^+$ and $\eta_n^R \in \mathcal{B}(\mathcal{K}_n^R \otimes \mathcal{H}_E)^+$.

(iv) $\rho = \sigma \mathcal{T}^*(\mathcal{T}(\sigma)^{-1} \mathcal{T}(\rho))$.

How do these Quantum Markov Chains Relate?

Theorem (Bluhm, C., Costa Rico, Jencova, '25)

Assume that $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$ is such that ρ_B is invertible. Define the state

$$\eta_{ABC} := \frac{1}{d_B} \rho_B^{-1/2} \rho_{ABC} \rho_B^{-1/2}.$$

Then, the following are equivalent:

(i) ρ_{ABC} is a BS-QMC.

(ii) $\rho_{ABC} = \rho_{AB} \rho_B^{-1} \rho_{BC}$.

(iii) The marginals η_{AB} and η_{BC} commute, and we have $\rho_{ABC} = d_B^2 \rho_B^{1/2} \eta_{AB} \eta_{BC} \rho_B^{1/2}$.

(iv) η_{ABC} is a QMC.

(v) There are Hilbert spaces $\mathcal{H}_{B_n^L}$, $\mathcal{H}_{B_n^R}$ and a unitary $U_B : \mathcal{H}_B \to \bigoplus_{n=1}^{N} (\mathcal{H}_{B_n^L} \otimes \mathcal{H}_{B_n^R})$, such that

$$\rho_{ABC} = \rho_B^{1/2} U_B^* \left(\bigoplus_n d_B \, p_n \, \tilde{\eta}_{AB_n^L} \otimes \tilde{\eta}_{B_n^R C} \right) U_B \rho_B^{1/2}$$

for some states $\tilde{\eta}_{AB_n^L}$ on $\mathcal{H}_{AB_n^L}$ and $\tilde{\eta}_{B_n^R C}$ on $\mathcal{H}_{B_n^R C}$ and a probability distribution $\{p_n\}$.

In particular:

$$I_\rho(A:C|B) = 0$$

$$\Updownarrow$$

$$\widehat{I}_\eta^{\mathrm{os,ts,rev}}(A:C|B) = 0$$

How do these Quantum Markov Chains Relate?

Theorem (Bluhm, C., Costa Rico, Jencova, '25)

Assume that $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$ is such that ρ_B is invertible. Define the state

$$\eta_{ABC} := \frac{1}{d_B}\rho_B^{-1/2}\rho_{ABC}\rho_B^{-1/2}.$$

Then, the following are equivalent:

(i) ρ_{ABC} is a BS-QMC.

(ii) $\rho_{ABC} = \rho_{AB}\rho_B^{-1}\rho_{BC}$.

(iii) The marginals η_{AB} and η_{BC} commute, and we have $\rho_{ABC} = d_B^2\rho_B^{1/2}\eta_{AB}\eta_{BC}\rho_B^{1/2}$.

(iv) η_{ABC} is a QMC.

(v) There are Hilbert spaces $\mathcal{H}_{B_n^L}$, $\mathcal{H}_{B_n^R}$ and a unitary $U_B : \mathcal{H}_B \to \bigoplus_{n=1}^{N}(\mathcal{H}_{B_n^L} \otimes \mathcal{H}_{B_n^R})$, such that

$$\rho_{ABC} = \rho_B^{1/2}U_B^*\left(\bigoplus_n d_B\, p_n\, \tilde{\eta}_{AB_n^L} \otimes \tilde{\eta}_{B_n^R C}\right)U_B\rho_B^{1/2}$$

for some states $\tilde{\eta}_{AB_n^L}$ on $\mathcal{H}_{AB_n^L}$ and $\tilde{\eta}_{B_n^R C}$ on $\mathcal{H}_{B_n^R C}$ and a probability distribution $\{p_n\}$.

In particular:

$$I_\rho(A:C|B) = 0$$

$$\Updownarrow$$

$$\widehat{I}_\eta^{\mathrm{os,ts,rev}}(A:C|B) = 0$$

$$\widehat{I}_\rho^{\mathrm{os,ts}}(A:C|B) = 0 \iff \rho_{ABC} = \rho_{BC}\rho_B\rho_{AB} \iff \rho_{ABC} = \rho_B^{1/2}(\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2})^{1/2}\rho_B^{-1/2}\rho_{BC}\rho_B^{-1/2}(\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2})^{1/2}\rho_B^{1/2}$$

How do these Quantum Markov Chains Relate?

Theorem (Bluhm, C., Costa Rico, Jencova, '25)

Let ρ_{ABC} be a BS-QMC and let η_{ABC} be the corresponding QMC. Let $\rho_{AB}^{1/2}\rho_B^{-1/2} = d_B^{1/2} W_{AB} \eta_{AB}^{1/2}$ be the polar decomposition, with W_{AB} unitary. Then, the following are equivalent.

