OTOC spectroscopy: detecting quantum advantage of quantum chaos through the lens of quantum algorithms

Keisuke Fujii *"Out-of-Time-Order Correlator Spectroscopy."* arXiv:2511.22654 (2025).

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QUANTUM INNOVATION 2025

29th July, 2nd Aug. 2025

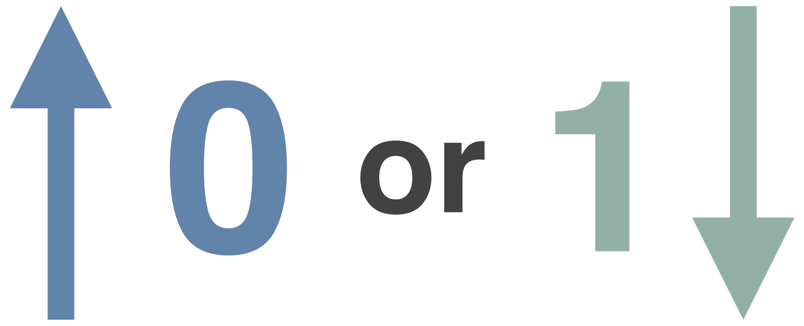
Outline

- Introduction to Quantum Computing and Progress Toward Quantum Advantage
- What is OTOC
- What is Quantum Singular Value Transformation
- Quantum Algorithmic Understanding of the higher order OTOC
- Summary

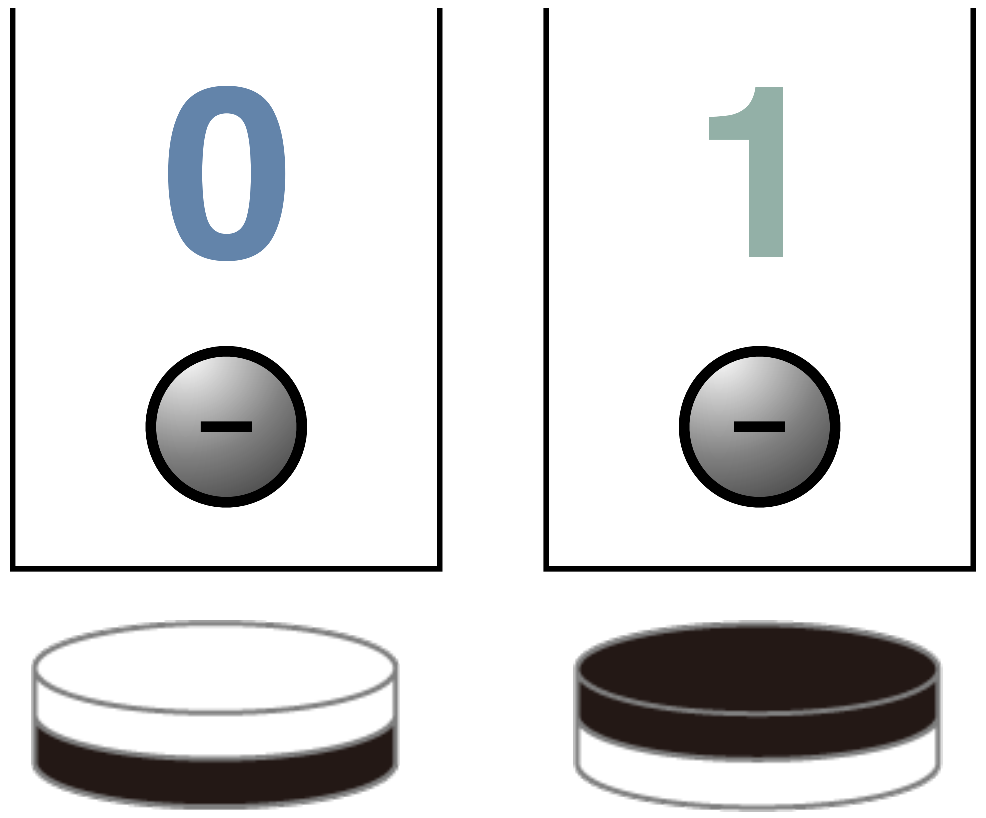
Classical bit and Quantum bit

The minimum unit of information in the “classical" or “quantum” world

Classical bit



$$x \in \{0, 1\}$$



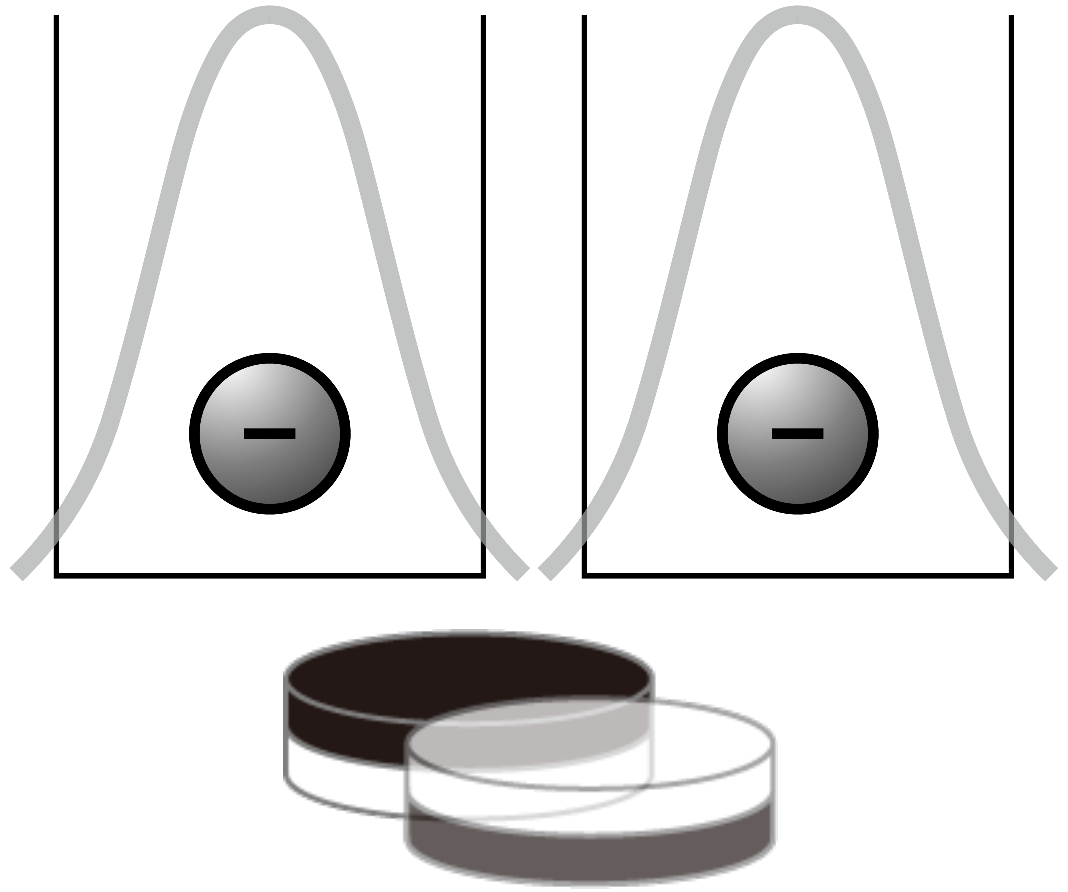
Quantum bit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Superimposed state of 0 and 1

complex vector space

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$



Whether it is 0 or 1 has yet to be determined.

measurement 0 or 1 ?

$$p_0 = |\alpha|^2$$

$$p_1 = |\beta|^2$$

Probability is given by squared abs. of complex Amplitudes.

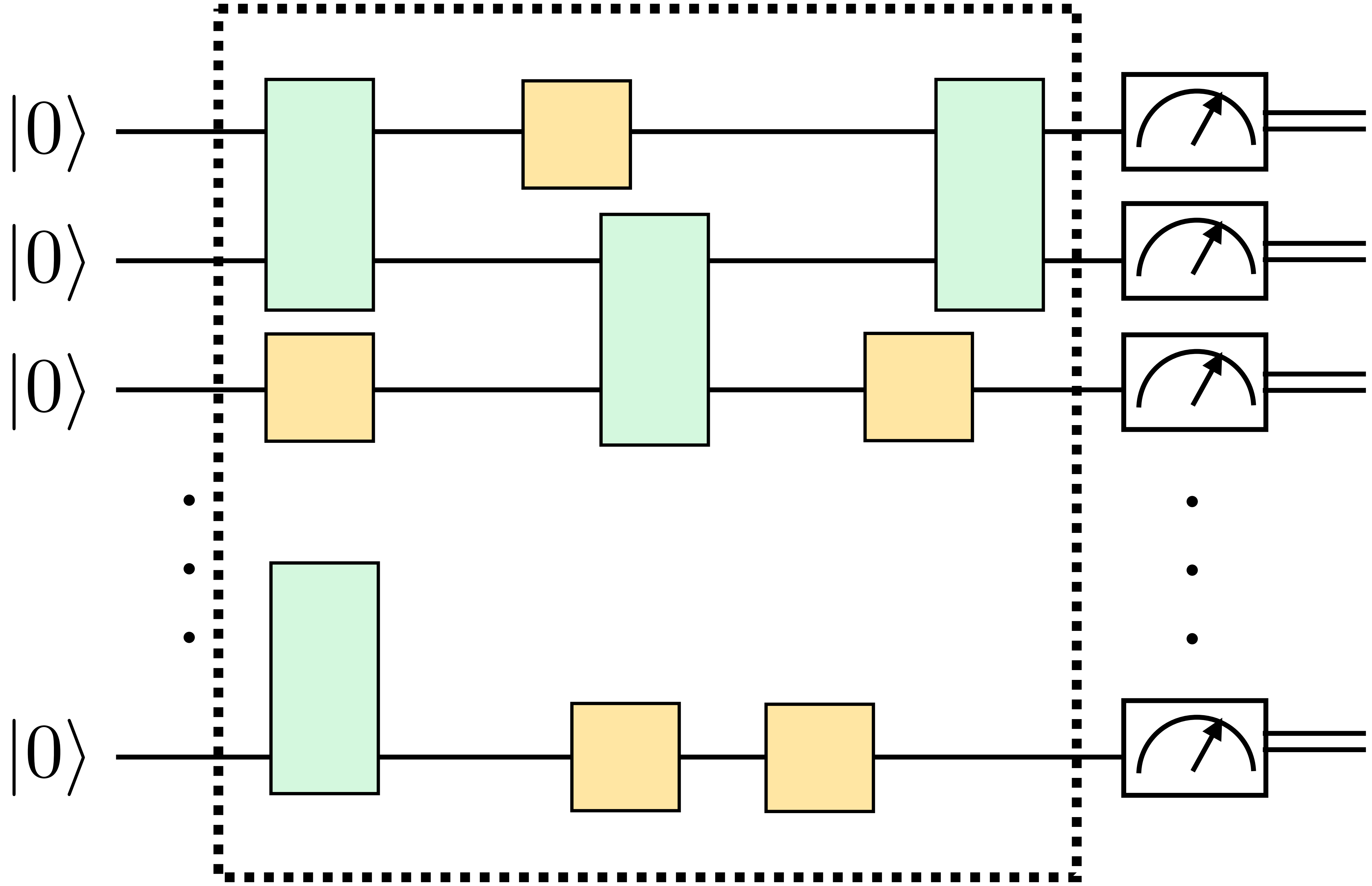
Basics of Quantum Computing

n-qubit system: a state vector $|\psi\rangle \in \mathbf{C}^{2^n}$

$$|\psi\rangle = \begin{pmatrix} \psi_{0\ldots0} \\ \psi_{0\ldots1} \\ \vdots \\ \psi_{1\ldots1} \end{pmatrix}$$

or density matrix $\rho = |\psi\rangle\langle\psi|$
(quantum analog of prob. dist.)

$\rho_{\text{in}} = |0\ldots0\rangle\langle 0\ldots0|$



$\rho_{\text{out}} = U\rho_{\text{in}}U^\dagger$ (Schrodinger pic.)

Quantum circuit U written as a $2^n \times 2^n$ unitary matrix

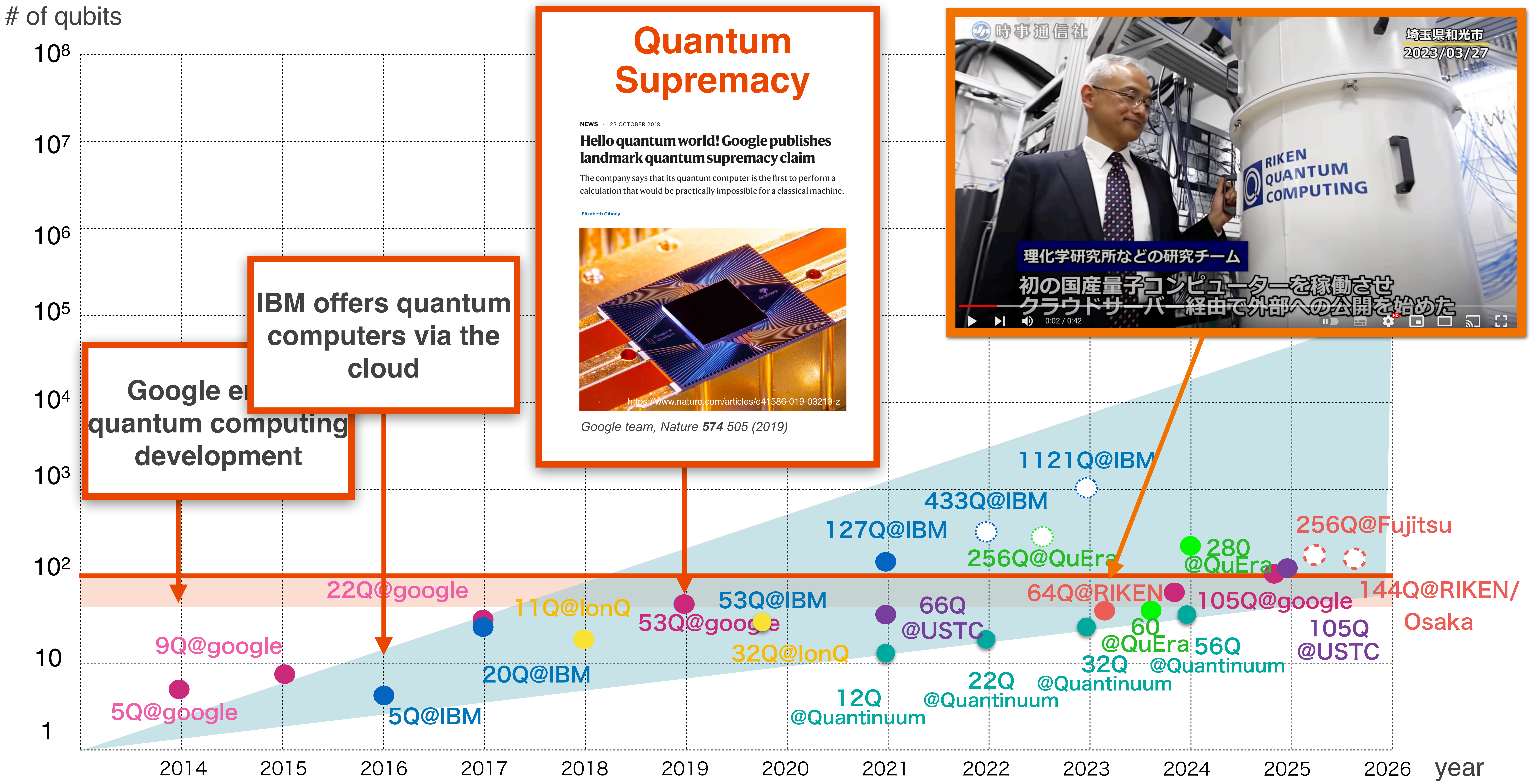
Sample a bitstring x with prob. dist. $p_x = |\psi_x|^2$.

Estimation of expectation value:

$$\langle 0\ldots0|U^\dagger M U|0\ldots0\rangle = \mathrm{Tr}[U\rho_{\text{in}}U^\dagger M]$$

- **Prime factoring**
- **Hamiltonian simulation**
- **Ground state energy estimation**
- **Linear system solver**

Progress in quantum computing to date

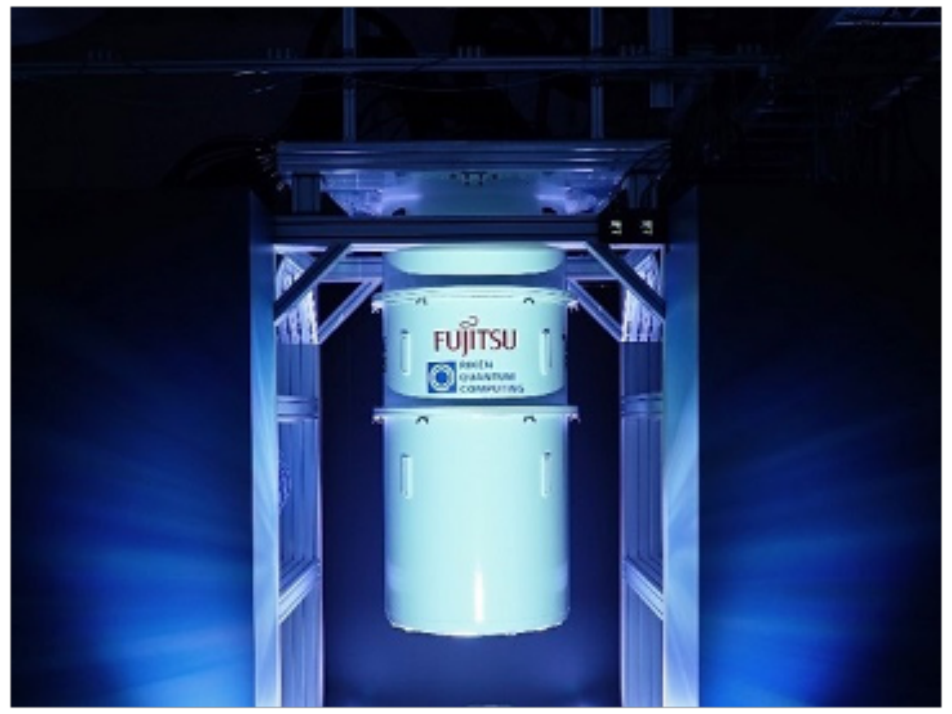


Japan's domestically developed quantum computer

- RIKEN: 1st domestically produced superconducting quantum computer released (March. 2023)
- Fujitsu: 2nd domestically produced superconducting quantum computer released (Oct. 2023)
- UOsaka： 3rd domestically produced superconducting quantum computer released (Dec. 2023)
- Fujitsu: 256 Qubit Superconducting Quantum Computer Released (Apr. 2025)
- UOsaka: "Pure Domestic" Quantum Computer Released (Jul. 2025)

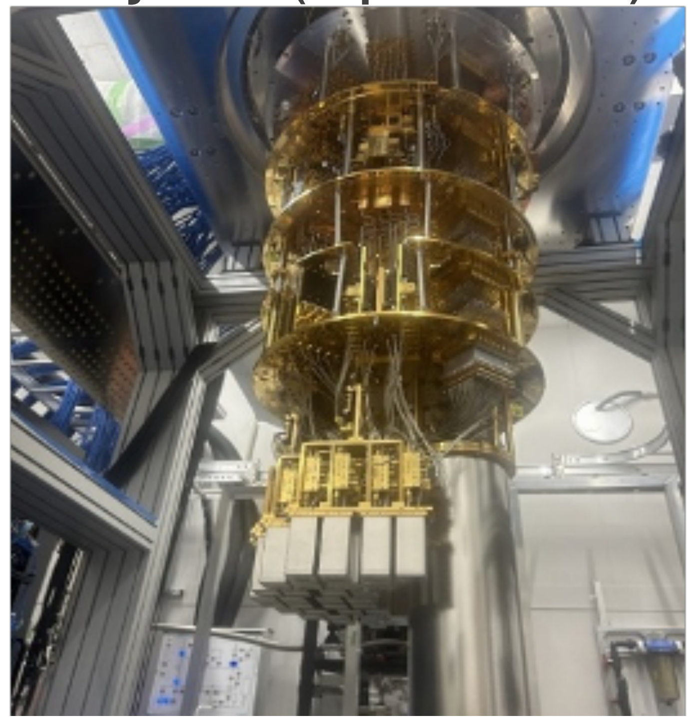
RIKEN (Mar. 2023)

Fujitsu (Oct. 2023)



UOsaka (Dec. 2023)

Fujitsu (Apr. 2025)



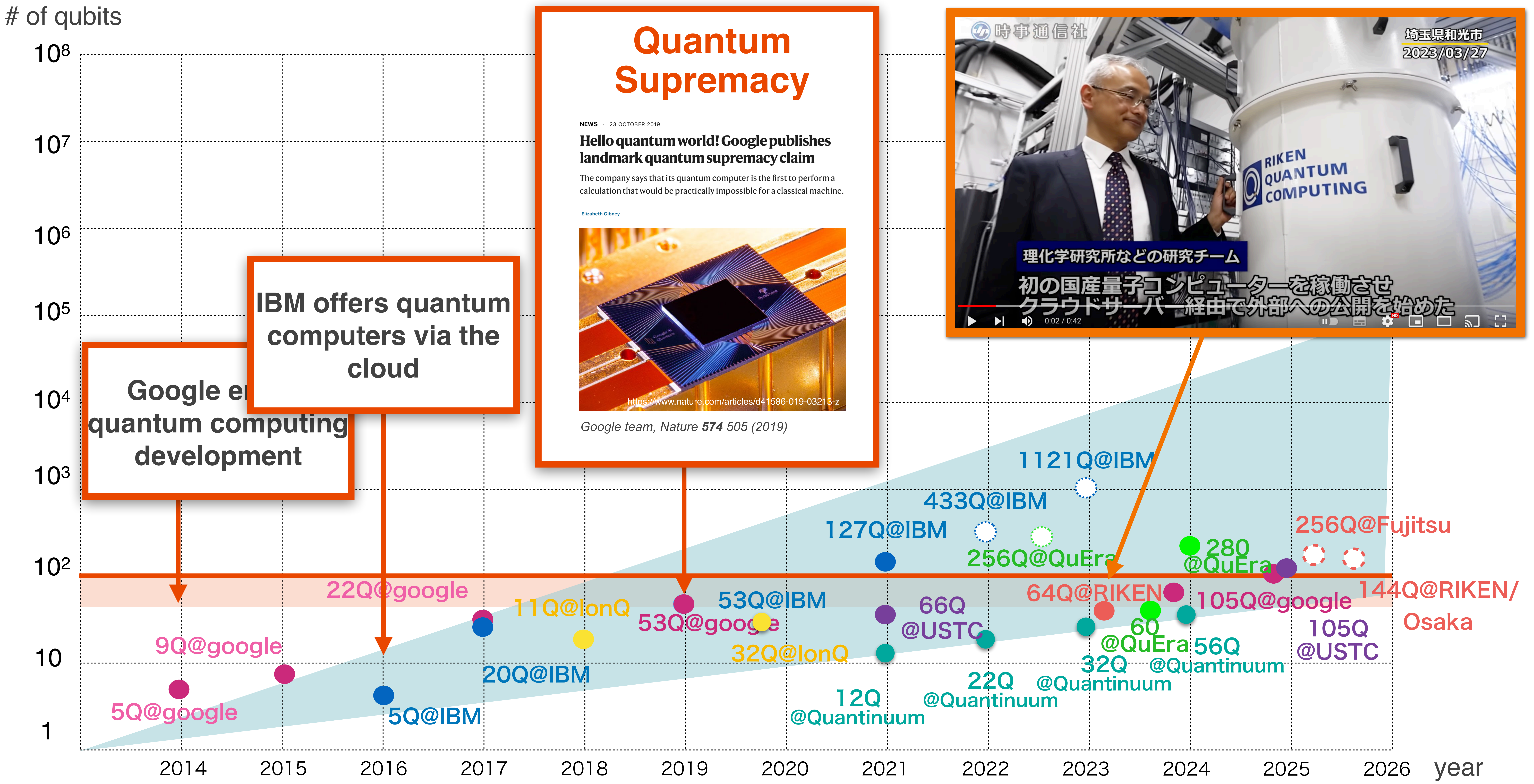
UOsaka (Jul. 2025)

Expo 2025 Osaka, Kansai, Japan

2025.8.14 [Thu] - 20 [Wed]
We had an exhibition called Entangled Moment at the Expo 2025 Osaka, Kansai, Japan.

