Cosmological model with wormhole and cosmological horizons

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Motivation

- Cosmological black hole
  – More practical solution, especially for expanding universe
  – Several solution for black hole embedded in expanding universe
  – Unification of global and local physics
- Cosmological wormhole
  – Exotic matter for wormhole
  – Wormhole can be generated in the very early universe and expands to macroscopic scale.
  – Wormhole can have a role in early universe or expansion of the universe
  – We need exact solution satisfying Einstein Field Eq.
  – What is happening in the universe with wormhole?

Models

- Black Hole Cosmology
  - Kottler (1918): SdS
  - McVittie (1933): black ole in FLRW
  - Sultana-Dyer (2005): conformal transform of Schwarzschild
  - Faraoni-Jacques (2007): generalized McVittie

- Wormhole Cosmology
  - Hochberg, Kepart (1993): two copies of FLRW & paste
  - Roman (1993): wormhole in inflation
  - SWK (1996): wormhole in FLRW
  - Mirza, Eshaghi, Dehdashti (2006): wormhole in FLRW

Isotropic Wormhole solution 1

- Morris-Thorne type wormhole

$$ds^2 = e^{2\Phi}dt^2 - \frac{1}{1-b(r)/r}dr^2 - r^2 d\Omega^2,$$

- Isotropic form of the solution

$$ds^2 = A^2 d\tilde{t}^2 - B^2(d\tilde{r}^2 + \tilde{r}^2 d\Omega^2).$$

$$A(\tilde{t},\tilde{r}) = e^{\Phi(r)}, \tilde{t} = t, B\tilde{r} = r \qquad B = re^{-\int \frac{dr}{\sqrt{r^2 - b(r)r}}}.$$

- Before the detailed form of $b(r)$ is not known, it is very hard to understand the spacetime structure

Isotropic Wormhole solution 2

- For simple case, $A = 1, b = \frac{{b_0}^2}{r}$ $(r > b_0)$

$$B = \frac{2}{1 + \sqrt{1 - \frac{b_0^2}{r^2}}}$$

$$\tilde{r} = \frac{r}{B} = \frac{1}{2}(r + \sqrt{r^2 - b_0^2}) \qquad (b_0/2 < \tilde{r} < \infty).$$

$$r = \tilde{r} + \frac{b_0^2}{4\tilde{r}},$$

$$B = \frac{r}{\tilde{r}} = \left(1 + \frac{b_0^2}{4\tilde{r}^2}\right)$$

$$ds^2 = d\tilde{t}^2 - \left(1 + \frac{b_0^2}{4\tilde{r}^2}\right)^2 (d\tilde{r}^2 + \tilde{r}^2 d\Omega^2)$$

Isotropic Wormhole solution 3

Matter solutions

Negative energy density

$$\frac{b'}{8\pi r^2} = \rho_w = -\frac{b_0^2}{8\pi r^4}$$
$$\frac{b}{8\pi r^3} = \tau_w = \frac{b_0^2}{8\pi r^4}$$
$$\frac{b - b'r}{16\pi r^3} = P_w = \frac{b_0^2}{8\pi r^4}$$

$$T_{\tilde{\mu}}{}^{\tilde{\nu}} = \frac{\partial x^\alpha}{\partial x^{\tilde{\mu}}}\frac{\partial x^{\tilde{\nu}}}{\partial x^\beta} T_\alpha{}^\beta$$

$$G^0{}_0 = -\frac{4^4 b_0^2 \tilde{r}^4}{(b_0^2 + 4\tilde{r}^2)^4} = 8\pi\rho_w$$
$$G^1{}_1 = \frac{4^4 b_0^2 \tilde{r}^4}{(b_0^2 + 4\tilde{r}^2)^4} = 8\pi\tau_w$$
$$G^2{}_2 = -\frac{4^4 b_0^2 \tilde{r}^4}{(b_0^2 + 4\tilde{r}^2)^4} = 8\pi P_w$$
$$G^3{}_3 = -\frac{4^4 b_0^2 \tilde{r}^4}{(b_0^2 + 4\tilde{r}^2)^4} = 8\pi P_w.$$

ρ_w, τ_w, P_w are wormhole energy density, tension, and pressure

Wormhole solution embedded in FLRW universe 1

- Isotropic form of FRLW universe $$ds^2 = dt^2 - \frac{a^2(t)}{(1+kr^2)^2}(dr^2 + r^2 d\Omega^2)$$