(i) ρ_{ABC} is a QMC.

(ii) There is a decomposition as in Theorem 3.1.3 (v), such that also

$$\rho_B = U_B^* \left(\bigoplus_n p_n \tilde{\rho}_{B_n^L} \otimes \tilde{\rho}_{B_n^R} \right) U_B$$

for some $\tilde{\rho}_{B_n^L} \in \mathcal{S}(\mathcal{H}_{B_n^L})$, $\tilde{\rho}_{B_n^R} \in \mathcal{S}(\mathcal{H}_{B_n^R})$ and a probability distribution $\{p_n\}$.

(iii) $[\rho_B^{it}\eta_{AB}\rho_B^{-it}, \eta_{BC}] = 0$ for all $t \in \mathbb{R}$.

(iv) $W_{AB}\eta_{BC}W_{AB}^* = \eta_{BC}$.

(v) $d_B^{-1}\rho_{AB}^{-1/2}\rho_{ABC}\rho_{AB}^{-1/2} = \eta_{BC}$.

Other Divergence measures

Quantum Rényi Divergences

Classical Information

Rényi divergences

$$D_\alpha(p\|q) = \frac{1}{\alpha - 1} \log \left(\sum_{x=1}^{n} \frac{p_x^\alpha}{q_x^{\alpha-1}} \right)$$

$$\alpha \in (0,1) \cup (1,\infty)$$

Quantum Information

Quantum Rényi Divergences

Classical Information

Rényi divergences

$$D_\alpha(p\|q) = \frac{1}{\alpha-1}\log\left(\sum_{x=1}^{n}\frac{p_x^\alpha}{q_x^{\alpha-1}}\right)$$

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Quantum Information

Petz Rényi divergences

$$\overline{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log\mathrm{Tr}[\rho^\alpha\sigma^{1-\alpha}]$$

$$\alpha \in (0,1)\cup(1,2)$$

Quantum Rényi Divergences

Classical Information

Rényi divergences

$$D_\alpha(p\|q) = \frac{1}{\alpha - 1}\log\left(\sum_{x=1}^{n}\frac{p_x^\alpha}{q_x^{\alpha-1}}\right)$$

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$$\overline{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log\mathrm{Tr}[\rho^\alpha\sigma^{1-\alpha}]$$

$$\alpha \in (0,1)\cup(1,2)$$

Sandwiched Rényi divergences

$$\widetilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log\mathrm{Tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha]$$

$$\alpha \in (1/2,1)\cup(1,\infty)$$

Quantum Rényi Divergences

Classical Information

Rényi divergences

$$D_\alpha(p\|q) = \frac{1}{\alpha - 1}\log\left(\sum_{x=1}^{n}\frac{p_x^\alpha}{q_x^{\alpha-1}}\right)$$

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$$\alpha \in (1/2,1)\cup(1,\infty)$$

Geometric Rényi divergences

$$\widehat{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log\mathrm{Tr}[\sigma(\sigma^{-1/2}\rho\sigma^{-1/2})^\alpha]$$

$$\alpha \in (1,2)$$

Quantum Rényi Divergences

Relative entropies

Umegaki relative entropy

$$D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$$

$\alpha \to 1$

Belavkin-Staszewski relative entropy

$$\widehat{D}(\rho\|\sigma) = \mathrm{Tr}[\rho\log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$$

$\alpha \to 1$

Rényi divergences

Petz Rényi divergences

$$\overline{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log\mathrm{Tr}[\rho^\alpha\sigma^{1-\alpha}]$$

$$\alpha \in (0,1)\cup(1,2)$$

Sandwiched Rényi divergences

$$\widetilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log\mathrm{Tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha]$$

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$$\widehat{D}(\rho\|\sigma) = \mathrm{Tr}[\rho\log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$$

$\alpha \to 1$

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Rényi divergences

Petz Rényi divergences

$$\overline{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log\mathrm{Tr}[\rho^\alpha\sigma^{1-\alpha}]$$

$$\alpha \in (0,1)\cup(1,2)$$

Sandwiched Rényi divergences

$$\widetilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log\mathrm{Tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha]$$

$$\alpha \in (1/2,1)\cup(1,\infty)$$

Geometric Rényi divergences

$$\widehat{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log\mathrm{Tr}[\sigma(\sigma^{-1/2}\rho\sigma^{-1/2})^\alpha]$$

$$\alpha \in (1,2)$$

Comparison of Quantum Rényi Divergences

Divergence measures

Umegaki relative entropy

$$D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$$

$\alpha \to 1$ $\alpha \to 1$

Petz Rényi divergences

$$\overline{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log\mathrm{Tr}[\rho^\alpha\sigma^{1-\alpha}]$$

$\alpha \in (0,1)\cup(1,2)$

Sandwiched Rényi divergences

$$\widetilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log\mathrm{Tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha]$$