https://qiqb.osaka-u.ac.jp/en/home

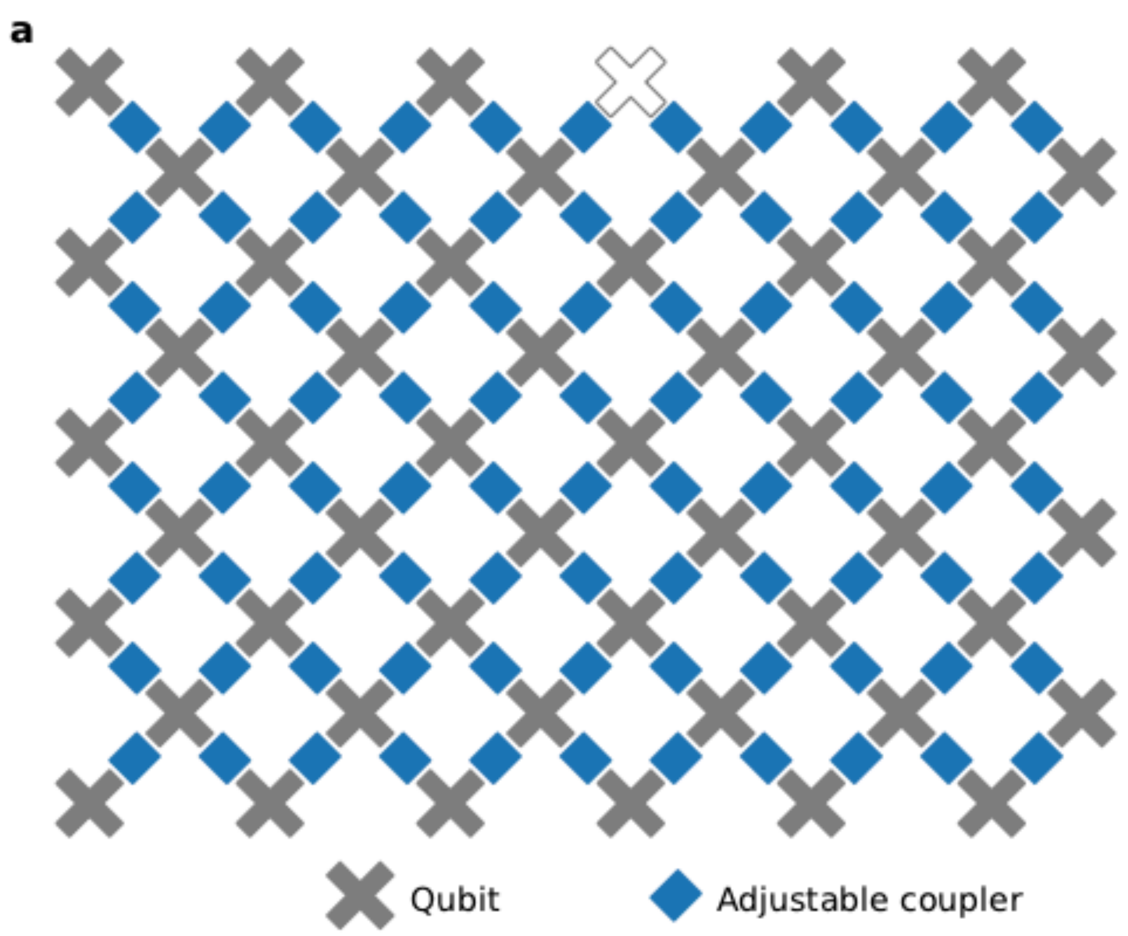
Progress in quantum computing to date

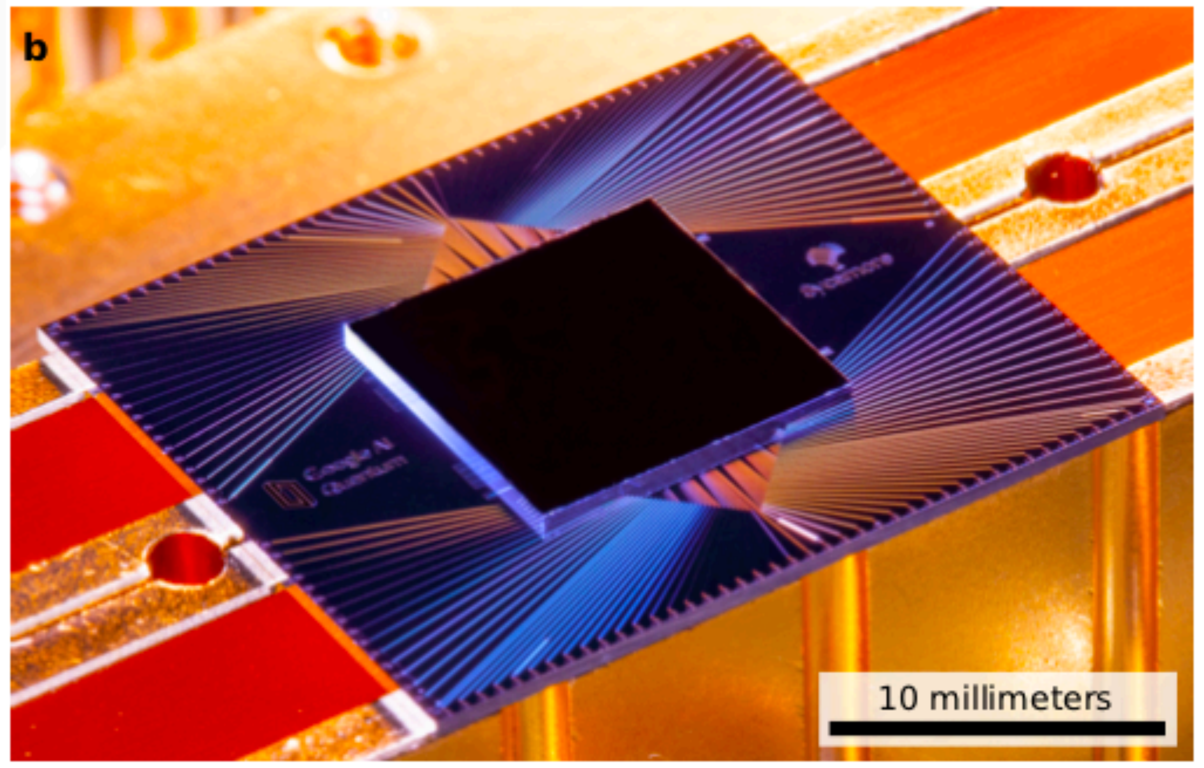


Google's quantum supremacy and beyond

53 Qubits

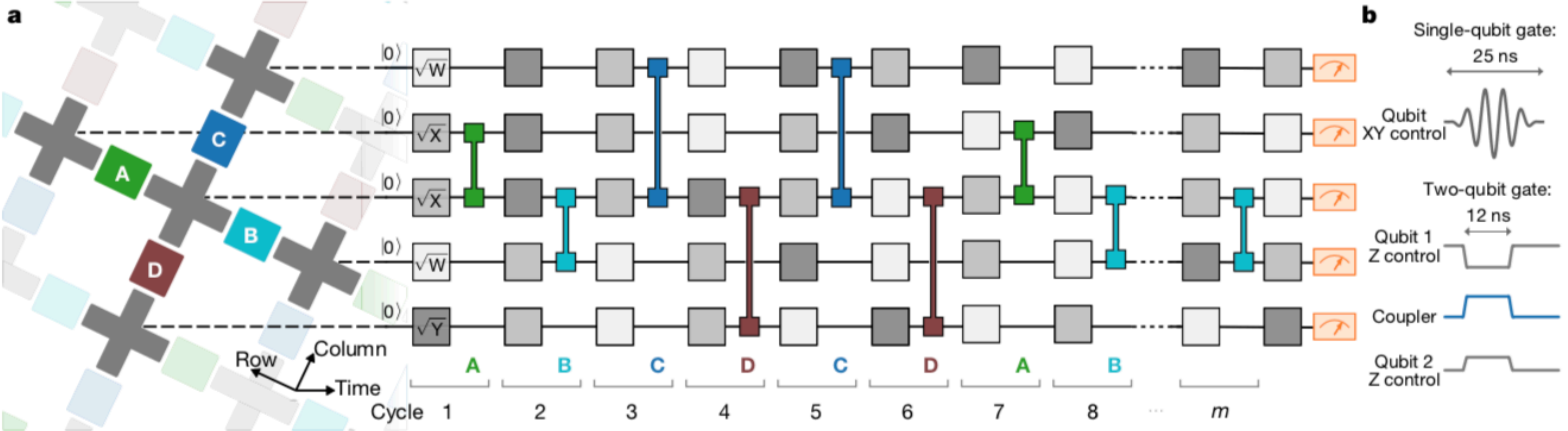
~ 8×10^{15} dimension





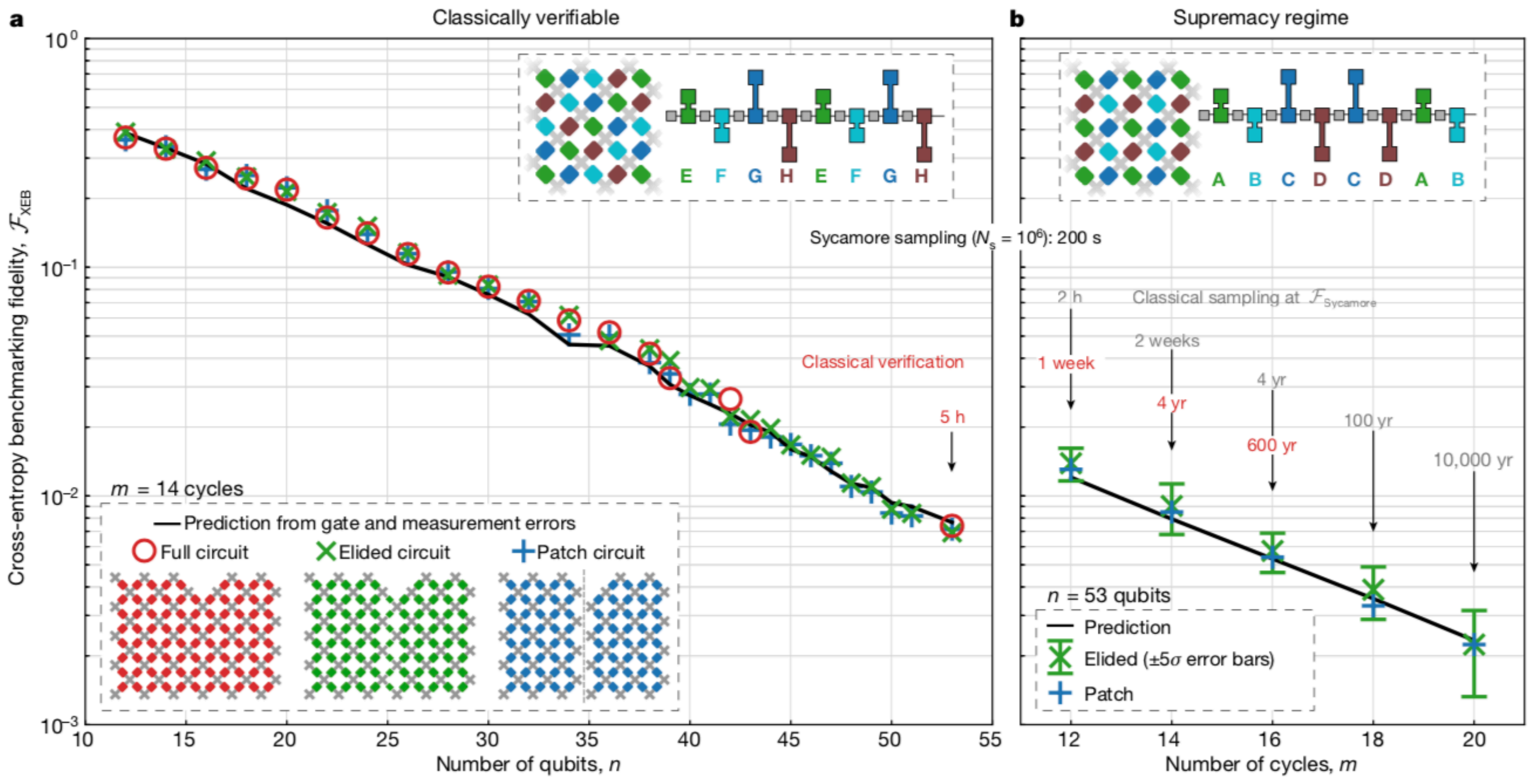
Google team, Nature **574** 505 (2019)

Random quantum circuit on qubit arrays



2Q gate **1Q gate** **Measurement** **Total fidelity**

$$(0.9938)^{430} \times (0.9984)^{1113} \times (0.962)^{53} = 0.0015$$

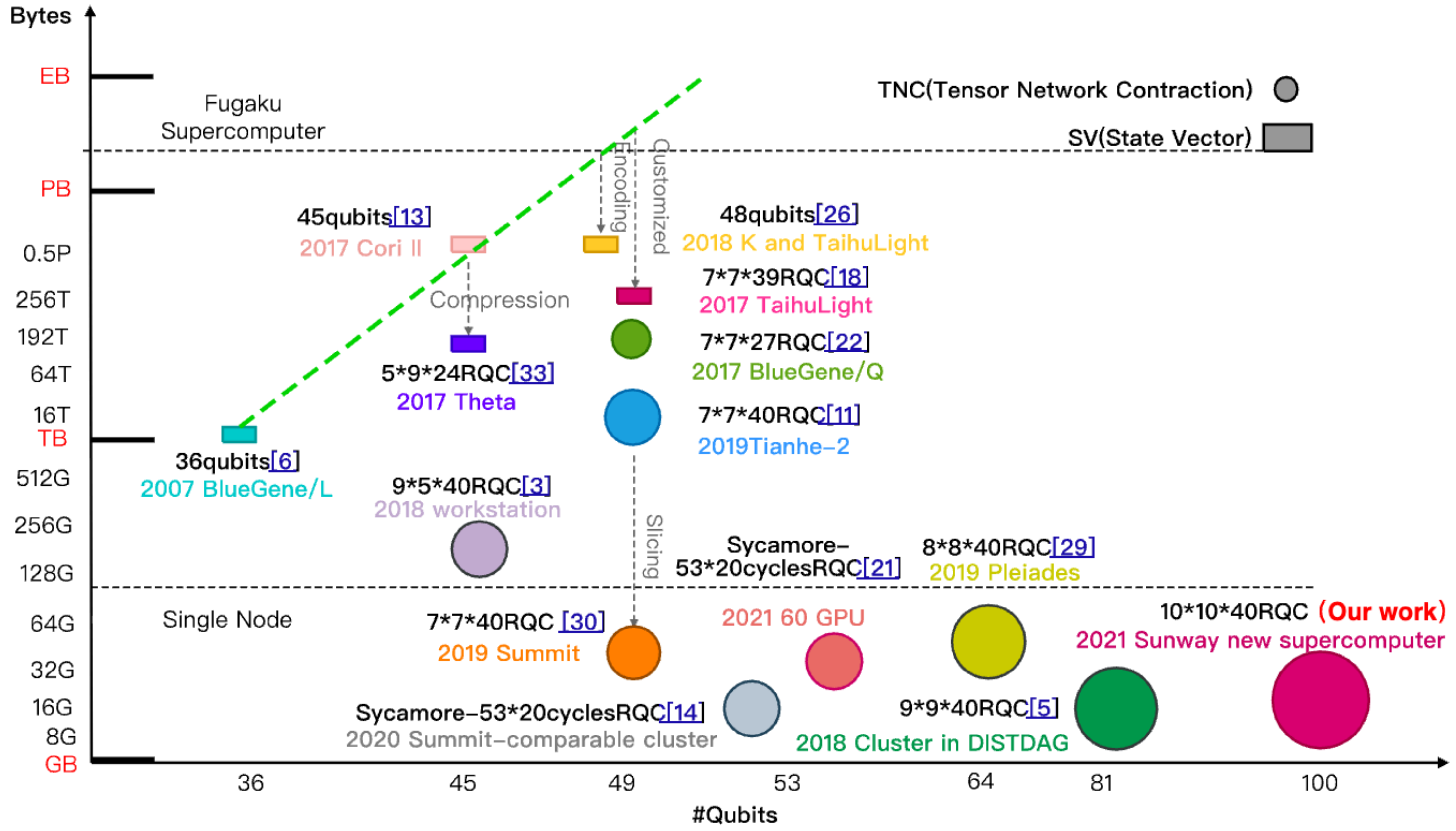


Quantum Computer:
200sec
Super computer:
10,000year

The fidelity is well approximated by the product of the gate fidelities here.

Exascale Classical Super Computer Strikes Back

Sunway Supercomputer, arXiv:2110.14502



Google's revenge

Article

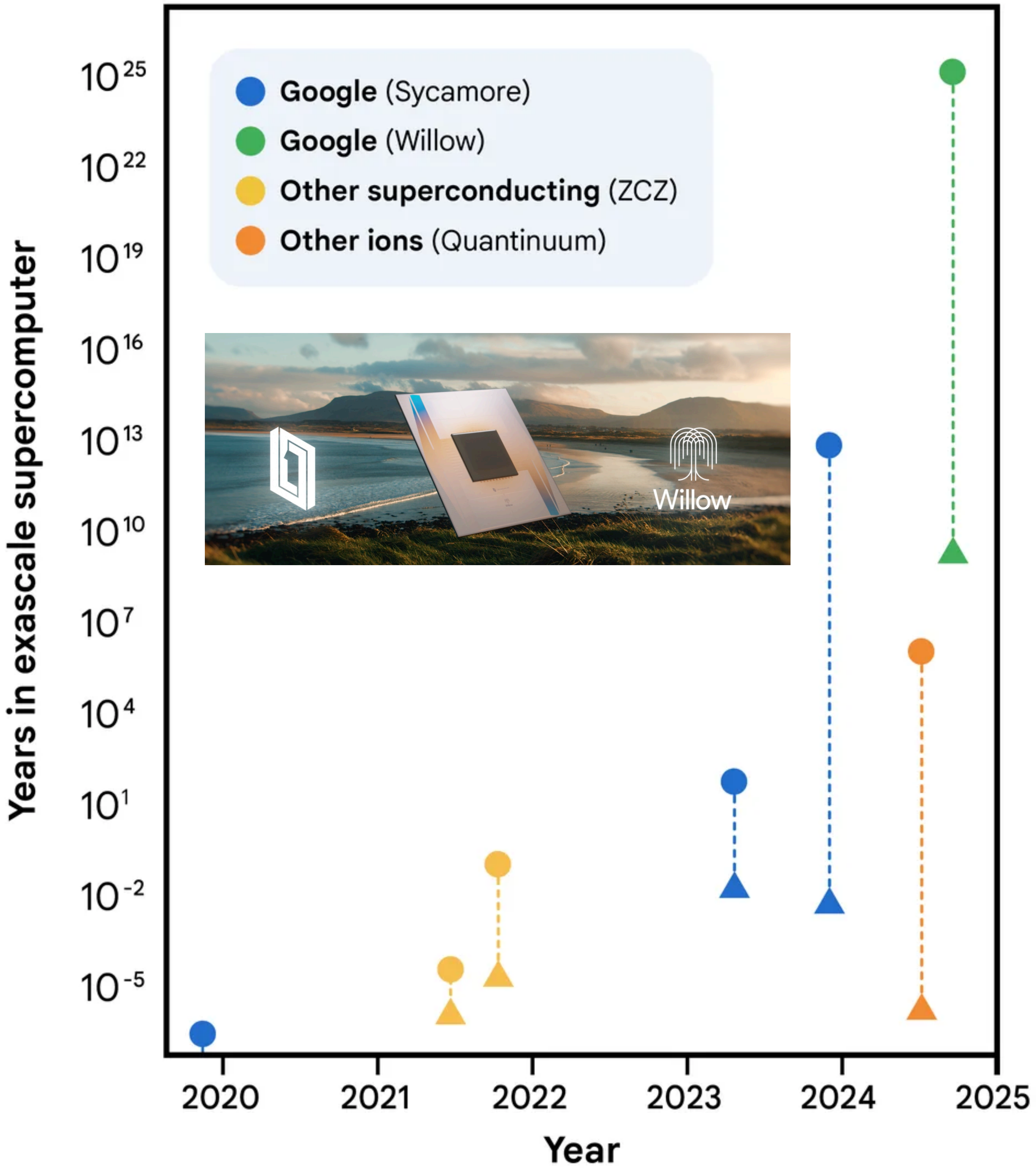
Phase transitions in random circuit sampling

https://doi.org/10.1038/s41586-024-07998-6
Received: 9 January 2024
Accepted: 28 August 2024
Published online: 9 October 2024
Open access
Check for updates

Undesired coupling to the surrounding environment destroys long-range correlations in quantum processors and hinders coherent evolution in the nominally available computational space. This noise is an outstanding challenge when leveraging the computation power of near-term quantum processors[1]. It has been shown that benchmarking random circuit sampling with cross-entropy benchmarking can provide an estimate of the effective size of the Hilbert space coherently available[2–8]. Nevertheless, quantum algorithms' outputs can be trivialized by noise, making them susceptible to classical computation spoofing. Here, by implementing an algorithm for random circuit sampling, we demonstrate experimentally that two phase transitions are observable with cross-entropy benchmarking, which we explain theoretically with a statistical model. The first is a dynamical transition as a function of the number of cycles and is the continuation of the anti-concentration point in the noiseless case. The second is a quantum phase transition controlled by the error per cycle; to identify it analytically and experimentally, we create a weak-link model, which allows us to vary the strength of the noise versus coherent evolution. Furthermore, by presenting a random circuit sampling experiment in the weak-noise phase with 67 qubits at 32 cycles, we demonstrate that the computational cost of our experiment is beyond the capabilities of existing classical supercomputers. Our experimental and theoretical work establishes the existence of transitions to a stable, computationally complex phase that is reachable with current quantum processors.