$a(t)$ is the scale factor and $k = 1/4R^2$ is the curvature

- Let's start from the metric form for wormhole in FRLW universe

$$ds^2 = e^{\zeta(r,t)}dt^2 - e^{\nu(r,t)}(dr^2 + r^2 d\Omega^2)$$

- Assume the matter distribution as

$$a(t)\rho(r,t) = a(t)\rho_c(t) + \rho_w(r)$$

$$a(t)p_1(r,t) = a(t)p_{1c}(t) + p_{1w}(r)$$

$$a(t)p_2(r,t) = a(t)p_{2c}(t) + p_{2w}(r)$$

$$a(t)p_3(r,t) = a(t)p_{3c}(t) + p_{3w}(r)$$

Wormhole solution embedded in FLRW universe 2

- Einstein field equation

$$G^0{}_0 = -\frac{1}{4r}\{[(8\nu' + 4\nu'' r + \nu'^2 r)e^{-\nu+\zeta} - 3\dot{\nu}^2 r]e^{-\zeta}\}$$

$$G^0{}_1 = \frac{1}{2}(-2\dot{\nu}' + \dot{\nu}\zeta')e^{-\zeta}$$

$$G^1{}_1 = -\frac{1}{2r}\{[r(-2\ddot{\nu} + (-\frac{3}{2}\dot{\nu} + \dot{\zeta})\dot{\nu})e^{-\zeta+\nu} + 2\nu' + 2\zeta' + \zeta'\nu' r + \frac{1}{2}r\nu'^2]e^{-\nu}\}$$

$$G^1{}_0 = -\frac{1}{2}(-2\dot{\nu}' + \dot{\nu}\zeta')e^{-\nu}$$

$$G^2{}_2 = G^3{}_3 = \frac{1}{4r}\{[-2\zeta' - 2\nu' - 2\nu'' r - 2\zeta'' r - \zeta'^2 r]e^{-\nu} - 2r(-2\ddot{\nu} - \frac{3}{2}\dot{\nu}^2 + \dot{\zeta}\dot{\nu})e^{-\zeta}\}$$

For the case of ultra-static observer, $e^{\zeta} = 1$

- Exact solution is $\quad G^1{}_0 = 0 \qquad \dot{\nu}' = 0$

Scale factor

$$\nu(r,t) = \alpha(t) + \beta(r) \quad \text{or} \quad e^{\nu(t,r)} = e^{\alpha(t)} e^{\beta(r)}$$

Wormhole solution embedded in FLRW universe 3

- Solution for spatial part
  - Boundary conditions

$$e^{\beta(r)} = \begin{cases} (1+\frac{b_0^2}{4r^2})^2 & (k=0 \text{ or } r \to b_0/2) \\ (1+kr^2)^{-2} & (b_0 = 0 \text{ or } r \to \infty) \end{cases}$$

When we compare ${G_1}^1$ and ${G_2}^2$

$$2\kappa(p_2 - p_1) = [\nu'' - \frac{1}{r}\nu' - \frac{1}{2}(\nu')^2]e^{-\nu}$$

Matter distribution

$$[\beta'' - \frac{1}{r}\beta' - \frac{1}{2}(\beta')^2]e^{-\beta} = 2\kappa a(t)(p_2 - p_1) = 2\kappa(p_{2w} - p_{1w})$$

  - Inhomogeneous differential equation

$$\beta = \beta_c + \beta_w \qquad \beta_c = -2\ln(kr^2+1) \qquad \beta_w = 2\ln\left(1+\frac{b_0^2}{4r^2}\right)$$

$$e^{\nu(r,t)} = e^{\alpha(t)+\beta(r)} = \frac{e^{\alpha(t)}}{(kr^2+1)^2}\left(1+\frac{b_0^2}{4r^2}\right)^2$$

Wormhole solution embedded in FLRW universe 4

- Final metric

$$\boxed{ds^2 = dt^2 - \frac{a^2(t)}{(kr^2+1)^2}\left(1+\frac{b_0^2}{4r^2}\right)^2 (dr^2 + r^2 d\Omega^2).}$$

- Discussions
  - General solution (from wormhole to cosmological wormhole)

$$\left(1+\frac{b_0^2}{4r^2}\right) \quad \rightarrow \quad w(t_1, r_1)\left(1+\frac{q(t_1)}{4r^2}\right)$$

  - General solution (from FLRW to cosmological wormhole)

$$\frac{a(t)}{(kr^2+1)^2} \rightarrow \frac{a(t)}{(kr^2+1)^2} X(t,r)$$

  - Coupling term & Interaction: $\frac{1}{(1+kr^2)^2} \longleftrightarrow \frac{b_0^2}{4r^2}$
  - We can return to the original form with Inverse transformation

