Defining tensor modes at 2nd order

Guillem Domenech

ITP, Heidelberg

Based on: GD & Sasaki, arXiv:1709.09804

February 13th, GC2018, YITP

Introduction

- In cosmology, what tensor modes do we observe?
  Gauge transformation: $T' \longrightarrow T + \underbrace{ST + SS}_{\text{Second order mixing!}}$

- Why do we care?

  - GWs from 2nd order scalar source: $T_2 \sim S_1S_1$
    From inflation: $T_1 \sim H$ and $S_1 \sim H/\sqrt{\varepsilon}$ => $T_2 > T_1$ if $\varepsilon < P_S$
    (e.g. PBH, maybe detectable by LISA)
    Baumann '07, Wands '07, Sasaki '12
  - **Large interaction** between S & T might be important for Temperature Bispectrum in modified gravity!
    $\langle S'S'T'\rangle = \langle SST\rangle + \langle SSSS\rangle + \ldots$
    GD, Hiramatsu, Lin, Sasaki, Shiraishi, Wang '17

What to do?

- Final goal: find what combination of metric perturbations appears in the **observables**, e.g. B-modes of CMB photons (at 2nd order).

Naruko et al ‘13

- First step: find **(correct) gauge invariant** variables.
  - Lagrangian approach (significantly involved!)

    perturb metric, expand, fix gauge (or not if you are brave enough), integrate out lapse and shift, compare gauges (Lie derivative), …

    Malik & Wands ‘09
  - **Hamiltonian approach** (suitable, like in gauge theories)

    lapse and shift Lagrange multipliers, hamiltonian/momentum constraints, poisson algebra, counting of d.o.f., canonical transformations, quantisation, …

    Langlois ’94
    (linear order)

More on Hamiltonian

$$\frac{\partial}{\partial t}\det\Upsilon = \Upsilon^{ij}\frac{\partial}{\partial t}\Upsilon_{ij} = 0$$

- 3+1 decomposition

$$ds^2 = -N^2dt^2 + \mathrm{e}^{2\Psi}\Upsilon_{ij}\left(dx^i + N^idt\right)\left(dx^j + N^jdt\right)$$

lapse trace d.o.f. traceless d.o.f. shift

- Hamiltonian

Canonical scalar field

$$\mathcal{H} = \Pi^{ij}\dot{\Upsilon}_{ij} + \Pi_\Psi\dot{\Psi} + \Pi_\Theta\dot{\Theta} - \mathcal{L} = N\mathcal{H}_N + N^i\mathcal{H}_i$$

Hamiltonian constraint Momentum constraint

- **Important:**
  Poisson Algebra

$$\boxed{\{H[N], H[M]\} = \int d^3x\,\mathrm{e}^{-2\Psi}\left(ND^iM - MD^iN\right)\mathcal{H}_i}$$

Lie derivative

$$\boxed{Q'_A = Q_A + \mathcal{L}_\epsilon Q_A = Q_A + \{Q_A, \epsilon^\mu\mathcal{H}_\mu\}}$$

Cosmological perturbations

- Expand around background $\alpha \equiv \ln a,\ \pi_\alpha = -6a^3 H,\ \pi_\phi = a^3\dot{\phi}$

$$N = 1 + A \quad \text{and} \quad N_i = a^{-2}\partial_i B$$

$$\Psi = \alpha(t) + \psi(t, \mathbf{x}) \qquad Y_{ij} \equiv [\ln \Upsilon]_{ij} = \gamma_{ij} + 2D_{ij}E$$

transverse-traceless

traceless derivative

- Resulting system

$$\mathcal{L} = \pi_{ij}\dot{\gamma}_{ij} + \underline{\pi_E \dot{E}} + \underline{\pi_\psi \dot{\psi}} + \underline{\pi_\varphi \dot{\varphi}} - \mathcal{H}_2 - \mathcal{H}_3$$
$$- A\left(\mathcal{H}_{N,1} + \mathcal{H}_{N,2}\right) - \partial_i B\left(\mathcal{H}_{i,1} + \mathcal{H}_{i,2}\right) + O(4)$$

3 scalars, 1 dynamical

- Infinitesimal coordinate transformation

$$\psi \to \psi + \left\{\psi, \epsilon^b \mathcal{H}_{b,1}\right\} = \psi - \frac{\pi_\alpha}{6a^3}A + \frac{1}{3}\Delta B$$

$$\gamma_{ij} \to \gamma_{ij} + \widehat{TT}_{ij}{}^{kl}\left\{Y_{kl}, \epsilon^b \mathcal{H}_{b,1}\right\} = \gamma_{ij}$$

Choice of GI variable?