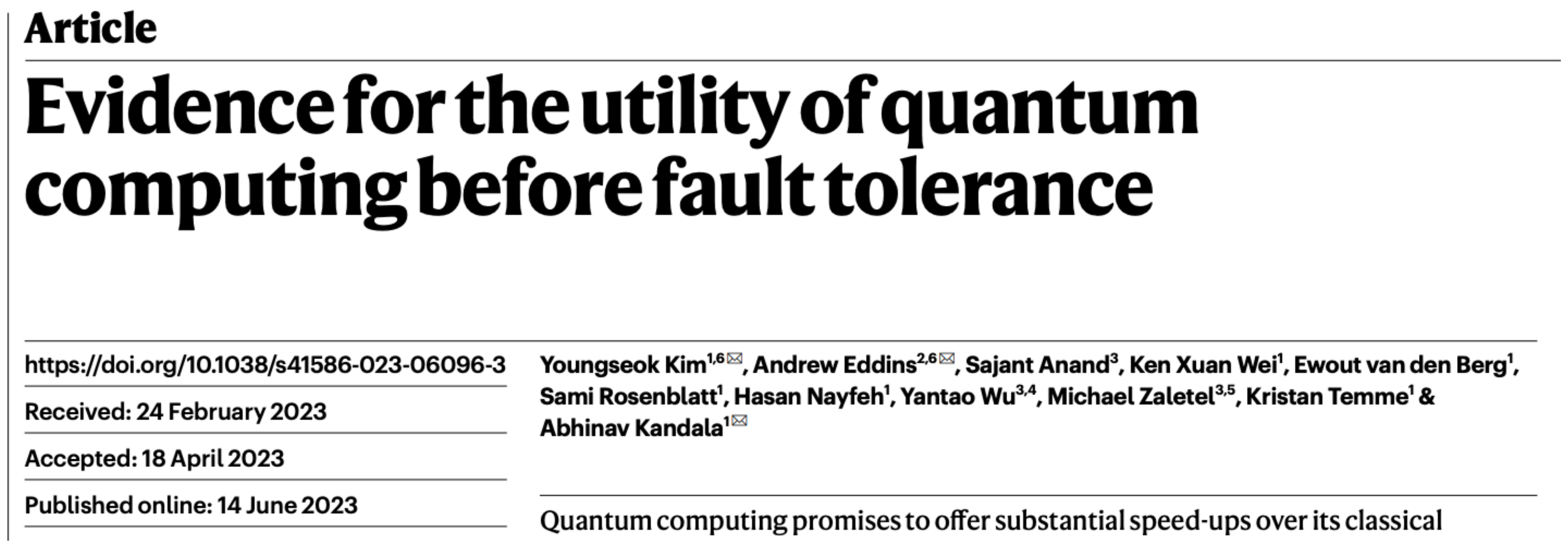
Exp.	1 amp.	1 million noisy samples		
	FLOPs	FLOPs	XEB fid.	Time
SYC-53 [9]	$6.44 \cdot 10^{17}$	$2.60 \cdot 10^{17}$	$2.24 \cdot 10^{-3}$	6.18 s
ZCZ-56 [10]	$6.24 \cdot 10^{19}$	$6.40 \cdot 10^{19}$	$6.62 \cdot 10^{-4}$	25.3 min
ZCZ-60 [11]	$1.32 \cdot 10^{21}$	$1.41 \cdot 10^{23}$	$3.66 \cdot 10^{-4}$	38.7 days
This work	$4.74 \cdot 10^{23}$	$6.27 \cdot 10^{25}$	$1.68 \cdot 10^{-3}$	47.2 yr

Google team, Nature **634** 328 (2023)



Google team, Nature **634** 328 (2023)

Toward Quantum Advantage: From Sampling to Expectation Value Estimation

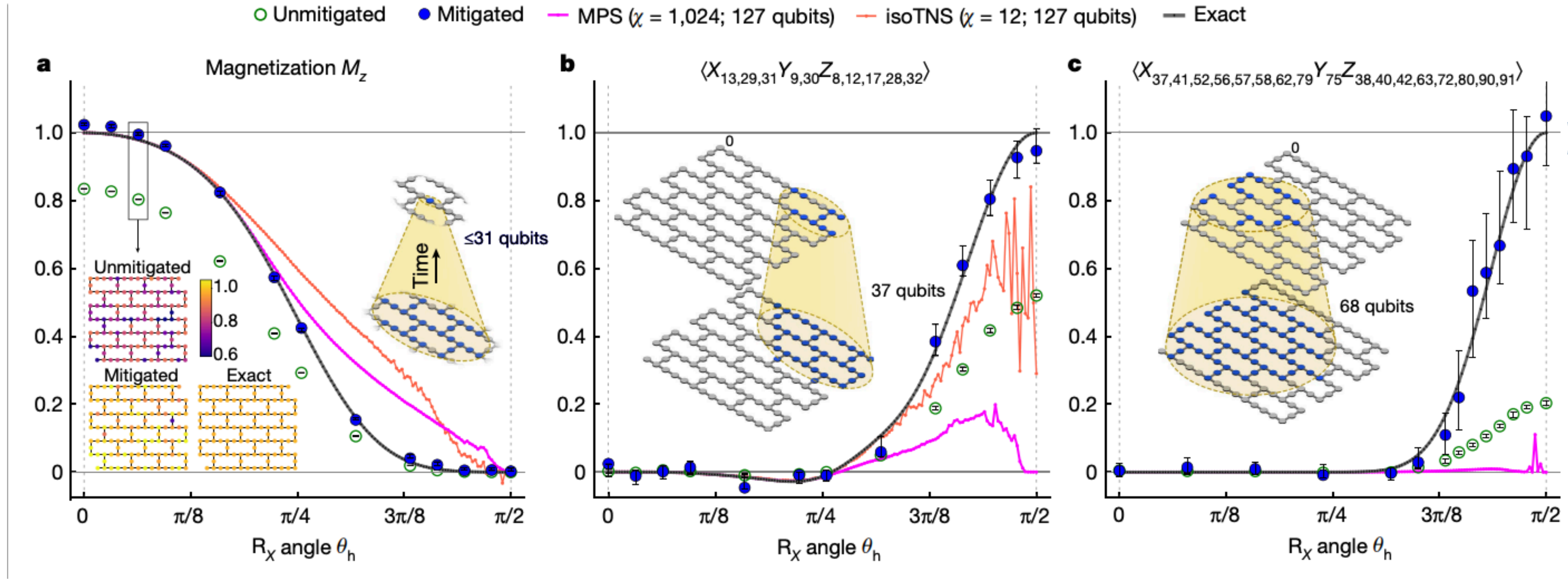

Article

Evidence for the utility of quantum computing before fault tolerance

https://doi.org/10.1038/s41586-023-06096-3
Received: 24 February 2023
Accepted: 18 April 2023
Published online: 14 June 2023

Youngseok Kim[1,6]✉, Andrew Eddins[2,6]✉, Sajant Anand[3], Ken Xuan Wei[1], Ewout van den Berg[1], Sami Rosenblatt[1], Hasan Nayfeh[1], Yantao Wu[3,4], Michael Zaletel[3,5], Kristan Temme[1] & Abhinav Kandala[1]✉

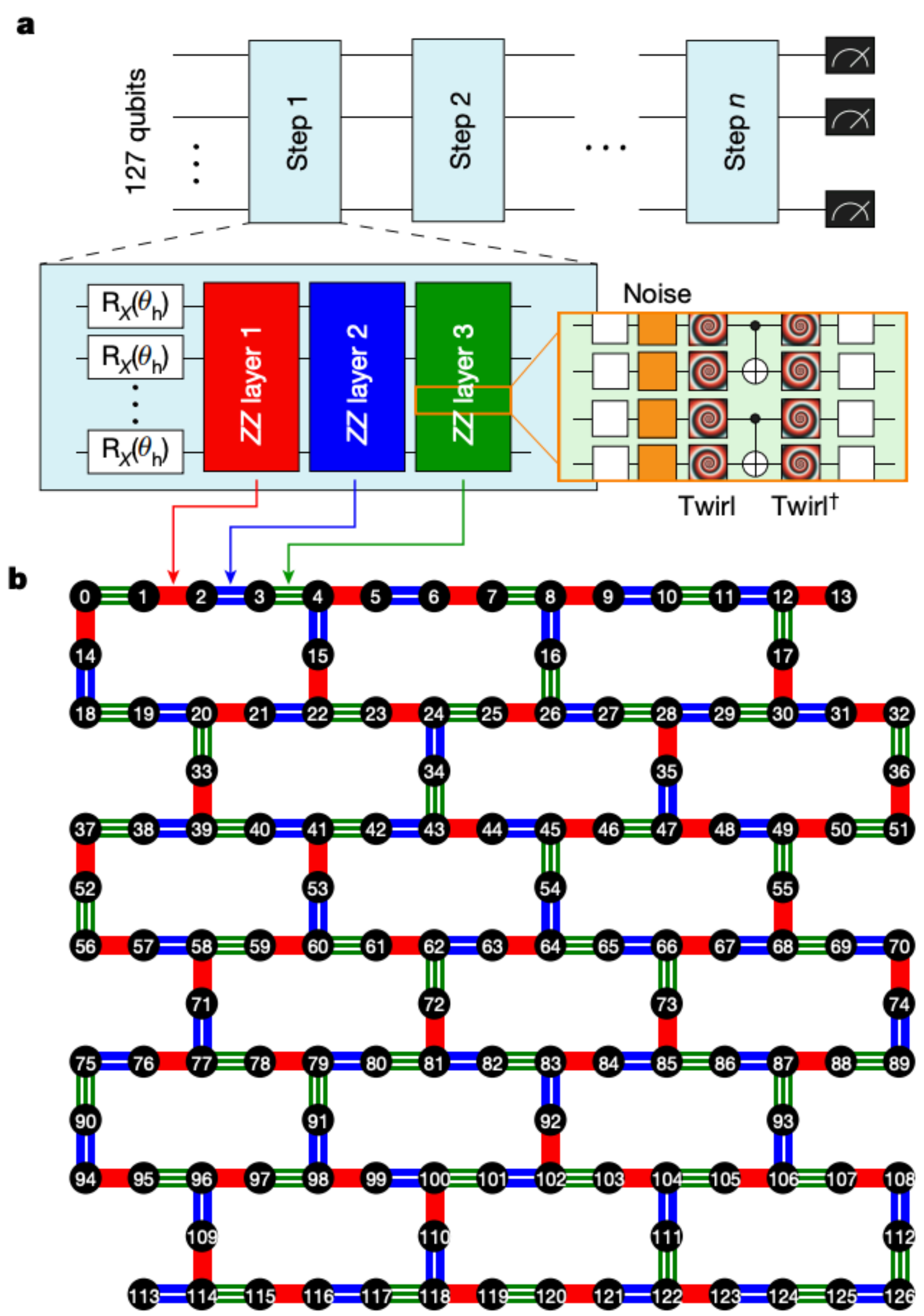
Quantum computing promises to offer substantial speed-ups over its classical



mitigated

law data

classical simlation

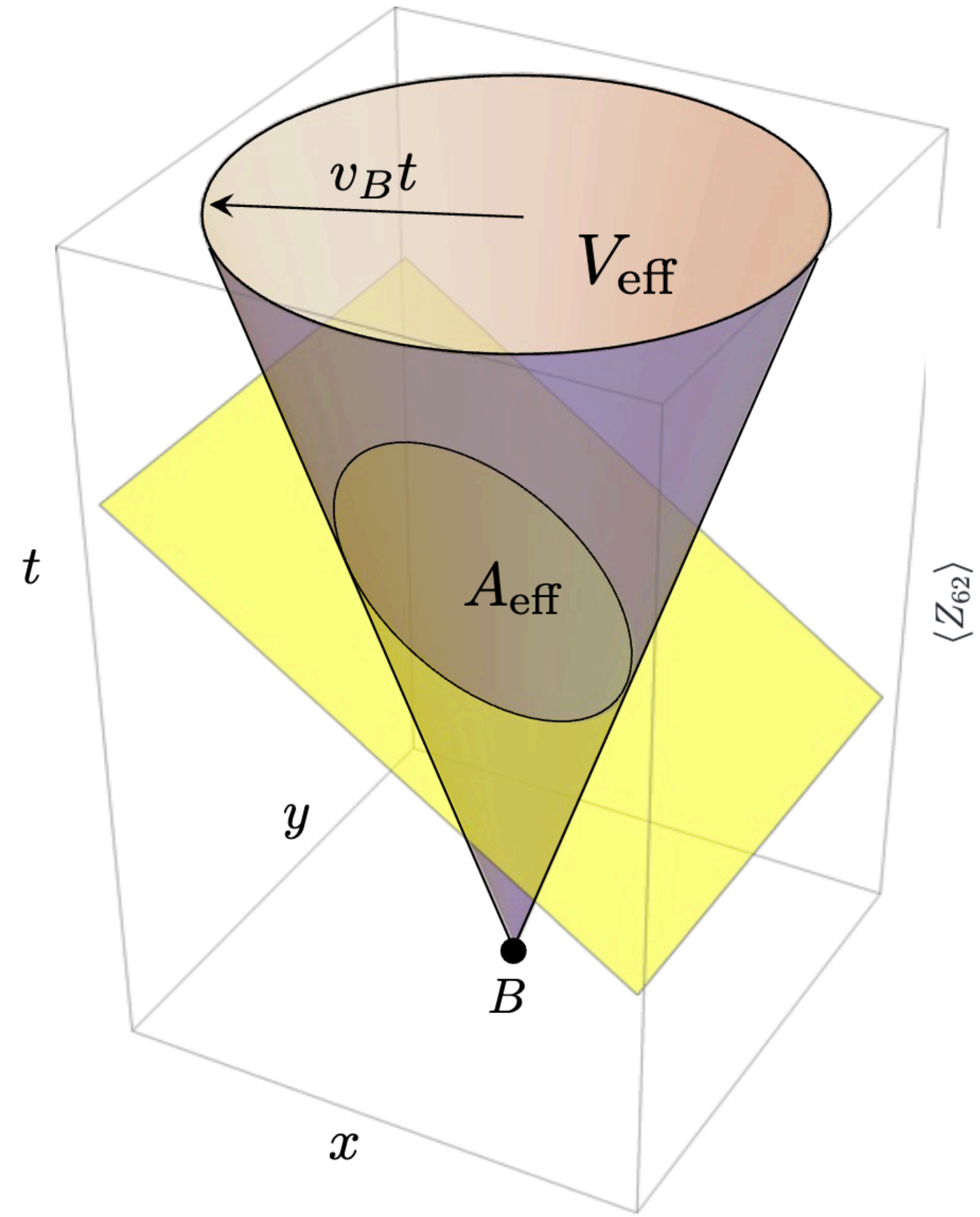


IBM, Nature **618** 500 (2023)

Tindall et al. *"Efficient tensor network simulation of IBM's Eagle kicked Ising experiment"* arXiv 2023 (PRX Quantum)
Begušić et al. "Fast and converged classical simulations of evidence for the utility of quantum computing before fault tolerance", arXiv 2023 (Science Advances)
K. Kechedzhi et al. "*Effective quantum volume, fidelity and computational cost of noisy quantum processing experiments*", arXiv 2023 (Future Generation Computer Systems)
Liao et al. *"Simulation of IBM's kicked Ising experiment with PEPO"* arXiv 2023
Patra et al. *"Efficient tensor network simulation of IBM's largest quantum processors"* arXiv 2023 (PRResearch)

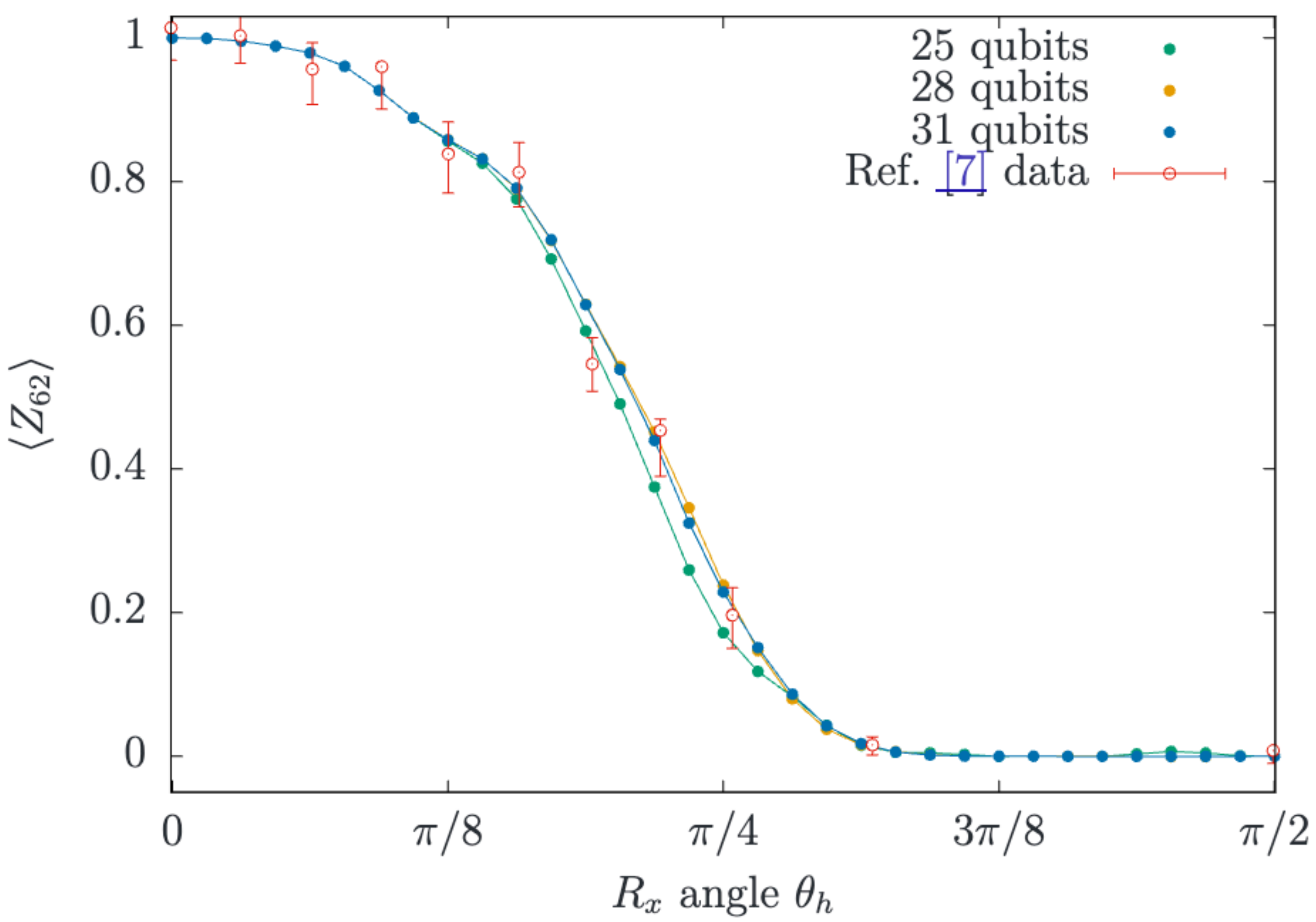
Toward Quantum Advantage: From Sampling to Expectation Value Estimation

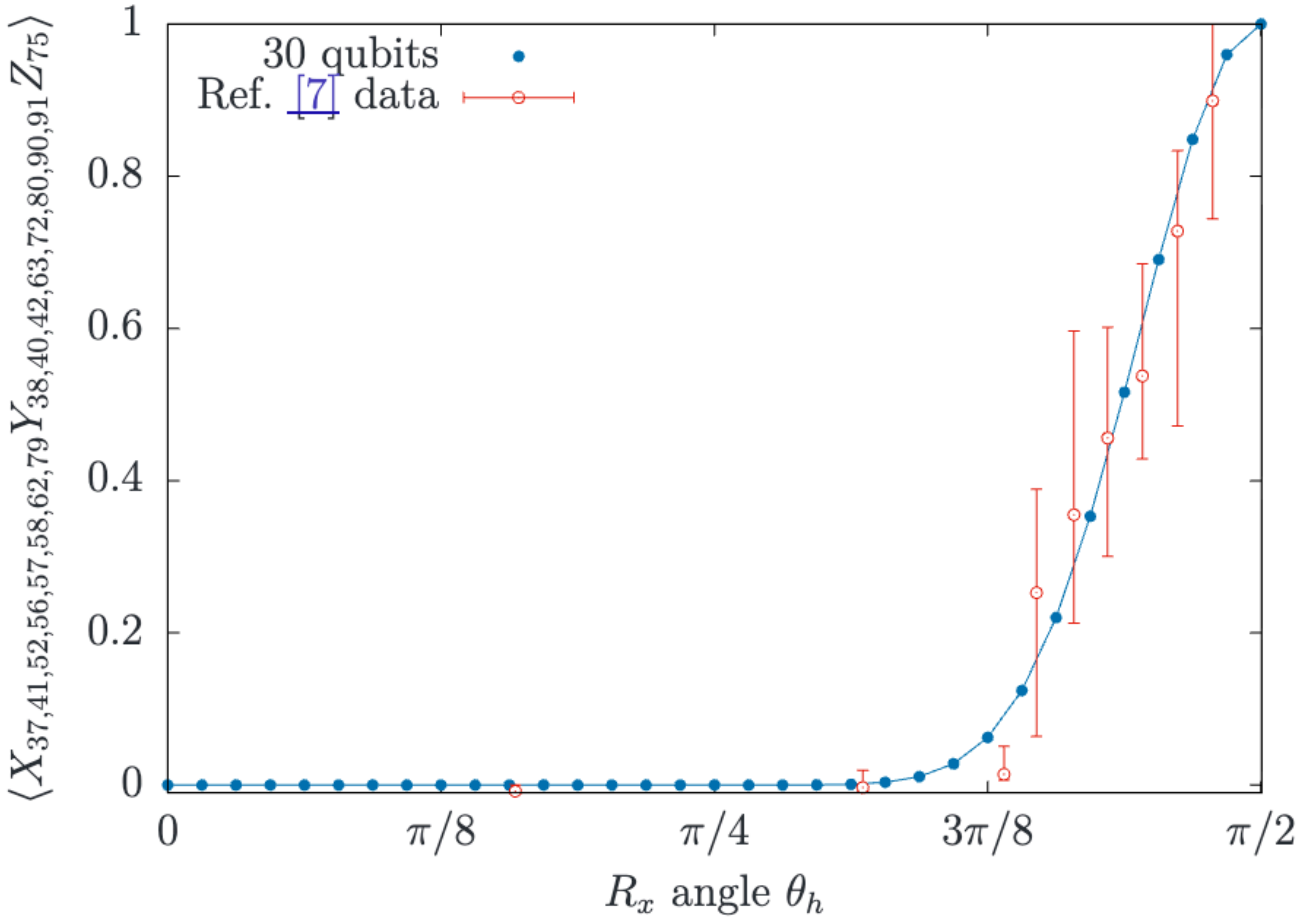
K. Kechedzhi et al. "*Effective quantum volume, fidelity and computational cost of noisy quantum processing experiments*", arXiv 2023
(Future Generation Computer Systems 2024)



・The actual regions that exhibit correlations are limited.

・Predictions can be made by simulating only the correlated regions required for estimating expectation values.





It is important to quantitatively understand how far apart in space meaningful correlations are established. → Out-of-time-order Correlator!!

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What is Out-of-Time-Order Correlator (OTOC)

What is Out-of-Time-Order Correlator

Let H be a many-body Hamiltonian and V, W be local unitary and hermitian (e.g. Pauli) operators.

The Heisenberg time evolution is defined as

$$W(t) = e^{iHt}We^{-iHt}.$$

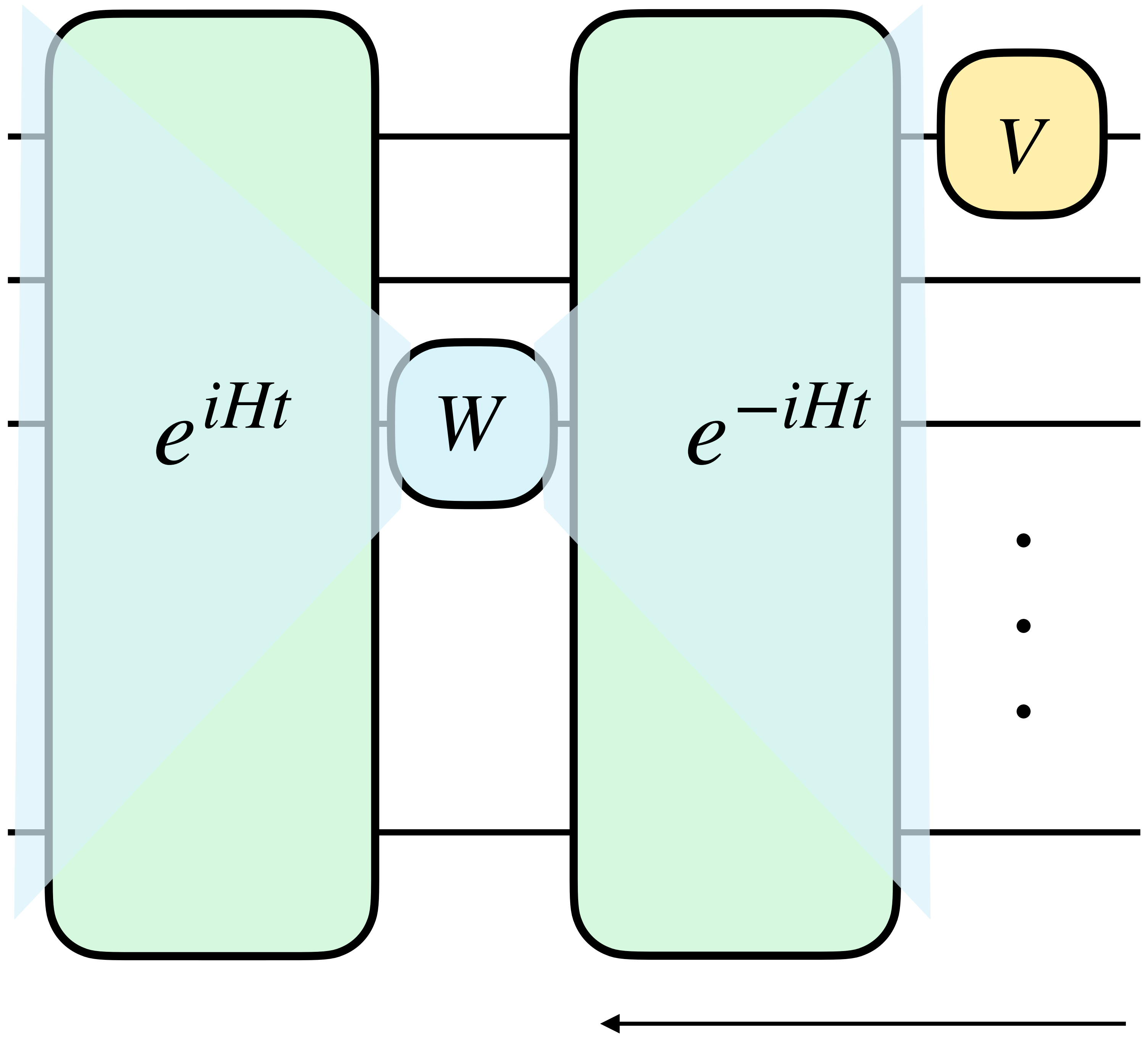
A standard out-of-time-order correlator[1,2] is

$$\text{OTOC}(t) = \langle\psi|W^\dagger(t)V^\dagger W(t)V|\psi\rangle.$$

This quantity measures how much $W(t)$ fails to commute with V.