Apparent horizons 1

- With new coordinate

$$R \equiv a\left(\frac{1+b_0^2/4r^2}{1+kr^2}\right)r = a(t)A(r)$$

$$ds^2 = h_{ab}dx^a dx^b + R^2 d\Omega^2, \qquad h^{ab}\partial_a R \partial_b R = 0.$$

$$\Delta \equiv 1 - \frac{H^2R^2}{r(R)^2J(R)^2} = 0 \qquad \frac{\dot{a}}{a} = H, \quad \frac{A'}{A} = J = \frac{1}{r} - \frac{b_0^2/2r^3}{1+b_0^2/4r^2} - \frac{2kr}{1+kr^2}$$

$$\boxed{ds^2 = -\underbrace{\left(1 - \frac{R^2H^2}{r^2J^2}\right)}_{\Delta} F^2 dT^2 + \frac{1}{r^2J^2}\left(1 - \frac{R^2H^2}{r^2J^2}\right)^{-1} dR^2 + R^2 d\Omega^2}$$

Apparent horizons 2

- Case 1: $b(t)<R_H/2$

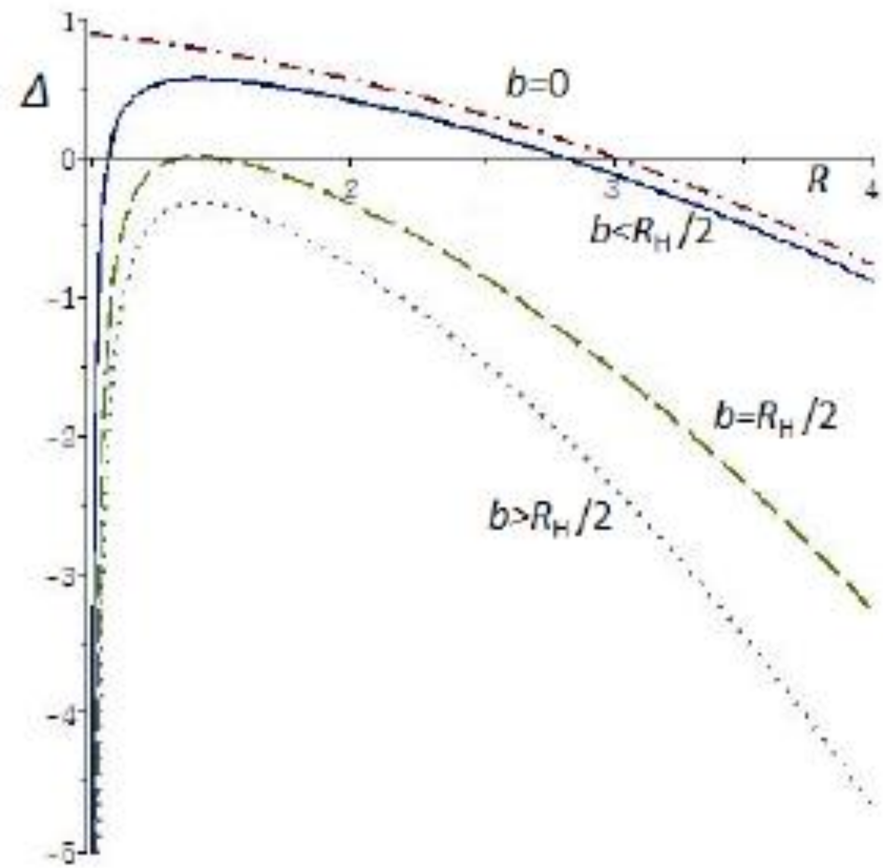
  Two horizons

- Case 2: $b(t)=R_H/2$

  1 horizon

- Case 1: $b(t)>R_H/2$

  0 horizon



Hawking temperature 2

- Radial null geodesic

$$\dot{R} = HR \pm \sqrt{(HR)^2 + (1 - b^2/R^2 - R^2/(R_+^2 + R_-^2)}$$

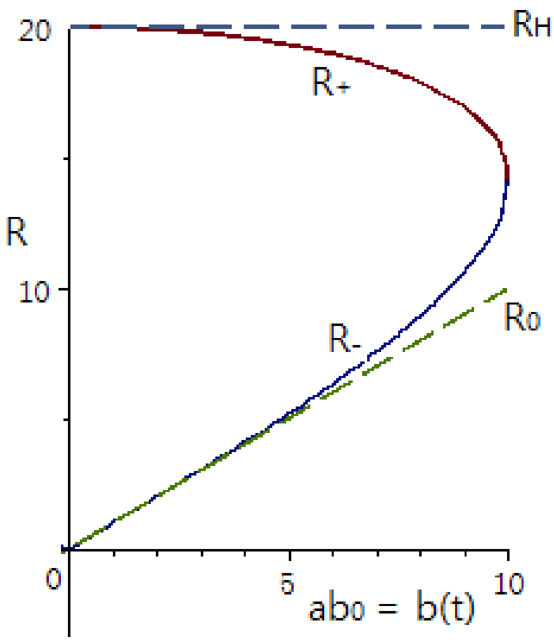
$$\mathrm{Im}\ \mathbf{S} = \mathrm{Im} \int_{R_i}^{R_f} p_R dR = \mathrm{Im} \int_{R_i}^{R_f} \int_0^{p_R} dp'_R dR \qquad \dot{R} = \frac{\partial \hat{H}}{\partial p_R} \frac{d\hat{H}}{dp_R} |_R$$

$$\begin{aligned} \mathrm{Im}\ \mathbf{S} &= \mathrm{Im} \int_{R_i}^{R_f} dR \int d\hat{H} \frac{1}{\dot{R}} \\ &= \mathrm{Im} \int_{R_i}^{R_f} dR \frac{\omega}{\dot{R}\sqrt{1 - b^2/R^2}} \\ &= -\omega \text{``}\mathrm{Im} \int_{R_i}^{R_f} \frac{dR}{\sqrt{1 - b^2/R^2}(\sqrt{(HR)^2 + (1 - b^2/R^2 - R^2/(R_+^2 + R_-^2))} - HR)} \\ &= \pi R_+ \omega. \end{aligned}$$

$$T = \frac{1}{2\pi R_+}$$

Apparent horizons 3