shear

$$\boxed{\pi_E = \frac{2}{3}\Delta\Delta\left(\dot{E} - B\right)}$$

Easy exercise

$$\boxed{\pi_{\mathcal{E}} \equiv -\partial_i \mathcal{H}_i}$$

$$\boxed{\pi_{\delta\phi} \equiv \frac{a^3}{\pi_\phi}\mathcal{H}_{N1}}$$

- Remember: constraints are gauge invariant! Let's do a canonical transformation so that

$$(\psi, \varphi, E, \pi_\psi, \pi_\varphi, \pi_E) \longrightarrow (\omega, \varphi, E, \pi_\omega, \pi_{\delta\phi}, \pi_{\mathcal{E}})$$

- Can you guess what is the **canonical variable ω**?

$$\omega = \psi + \frac{\pi_\alpha}{6\pi_\phi}\varphi - \frac{1}{3}\Delta E$$

Mukhanov-Sasaki variable

- We implicitly chose the uniform-ϕ slicing (φ=0). We could have used the Newtonian slicing (π_E=0), etc.

- The resulting hamiltonian is

2nd order hamiltonian curvature perturbation

$$\mathcal{H}_2^{\rm red} = \mathcal{H}_2 + \delta\mathcal{H} = \boxed{2a^{-3}\pi_{ij}\pi_{ij} - \frac{a}{8}\gamma_{ij}\Delta\gamma_{ij} + \frac{\pi_\omega^2}{4a^3\epsilon} - a\epsilon\omega\Delta\omega} + \frac{3}{4a^3}\left(\Delta^{-1}\pi_{\mathcal{E}}\right)^2 + \frac{\pi_{\delta\phi}^2}{2a^3}$$
$$- \pi_{\mathcal{E}}\left(\frac{\Delta^{-1}\pi_\omega}{2a^3} + \frac{6a}{\pi_\alpha}\omega - \frac{a}{\pi_\phi}\varphi\right) + \pi_{\delta\phi}\left(\frac{\pi_\alpha}{6a^3\pi_\phi}\pi_\omega + \frac{\pi_\alpha}{2a^3}\varphi - \frac{a^3 V_\phi}{\pi_\phi}\varphi\right),$$

The rest are higher order terms —> H_3

Next order

- We have

$$\mathcal{L} = \pi_{ij}\dot{\gamma}_{ij} + \pi_\omega \dot{\omega} + \pi_{\mathcal{E}}\dot{E} + \pi_{\delta\phi}\dot{\varphi} - \mathcal{H}_2^{\mathrm{red}} - \mathcal{H}_3 \\ - A\left(\frac{\pi_\phi}{a^3}\pi_{\delta\phi} + \mathcal{H}_{N,2}\right) - B\left(\pi_{\mathcal{E}} - \partial_i \mathcal{H}_{i,2}\right) + O(4)$$

- 2nd order transformation

Bruni et al '97

$$q_a \to q_a + \mathrm{e}^{\pounds_\epsilon} q_a = q_a + \{q_a, \epsilon^\mu \mathcal{H}_{\mu,1+2}\} + \frac{1}{2}\{\{q_a, \epsilon^\mu \mathcal{H}_{\mu,2}\}, \epsilon^\nu \mathcal{H}_{\nu,1}\} + O(3)$$

- For tensor modes

Not good! Does not coincide with φ =0

$$\gamma_{ij}^{GI} \equiv \gamma_{ij} + \widehat{TT}_{ij}{}^{ab}\left\{\frac{9}{4a^4}\partial_{(a}\Delta^{-2}\pi_E\partial_{b)}\Delta^{-2}\pi_E - \frac{4}{\pi_\phi}\varphi\pi_{ab} + \partial_k\partial_{(a}E\partial_{b)}\partial_k E - \partial_k\gamma_{ab}\partial_k E\right\}$$

- Removing unwanted terms…

$$\boxed{\begin{aligned}\gamma_{ij}^{C} \equiv \gamma_{ij} + \widehat{TT}_{ij}{}^{ab}\Bigg\{ & -\frac{4}{\pi_\phi}\varphi\pi_{ab} + \partial_k\partial_{(a}E\partial_{b)}\partial_k E - \partial_k\gamma_{ab}\partial_k E + \frac{a^4}{\pi_\phi^2}\partial_a\varphi\partial_b\varphi \\ & - \frac{12a^4}{\pi_\phi\pi_\alpha}\partial_a\varphi\partial_b\omega - \frac{1}{\pi_\phi}\partial_a\varphi\partial_b\Delta^{-1}\pi_\omega\Bigg\},\end{aligned}}$$

Maybe the right canonical definition?

Let's check!