At early times, local operators approximately commute and the OTOC remains close to its initial value 1.

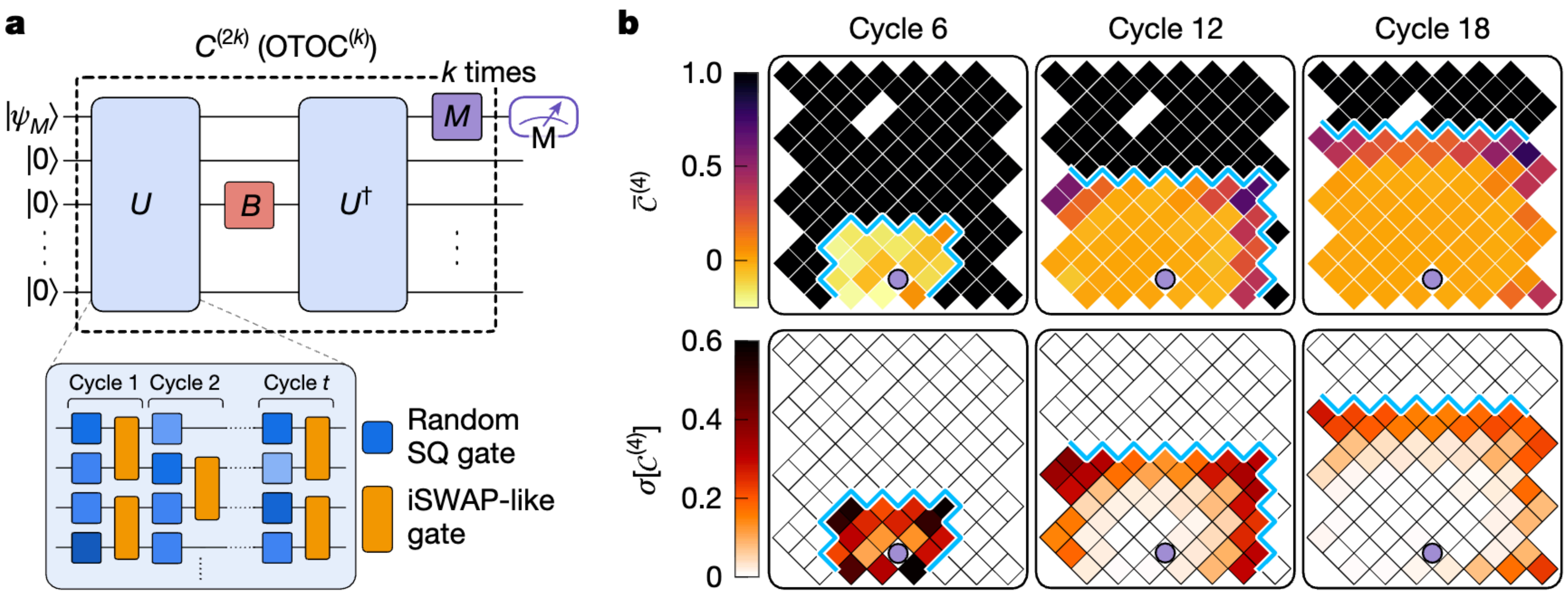
As time increases, $W(t)$ spreads and becomes nonlocal. When it reaches the support of V, the OTOC decays.



[1] AI Larkin, YN Ovchinnikov - Sov Phys JETP, 1969
[2] J Maldacena, SH Shenker, D Stanford - Journal of High Energy Physics, 2016

Google's quantum echo experiment:

Google Quantum AI team, *"Observation of constructive interference at the edge of quantum ergodicity."* Nature, 2025.



- Experimental measurement of high-order OTOCs on a programmable quantum processor.
- Real-space visualization of operator growth and scrambling fronts.
- Enhanced robustness of higher-order OTOCs against noise and decoherence: the higher-order OTOCs decay more slowly and retain meaningful information at longer circuit depths than lower-order correlators.

Definition of Higher order OTOCs

Let U be a unitary operator, B_i and M_j be local hermitian and unitary (i.e., Pauli) operators acting on ith and jth sites, respectively.

Echo operator:

$$C_{i,j} = U^\dagger B_i U M_j$$

The k-th order OTOC is defined by

$$\mathrm{OTOC}^{(k)}_{i,j} = \langle \psi_{\mathrm{ref}} | C^{2k}_{i,j} | \psi_{\mathrm{ref}} \rangle,$$

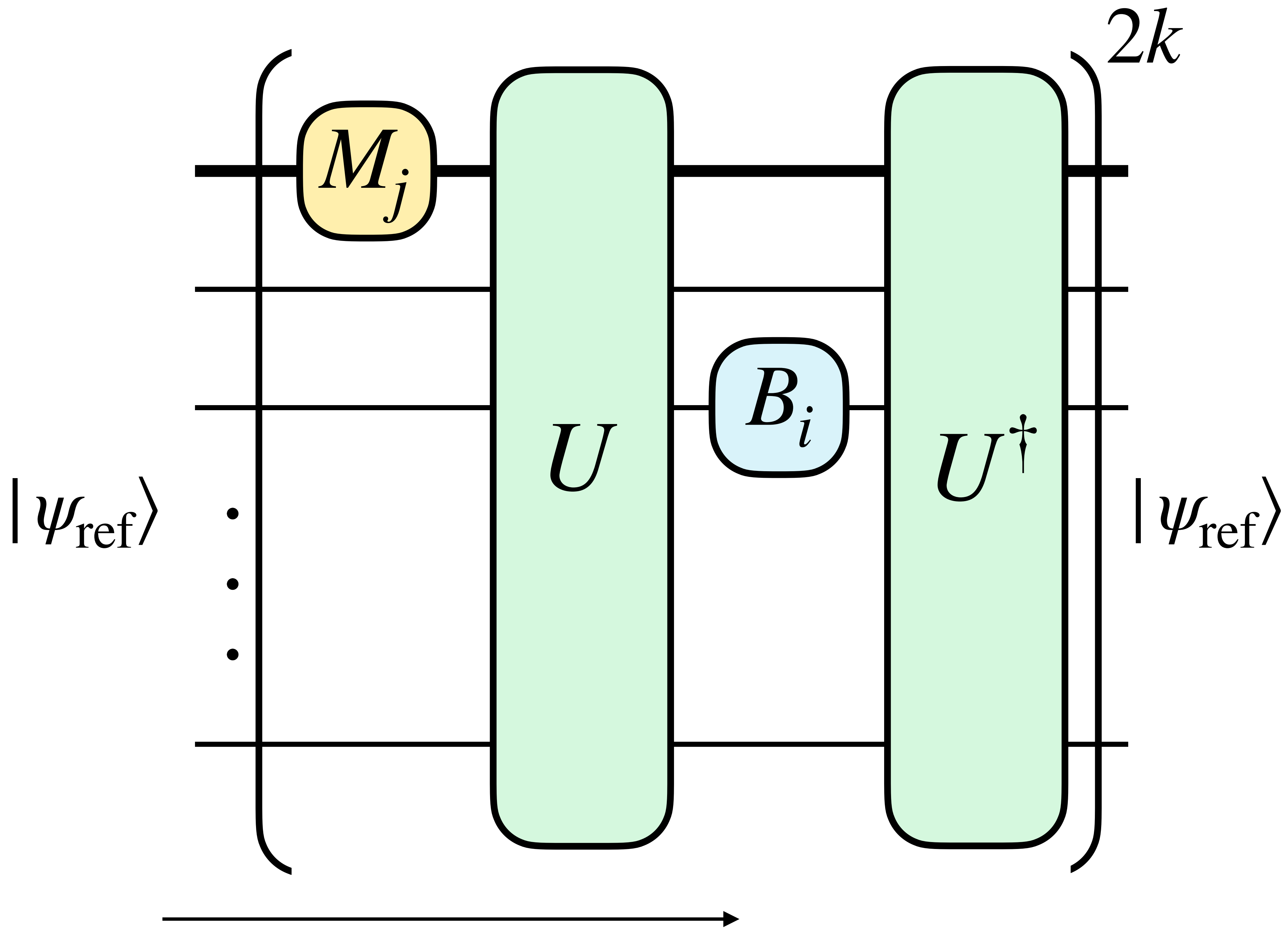
where $|\psi_{\mathrm{ref}}\rangle = |0^N\rangle$ for simplicity.

Specifically, standard OTOC is given by $k = 1$:

$$\mathrm{OTOC}^{(1)}_{i,j} = \langle \psi_{\mathrm{ref}} | \left(U^\dagger B_i U M_j \right)^2 | \psi_{\mathrm{ref}} \rangle$$

For $k = 1/2$, it reduces to time order correlator:

$$\mathrm{OTOC}^{(1/2)}_{i,j} = \langle \psi_{\mathrm{ref}} | \left(U^\dagger B_i U M_j \right) | \psi_{\mathrm{ref}} \rangle$$



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Echo operator:

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Can we understand the property of the higher order OTOCs from quantum algorithmic viewpoint?

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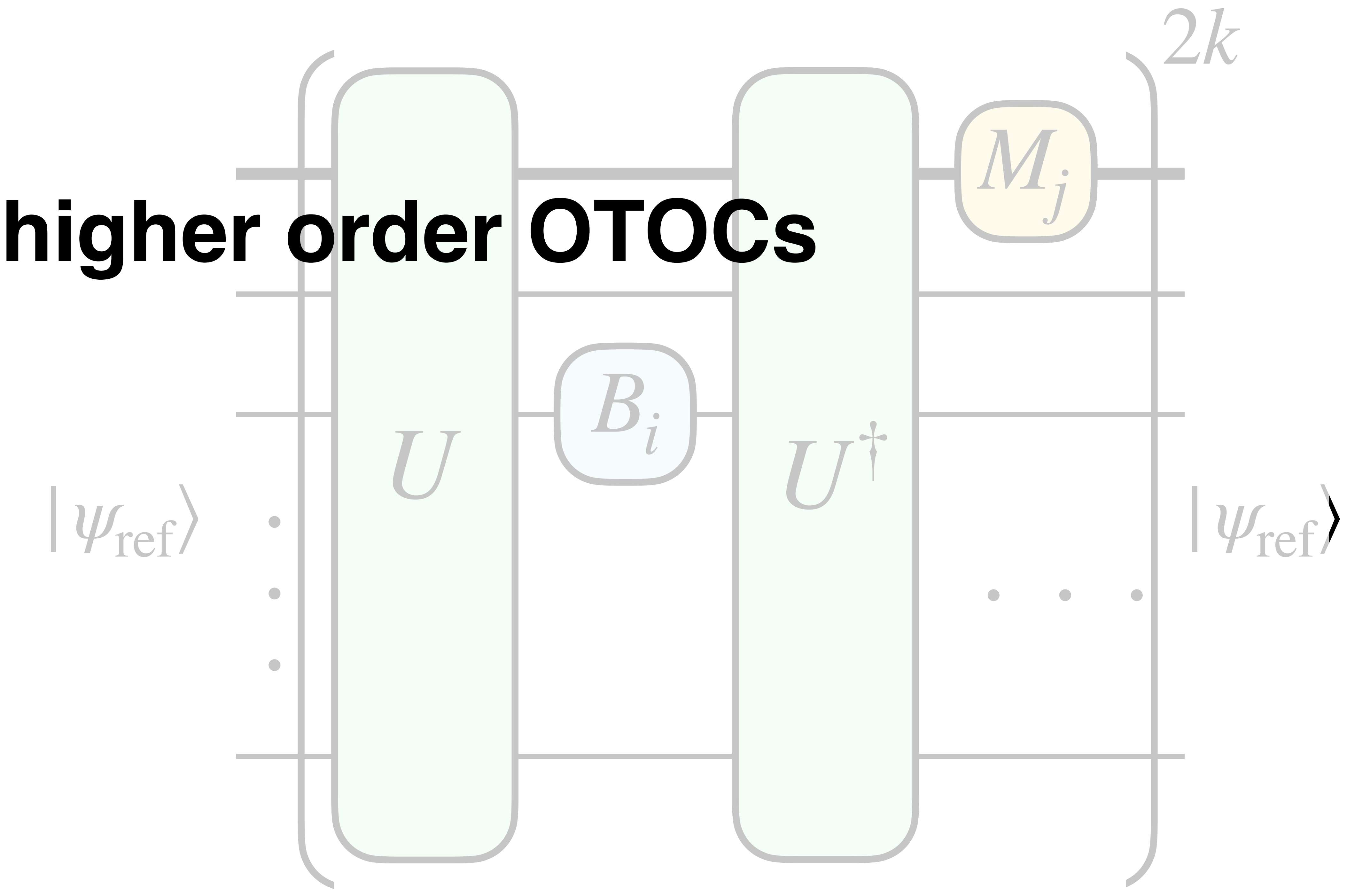

Motivation of this work

Keisuke Fujii *"Out-of-Time-Order Correlator Spectroscopy."* arXiv:2511.22654 (2025).

- Out-of-time-order correlators (OTOCs) have become a standard diagnostic of quantum dynamics, especially in the context of scrambling and quantum chaos.

- They quantify how initially local operators spread over the system under time evolution, and are widely used in condensed matter physics, high-energy physics, and quantum information theory.

- Despite their popularity, the interpretation of higher-order OTOCs remains unclear.

    - In particular, it is not well understood what kind of information they extract, and why the higher order OTOCs are more sensitive.

- This work provides a unified and algorithmic interpretation of OTOCs, based on quantum singular value transformation.

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Quantum Singular Value Transformation (QSVT)

A relatively new quantum algorithmic design principle

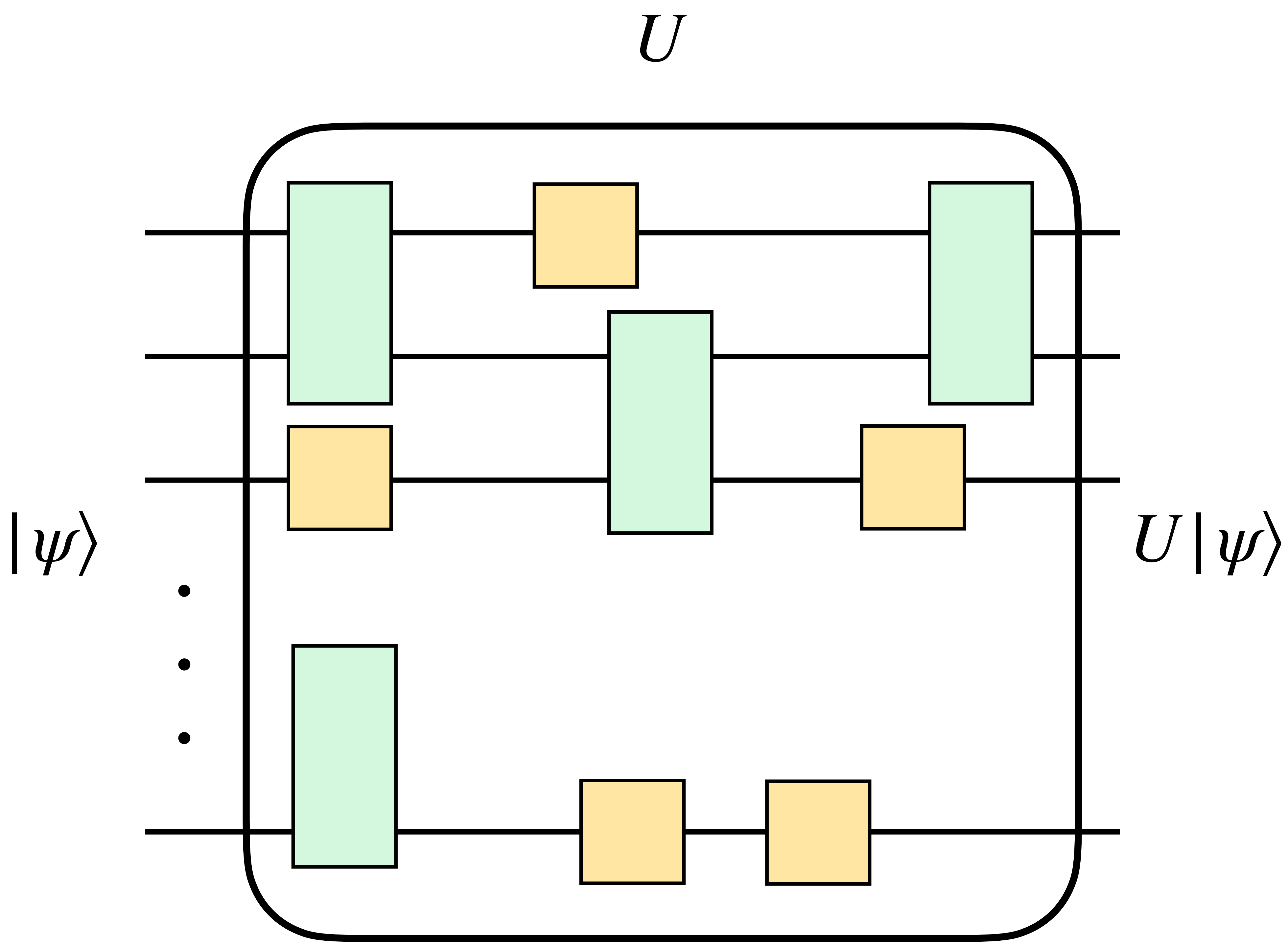
Quantum mechanics is built on linear algebra, and well suited for linear-algebraic operations on extremely high-dimensional matrices.

The operators that can be executed naturally on a quantum computer are unitary. → **block encoding**

Block Encoding: A matrix A ($\|A\| \leq 1$) is block-encoded in a unitary U as

$$U = \begin{pmatrix} A & * \\ * & * \end{pmatrix}, \quad A = (\langle 0|_a \otimes I_{\text{rest}}) U (|0\rangle_a \otimes I_{\text{rest}})\,.$$

QSVT is a quantum algorithm that uses U and $U^\dagger$ to perform polynomial transformations of the singular values of A.



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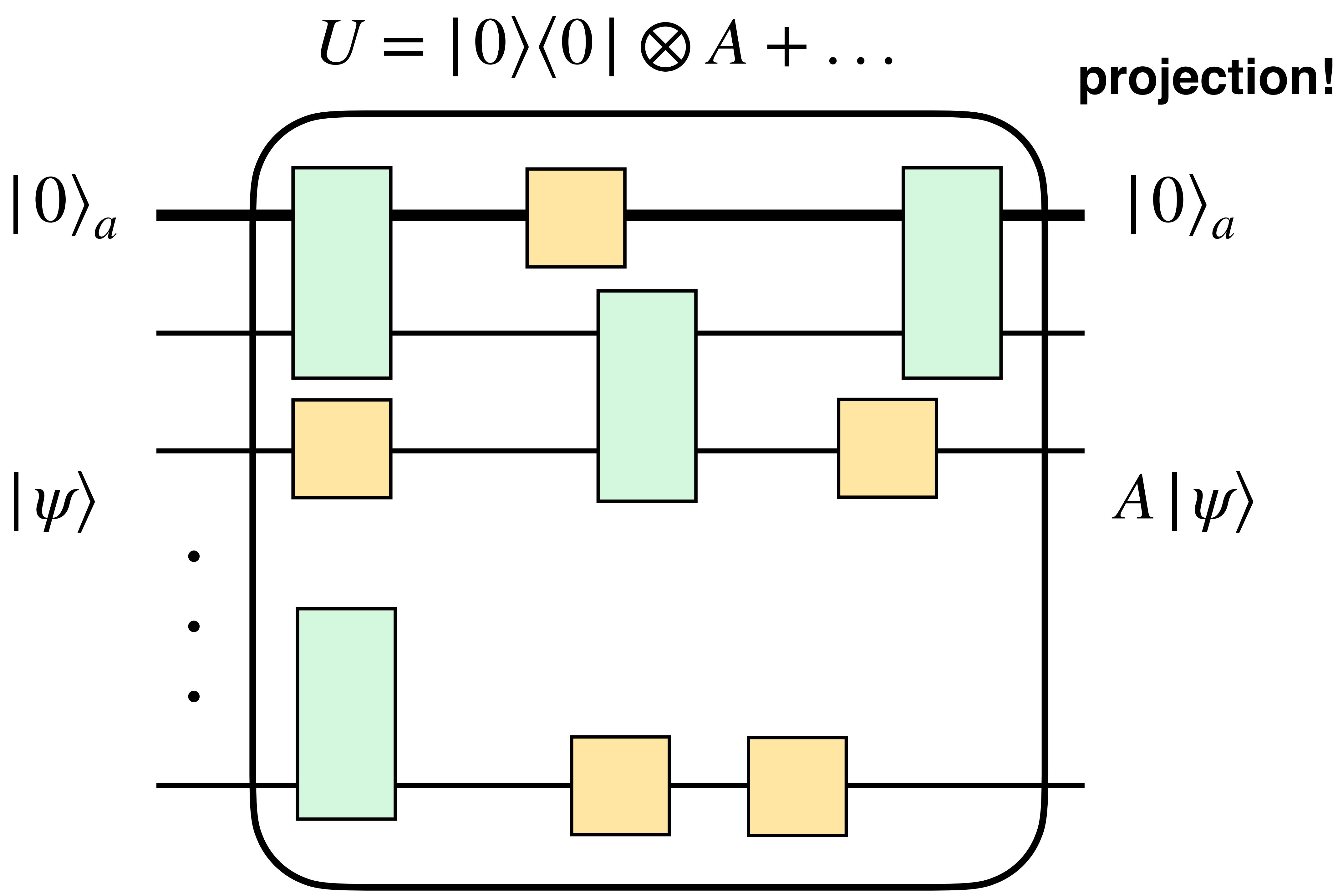

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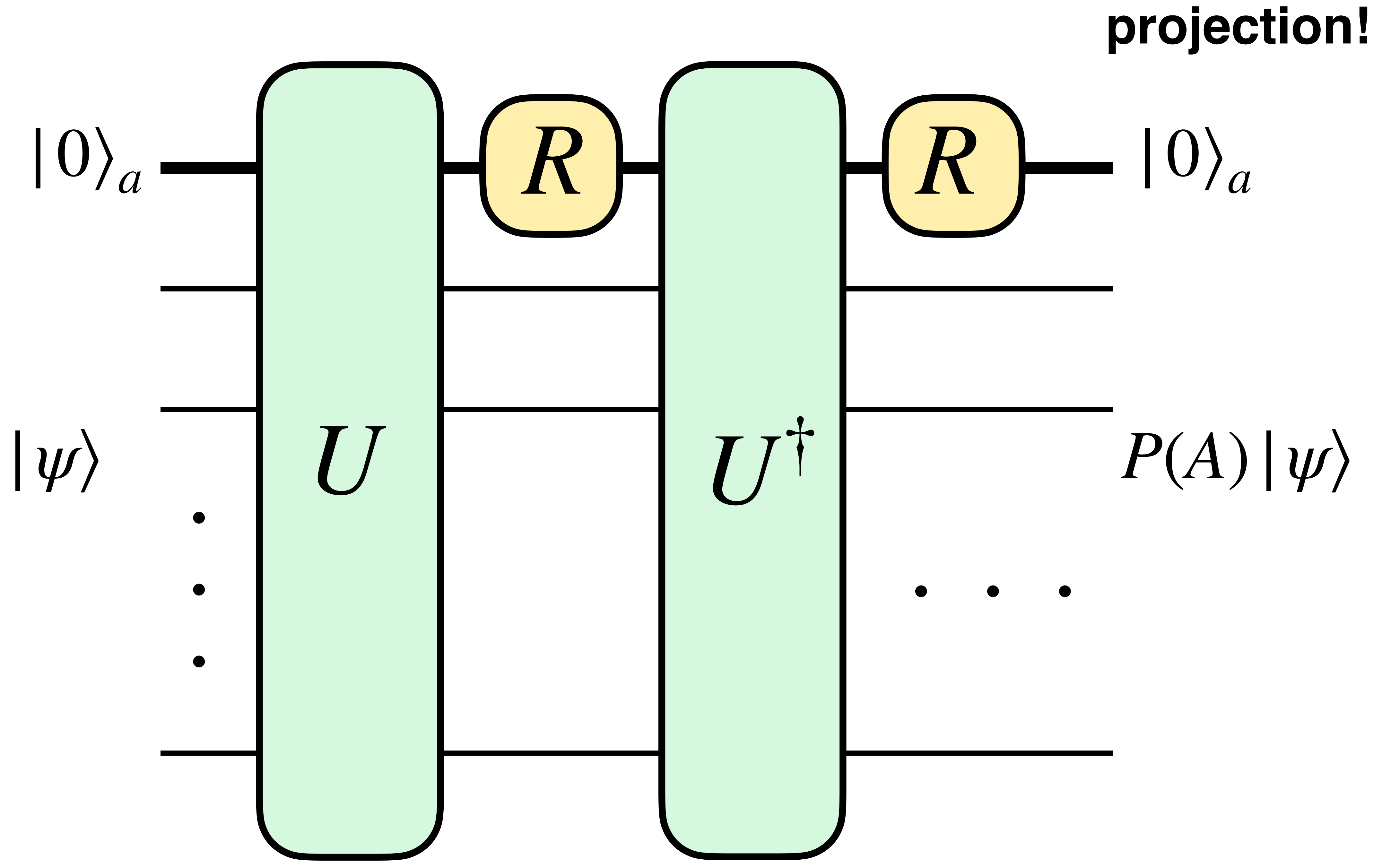
QSVT is a quantum algorithm that uses U and $U^\dagger$ to perform polynomial transformations of the singular values of A.