$\alpha \in (1/2,1)\cup(1,\infty)$

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$$\widehat{D}(\rho\|\sigma) = \mathrm{Tr}[\rho\log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$$

$\alpha \to 1$

Geometric Rényi divergences

$$\widehat{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log\mathrm{Tr}[\sigma(\sigma^{-1/2}\rho\sigma^{-1/2})^\alpha]$$

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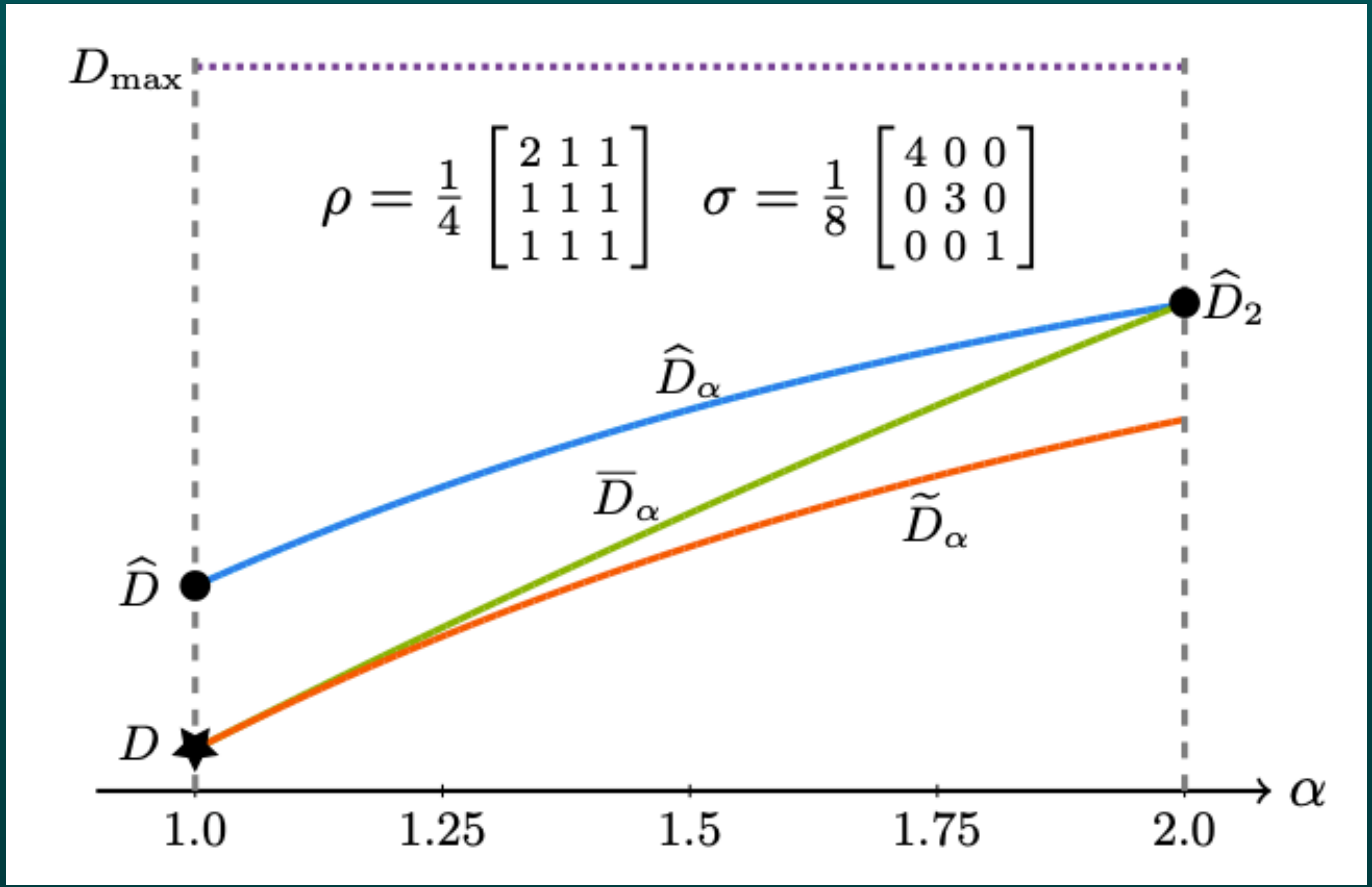

[Fang, Fawzi, '21]

DATA PROCESSING INEQUALITIES FOR QUANTUM RÉNYI DIVERGENCES

RELATIVE ENTROPIES

Umegaki relative entropy

$\alpha \to 1$ $$D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$$ $\alpha \to 1$

Petz Rényi divergences

$$\overline{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log\mathrm{Tr}[\rho^\alpha\sigma^{1-\alpha}]$$