Reducing the Hamiltonian

- For the **Mukhanov-Sasaki variable** we find

$$\zeta \equiv \omega + \frac{\eta}{8\epsilon}\varphi^2 - \frac{1}{2\epsilon\pi_\phi}\pi_\omega\varphi - \partial_i\omega\partial_i E + \frac{\Delta^{-1}}{4}\left(\partial_k\gamma_{ij}\partial_i\partial_j\partial_k E\right) + \frac{\Delta^{-1}}{\pi_\phi}\left(\pi_{ij}\partial_i\partial_j\varphi\right)$$
$$+\frac{1}{4}\left(\delta_{ij} - \partial_i\partial_j\Delta^{-1}\right)\left(\partial_i\partial_k E\partial_j\partial_k E + \frac{a^4}{\pi_\phi^2}\partial_i\varphi\partial_j\varphi - \frac{12a^4}{\pi_\phi\pi_\alpha}\partial_i\varphi\partial_j\omega - \frac{1}{\pi_\phi}\partial_i\varphi\partial_j\Delta^{-1}\pi_\omega\right)$$

- Plugging in and solving the constraints…

$$\mathcal{H}_3^{\text{red}} = \mathcal{H}_3 + \delta_3\mathcal{H}_2^{\text{red}} + \delta\mathcal{H} = -6a^{-3}\zeta\pi_{ij}^C\pi_{ij}^C - \frac{\pi_\alpha}{3a^3\pi_\phi^2}\pi_\zeta\pi_{ij}^C\pi_{ij}^C - \frac{a\pi_\alpha}{64\pi_\phi^2}\pi_\zeta\partial_k\gamma_{ij}^C\partial_k\gamma_{ij}^C$$
$$+\frac{a}{8}\zeta\partial_k\gamma_{ij}^C\partial_k\gamma_{ij}^C + \pi_{ij}^C\partial_k\gamma_{ij}^C\partial_k\chi + \ldots$$

Exactly matches the action in the uniform-ϕ gauge

We have found canonical scalar and tensor gauge invariant variables!

Can we change gauges?

Newtonian potential

- Let’s go to the Newtonian gauge $\boxed{\theta \equiv \psi - \frac{1}{3}\Delta E + \frac{\pi_\alpha}{4a^4}\Delta^{-2}\pi_E}$

$$\gamma^C_{ij} = \gamma^N_{ij} + \widehat{TT}_{ij}{}^{ab}\left\{\frac{2}{\dot{\phi}^2}\left(\dot{\Phi} + H\Phi\right)\dot{\gamma}^N_{ab} - \boxed{\frac{4}{a^2\dot{\phi}^4}\partial_a\left(\dot{\Phi} + H\Phi\right)\partial_b\left(\dot{\Phi} + H\Phi\right)}\right\}$$

Contribution to SST bispectrum

Contribution to SSSS trispectrum

$$\zeta = \boxed{\left(1 + \frac{\pi_\alpha^2}{18\pi_\phi^2}\right)\Phi + \frac{\pi_\alpha}{12a^4}\Delta^{-1}\pi_\Phi} + \frac{\Delta^{-1}}{\pi_\phi}\left(\pi^N_{ij}\partial_i\partial_j\Xi\right) + \boxed{\Delta^{-1}\left(\gamma^N_{ij}\partial_i\partial_j\Phi\right)}$$
$$- \frac{\Delta^{-1}}{a^4}\left(\pi^N_{ij}\pi^N_{ij}\right) - \frac{\Delta^{-1}}{16}\left(\partial_k\gamma^N_{ij}\partial_k\gamma^N_{ij}\right) + O(\Phi^2)$$

Linear relation

- Second order GWs

$$\ddot{\gamma}^N_{ij} + 3H\dot{\gamma}^N_{ij} - a^{-2}\Delta\gamma^N_{ij} = \widehat{TT}_{ij}{}^{ab}\left\{4a^{-2}\partial_a\Phi\partial_b\Phi + \frac{8}{a^2\dot{\phi}^2}\partial_a\left(\dot{\Phi} + H\Phi\right)\partial_b\left(\dot{\Phi} + H\Phi\right)\right\}$$