$$U = |0\rangle\langle 0| \otimes A + \ldots$$



→ **polynomial transformation of singular value**

Qubitization

A relatively new quantum algorithmic design principle [Gilyén et al. STOC (2019)]

In general, the action of a block-encoding U is complex, but it can be understood more simply by considering the two-dimensional invariant subspaces (**qubitization**) labeled by the singular values.

$$A = \sum_l \lambda_l |\psi_l\rangle\langle\phi_l|$$

The whole Hilbert space can be decomposed into a direct sum of **qubitized** subspaces labeled by the singular values:

$$\mathscr{H}_l = \text{span}(|0\rangle|\phi_l\rangle, |\phi_l^\perp\rangle\rangle)$$

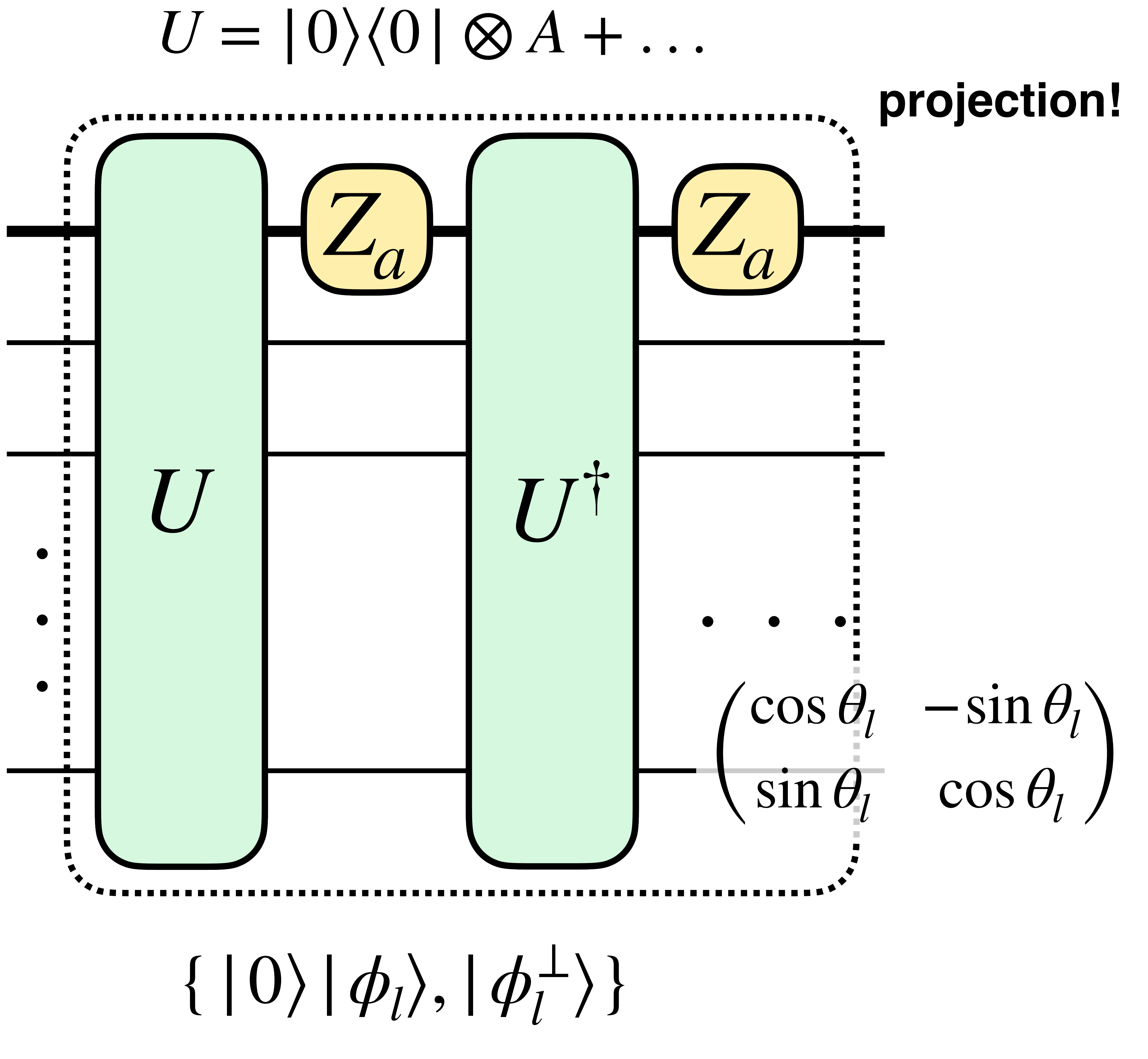
and

$$\tilde{\mathscr{H}}_l = \text{span}(|0\rangle|\psi_l\rangle, |\psi_l^\perp\rangle)$$

The action of U is a single qubit rotation from $\mathscr{H}_l$ to $\tilde{\mathscr{H}}_l$.

$$\begin{pmatrix} \lambda_l & \sqrt{1-\lambda_l^2} \\ \sqrt{1-\lambda_l^2} & -\lambda_l \end{pmatrix} = Ze^{-i(\theta_l/2)Y} \quad \rightarrow \quad Z_a U^\dagger Z_a U = \bigoplus_l e^{-i\theta_l Y}$$

where $\theta_l = 2\arccos(\lambda_l)$



Quantum Signal Processing

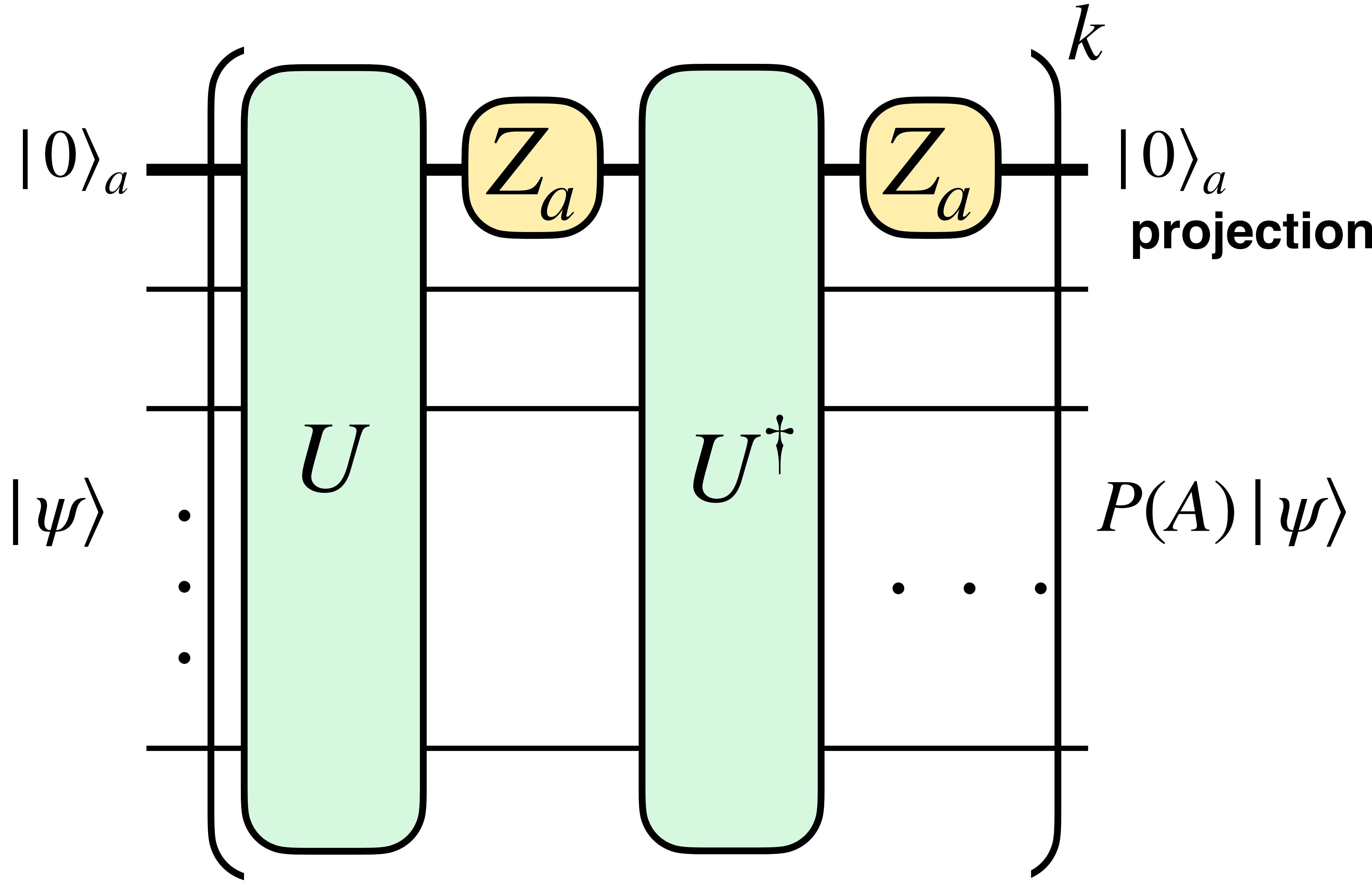
A relatively new quantum algorithmic design principle [Gilyén et al. STOC (2019)]

$$(Z_a U^\dagger Z_a U)^k = \bigoplus_l e^{-ik\theta_l Y}$$

By projecting the ancilla qubit, we have

$$\langle 0|_a (Z_a U^\dagger Z_a U)^k |0\rangle_a = \sum_l \cos(k\theta_l)|\phi_l\rangle\langle\phi_l|$$
$$= T_{2k}(\lambda_l)|\phi_l\rangle\langle\phi_l|$$

$$U = |0\rangle\langle 0| \otimes A + \dots$$



$$T_{2k}(A) = \sum_l T_{2k}(\lambda_l)|\phi_l\rangle\langle\phi_l|$$

More generally, if we replace Z_a by $e^{-i(\phi/2)Z_a}$

$$\langle 0|_a \prod_{k=1}^{d} (e^{-i(\phi_{2k+1}/2)Z_a} U^\dagger e^{-i(\phi_{2k}/2)Z_a} U)|0\rangle_a = P(\lambda_l)|\phi_l\rangle\langle\phi_l|$$

where $P()$ is a degree $2d$ polynomial designed by angles $\{\phi_i\}_{i=1}^{2d}$

→Quantum Singular Value Transforamtion

Quantum Signal Processing

A relatively new quantum algorithmic design principle

$$(Z_a U^\dagger Z_a U)^k = \bigoplus_l e^{-ik\theta_l Y}$$

By projecting the ancilla qubit, we have

$$\langle 0|_a (Z_a U^\dagger Z_a U)^k |0\rangle_a = \sum_l \cos(k\theta_l) |\phi_l\rangle\langle\phi_l|$$
$$= T_{2k}(\lambda_l) |\phi_l\rangle\langle\phi_l|$$

More generally, if we replace Z_a by $e^{-i(\phi/2)Z_a}$

$$\langle 0|_a \prod_{k=1}^{d} (e^{-i(\phi_{2k+1}/2)Z_a} U^\dagger e^{-i(\phi_{2k}/2)Z_a} U) |0\rangle_a = P(\lambda_l) |\phi_l\rangle\langle\phi_l|$$

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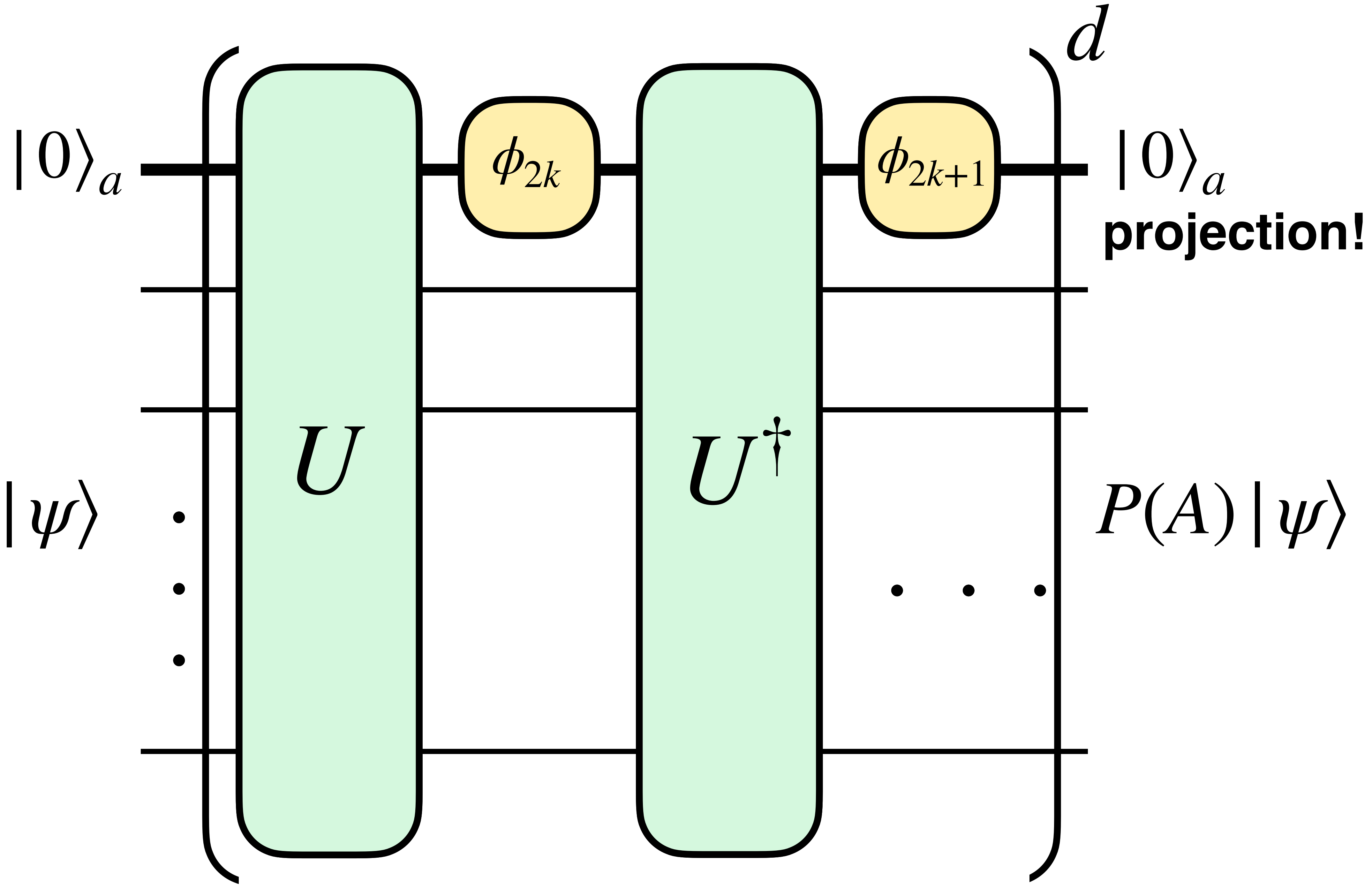
→Quantum Singular Value Transforamtion

$$U = |0\rangle\langle 0| \otimes A + \ldots$$



$$P(A) = \sum_l T_{2k}(\lambda_l) |\phi_l\rangle\langle\phi_l|$$

Applications of QSVT

A relatively new quantum algorithmic design principle

Hamiltonian simulation [Low-Chuang PRL 2017; Gilyén et al. STOC (2019)]:

$$A = H, \quad P(x) = e^{-ixt}$$

Eigen filtering [Gilyén et al. STOC (2019)]:

$$A = H, \quad \Theta(x - \epsilon) \text{ Heaviside step function}$$

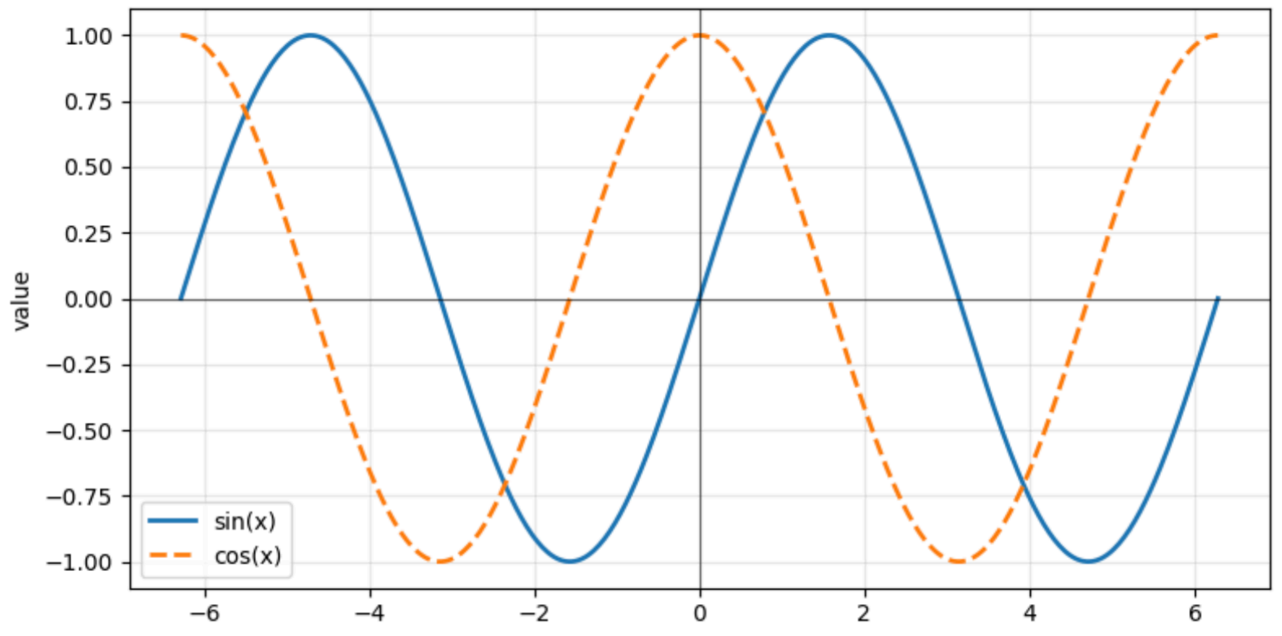
Gibbs state preparation [Gilyén et al. STOC (2019)]:

$$A = H, \quad P(x) = e^{-\beta x}$$

Matrix inverse [Child et al. SIAM'17; Gilyén et al. STOC (2019)]:

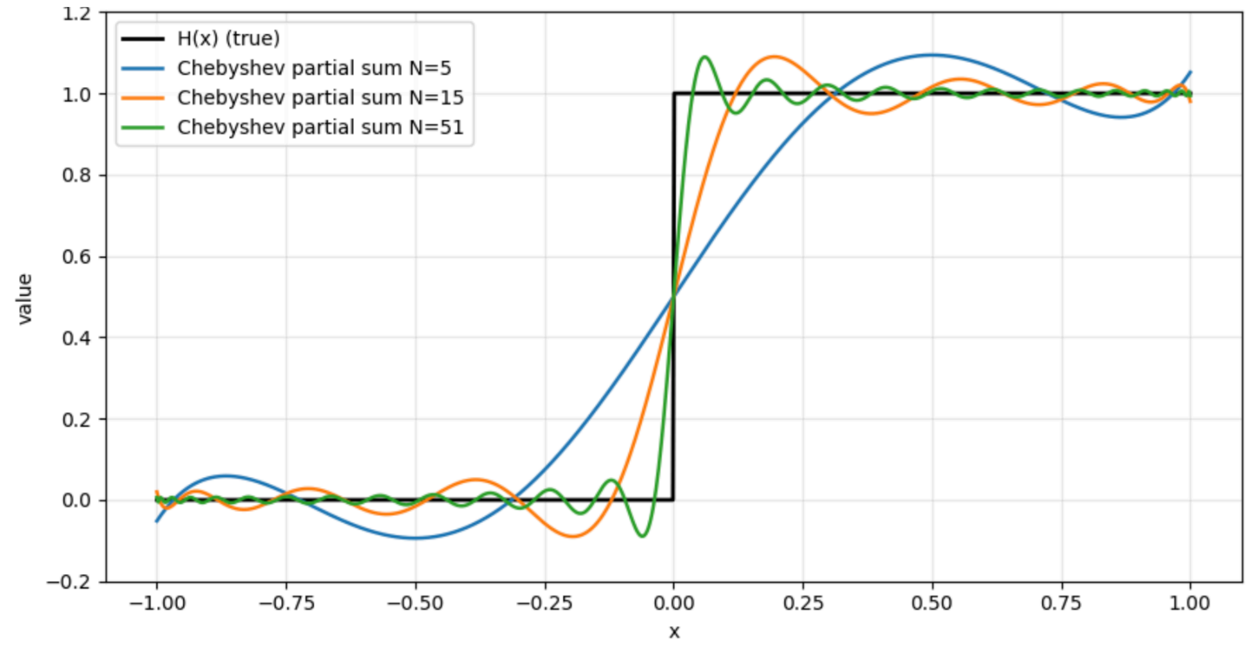
$$A, \quad P(x) = 1/x$$

See John M. Martyn et al. *"Grand unification of quantum algorithms."* PRX Quantum (2021)



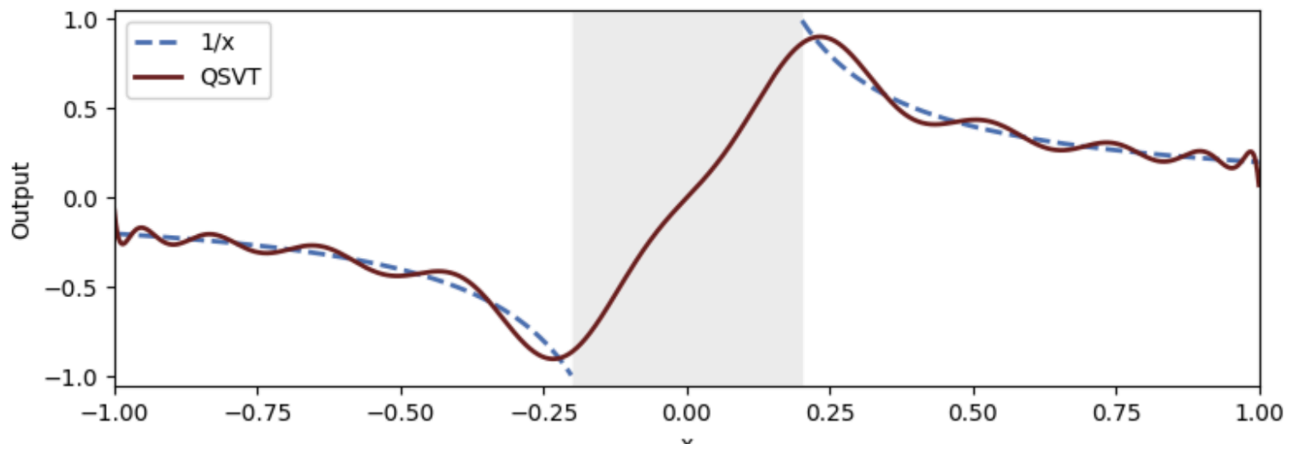
exponential

$e^{-ix} = \cos x + i \sin x$



Heaviside step function

$\Theta(x)$



inverse

$1/x$

Definition of Higher order OTOCs

Let U be a unitary operator, B_i and M_j be local hermitian and unitary (i.e., Pauli) operators acting on ith and jth sites, respectively.