$\alpha \in (0,1) \cup (1,2)$

Sandwiched Rényi divergences

$$\widetilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log\mathrm{Tr}[(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha]$$

$\alpha \in (1/2,1) \cup (1,\infty)$

Belavkin-Staszewski relative entropy

$$\widehat{D}(\rho\|\sigma) = \mathrm{Tr}[\rho\log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$$

$\alpha \to 1$

Geometric Rényi divergences

$$\widehat{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log\mathrm{Tr}[\sigma(\sigma^{-1/2}\rho\sigma^{-1/2})^\alpha]$$

$\alpha \in (1,2)$

Multiple results of strengthened DPI for Petz and sandwiched Rényi divergences:

Recoverability for optimized quantum *f*-divergences

Li Gao[1,*] and Mark M Wilde[2,3]

(also involving rotated Petz recovery maps)

DPIs for Sandwiched Rényi Divergences: Particular case

Sandwiched Rényi divergences

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Sandwiched Rényi Conditional Mutual Information

$$\widetilde{I}_\alpha(A:C|B)_\rho = \widetilde{H}_\alpha(C|B)_\rho - \widetilde{H}_\alpha(C|AB)_\rho$$

with $\widetilde{H}_\alpha(A|B)_\rho = \frac{1}{1-\alpha}\log\mathrm{Tr}[((1_A\otimes\rho_B)^{\frac{1-\alpha}{2\alpha}}\rho_{AB}(1_A\otimes\rho_B)^{\frac{1-\alpha}{2\alpha}})^\alpha]$

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[Bluhm, C., Gondolf, Möbus '23]

For $\alpha \in (1/2,1)$,

$$\begin{aligned}\frac{\alpha}{1-\alpha}\log\left(1+\left(\mathcal{K}\left\|\rho_{ABC}-\rho_{BC}^{\frac{1}{2}+it}\rho_B^{-\frac{1}{2}-it}\rho_{AB}\rho_B^{-\frac{1}{2}-it}\rho_{BC}^{\frac{1}{2}+it}\right\|_1\right)^{\frac{1}{1-\frac{1}{2\alpha}-\varepsilon}}\right)&\\ \leq \widetilde{I}_\alpha(A:C|B)_\rho&\\ \leq c\left(\alpha,\|\rho_{ABC}^{-1}\|_\infty^{-1},d_C,d_{ABC}\right)\left\|\rho_{ABC}-\rho_{BC}^{1/2}\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}\rho_{BC}^{1/2}\right\|_1^{1/2}&,\end{aligned}$$

for any $\varepsilon\in(0,1-\frac{1}{2\alpha})$, with

$$\mathcal{K} = \left(\left(\frac{\pi}{e\varepsilon\sin(\pi\frac{1-\alpha}{\alpha})}\right)^{1/2}+8\right)\frac{\pi}{2\cosh(\pi t)}.$$

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[Bluhm, C., Gondolf, Möbus '23]

For $\alpha \in (1,\infty)$,

$$\begin{aligned}\frac{\alpha}{\alpha-1}\log\left(1+\left(\mathcal{K}'\left\|\rho_{ABC}-\rho_{BC}^{\frac{1}{2}+it}\rho_B^{-\frac{1}{2}-it}\rho_{AB}\rho_B^{-\frac{1}{2}-it}\rho_{BC}^{\frac{1}{2}+it}\right\|_1\right)^{\frac{1}{\frac{1}{2\alpha}-\varepsilon}}\right)\\ \leq \widetilde{I}_\alpha(A:C|B)_\rho\\ \leq c\left(\alpha,\|\rho_{ABC}^{-1}\|_\infty^{-1},d_C,d_{ABC}\right)\left\|\rho_{ABC}-\rho_{BC}^{1/2}\rho_B^{-1/2}\rho_{AB}\rho_B^{-1/2}\rho_{BC}^{1/2}\right\|_1^{1/2},\end{aligned}$$

for any $\varepsilon \in (0, \frac{1}{2\alpha})$, with

$$\mathcal{K}' = d_C^{\frac{2(1-\alpha)}{\alpha}}\left(\left(\frac{\pi}{e\varepsilon\sin(\pi\frac{\alpha-1}{\alpha})}\right)^{1/2}+8\right)\frac{\pi}{2\cosh(\pi t)}.$$

Conclusions

- We have reviewed entropies and divergence measures in classical and quantum information theory.
- We have recalled results of strengthened and reversed data processing inequalities, as well as their relation to approximate recoverability.
- We have studied the particular case of tripartite Hilbert spaces and applied our results to the study of decay of correlations, as well as learning of Gibbs states.
- We have reviewed Quantum Markov Chains and introduced the analogous notion for the Belavkin-Staszewski relative entropy.
- We have recalled some quantum Rényi divergences and briefly explored their DPIs.

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Thanks for your attention!