Baumann ‘07, Wands ‘07, Sasaki ‘12

Summary

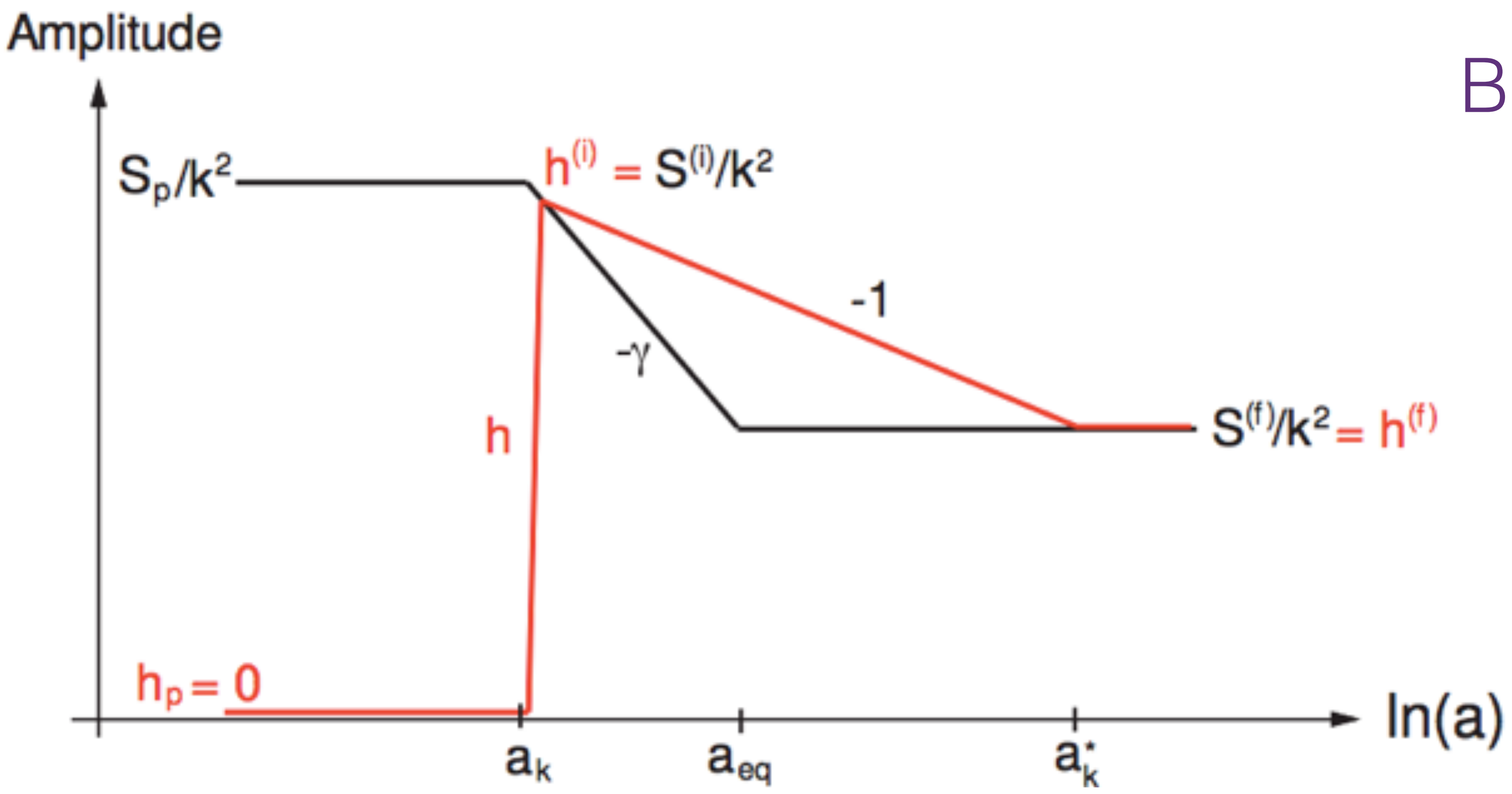
- Definition of observable **tensor modes ambiguous** at 2nd order in perturbation theory.

- As a first step we found the (**canonical**) scalar and **tensor gauge invariant** variables.

- The **Hamiltonian** approach is suitable for such task. Constraints, canonical transformations to **reduce the system**.

- We can easily move from gauge to gauge. Important for **2nd order GWs** and for **SST** interactions.

Scalar induced tensor modes



Baumann '07

$$k_{\rm eq} \equiv a_{\rm eq}\, H_{\rm eq} \approx 0.00974\,{\rm Mpc}^{-1}\,, \qquad M_{\rm eq} \equiv \frac{c^3}{2\,G\,H_{\rm eq}} \approx 6.67 \times 10^{50}\,{\rm g}$$

$$k_{\rm BH} = a\,H \simeq \frac{k_{\rm eq}}{2^{1/4}} \left(\frac{g_*}{g_{*{\rm eq}}}\right)^{-1/12} \left(\frac{M}{M_{\rm eq}}\right)^{-1/2} \qquad M \equiv c^3/(2\,G\,H(t))$$

$$\Omega_{GW} = \frac{1}{1+z_{eq}} P_h(k)$$