Echo operator:

$$C_{i,j} = U^\dagger B_i U M_j$$

The k-th order OTOC is defined by

$$\mathrm{OTOC}^{(k)}_{i,j} = \langle \psi_{\mathrm{ref}} | C^{2k}_{i,j} | \psi_{\mathrm{ref}} \rangle,$$

where $|\psi_{\mathrm{ref}}\rangle = |0^N\rangle$ for simplicity.

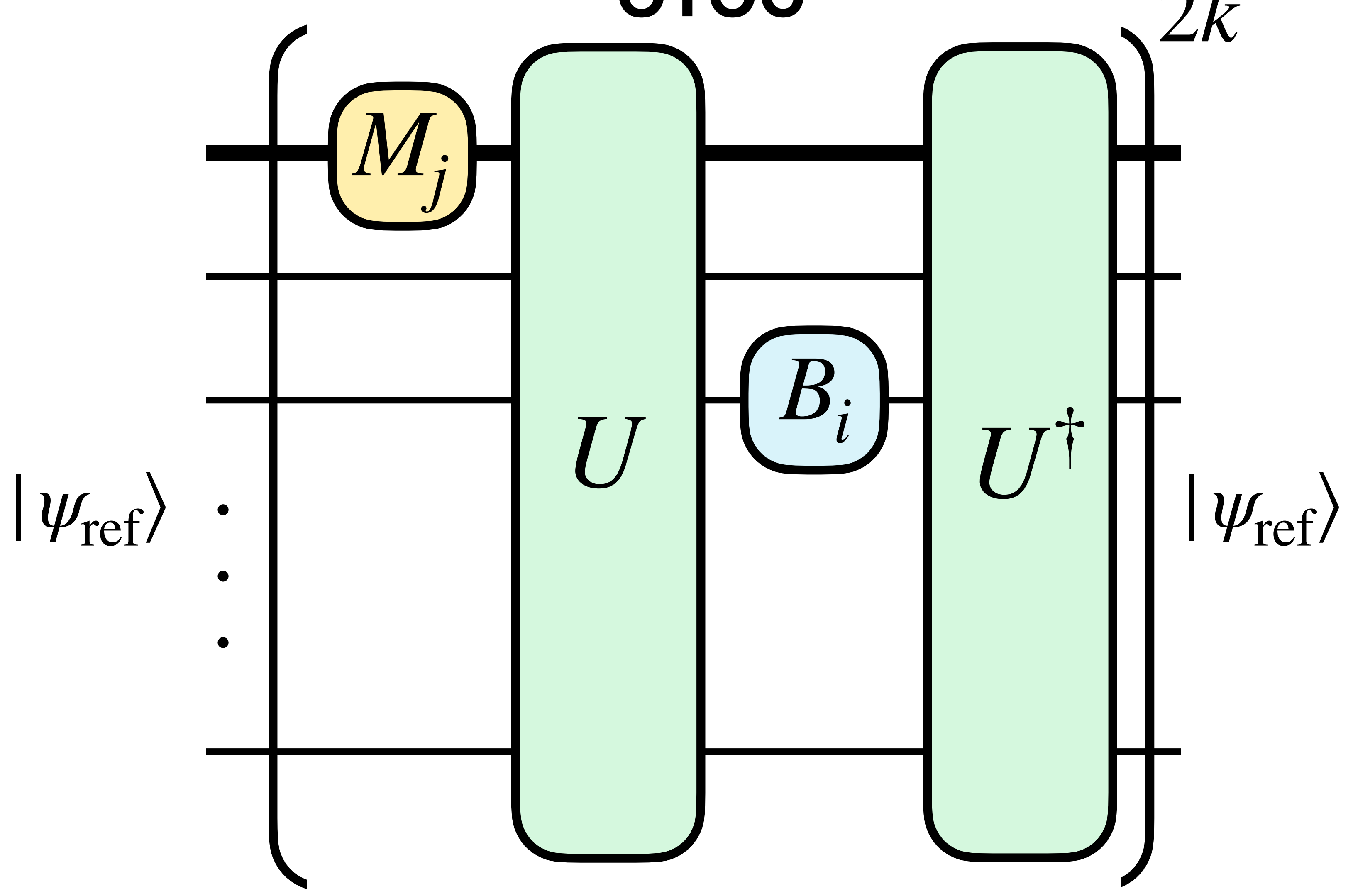
Specifically, standard OTOC is given by $k = 1$:

$$\mathrm{OTOC}^{(1)}_{i,j} = \langle \psi_{\mathrm{ref}} | \left(U^\dagger B_i U M_j \right)^2 | \psi_{\mathrm{ref}} \rangle$$

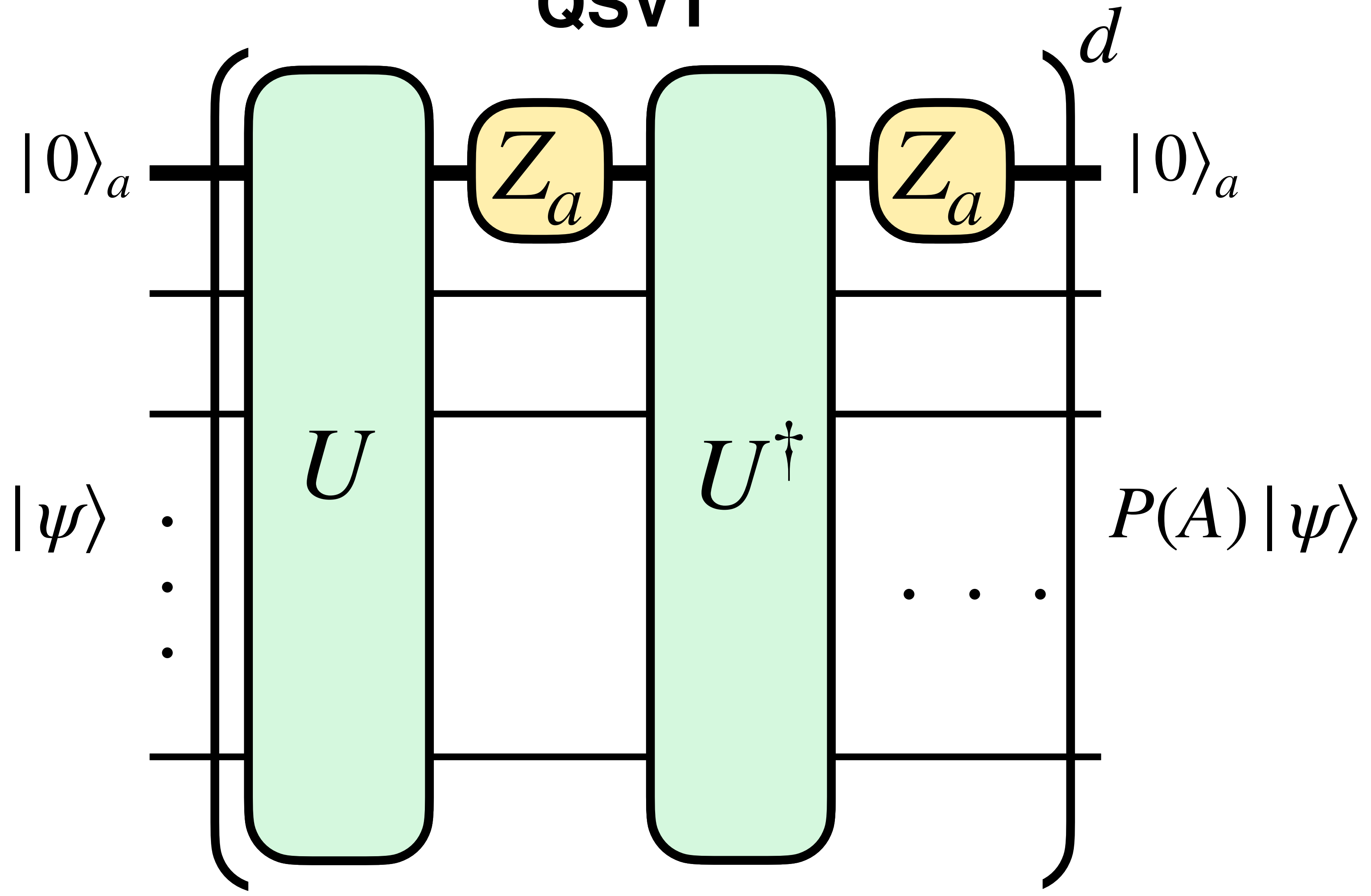
For $k = 1/2$, it reduces to time order correlator:

$$\mathrm{OTOC}^{(1/2)}_{i,j} = \langle \psi_{\mathrm{ref}} | \left(U^\dagger B_i U M_j \right) | \psi_{\mathrm{ref}} \rangle$$

OTOC



QSVT



Outline

- Introduction to Quantum Computing and Progress Toward Quantum Advantage
- What is OTOC
- What is Quantum Singular Value Transformation
- Quantum Algorithmic Understanding of the higher order OTOC
- Summary

Spatially Resolved Truncated Propagator

Let us define a spatially resolved truncated propagator:

$$A_{i,j} = (\langle 0_i | \otimes I_{\text{rest}})\, U \,(|0_j\rangle \otimes I_{\text{rest}}).$$

Equivalently, by introducing swap operator and defining $\tilde{U} = US_{i\leftrightarrow j}$

$$A_{i,j} = (\langle 0_i | \otimes I_{\text{rest}})\, \tilde{U} \,(|0_i\rangle \otimes I_{\text{rest}}).$$

Now we can regard $\tilde{U}$ as a block-encoding of A i.e.,

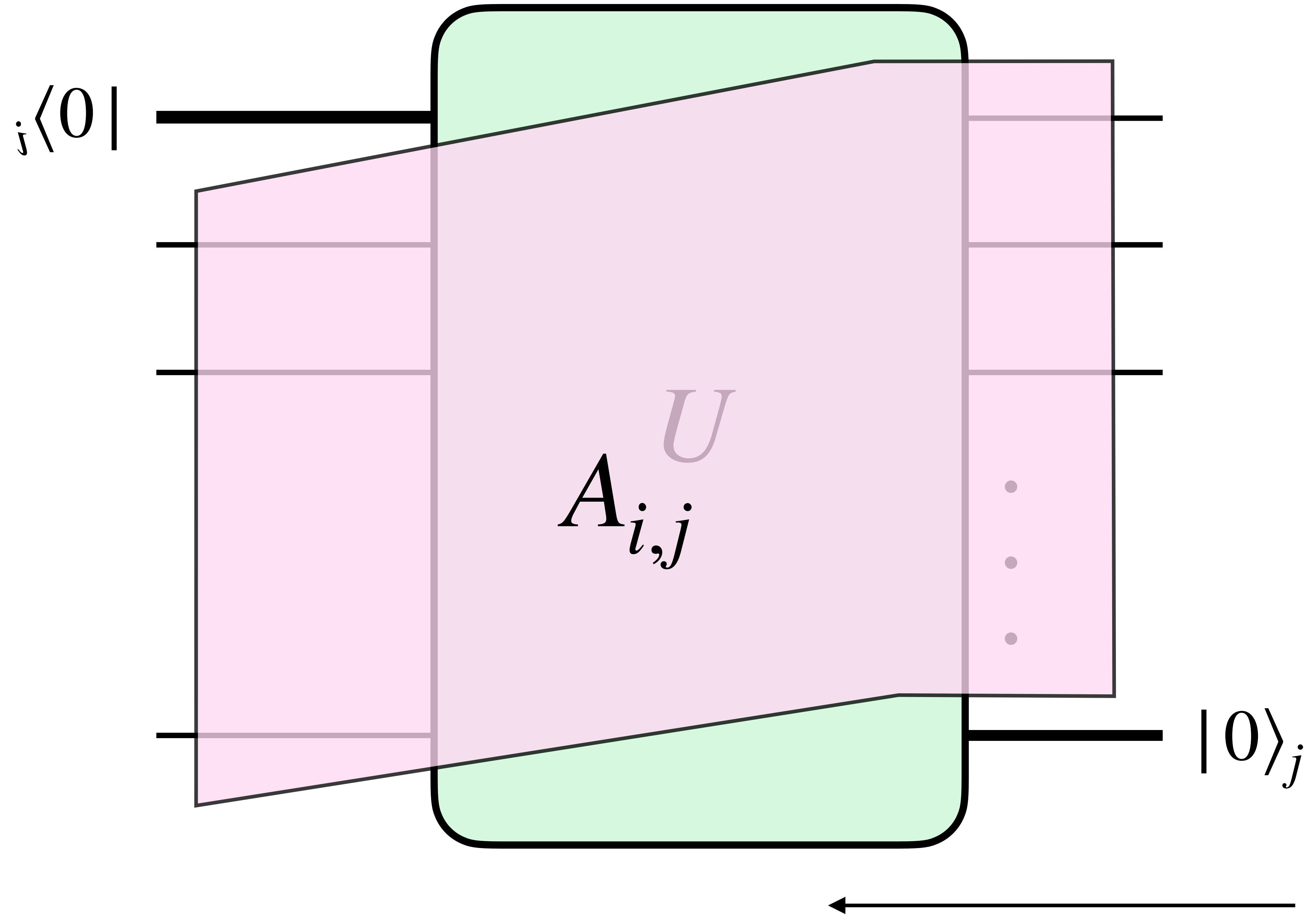
$$\tilde{U} = \begin{pmatrix} A & B' \\ C' & D \end{pmatrix}, \qquad A = \langle 0_i | \tilde{U} | 0_i \rangle .$$

Then OTOC can be rewritten as

$$\text{OTOC}^{(k)}_{i,j} = \langle \psi_{\text{ref}} | \left(\tilde{U}^\dagger B_i \tilde{U} M_i \right)^{2k} | \psi_{\text{ref}} \rangle.$$

→**Qubitization and Quantum Signal Processing!!**

Spatially resolved Truncated Propagator



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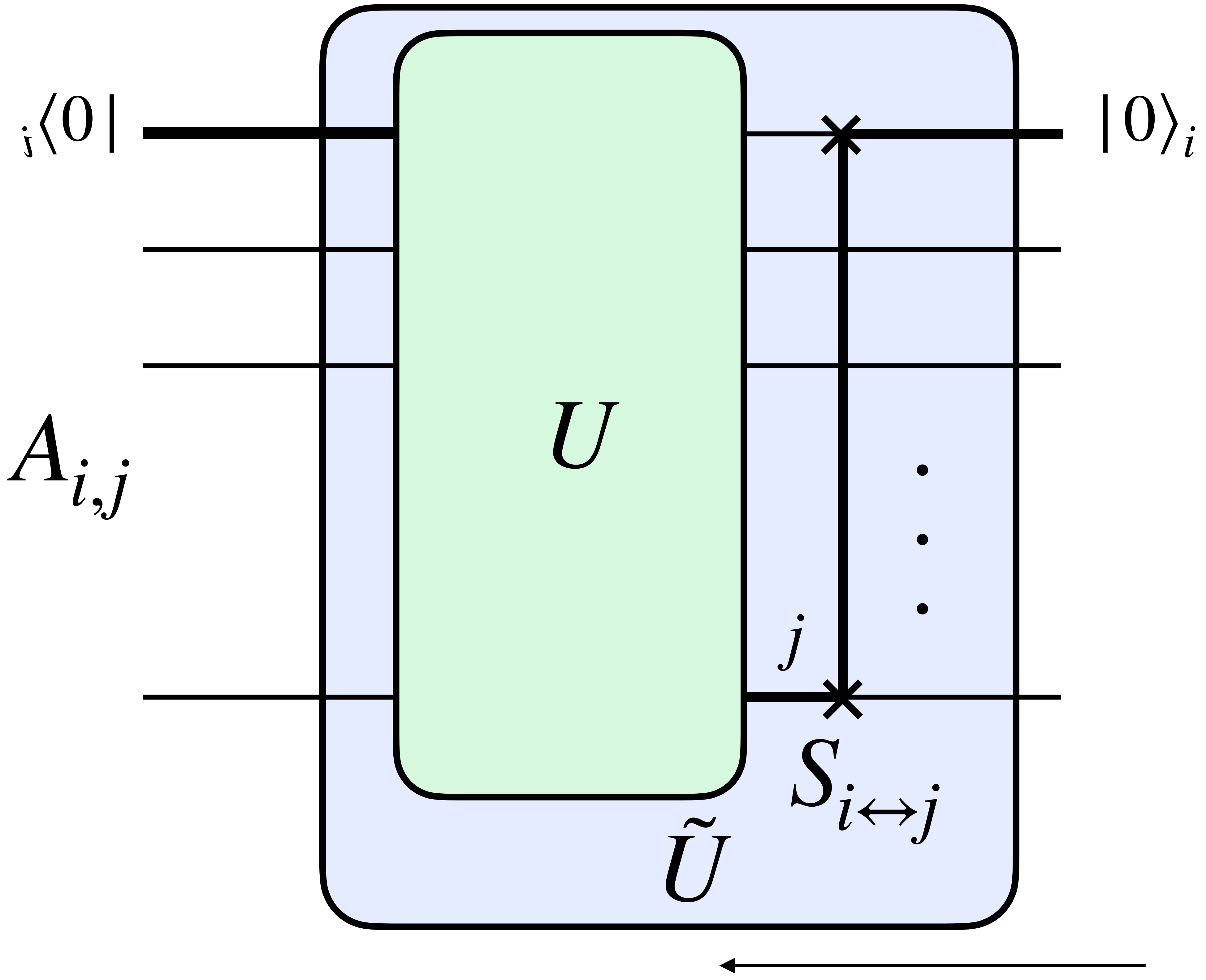

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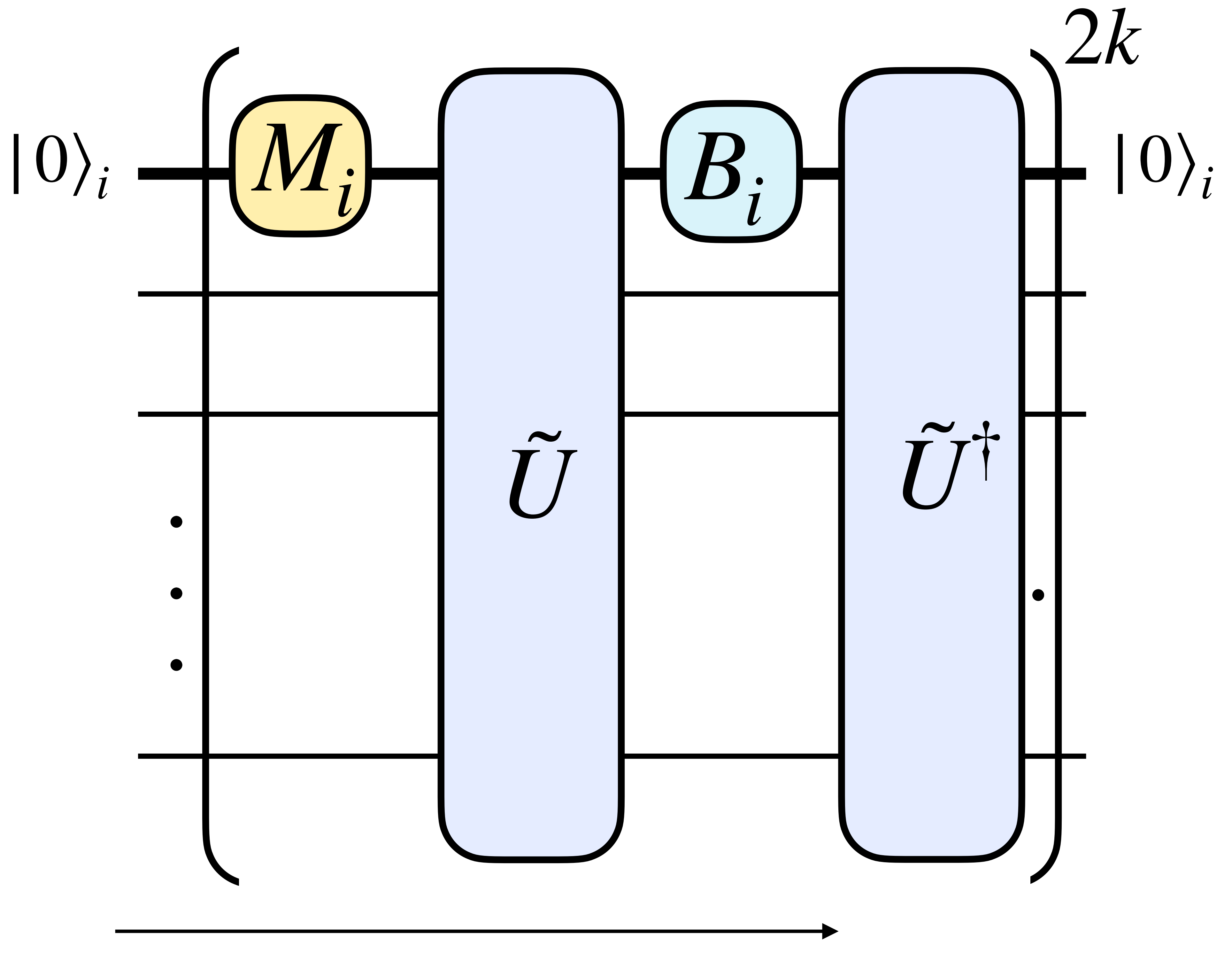


QSVT formulation of OTOC

Singular value decomposition of A

$$A\,|\phi_l\rangle = \lambda_l|\psi_l\rangle \quad \text{or} \quad A = \sum_l \lambda_l |\psi_l\rangle\langle\phi_l|,$$

where we define a phase of a singular value:

$$\lambda_l = \cos(\theta_l/2), \quad \theta_l \in [0,\pi].$$

The action of $\tilde{U}$ on the invariant subspace (qubitized space):

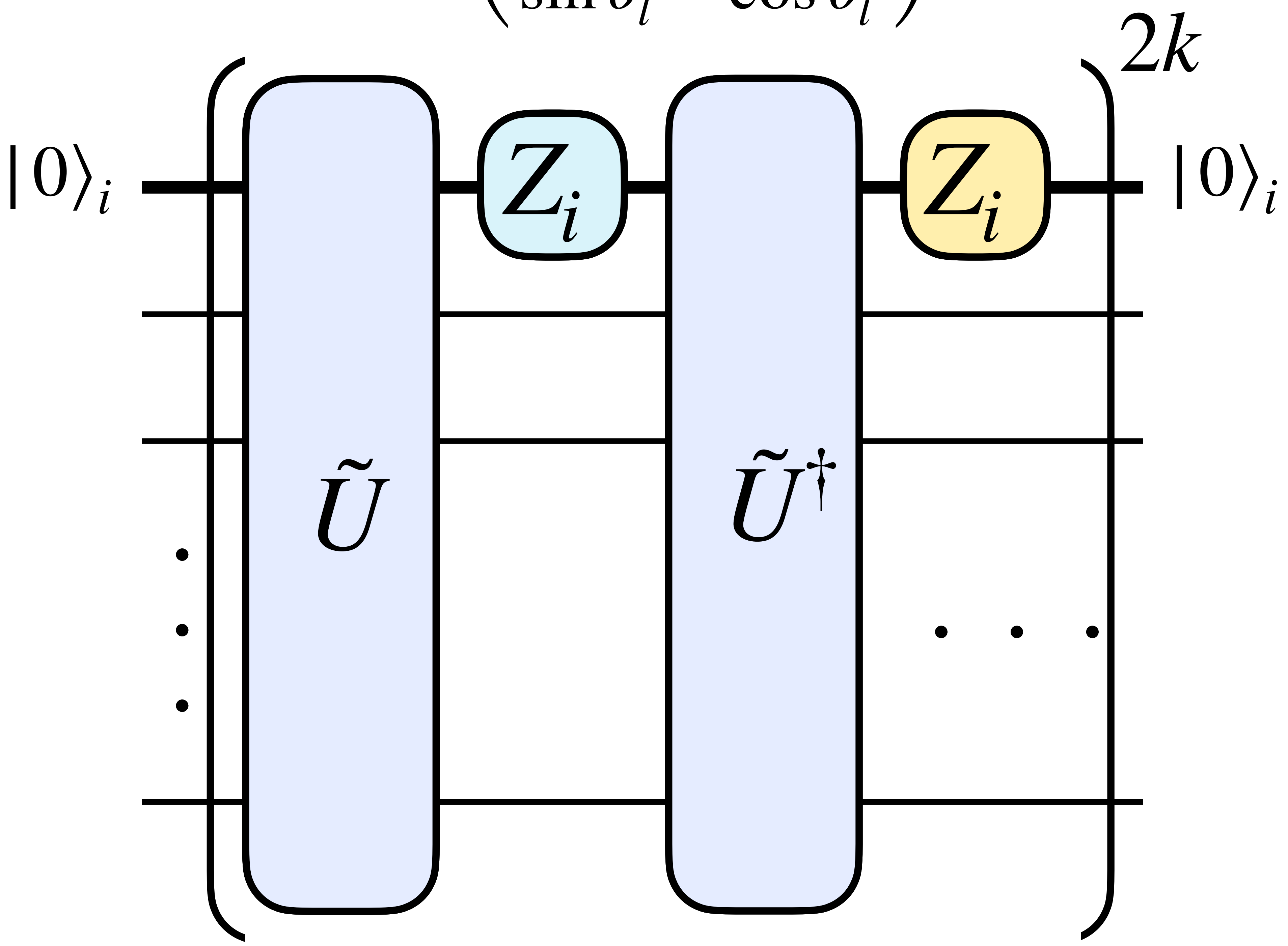
$$\tilde{U}_{\lambda_l} = Ze^{-i(\theta_l/2)Y}$$

Suppose $B = M = Z$,

$$C_{\lambda_l} = e^{-i\theta_l Y}, \qquad C_{\lambda_l}^{2k} = e^{-i2k\theta_l Y}.$$

→**Qubitization and Quantum Signal Processing!!**

$$C_{\lambda_l} = \begin{pmatrix} \cos\theta_l & -\sin\theta_l \\ \sin\theta_l & \cos\theta_l \end{pmatrix}$$



$$\langle 0|_i C_{\lambda_l}^{2k}|0\rangle_i = \sum_l \cos(2k\theta_l)|\phi_l\rangle\langle\phi_l|$$

$$= \sum_l T_{4k}(\lambda_l)|\phi_l\rangle\langle\phi_l|$$

QSVT formulation of OTOC

Theorem: Let U be the given dynamics and take $B_i = Z_i$ and $M_j = Z_j$ on the probe qubit.