- $k = 0,$ $R = ar\left(1+\frac{b_0^2}{4r^2}\right) = ar + \frac{ab_0^2}{4r}$ $r = \frac{R}{2a} \pm \sqrt{\left(\frac{R}{2a}\right)^2 - \left(\frac{b_0^2}{4}\right)}$



$$\boxed{R_\pm = \frac{1}{\sqrt{2}H}[1 \pm \sqrt{1-(2b(t)H)^2}]^{1/2} < R_H = \frac{1}{H}}$$

$$R_0 < R_- < R_+ < R_H$$

$$1 - \frac{2M_{\rm MSH}}{R} = 1 - \frac{H^2R^2}{r^2J^2}$$

Misner-Sharp-Hermandez mass: $M_{\rm MSH} = H^2\frac{R^3}{2(1-b^2/r^2)} = \frac{4\pi}{3}R^3\rho_c[1-(b/R)^2]^{-1}$

Hawking Temperature 1

$$ds^2 = -\left(1 - \frac{R^2/(R_+^2 + R_-^2)}{1 - b^2/R^2}\right) dt^2 - \frac{2HR}{1 - b^2/R^2} dtdR + \frac{1}{1 - b^2/R^2} dR^2 + R^2 d\Omega^2$$

- Kodama vector

$$K^a \equiv -\varepsilon^{ab}\partial_b R = \sqrt{1 - \frac{b^2}{R^2}} \left(\frac{\partial}{\partial t}\right)^a$$

- Hamilton-Jacobi equation

$$\omega = -K^a \partial_a \mathrm{S} = -\sqrt{1 - \frac{b^2}{R^2}} \partial_t \mathrm{S}, \qquad k_R = \left(\frac{\partial}{\partial R}\right)^a \partial_a \mathrm{S} = \partial_R \mathrm{S}$$

$$\begin{aligned} \mathrm{Im}\,\mathrm{S} &= \mathrm{Im} \int \frac{-HR \pm \sqrt{H^2R^2 + \lambda[1 - \frac{m^2}{\omega^2}(1 - b^2/R^2)]}}{\sqrt{1 - b^2/R^2}(R_+^2 - R^2)(R^2 - R_-^2)} R^2(R_+^2 + R_-^2)\omega \\ &= \pi R_+ \omega. \end{aligned}$$

$$T = \frac{1}{2\pi R_+}$$

Summary & future work

- Exact solution of wormhole embedded in FRLW universe
  - Isotropic form of MT wormhole is found.
  - It satisfies Einstein's equation.
  - Coupling term and no interaction of global & local structure.
- Apparent horizons are found.
  - Two horizons – cosmological horizon, wormhole trapping horizon (wormhole throat size) < Hubble horizon
- Hawking temperature is calculated.