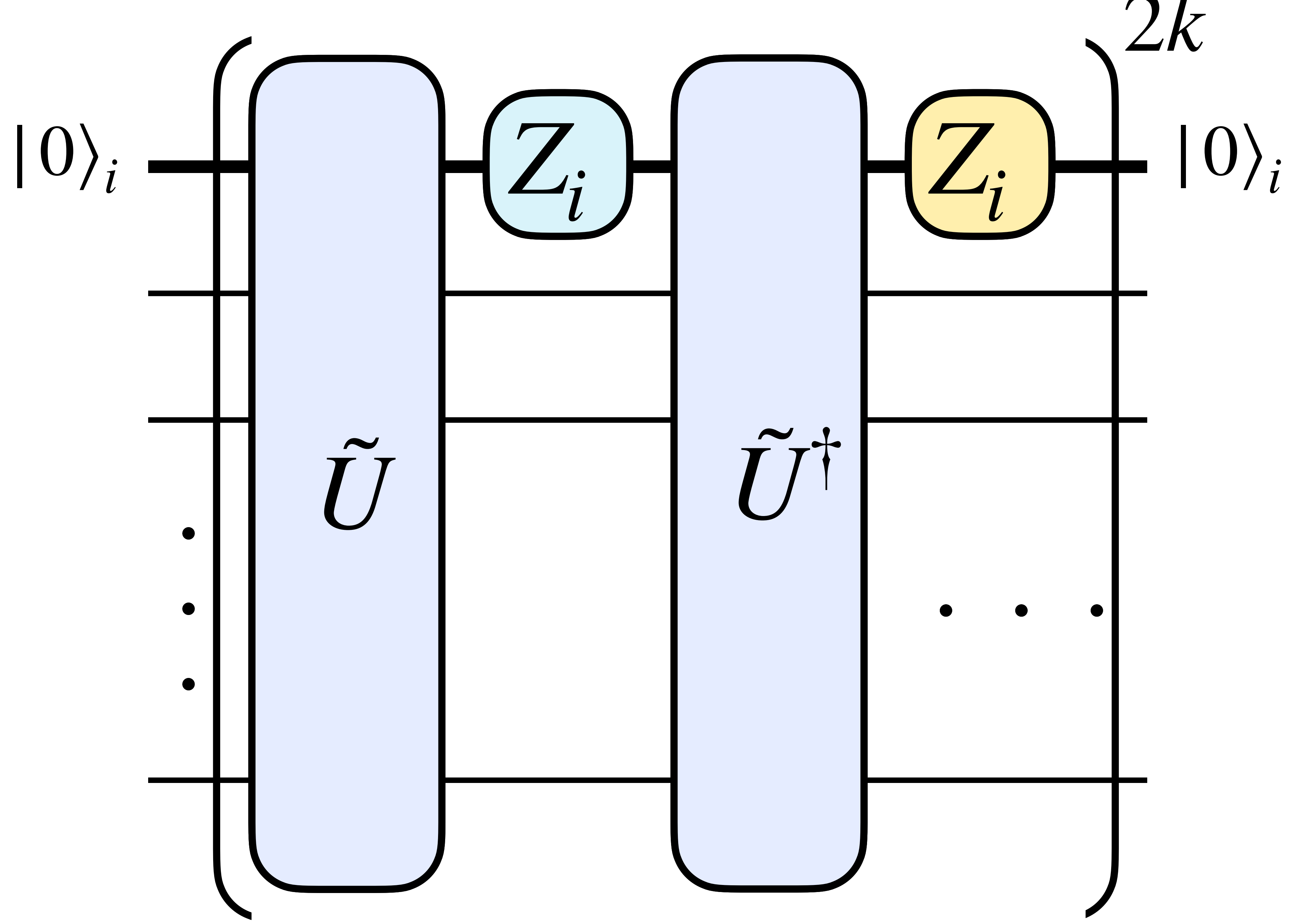
Then the k-th-order OTOC associated with this echo sequence satisfies

$$\mathrm{OTOC}^{(k)}(U, Z_i, Z_j) = \sum_l |\alpha_l|^2 \cos(2k\theta_l)$$
$$= \sum_l |\alpha_l|^2 T_{4k}(\lambda_l),$$

where λ_l are the singular values of $A = \langle 0_i | \tilde{U} | 0_i \rangle$ with $\tilde{U} = U S_{j\leftrightarrow i}$ (or equivalently $A_{i,j} = \langle 0_i | U | 0_j \rangle$) and $\alpha_l = \big(\langle 0_i | \otimes \langle \phi_l |\big) | 0^n \rangle$.

The k-th order OTOC is the $4k$ order Chebyshev moment of singular value λ and $2k$-th Fourier mode of phase θ of the spatially resolved truncated propagator.

$$C_{\lambda_l} = \begin{pmatrix} \cos\theta_l & -\sin\theta_l \\ \sin\theta_l & \cos\theta_l \end{pmatrix}$$



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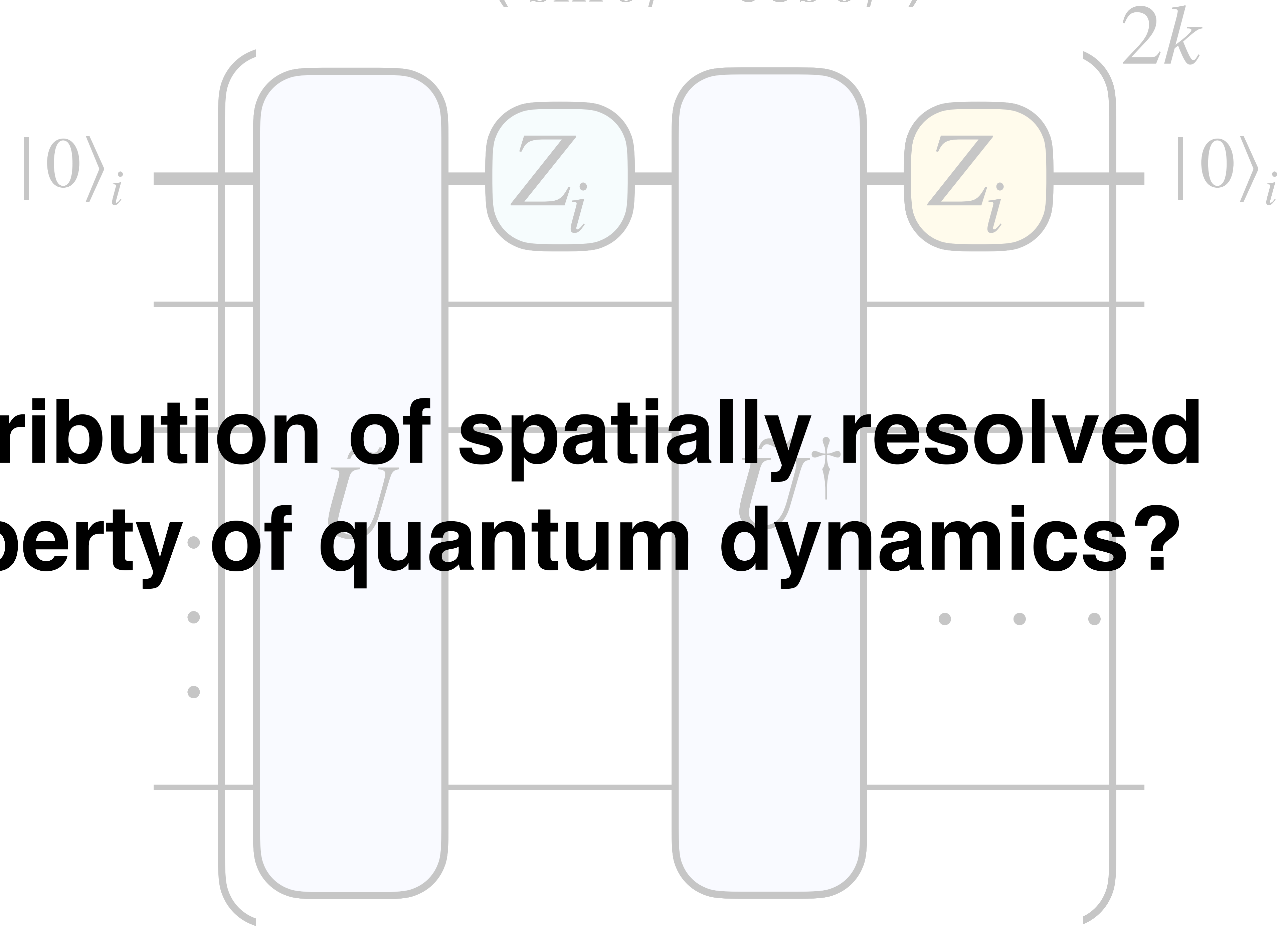


$$\langle 0 |_i C_{\lambda_l}^{2k} | 0 \rangle_i = \sum_l \cos(2k\theta_l) | \phi_l \rangle \langle \phi_l |$$

$$= \sum_l T_{4k}(\lambda_l) | \phi_l \rangle \langle \phi_l |$$

It is natural to ask, singular value distribution of spatially resolved truncated propagator encode the property of quantum dynamics?

Singular value (phase) distribution of Spatially Resolved Truncated Propagator

Phase distribution (singular value distribution)

Define a distribution function of phase $\theta_l^{(i,j)} = 2\arccos\lambda_l^{(i,j)}$:

$$p_{i,j}(\theta) := \sum_l |\alpha_l^{(i,j)}|^2 \delta\left(\theta - \theta_l^{(i,j)}\right).$$

kth order OTOC:

$$\mathrm{OTOC}^{(k)}(U, Z_i, Z_j) = \int_0^\pi d\theta \cos(2k\theta) p_{i,j}(\theta)$$

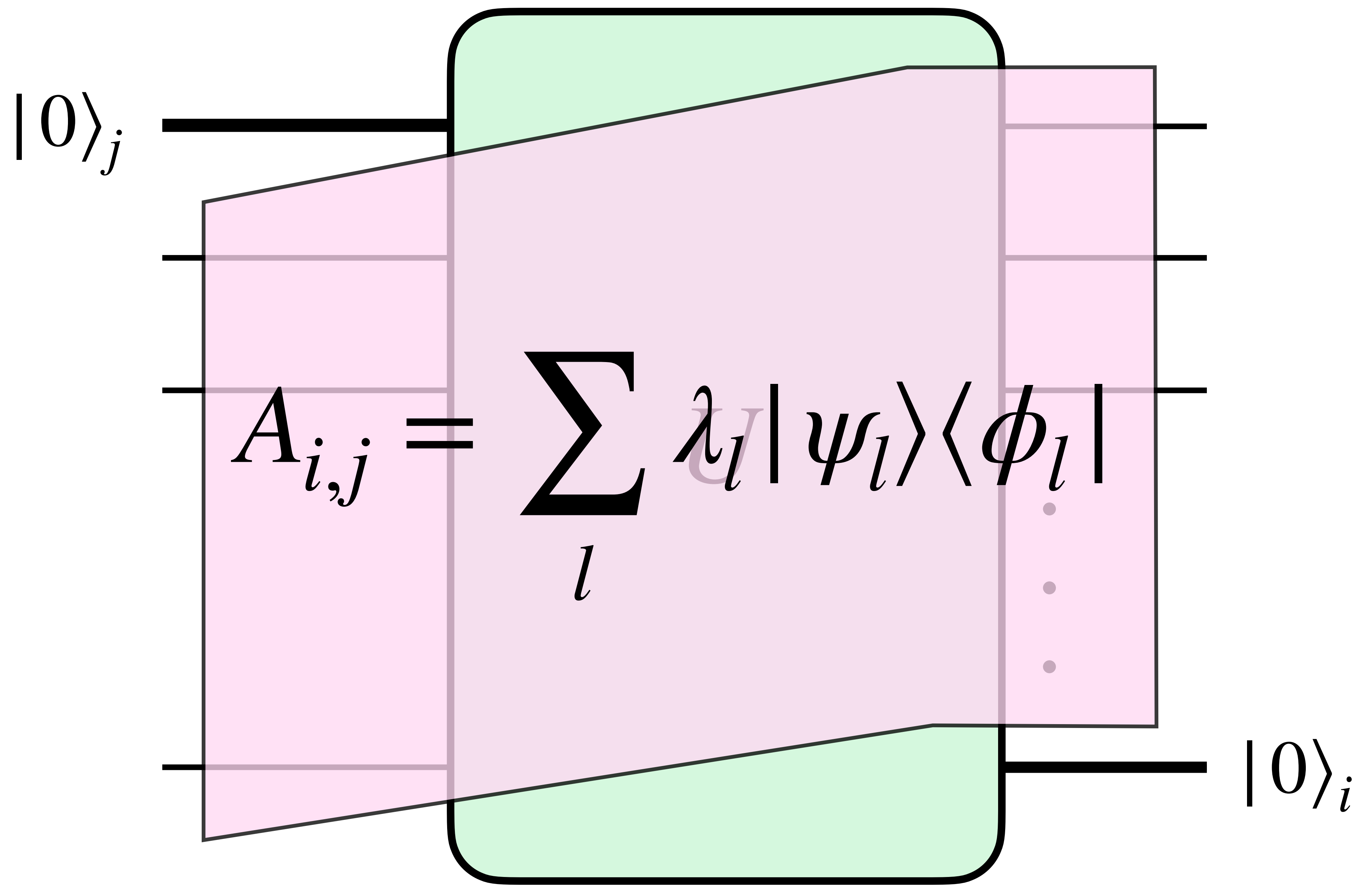
→$2k$th Fourier mode of the phase distribution.

Let us define an intrinsic phase distribution

$$\tilde{p}_{i,j}(\theta) = \sum_l \delta(\theta - \theta_l^{(i,j)}),$$

which is determined only by $A_{i,j}$ (w/o depending on $|\psi_{\mathrm{ref}}\rangle$).

Spatially resolved Truncated Propagator



$0 \le \lambda_l \le 1$

information transmission rate

Phase distributions and sensitivities of OTOC

- Haar random unitary U:

$$\langle 0_i | U | 0_j \rangle$$: truncated Haar random unitary[1,2,3]

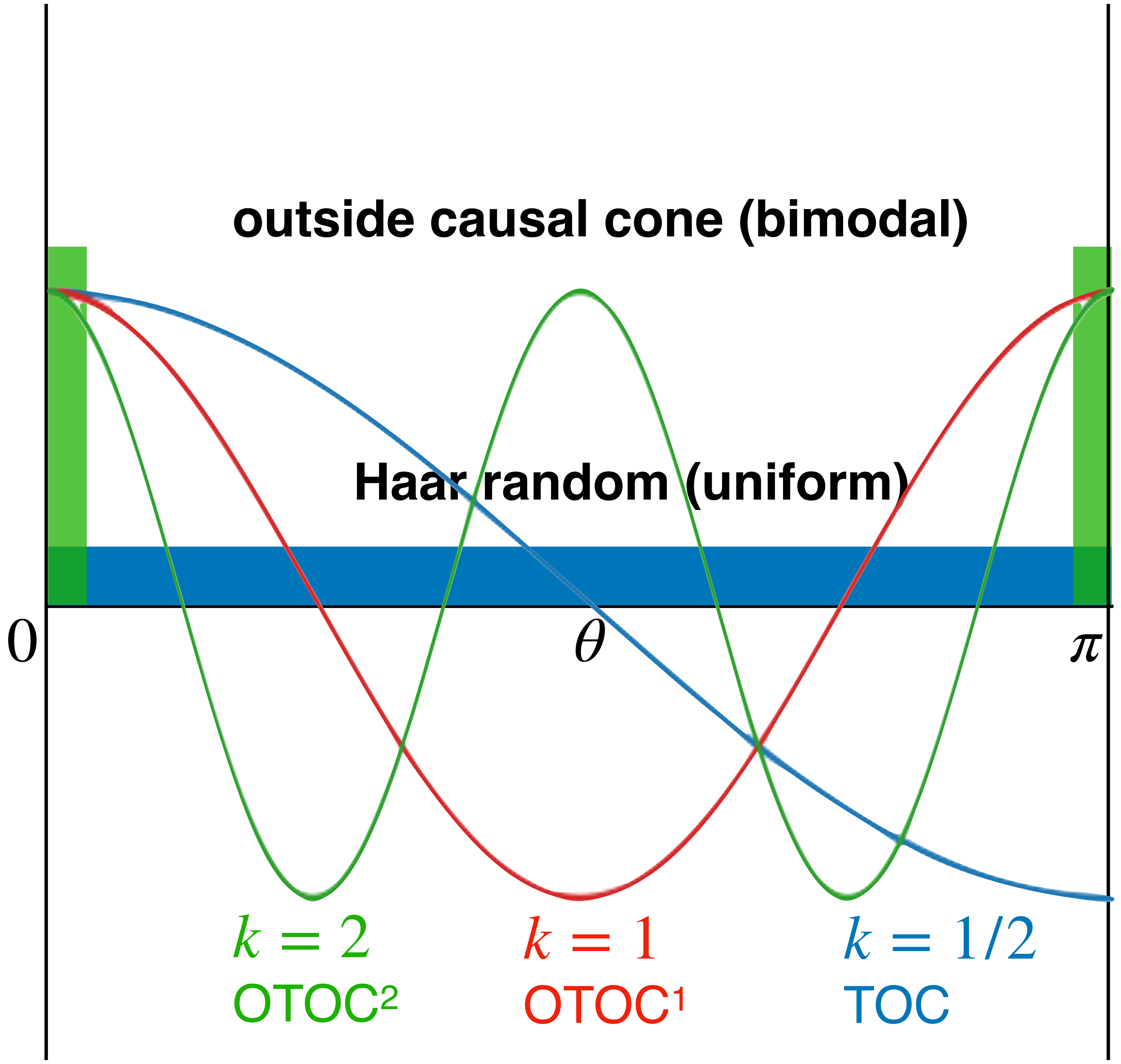
$$\tilde{p}_{i,j}(\theta) = \frac{1}{\pi} \quad \text{(uniform)}$$

- Outside causal cone for $U(t) = e^{-iHt}$:

$$A_{i,j}A_{i,j}^\dagger = \langle 0_i | U(t) P_j U^\dagger(t) | 0_i \rangle$$

If $[P_j(t), P_i] \simeq 0$, $A_{i,j}A_{i,j}^\dagger \simeq P_j(t) \otimes I_{\text{rest}}$, so the singular values or phases are bimodal:

$$\lambda = 0,1, \quad \theta = 0,\pi \qquad \text{(bimodal)}$$



$\mathrm{OTOC}^{(k)}(U, Z_i, Z_j) \to 2k$th Fourier mode

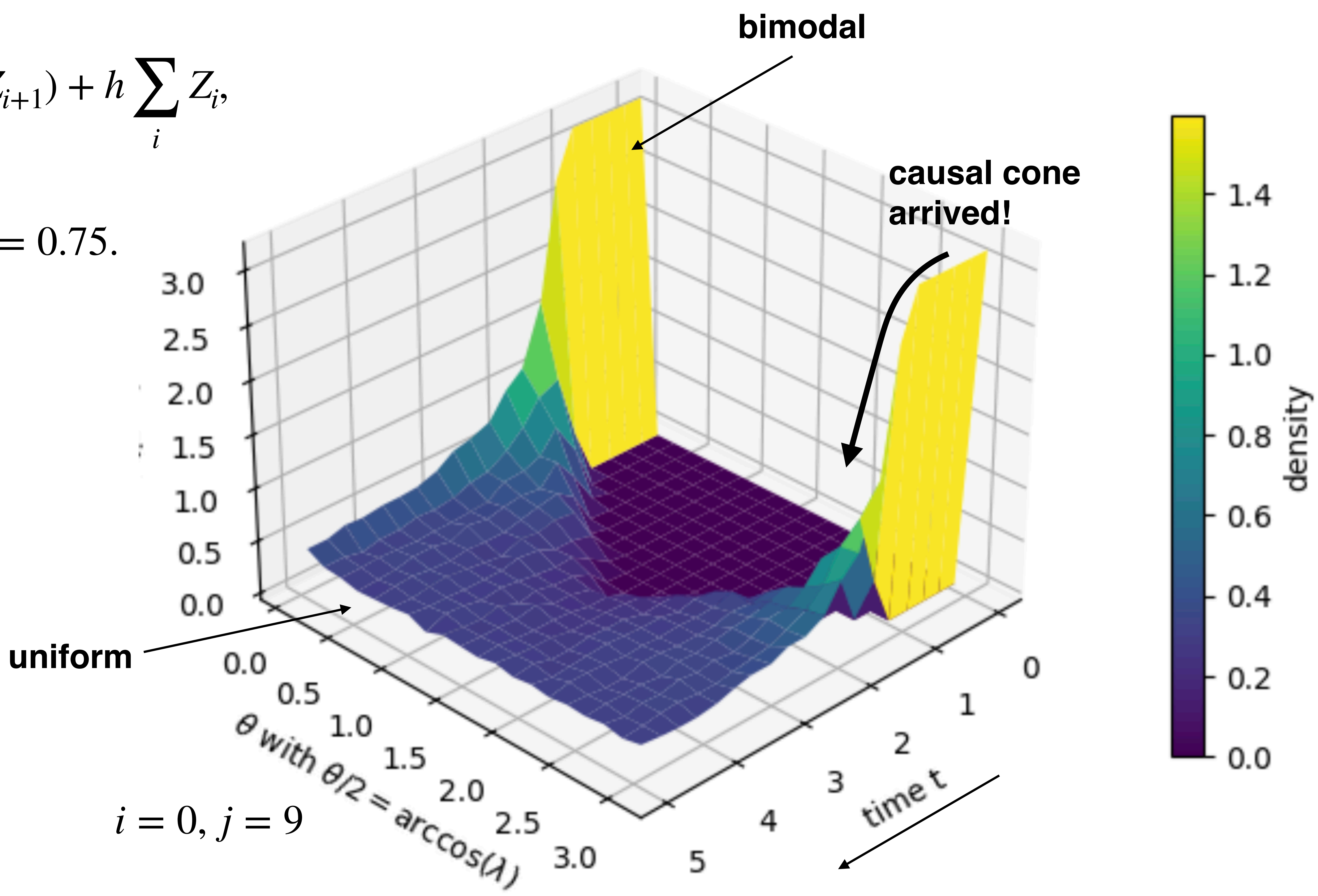
[1] K Zyczkowski, HJ Sommers - Journal of Physics A: Mathematical and General, 2000
[2] B Collins - Probability theory and related fields, 2005
[3] Z Dong, T Jiang, D Li - Journal of Mathematical Physics, 2012

Phase distributions and characteristics of Hamiltonian

Chaotic Hamiltonian:

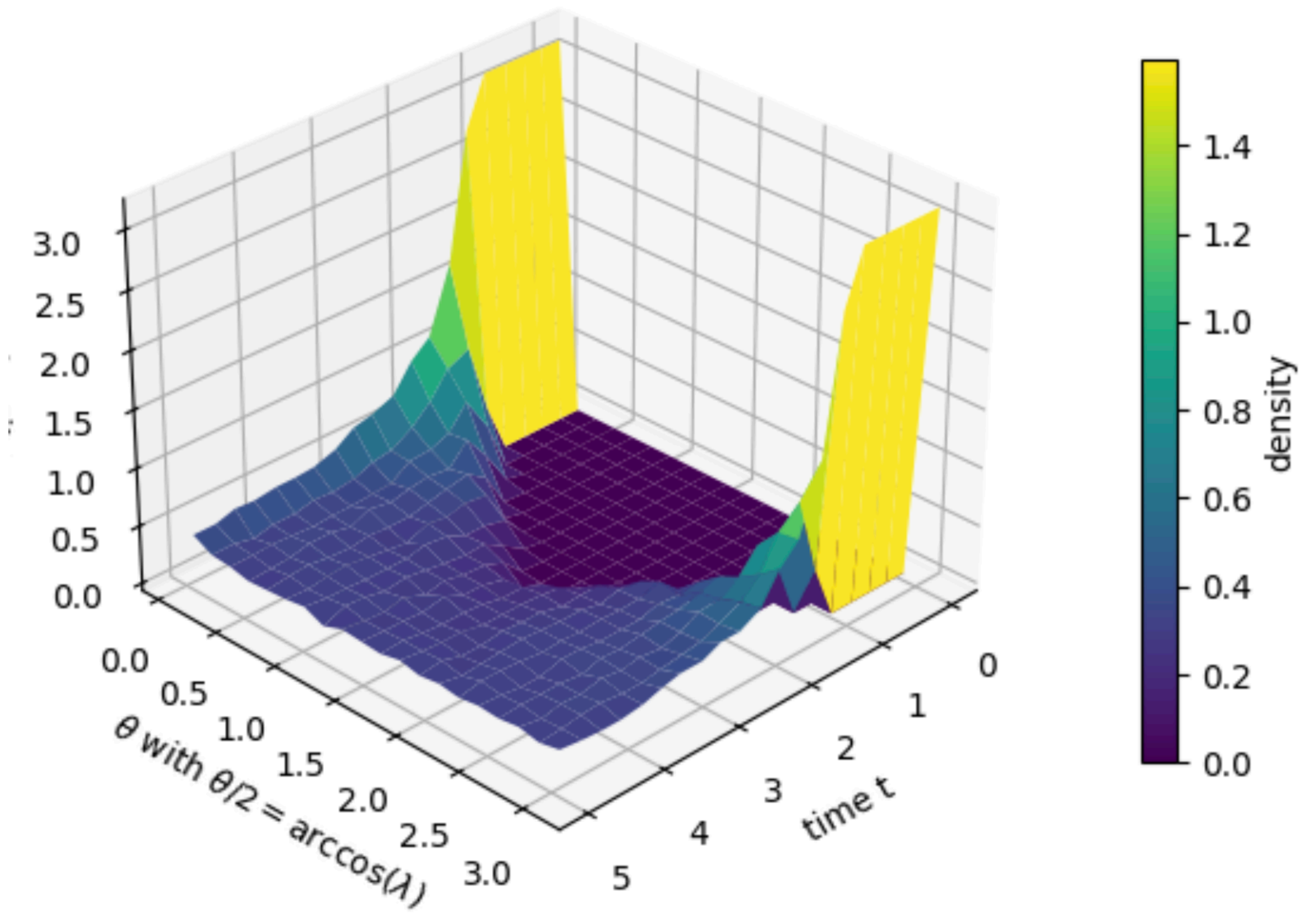
$$H_{\text{chaos}} = \sum_i (J_x X_i X_{i+1} + J_y Y_i Y_{i+1} + J_z Z_i Z_{i+1}) + h \sum_i Z_i,$$

where we specifically choose
$J_x = -0.4, J_y = -2.0, J_z = -1.0$, and $h = 0.75$.

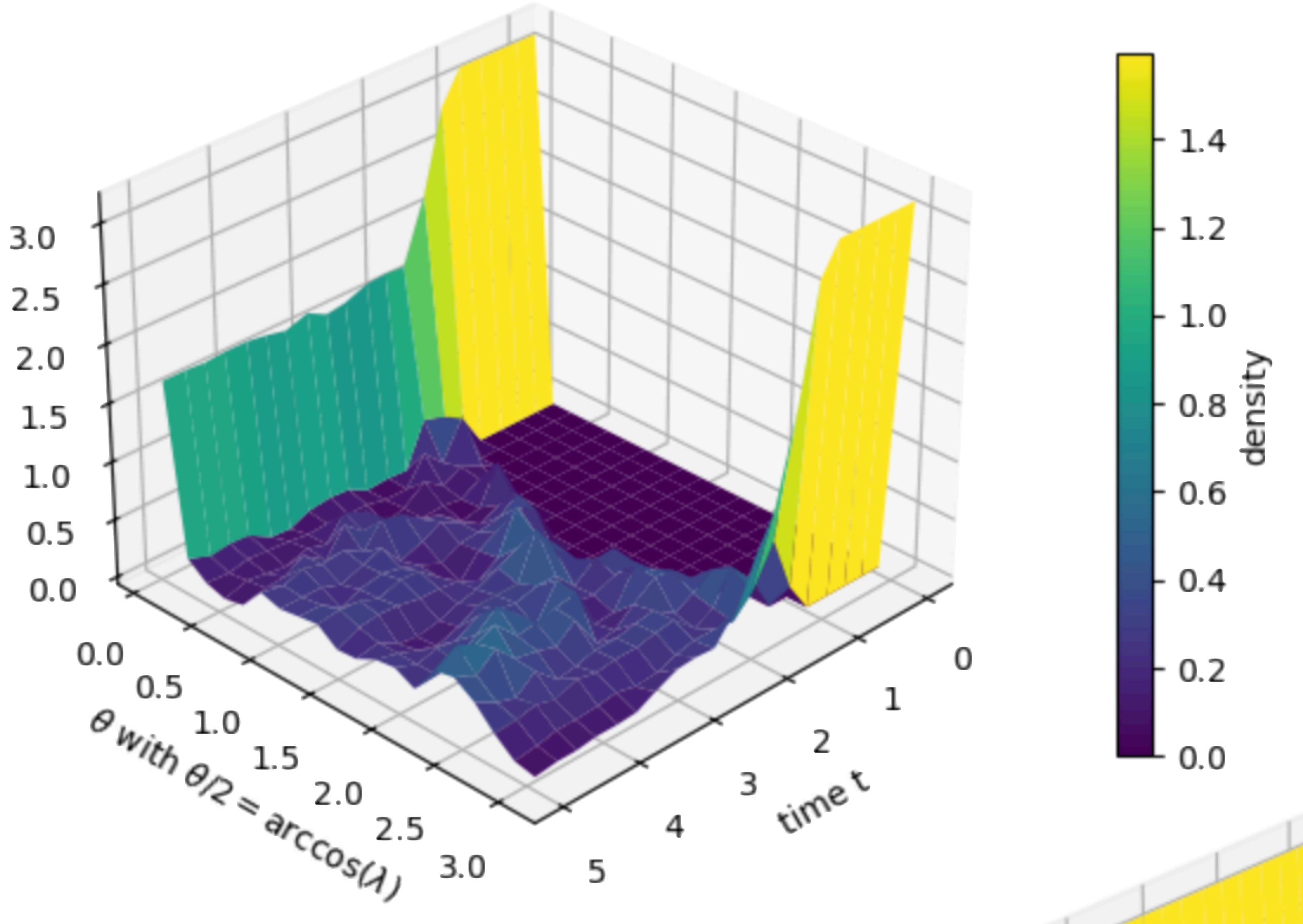


$i = 0,\ j = 9$

Phase distributions and characteristics of Hamiltonian

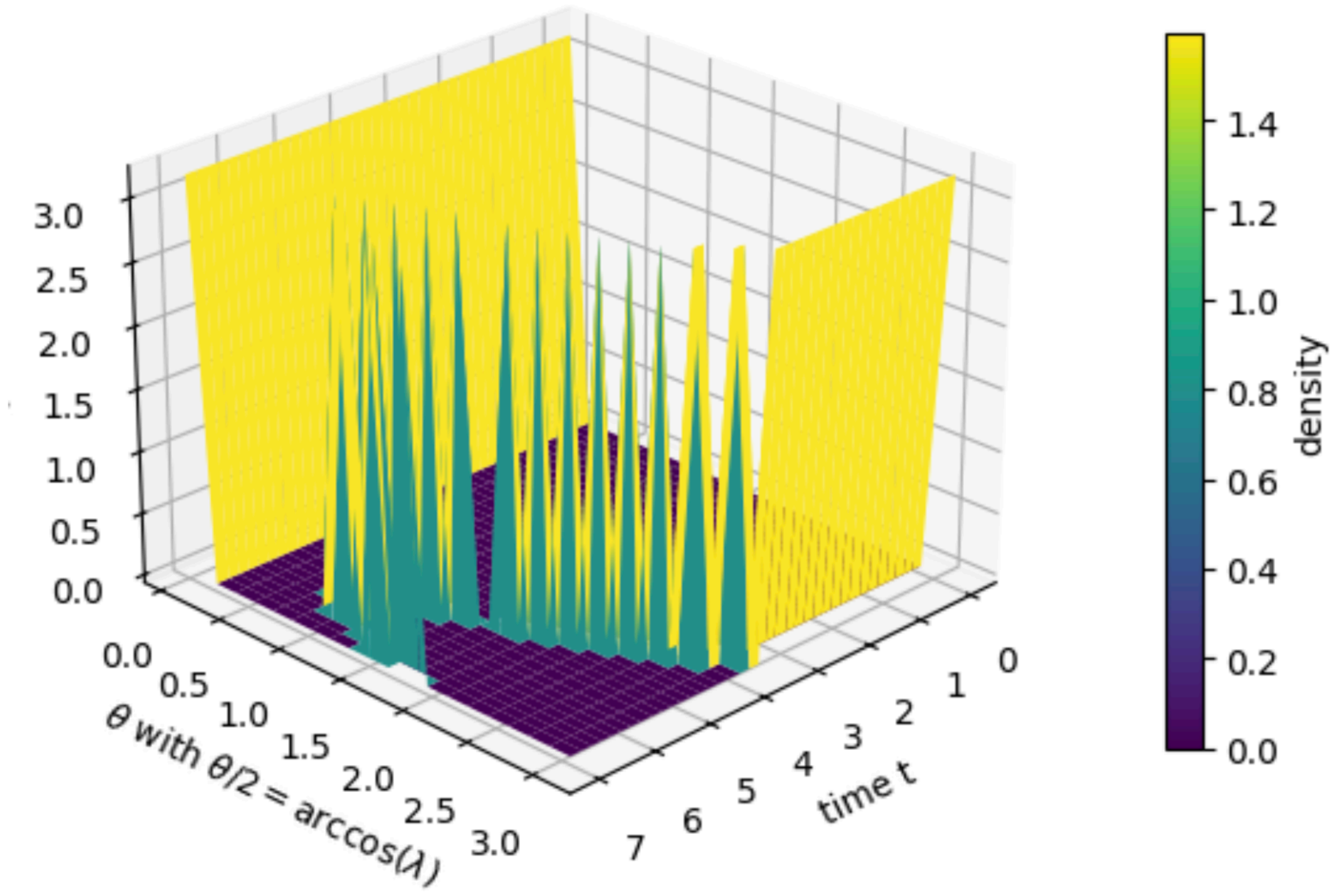


Chaotic

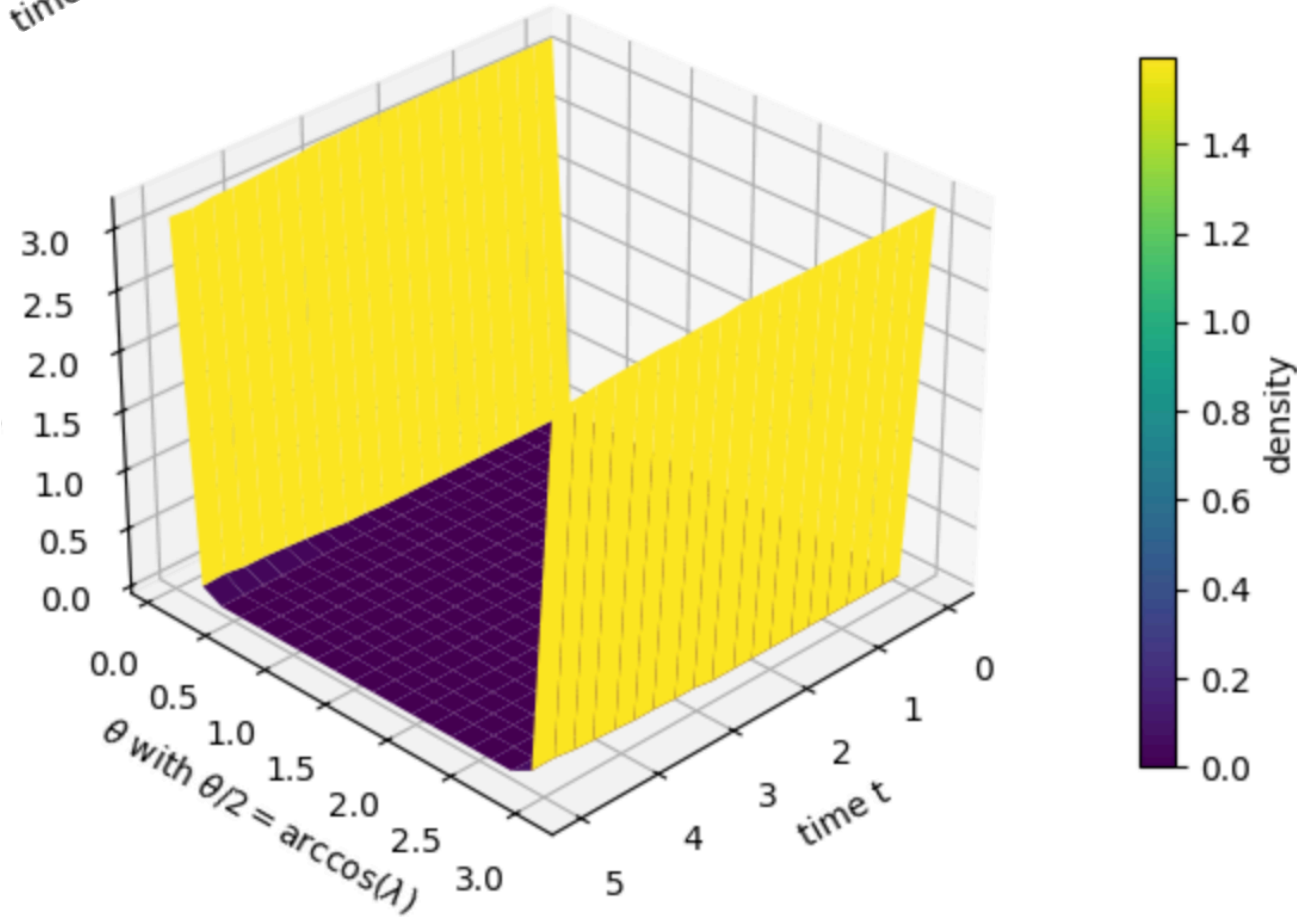


Integrable
(Bethe ansatz)

Integrable
(free-fermion)



MBL



Behavior of OTOC$^{(k)}$ (Chebyshev moments)

Chaotic Hamiltonian:

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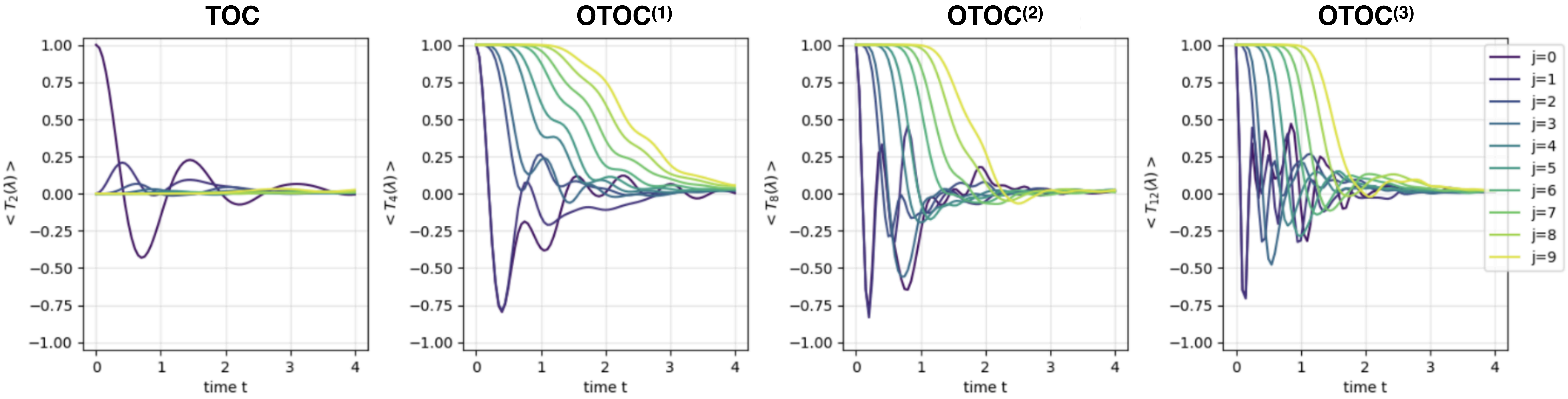


The higher order OTOC can sensitively detect change of phase distribution from bimodal to uniform.

Generalization: OTOC spectroscopy

Generalization: OTOC spectroscopy

Let U be the given dynamics and take $B_i = Z_i$ and $M_j = Z_j$ on the probe qubit.

A generalized echo operator:

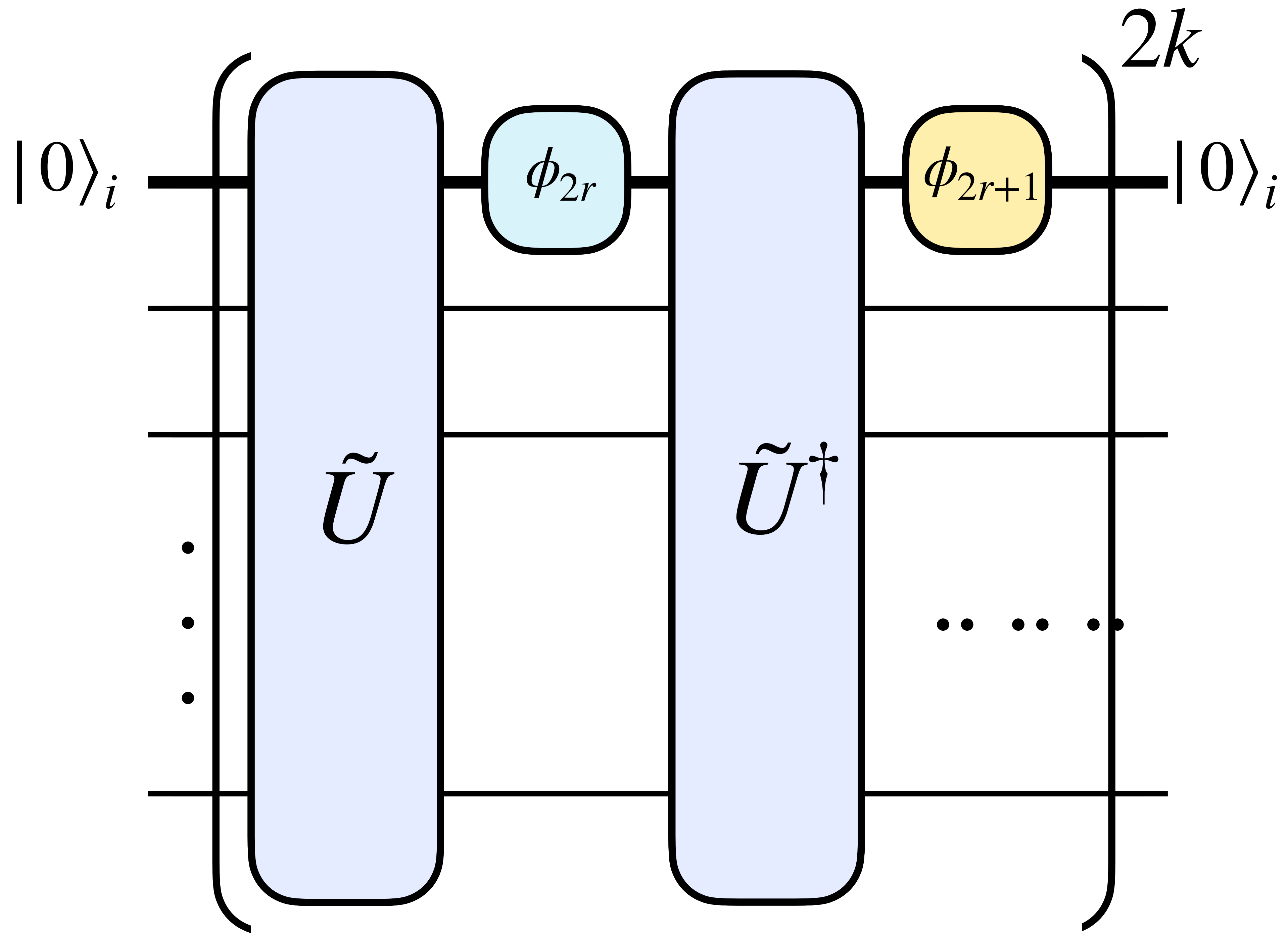
$$C^{(r)}_{i,j}(t) := Z_i(\phi_{2r+1})\, U^\dagger(t)\, Z_j(\phi_{2r}) U(t)$$

Then the k-th-order OTOC spectroscopy is defined by

$$\text{QSP-OTOC}^{(d)}_{i,j}(t, \vec{\phi}) := \langle \psi_{\text{ref}} | Z_i(\phi_{2d}) \prod_{r=1}^{d} C^{(r)}_{i,j}(t) | \psi_{\text{ref}} \rangle$$

$$= \sum_l |\alpha_l^{(i,j)}|^2 P_{d,\vec{\phi}}\left(\lambda_l^{(i,j)}(t)\right)$$

By tuning the probe angles, one can implement a polynomial transformation of degree 2d acting on the singular values, enabling the design of filters that selectively target specific singular values. → OTOC spectroscopy



$$\langle 0 |_i Z_i(\phi_{2d}) \prod_{r=1}^{d} C^{(r)}_{i,j}(t) | 0 \rangle_i = P_{d,\vec{\phi}}\left(\lambda_l^{(i,j)}(t)\right) |\phi_l\rangle\langle\phi_l|$$

Summary

- We are entering a stage where quantum advantage can be demonstrated by quantum computers, in the sense of showing superiority in computing meaningful quantities.
- However, for expectation value estimation, it is crucial to understand how far correlations spread across the system. Spatial and temporal evolution of correlations, as well as the detection of the edge of chaos via OTOCs, play a key role.
- Higher-order OTOCs, which are highly sensitive probes of the edge of chaos, can be cleanly understood within the framework of QSVT.
- By leveraging Quantum Signal Processing, one can apply specific filters to selectively probe characteristic features of quantum dynamics.
- This provides a unified framework connecting the understanding of quantum chaos and scrambling, advanced quantum algorithms, and the experimental verification of quantum advantage on real quantum hardware.